Title: Infinite Dimensional Optimisation Problems in Quantum Information â€” An operator algebra approach to the NPA Hierarchy

Speakers: Julius Zeiss

Series: Quantum Foundations

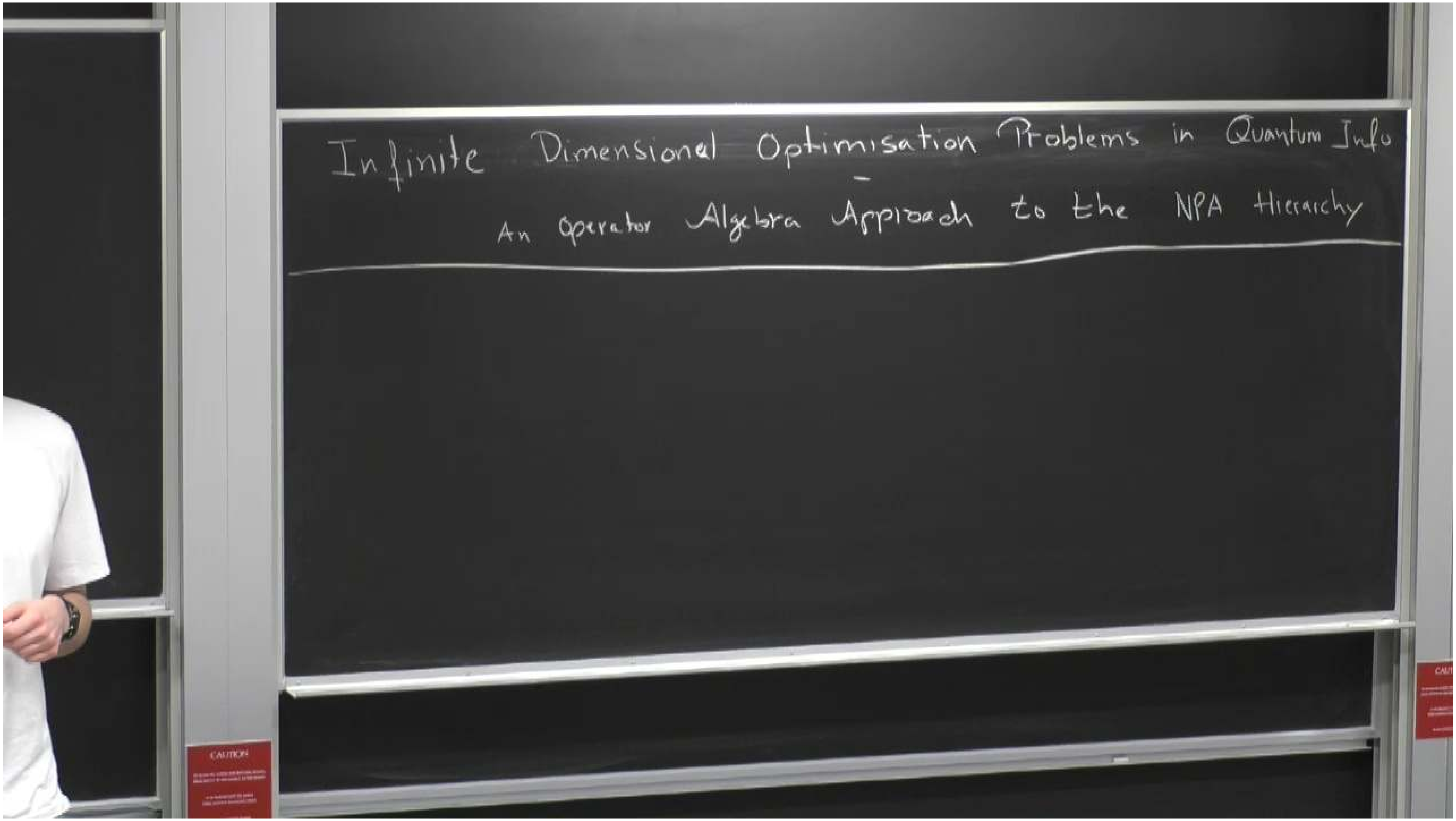
Date: February 14, 2023 - 2:00 PM

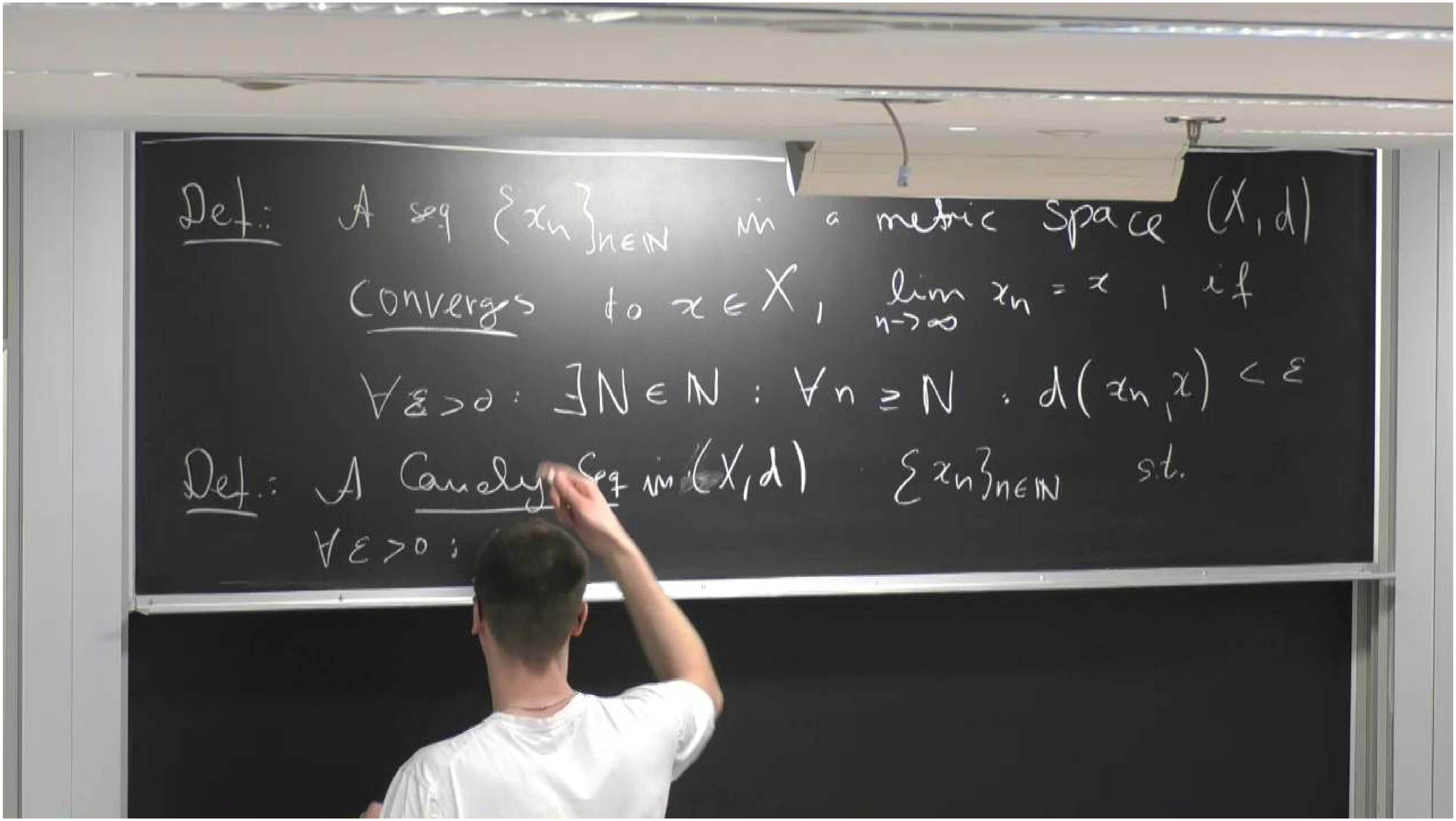
URL: https://pirsa.org/23020049

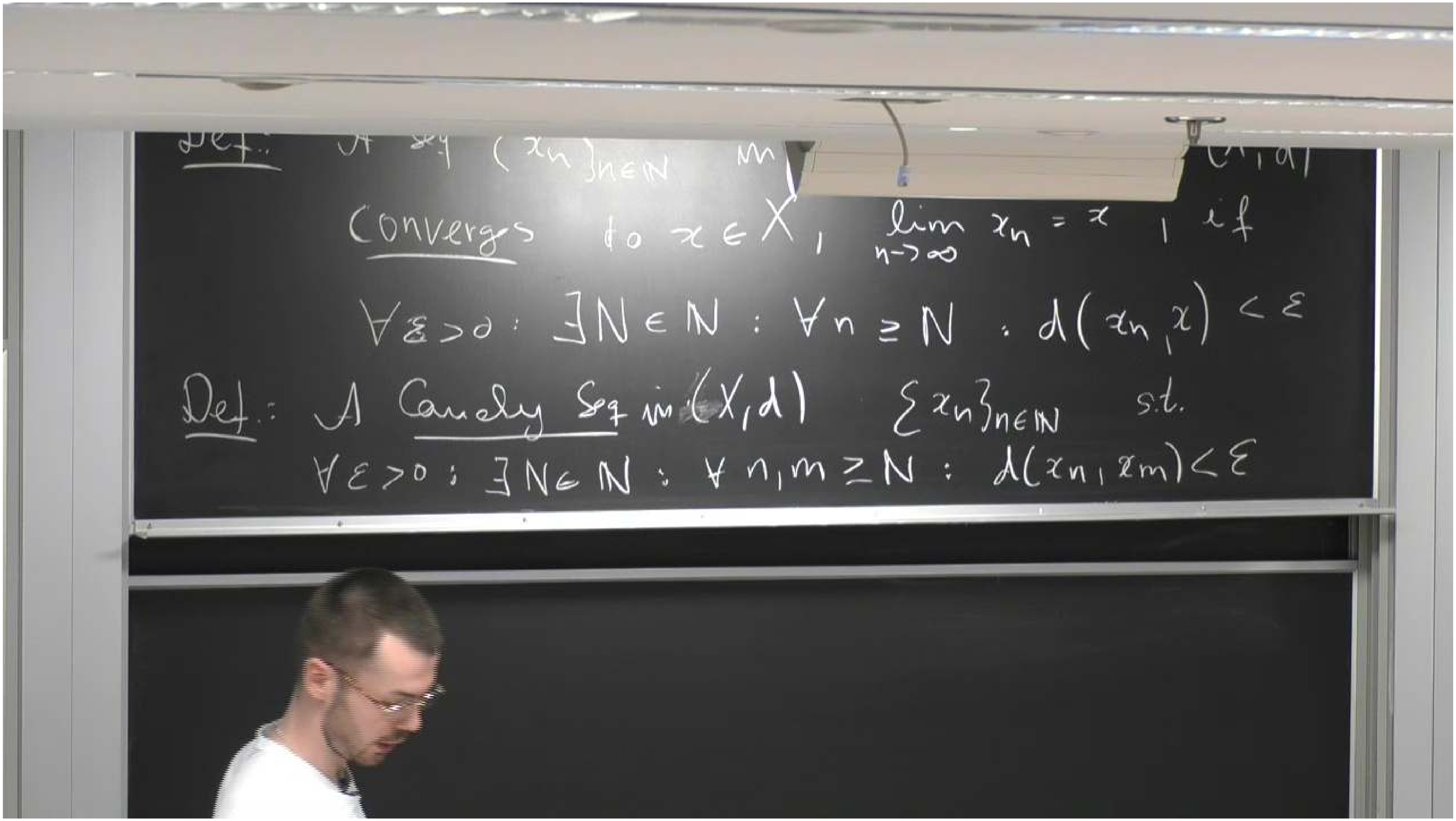
Abstract: The theory of polynomial optimisation considers a polynomial objective function subject to countable many polynomial constraints. In a seminal contribution Navascués, Pironio and Acín (NPA) generalised a previous result from Lassere, allowing for its application in quantum information theory by considering its non-commutative variant. Non-commutative variables are represented as bounded operators on potentially infinite dimensional Hilbert spaces. These infinite-dimensional non-commutative polynomials optimisation (NPO) problems are recast as a complete hierarchy of semidefinite programming (SDP) relaxations by a suitable partitioning of the underlying spaces.

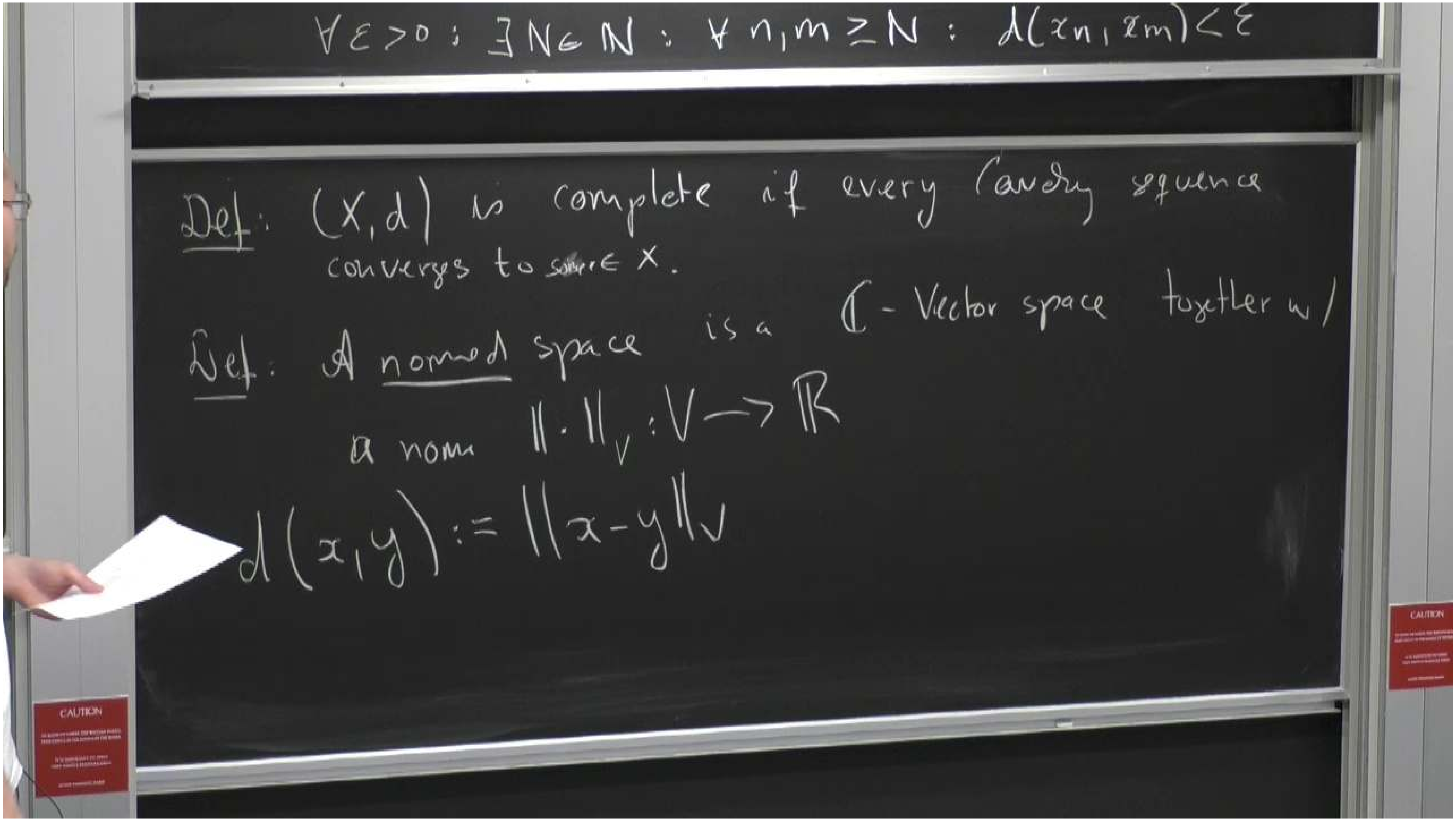
The reformulation into convex optimisation problems allows for numerical analysis. We focus on an operator theoretical approach to the NPA hierarchy and show its equiv-
alence to the original NPA hierarchy. To do so, we introduce the necessary mathematical preliminaries from operator algebra theory and semidefinite programming. We conclude by showing how certain relations on operators translate to SDP relaxations yielding drastically reduced problem sizes.

Zoom Link: https://pitp.zoom.us/j/98583295694?pwd=SlcvNG90RzFrODBKSHNaUi84bG9DZz09


Infinite Dimensional Optimisation Problems in Quantum Info
An Operator Algebra Approach to the NPA Hierarchy
CAUTION


Def.: A seq $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X, d)
converges to $x \in X$, $\lim_{n\to\infty} x_n = x$, if
$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) < \varepsilon$
Def.: A Cauchy seq in (X, d) $\{x_n\}_{n\in\mathbb{N}}$ s.t.
$\forall \varepsilon > 0 :$

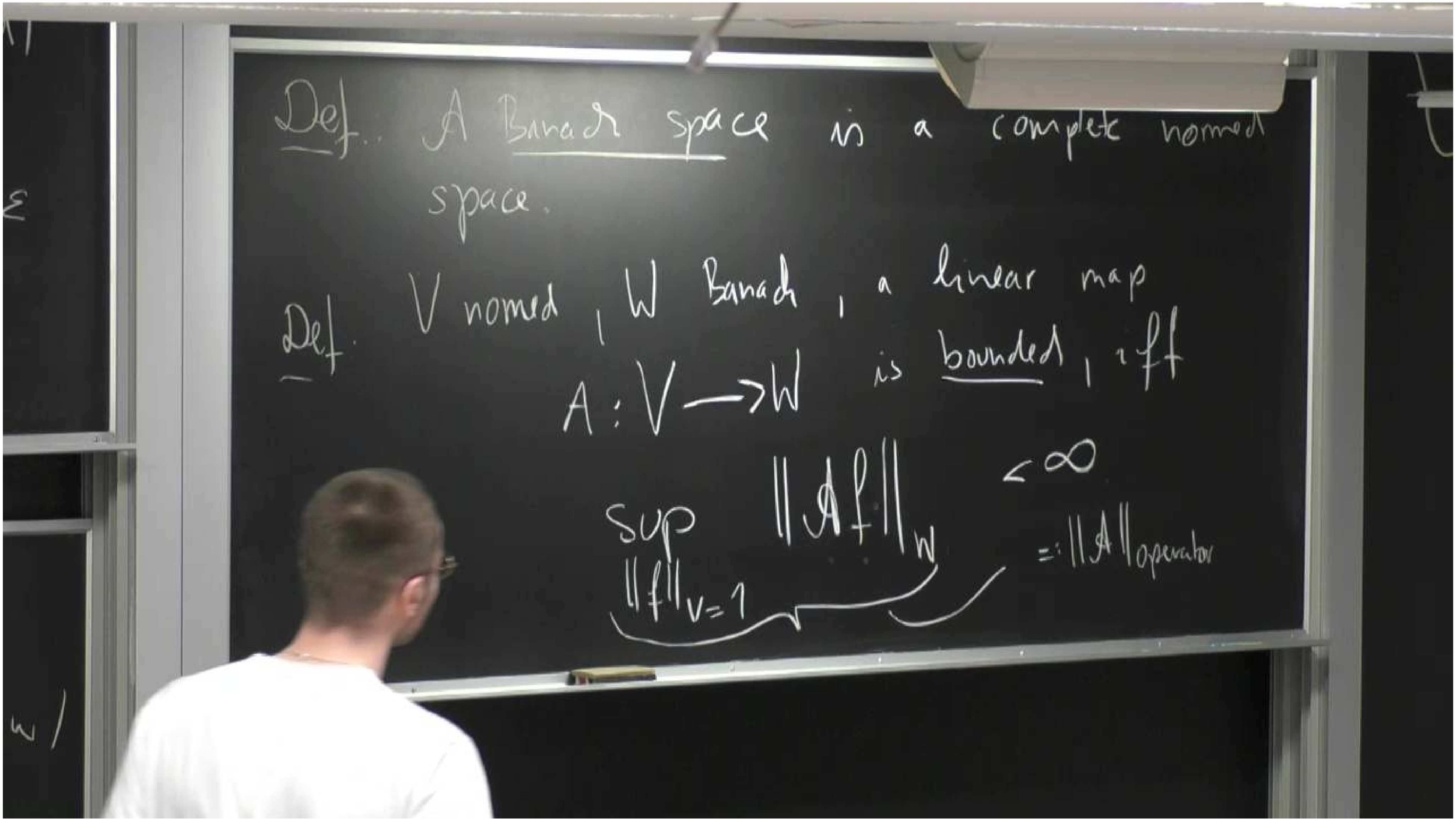

Def.: A seq $(x_n)_{n\in\mathbb{N}}$ in
Converges to $x \in X$, $\lim_{n\to\infty} x_n = x$, if
$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) < \varepsilon$
Def.: A Cauchy Seq in (X,d) $\{x_n\}_{n\in\mathbb{N}}$ s.t.
$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \varepsilon$


∀ε>0 : ∃N∈ℕ : ∀ n,m ≥ N : d(xn, xm) < ε
Def: (X,d) is complete if every Cauchy sequence converges to some ∈ X.
Def: A normed space is a ℂ - Vector space together w/ a norm ‖·‖V : V → ℝ
d(x,y) := ‖x-y‖V
CAUTION

Def.. A <u>Banach space</u> is a complete normed space.

Def. V normed, W Banach, a linear map $A : V \longrightarrow W$ is <u>bounded</u>, iff

$$\sup_{\|f\|_V = 1} \|Af\|_W < \infty$$


Def.. A Banach space is a complete normed space.
Def. V normed, W Banach, a linear map
$A : V \to W$ is bounded, iff
$\underbrace{\sup_{\|f\|_V = 1} \|Af\|_W}_{=: \|A\|_{operator}} < \infty$

Thm: $\left(\mathcal{L}(V, W), \cdot_{\mathbb{K}}, +_{\text{pointwise}}, \|\cdot\|_{op}\right)$

is a Banach space

BLT Thm: $A : \mathcal{D}_A \subseteq V \longrightarrow W$ such that $\mathcal{D}_A$ is dense in V, then we can uniquely extend A to $\hat{A} : V \longrightarrow W$ & $\|A\|_{\mathcal{D}_A} = \|\hat{A}\|_V$

Importantly, a (pre-) inner product induces a (semi-) norm via $\|v\|^2 := \langle v, v \rangle \quad \forall v \in V$

Def: A Hilbert space is a vector space equipped w/ a sequi-linear $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ s.t. it is complete in the norm it induces.

$\to$ Banach space

is complete in the norm it induces.

$\to$ Banach space

X normed, X^*

We can identify X w/ linear functionals on X^*

$\to$ induce the topology of X on X^*

<u>Def</u>: The weak-$*$ topology on X^* to X is the coarsest topology s.t.

$$T_x : X^* \longrightarrow \mathbb{K}$$
$$\phi \longrightarrow T_x(\phi) = \phi(x)$$

are continuous

$$T_x : X \longrightarrow \mathbb{K}$$ are

$$\phi \longrightarrow T_x(\phi) = \phi(x)$$

Def.: A net ϕ_λ in X^* is weak-* convergent if it is pointwise convergent

$$\phi_\lambda(x) \longrightarrow \phi(x) \quad \forall x \in X$$

Importantly, a (pre-) inner product induces

(semi-) norm via $\|v\|^2 := \langle v, v$

$\forall \varepsilon > 0 \; ; \; \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \varepsilon$

<u>Thm</u> Banach - Alaoglu:

If X is a Banach space then the closed unit ball in X^* is weak-* compact.

<u>Def</u>: (X, d) is complete if every Cauchy sequence converges to some $\in X$.

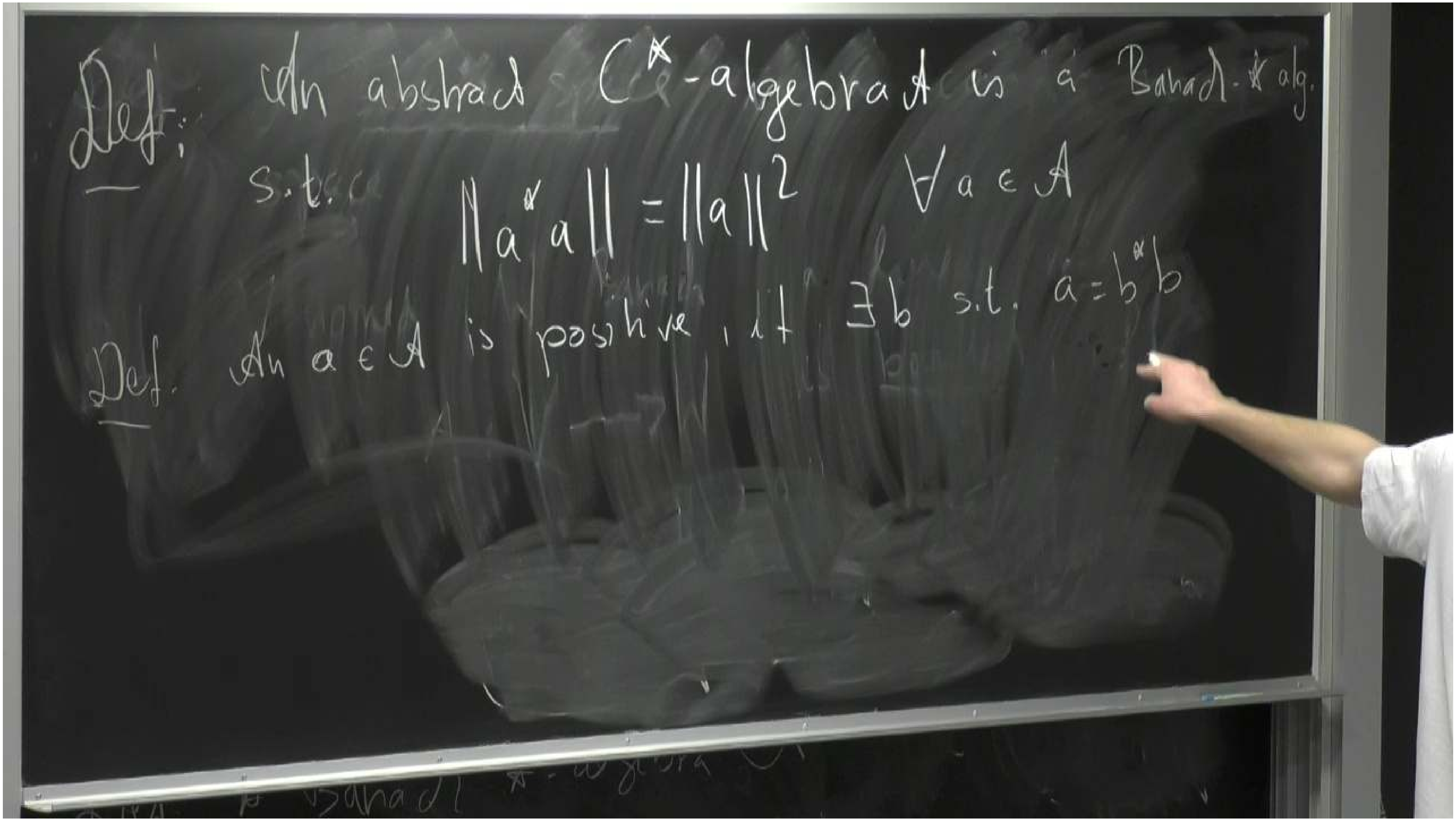
<u>Def</u>: A <u>normed</u> space is a $\mathbb{C}$ - Vector space together w/ a norm $\|\cdot\|_V : V \to \mathbb{R}$

$d(x, y) := \|x - y\|_V$

Def: A **Banach Algebra** $\mathcal{A}$ is a complex algebra which is a Banach space under norm which is submultiplicative

$$\|a \cdot b\| \leq \|a\| \|b\| \quad , \forall a, b \in \mathcal{A}$$

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution.


Def: An abstract C^*-algebra $\mathcal{A}$ is a Banach-$*$ alg.
s.t. $\|a^* a\| = \|a\|^2 \quad \forall a \in \mathcal{A}$
Def. An $a \in \mathcal{A}$ is positive, if $\exists b$ s.t. $a = b^* b$

Def: The weak-* topology on X^* to X is

Def: A repr. of a C^*-algebra $\mathcal{A}$ is a $*$-homomorphism

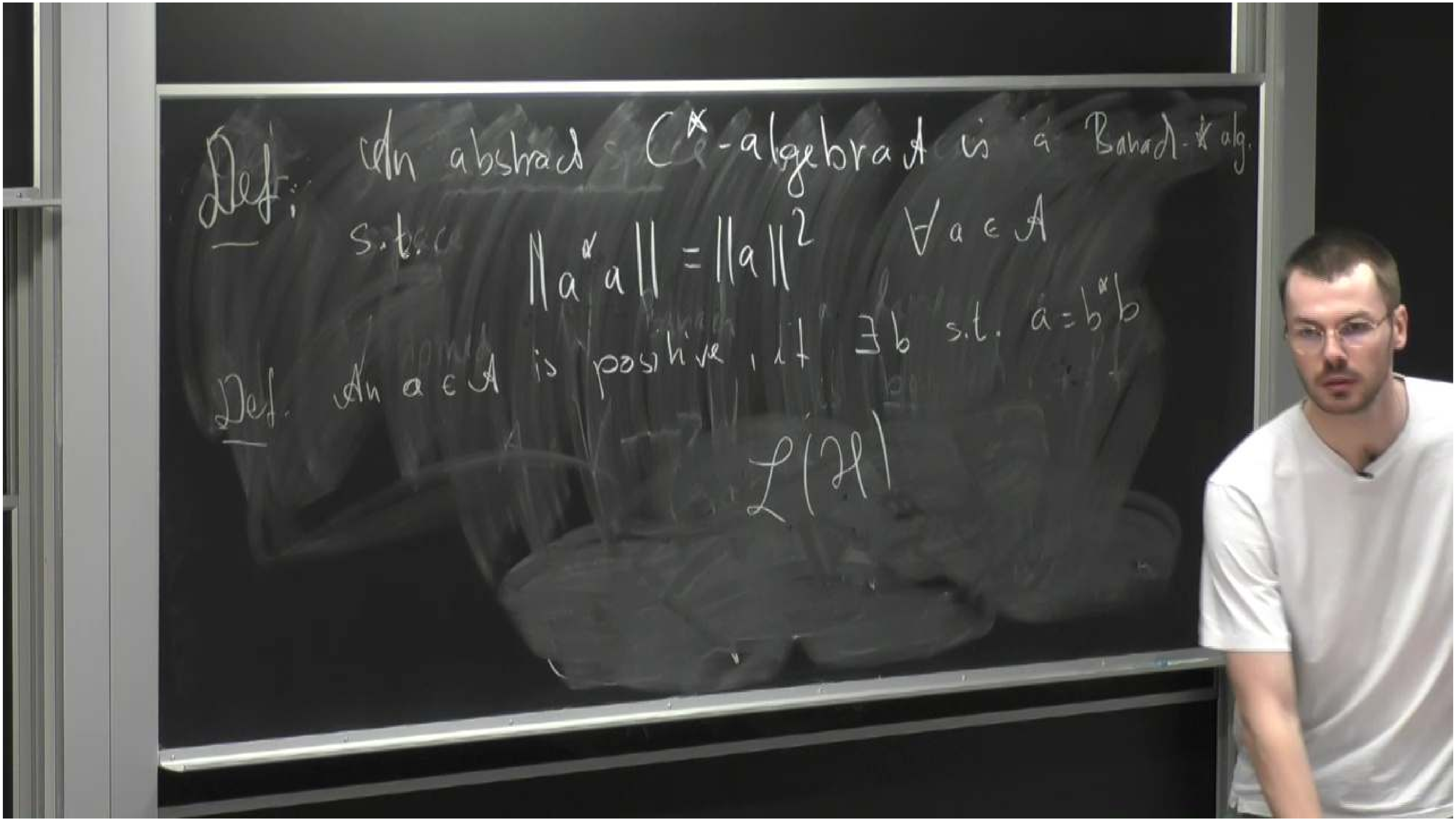
$$\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$$

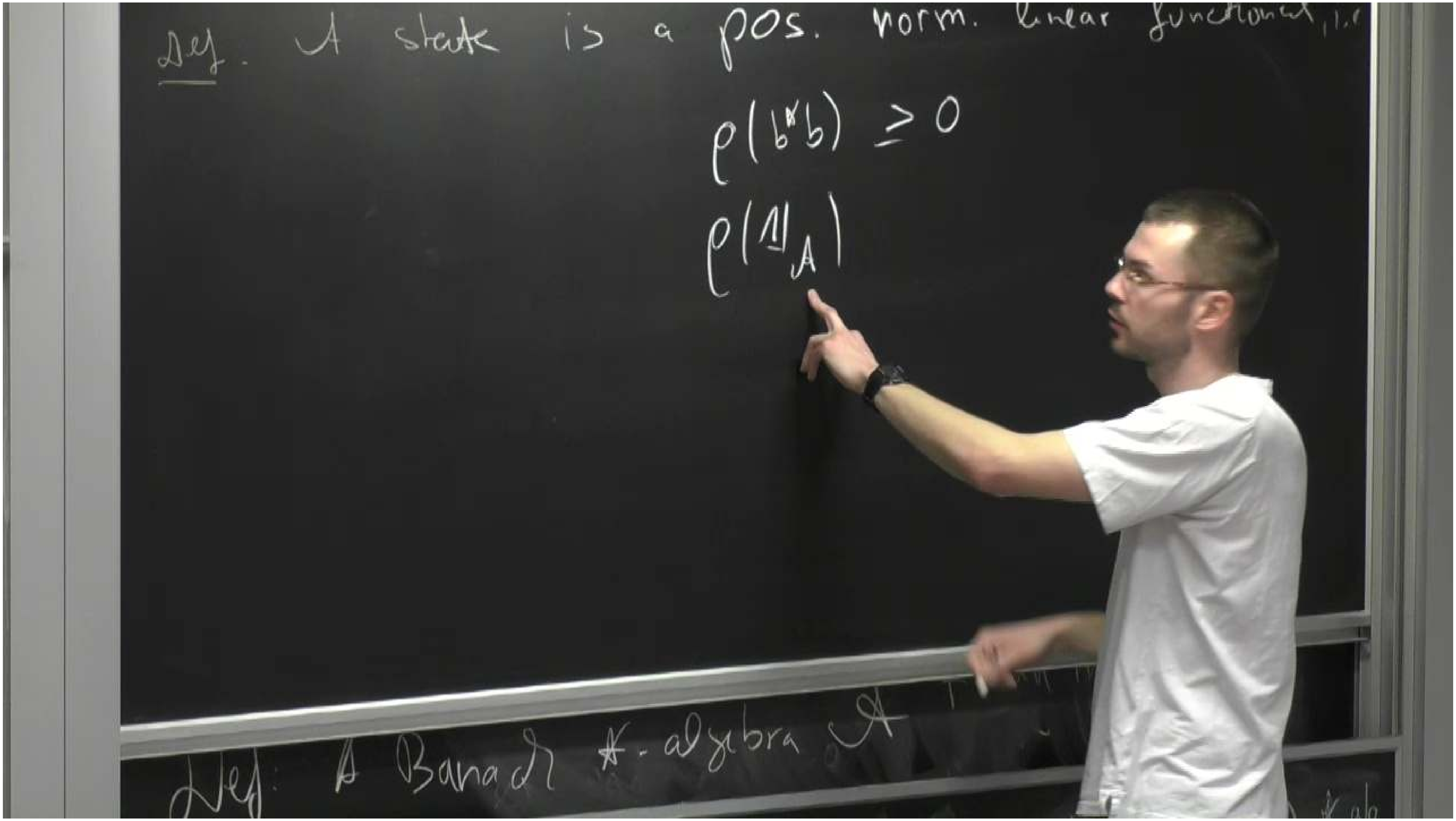
$$\|a\|_{\mathcal{A}} \geq \|\pi(a)\|_{B(\mathcal{H})}$$

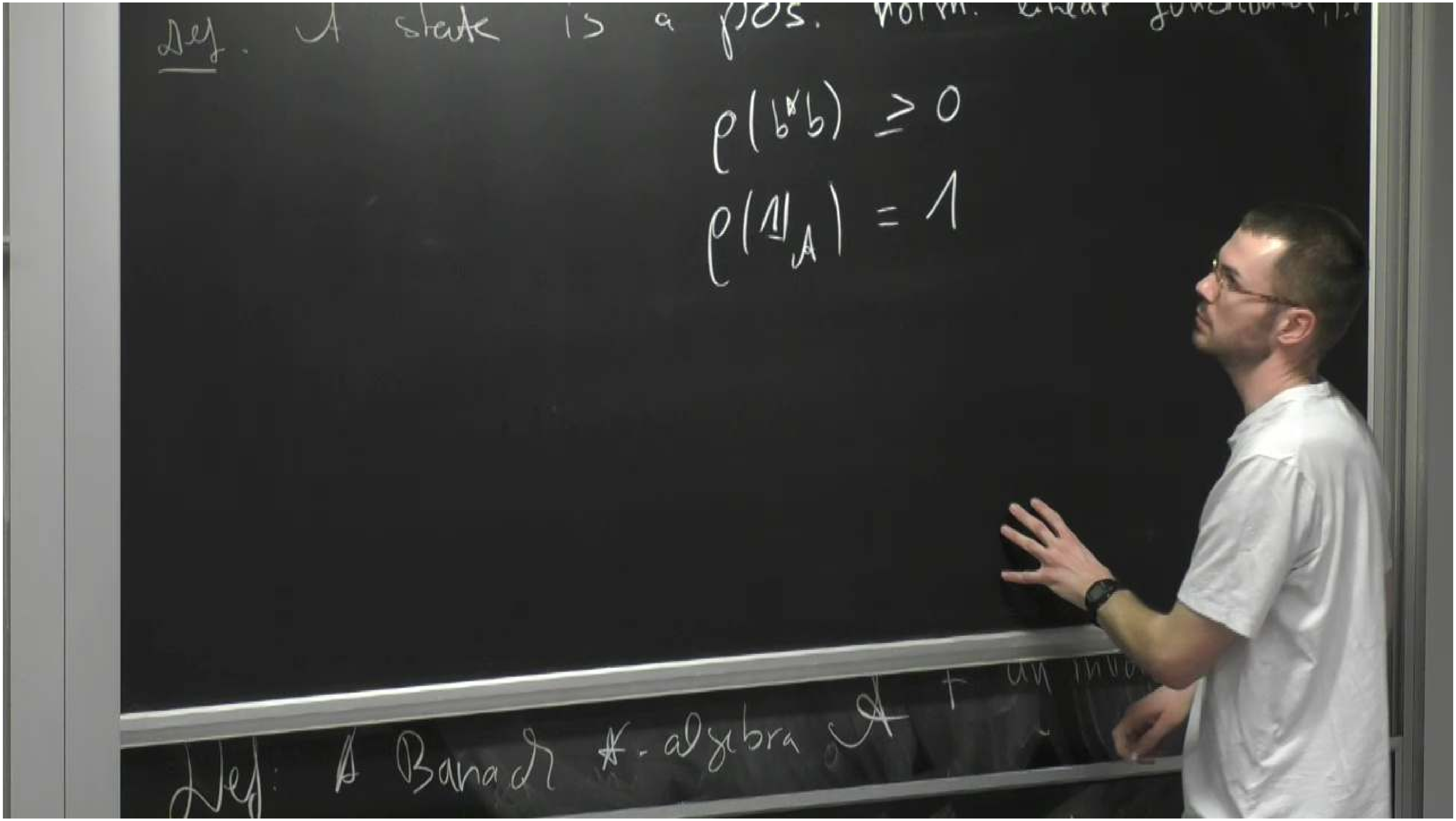
$$\|a\|_{\mathcal{A}} \geq \|\pi(a)\|_{B(\mathcal{H})}$$

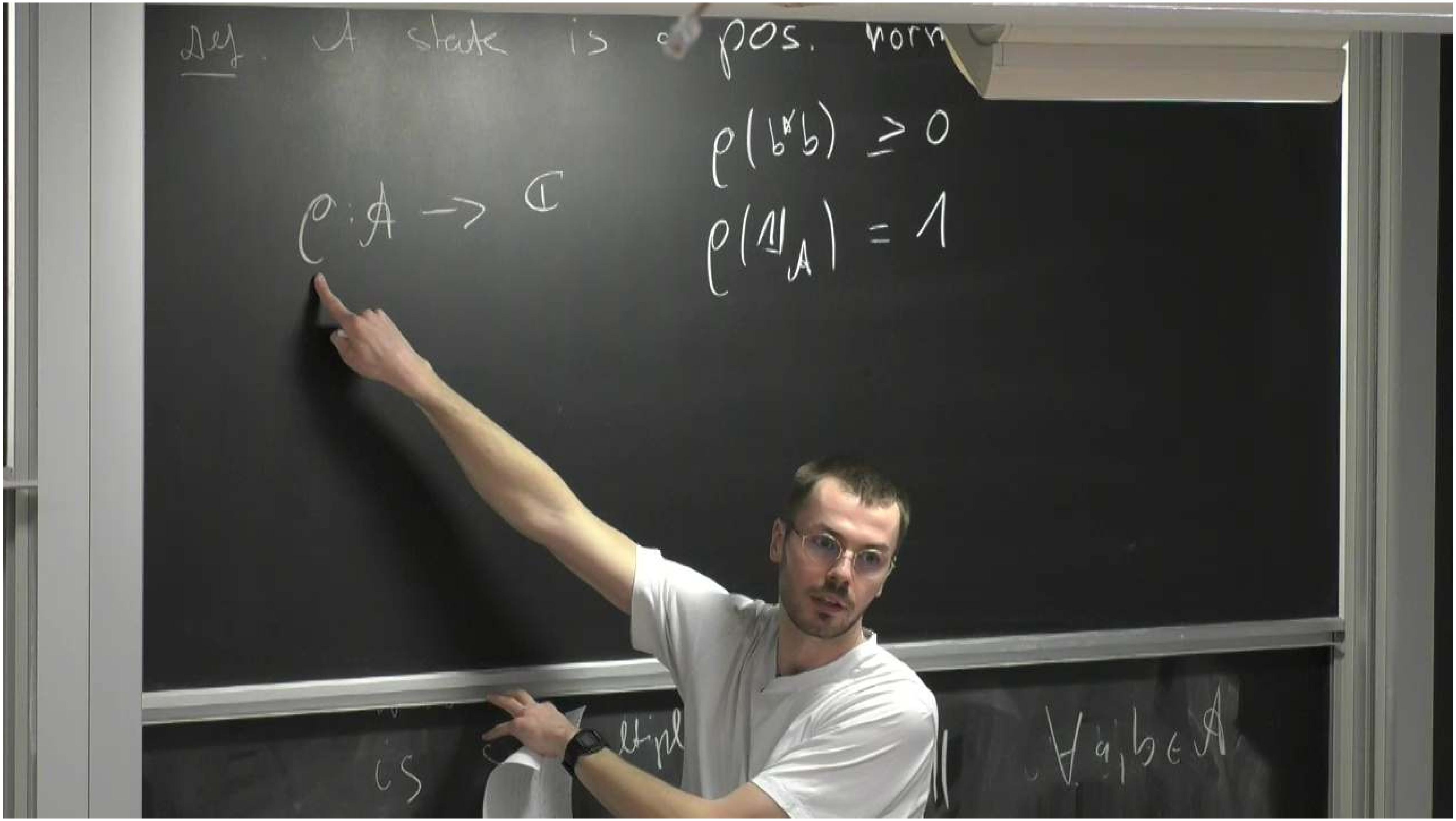
Def. π is cyclic iff $\exists\, \Omega \in \mathcal{H}$ s.t.

$$\overline{\pi(\mathcal{A})\Omega} = \mathcal{H}$$


Def: An abstract C*-algebra A is a Banach-*-alg.
s.t. ||a*a|| = ||a||^2 ∀ a ∈ A
Def. An a ∈ A is positive, if ∃ b s.t. a = b*b
L(H)


Def. A state is a pos. norm. linear functional, i.e.
ρ(b*b) ≥ 0
ρ(1_A)
Def: A Banach *-algebra A


Def. A state is a pos. norm. linear
ρ(b*b) ≥ 0
ρ(1_A) = 1
Def: A Banach *-algebra A


Def. A state is a pos. norm
ρ : A → ℂ
ρ(b*b) ≥ 0
ρ(1_A) = 1
is
∀ a,b ∈ A

$\rho(\mathbb{1}_{\mathcal{A}}) =$

$S(\mathcal{A}) \subseteq \mathcal{A}^*$

$\{ f \in \mathcal{A}^* : \sup_{u \in \mathcal{U}} | f(u) | \leq 1 \}$

Def: A Banach Algebra $\mathcal{A}$ is a complex alg... which is a Banach space under norm which

$\varphi(\mathbb{1}_{\mathcal{A}}) =$

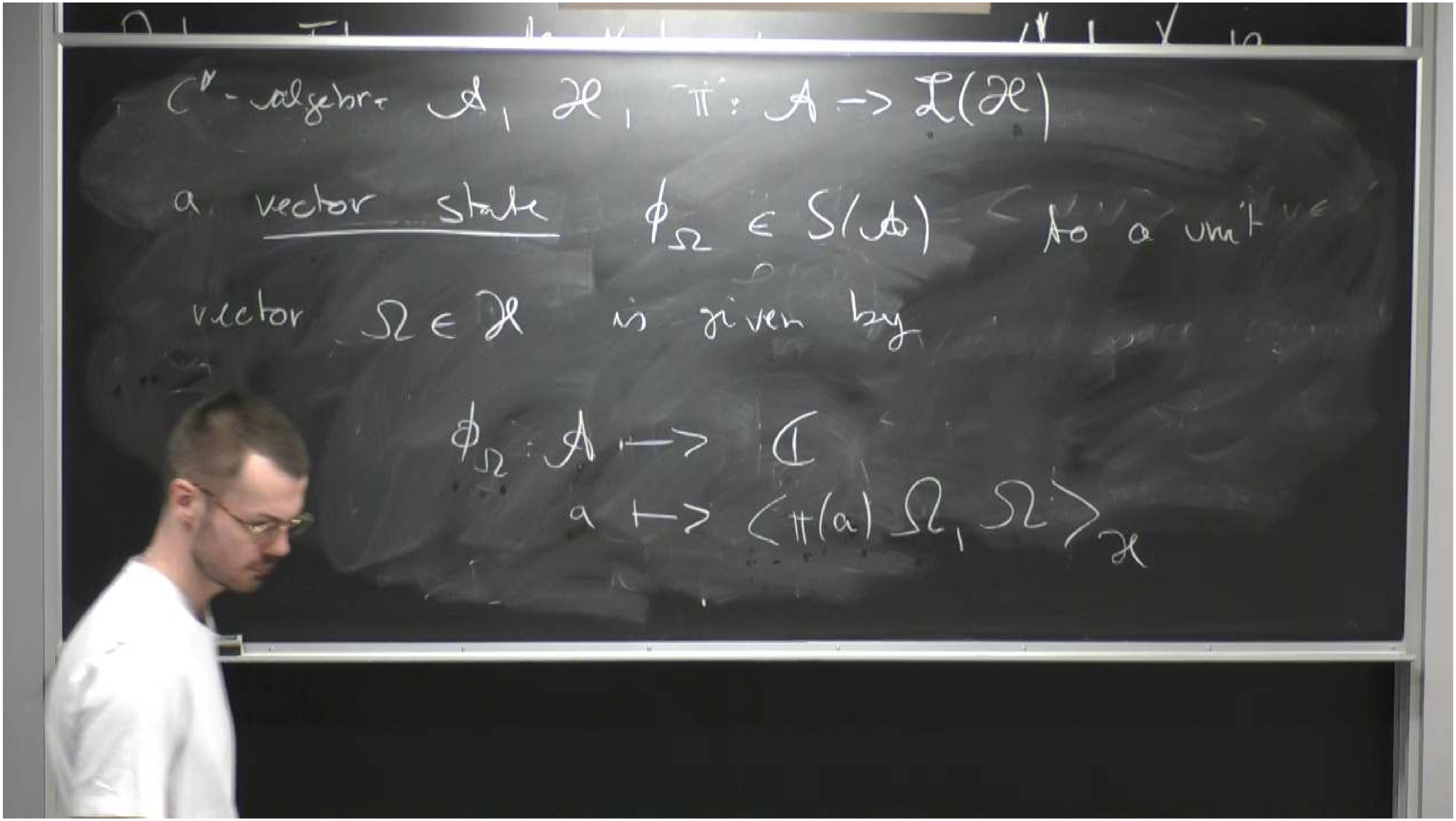
$S(\mathcal{A}) \subseteq \mathcal{A}^*$

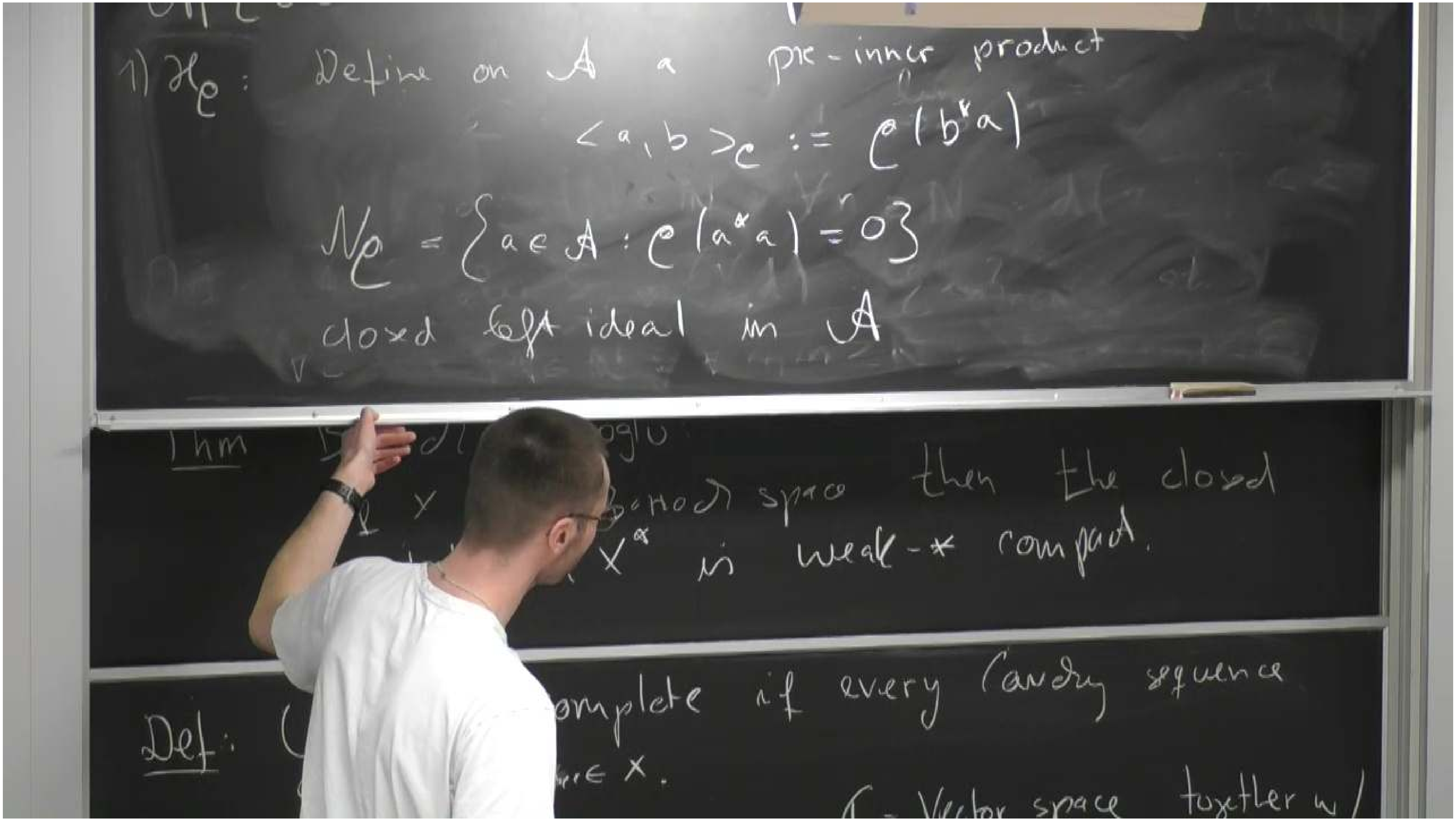
weak-* compact

$$\{ f \in \mathcal{A}^* : \sup_{u \in \mathcal{U}} | f(u) | \leq 1 \}$$

$$\cap \{ f \in \mathcal{A}^* : f(\mathbb{1}_{\mathcal{A}}) = 1 \}$$

Def A <u>Banach Algebra</u> $\mathcal{A}$ is a complex algebra which is a Banach space under norm which


C*-algebra $\mathcal{A}$, $\mathcal{H}$, $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
a vector state $\phi_{\Omega} \in S(\mathcal{A})$ to a unit vector $\Omega \in \mathcal{H}$ is given by
$\phi_{\Omega} : \mathcal{A} \to \mathbb{C}$
$a \mapsto \langle \pi(a)\Omega, \Omega \rangle_{\mathcal{H}}$


1) $\mathcal{H}_\rho$: Define on $\mathcal{A}$ a pre-inner product
$\langle a, b \rangle_\rho := \rho(b^* a)$
$N_\rho = \{a \in \mathcal{A} : \rho(a^* a) = 0\}$
closed left ideal in $\mathcal{A}$
Thm
Banach space then the closed
X^* is weak-* compact.
Def:
complete if every Cauchy sequence
$\in X$.
Vector space together w/

1) $\mathcal{H}_\varrho$: Define on $\mathcal{A}$ a p

$$\langle a, b \rangle_\varrho := \varrho(b^* a)$$

$$\mathcal{N}_\varrho = \{ a \in \mathcal{A} : \varrho(a^* a) = 0 \}$$

$(v, v) \geq 0$

$(v, v) = 0 \Leftrightarrow v = 0$

closed left ideal in $\mathcal{A}$

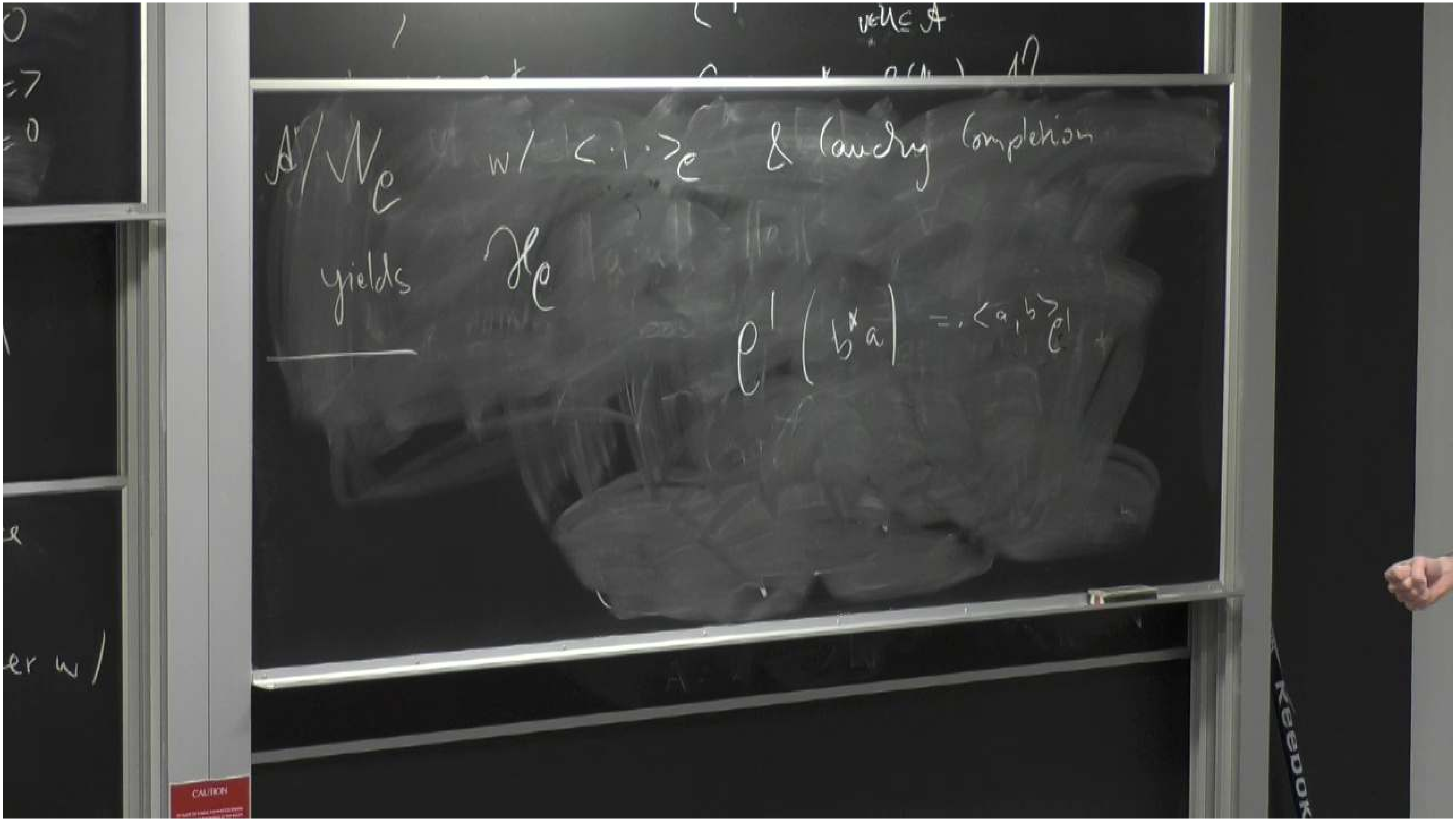
<u>Thm</u> Banach - Alaoglu:

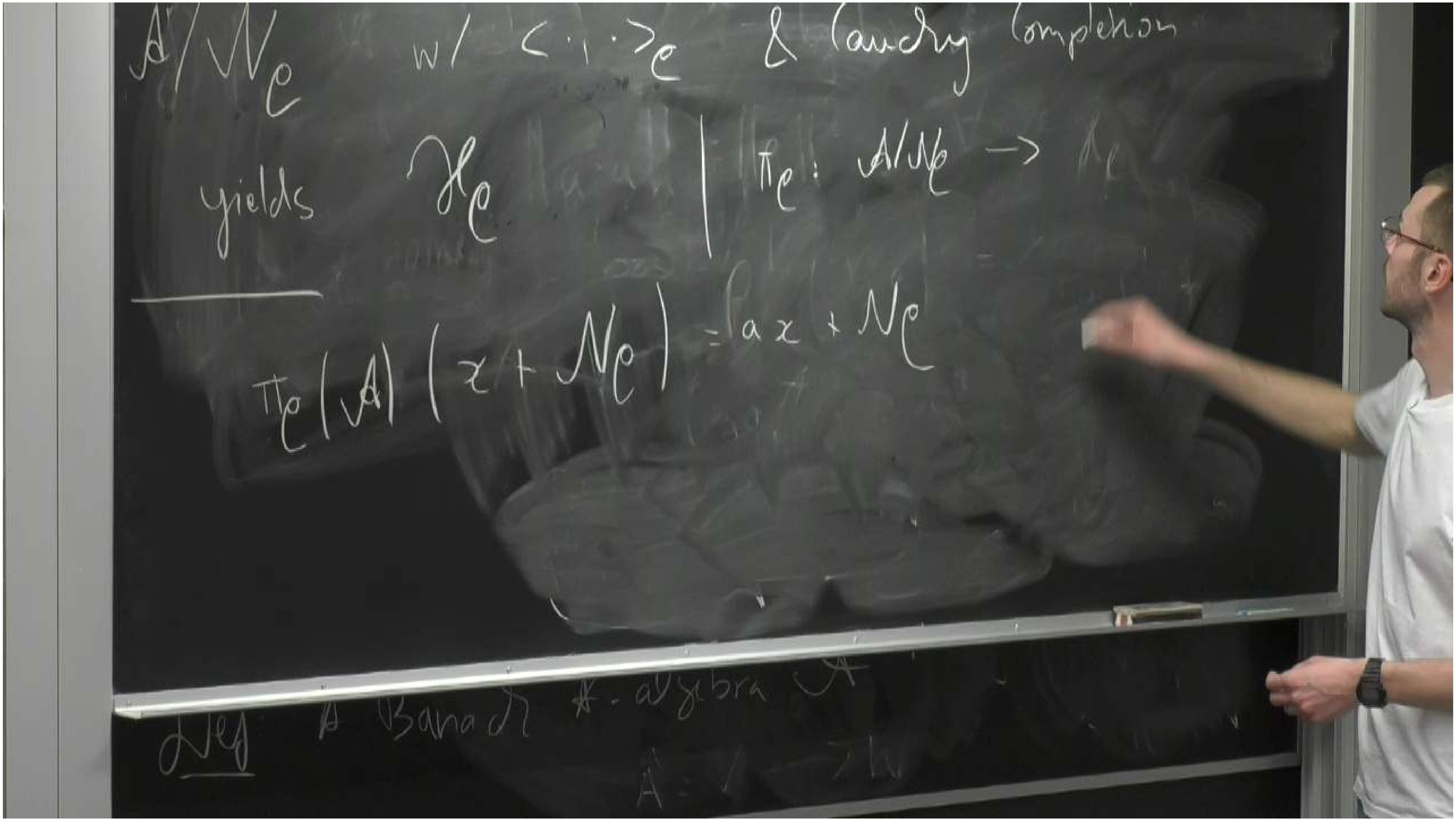
If X is a Banach space then the closed unit ball in X^* is weak-$*$ compact.

Def: (X, d) is complete if every Cauchy sequence

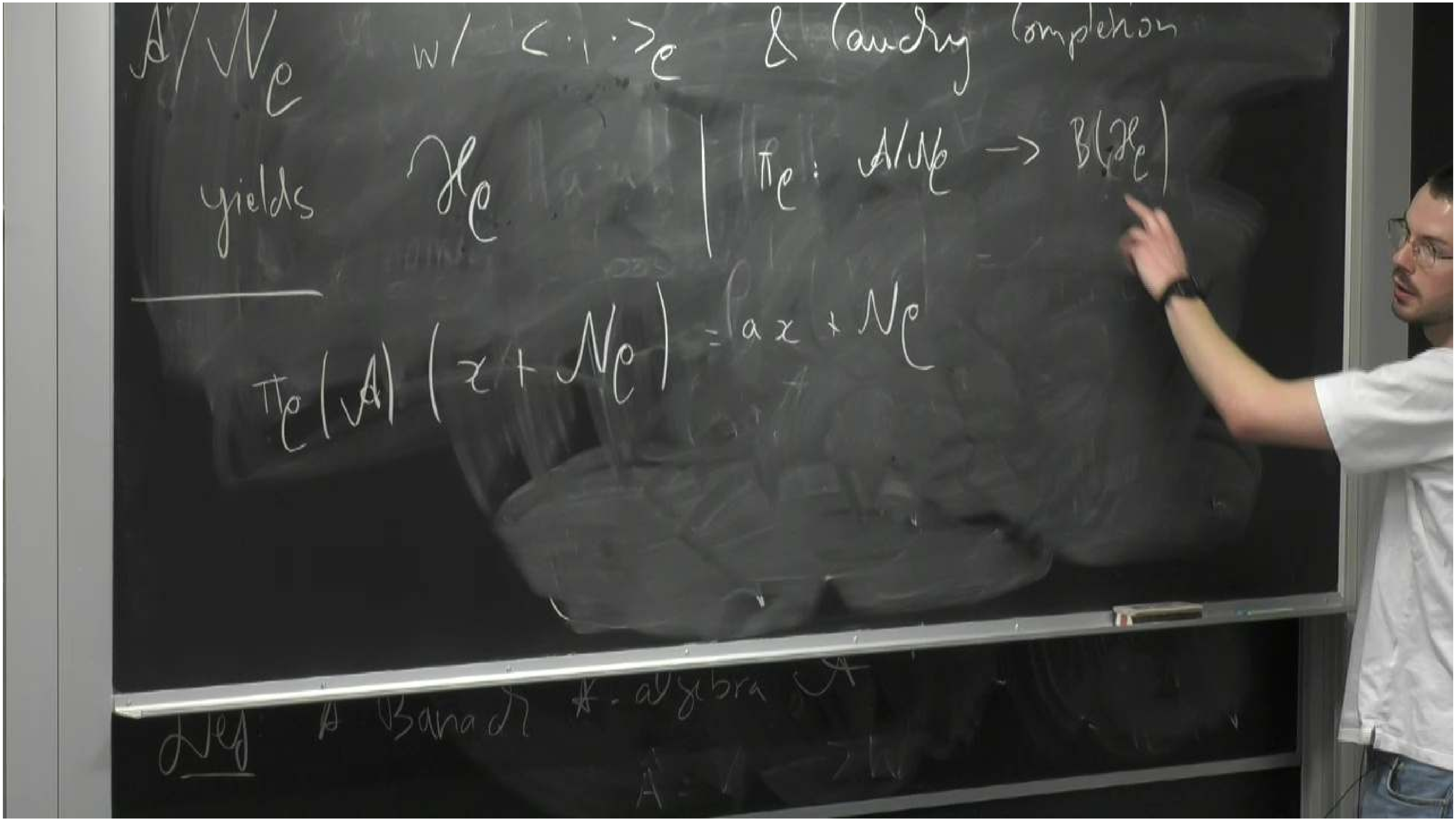
$\mathcal{A}/\mathcal{N}_\rho$ w/ $\langle \cdot , \cdot \rangle_\rho$ & Cauchy completion

yields $\mathcal{H}_\rho$


v∈U⊆A
A/N_ρ w/ ⟨·,·⟩_ρ & Cauchy completion
yields H_ρ
ρ'(b*a) =: ⟨a,b⟩_ρ'
er w/



$A/\mathcal{N}_\ell$ w/ $\langle\cdot,\cdot\rangle_\ell$ & Cauchy completion
yields $\mathcal{H}_\ell$
$\pi_\ell : A/\mathcal{N}_\ell \to$
$\pi_\ell(A)(x+\mathcal{N}_\ell) = ax + \mathcal{N}_\ell$
Def A Banach *-algebra A



$\mathcal{A}/\mathcal{N}_\varphi$ w/ $\langle\cdot,\cdot\rangle_\varphi$ & Cauchy completion
yields $\mathcal{H}_\varphi$
$\pi_\varphi : \mathcal{A}/\mathcal{N}_\varphi \to B(\mathcal{H}_\varphi)$
$\pi_\varphi(A)(x + \mathcal{N}_\varphi) = ax + \mathcal{N}_\varphi$
Def A Banach *-algebra $\mathcal{A}$

yields $\mathcal{H}_\rho$ | $\pi_\rho : \mathcal{A}/\mathcal{N}_\rho \to B(\mathcal{H}_\rho)$

$$\pi_\rho(\mathcal{A})(x + \mathcal{N}_\rho) = ax + \mathcal{N}_\rho$$

GNS cyclic vector is just image of $\mathbb{1}_\mathcal{A}$ in $\mathcal{H}_\rho$

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution.

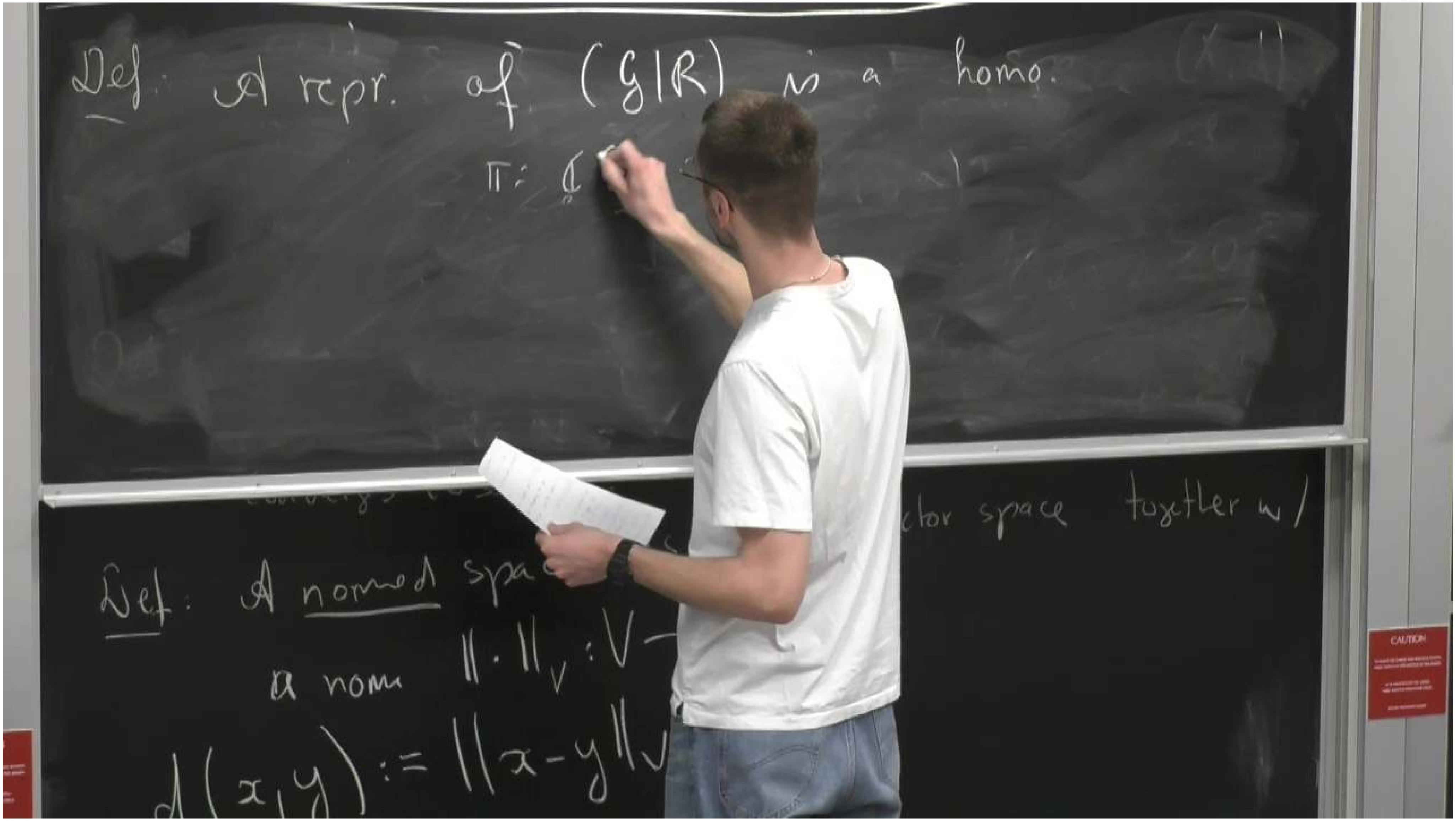
Def: free complex $*$-algebra $\mathbb{C}\langle \mathcal{G} \rangle$ generated by $\mathcal{G} = \{g_i\}_{i \in \mathbb{N}}$ is the set of finite complex linear comb. of words in g_i, g_i^*

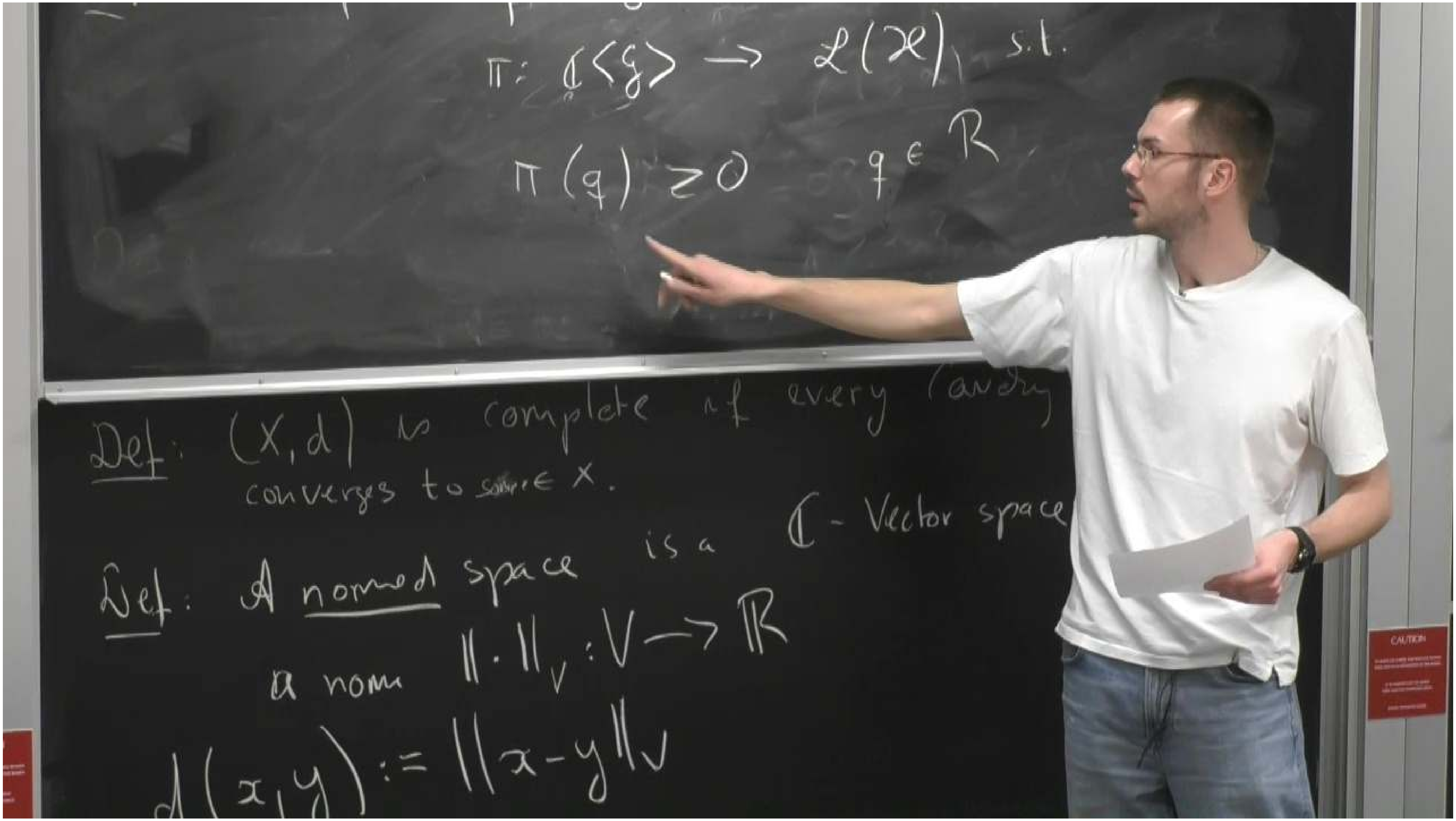
is the set of finite complex linear comb. of words in g_i, g_i^*

Choose a countable set $R \subset \mathbb{C}\langle \mathcal{G} \rangle$ corresponding to our constraints.

e.g. $g_i^2 - g_i = 0 \Rightarrow \{ g_i^2 - g_i, -g_i^2 + g_i \}$

$$\overline{\pi(\mathcal{A})\Omega} = \mathcal{H}$$


Def: A repr. of (G|R) is a homo.
π: G
ctor space together w/
Def: A normed spa
a norm ||·||_V : V
d(x,y) := ||x-y||_V


π: ℂ<g> → ℒ(ℋ) s.t.
π(q) ≥ 0 q ∈ R
Def: (X,d) is complete if every Cauchy
converges to some ∈ X.
ℂ - Vector space
Def: A normed space is a
a norm ‖·‖_V : V → ℝ
d(x,y) := ‖x-y‖_V

$\iff \langle \pi(q)\varphi, \varphi\rangle_{\mathcal{H}} \geq 0 \quad \forall \varphi \in \mathcal{H}$

If X is a Banach space then the closed unit ball in X^* is weak-$*$ compact.

Def: (X,d) is complete if every Cauchy sequence converges to some $\in X$.

Def: A normed space is a $\mathbb{C}$-Vector space together w/ a norm $\|\cdot\|_V : V \to \mathbb{R}$

$d(x,y) := \|x-y\|_V$

$$\|x\|_{C^*\langle g\rangle} = \sup\left\{ \|\pi(x)\|_{B(\mathcal{H})} : \pi \text{ is a repr. of } (g \mid R) \right\}$$

Need to ensure $\|x\|_{C^*\langle g\rangle} < \infty$ via R

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution,

$$\|x\|_{\mathbb{C}\langle g\rangle} = \sup\left\{ \|\pi(x)\|_{B(\mathcal{H})} : \pi \text{ is a repr. of } (g, R) \right\}$$

[Need to ensure $\|g_i\|_{\mathbb{C}\langle g\rangle} < \infty$ via R]

$g_i \in g$

GNS

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution,

$\| a \cdot b \| = \| \quad$

$\|x\|_{\mathbb{C}\langle g\rangle} = \sup\{ \|\pi(x)\|_{B(\mathcal{H})} \ldots (g \mid R) \}$

[Need to ensure $\|g_i\|_{\mathbb{C}\langle g\rangle} < \infty$ via R]

$g_i \in g$

Cauchy Compl of $\mathbb{C}\langle g\rangle$ w.r.t. $\|\cdot\|_{\mathbb{C}\langle g\rangle}$

$\longrightarrow$ Universal C^*-Algebra $C^*(G \mid R)$

$\forall a, b \in \mathcal{A}$

$\|a \cdot b\| \leq \|a\| \|b\|$

+ an involution.

Def: A Banach $*$-algebra $\mathcal{A}$

Thm Banach-Alaoglu:

then the closed

NPO: $p^* = \min_{\rho \in S(C^*(G|R))} \rho(p)$

NPO from NPA $p^* = \min_{\pi, \Omega} \langle \pi(p)\Omega, \Omega \rangle$

s.t. $\pi(q) \geq 0 \quad q \in R$