Title: Infinite Dimensional Optimisation Problems in Quantum Information â€" An operator algebra approach to the NPA Hierarchy

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Series: Quantum Foundations

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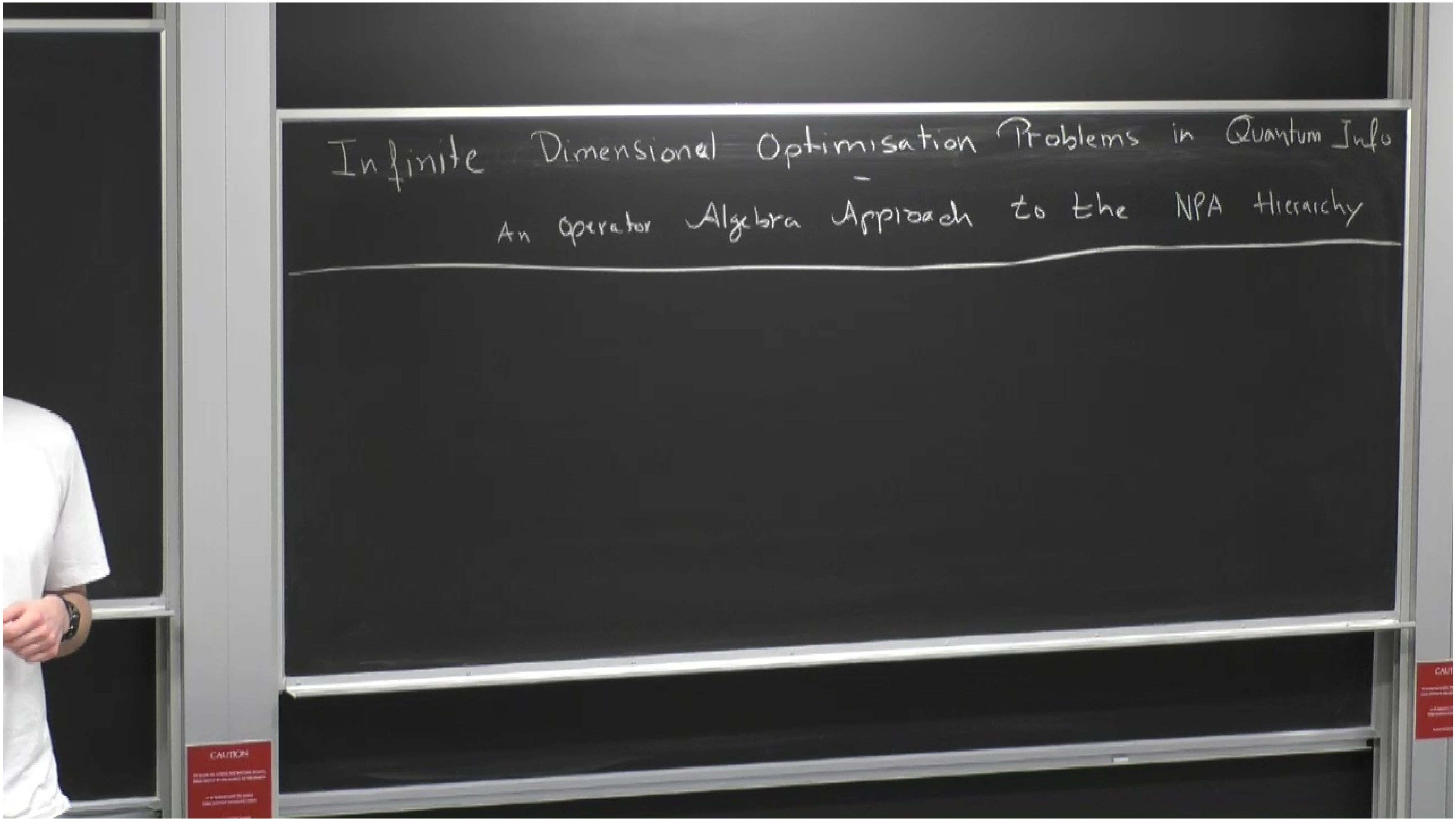
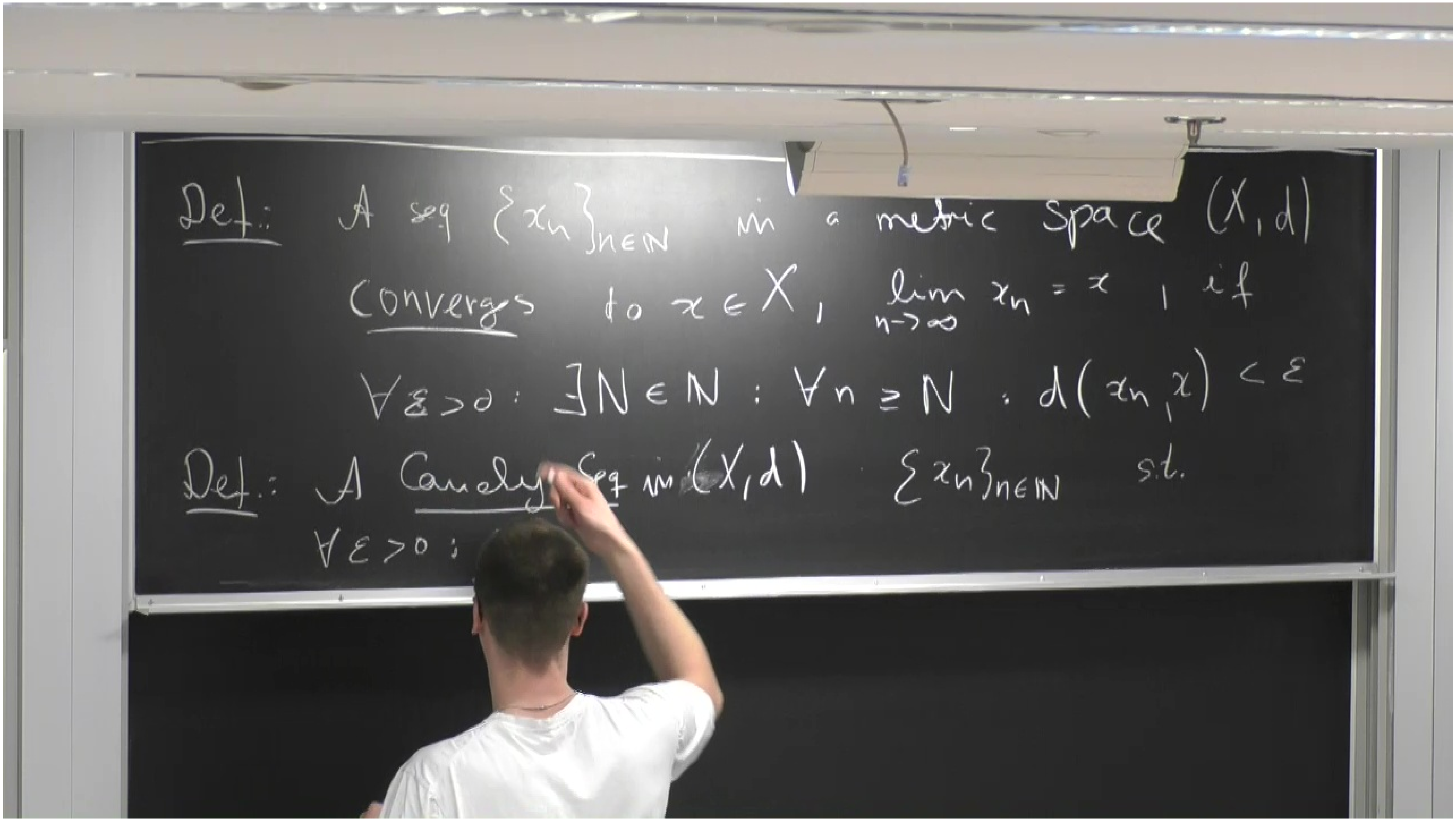
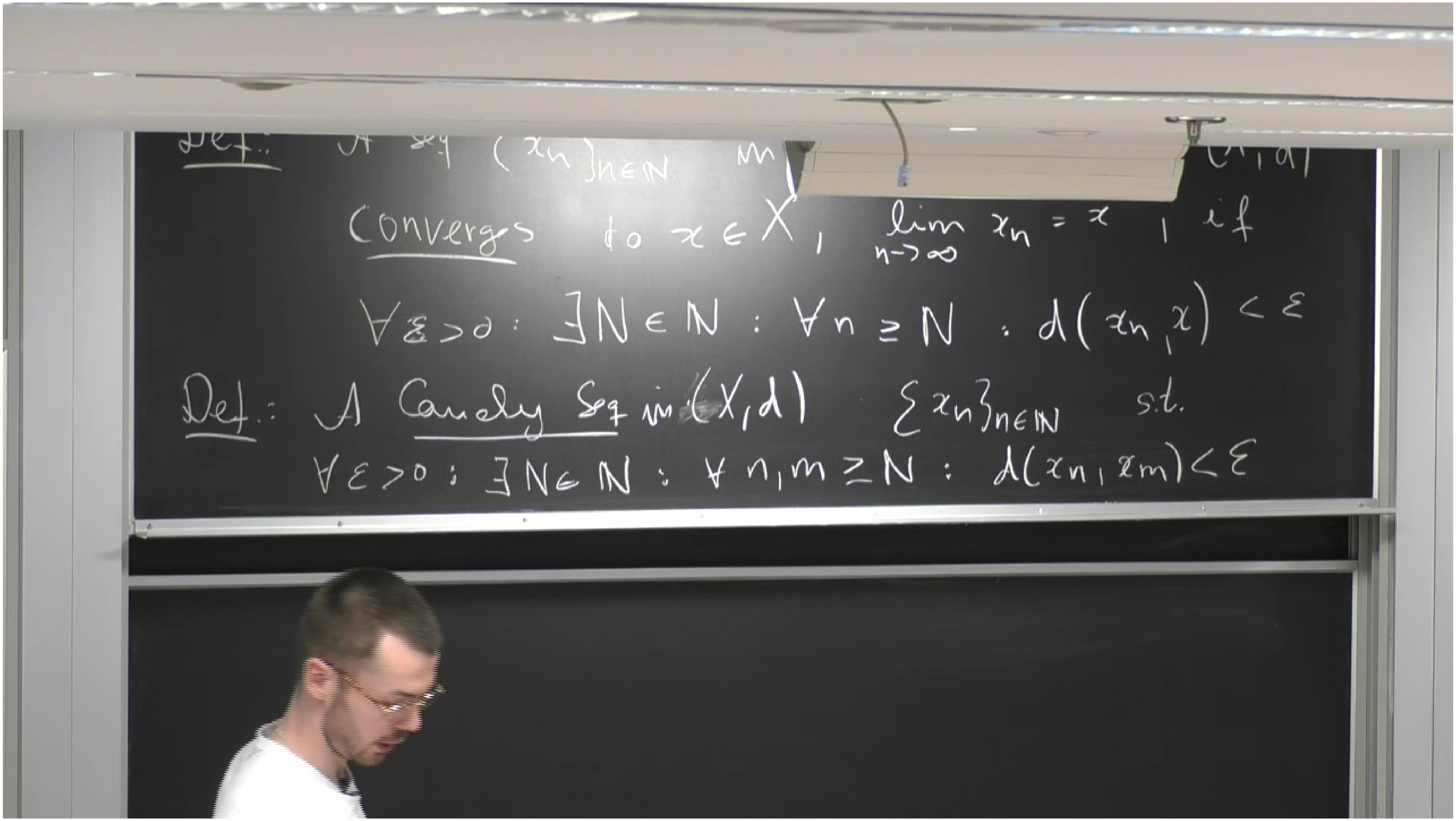
Abstract: The theory of polynomial optimisation considers a polynomial objective function subject to countable many polynomial constraints. In a seminal contribution Navascués, Pironio and Acín (NPA) generalised a previous result from Lassere, allowing for its application in quantum information theory by considering its non-commutative variant. Non-commutative variables are represented as bounded operators on potentially infinite dimensional Hilbert spaces. These infinite-dimensional non-commutative polynomials optimisation (NPO) problems are recast as a complete hierarchy of semidefinite programming (SDP) relaxations by a suitable partitioning of the underlying spaces.

The reformulation into convex optimisation problems allows for numerical analysis. We focus on an operator theoretical approach to the NPA hierarchy and show its equiv-
alence to the original NPA hierarchy. To do so, we introduce the necessary mathematical preliminaries from operator algebra theory and semidefinite programming. We conclude by showing how certain relations on operators translate to SDP relaxations yielding drastically reduced problem sizes.

Zoom Link: https://pitp.zoom.us/j/98583295694?pwd=SlcvNG90RzFrODBKSHNaUi84bG9DZz09


Infinite Dimensional Optimisation Problems in Quantum Info
An Operator Algebra Approach to the NPA Hierarchy


Def.: A seq $\{x_n\}_{n \in \mathbb{N}}$ in a metric space (X, d)
converges to $x \in X$, $\lim_{n \to \infty} x_n = x$, if
$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : d(x_n, x) < \varepsilon$
Def.: A Cauchy seq in (X, d) $\{x_n\}_{n \in \mathbb{N}}$ s.t.
$\forall \varepsilon > 0 :$



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<u>Def</u>: (X, d) is complete if every Cauchy sequence converges to some $\in X$.

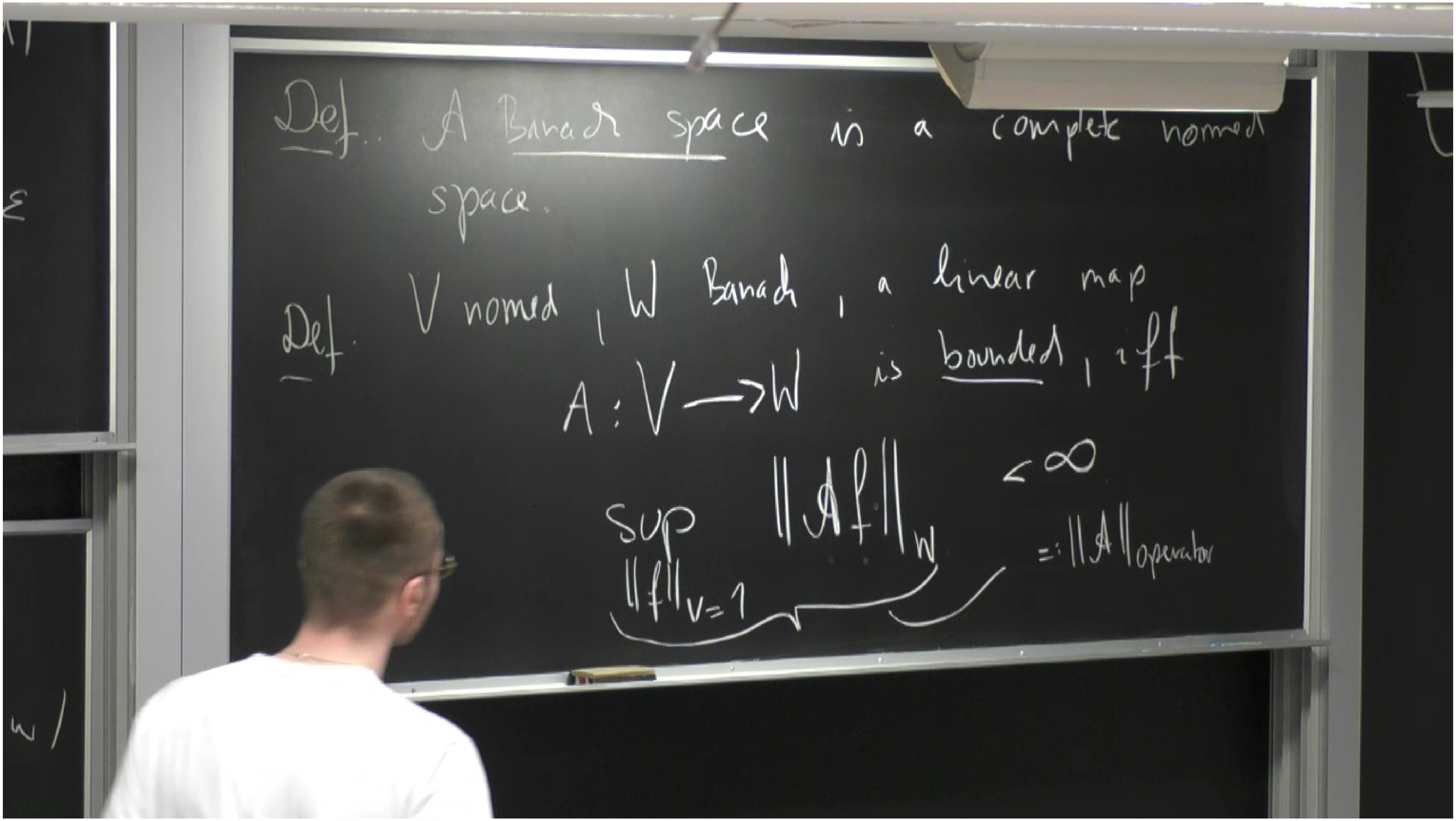
<u>Def</u>: A <u>normed</u> space is a $\mathbb{C}$-Vector space together w/ a norm $\|\cdot\|_V : V \to \mathbb{R}$

$$d(x, y) := \|x - y\|_V$$

Def. A **Banach space** is a complete normed space.

Def. V normed, W Banach, a linear map $A : V \longrightarrow W$ is **bounded**, iff

$$\sup_{\|f\|_V = 1} \|Af\|_W < \infty$$


Def.. A Banach space is a complete normed space.
Def. V normed, W Banach, a linear map
$A : V \longrightarrow W$ is bounded, iff
$\sup_{\|f\|_V = 1} \|Af\|_W < \infty$
$=: \|A\|_{operator}$

Thm: $\left(\mathcal{L}(V, W), \cdot_{\mathbb{K}}, +_{\text{pointwise}}, \|\cdot\|_{op} \right)$ is a Banach space

BLT Thm: $A : \mathcal{D}_A \subseteq V \longrightarrow W$ such that $\mathcal{D}_A$ is dense in V, then we can uniquely extend A to $\hat{A} : V \longrightarrow W$ & $\|A\|_{\mathcal{D}_A} = \|\hat{A}\|_V$

Importantly, a (pre-) inner product induces a (semi-) norm via $\|v\|^2 := \langle v, v \rangle \quad \forall v \in V$

<u>Def</u>: A <u>Hilbert space</u> is a vector space equipped w/ a sequi-linear $\langle \cdot , \cdot \rangle_{\mathcal{H}}$ s.t. it is complete in the norm it induces.

$\rightarrow$ Banach space

is complete in the norm it induces.

$\rightarrow$ Banach space

X normed, X^*

We can identify X w/ linear functionals on X^*

$\rightarrow$ induce the topology of X on X^*

Def: The weak-$*$ topology on X^* to X is the coarsest topology s.t.

$$T_x : X^* \longrightarrow \mathbb{K}$$
$$\phi \longrightarrow T_x(\phi) = \phi(x)$$

are continuous

$$T_x : X^{*} \longrightarrow \mathbb{K}$$
$$\phi \longrightarrow T_x(\phi) = \phi(x)$$

are

Def: A net ϕ_λ in X^* is weak-$*$ convergent if it is pointwise convergent

$$\phi_\lambda(x) \longrightarrow \phi(x) \quad \forall x \in X$$

Importantly, a (pre-) inner product induces

(semi-) norm via $\|v\|^2 := \langle v, v$

$\forall \varepsilon > 0 : \exists N \in \mathbb{N} : \forall n, m \geq N : d(x_n, x_m) < \varepsilon$

Thm Banach-Alaoglu:

If X is a Banach space then the closed unit ball in X^* is weak-$*$ compact.

Def: (X, d) is complete if every Cauchy sequence converges to some $\in X$.

Def: A normed space is a $\mathbb{C}$-vector space together w/ a norm $\|\cdot\|_V : V \to \mathbb{R}$

$d(x,y) := \|x-y\|_V$

Def: A **Banach Algebra** $\mathcal{A}$ is a complex algebra which is a Banach space under norm which is submultiplicative

$$\|a \cdot b\| \leq \|a\| \|b\| \quad \forall a, b \in \mathcal{A}$$

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution.

Def: An abstract C^*-algebra $\mathcal{A}$ is a Banach-$*$ alg. s.t.

$$\|a^* a\| = \|a\|^2 \quad \forall a \in \mathcal{A}$$

Def. An $a \in \mathcal{A}$ is positive, if $\exists\, b$ s.t. $a = b^* b$

Def: The weak-$*$ topology on X^* to X is

Def: A repr. of a C^*-algebra $\mathcal{A}$ is a $*$-homomorphism

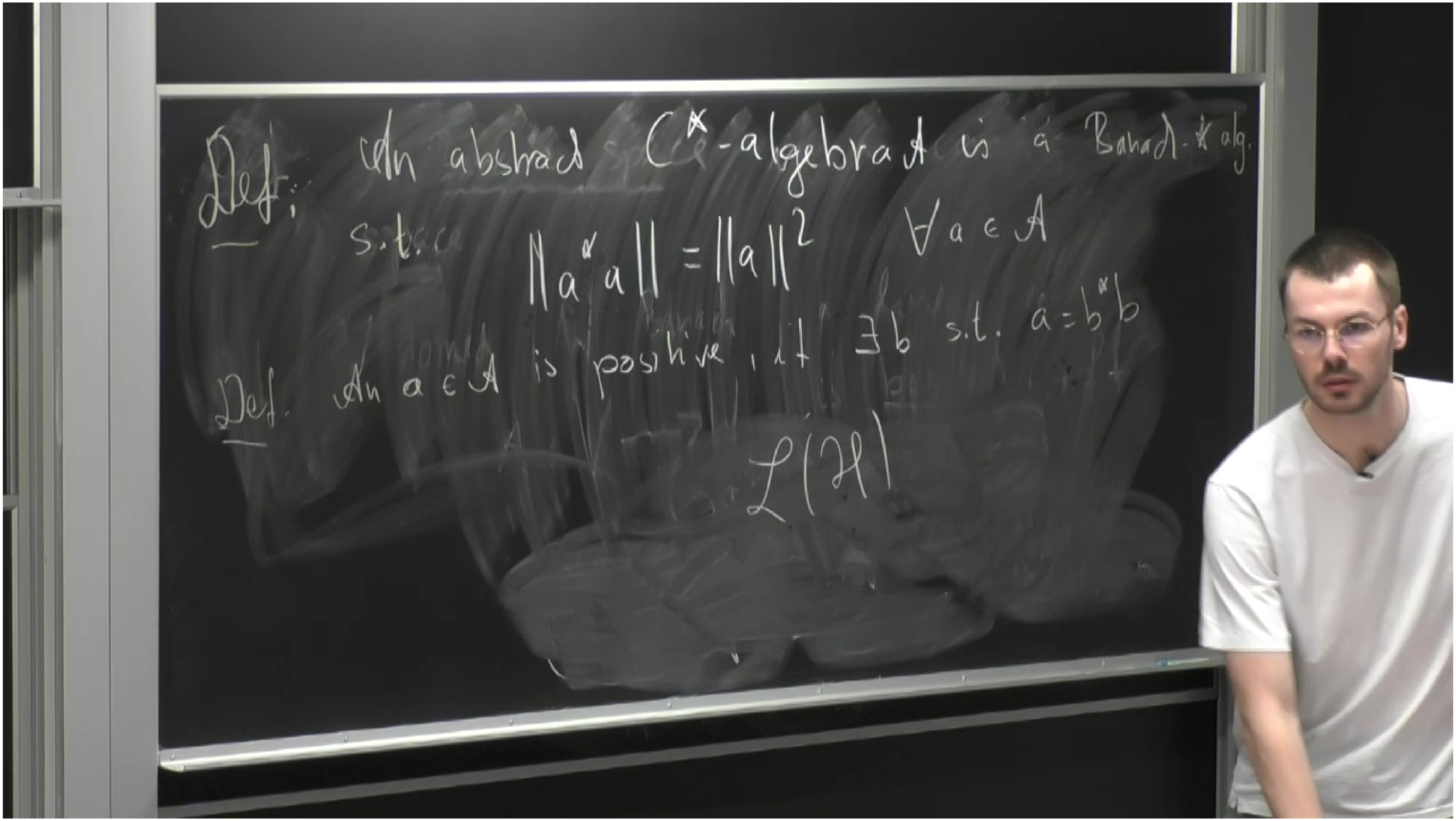
$$\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$$

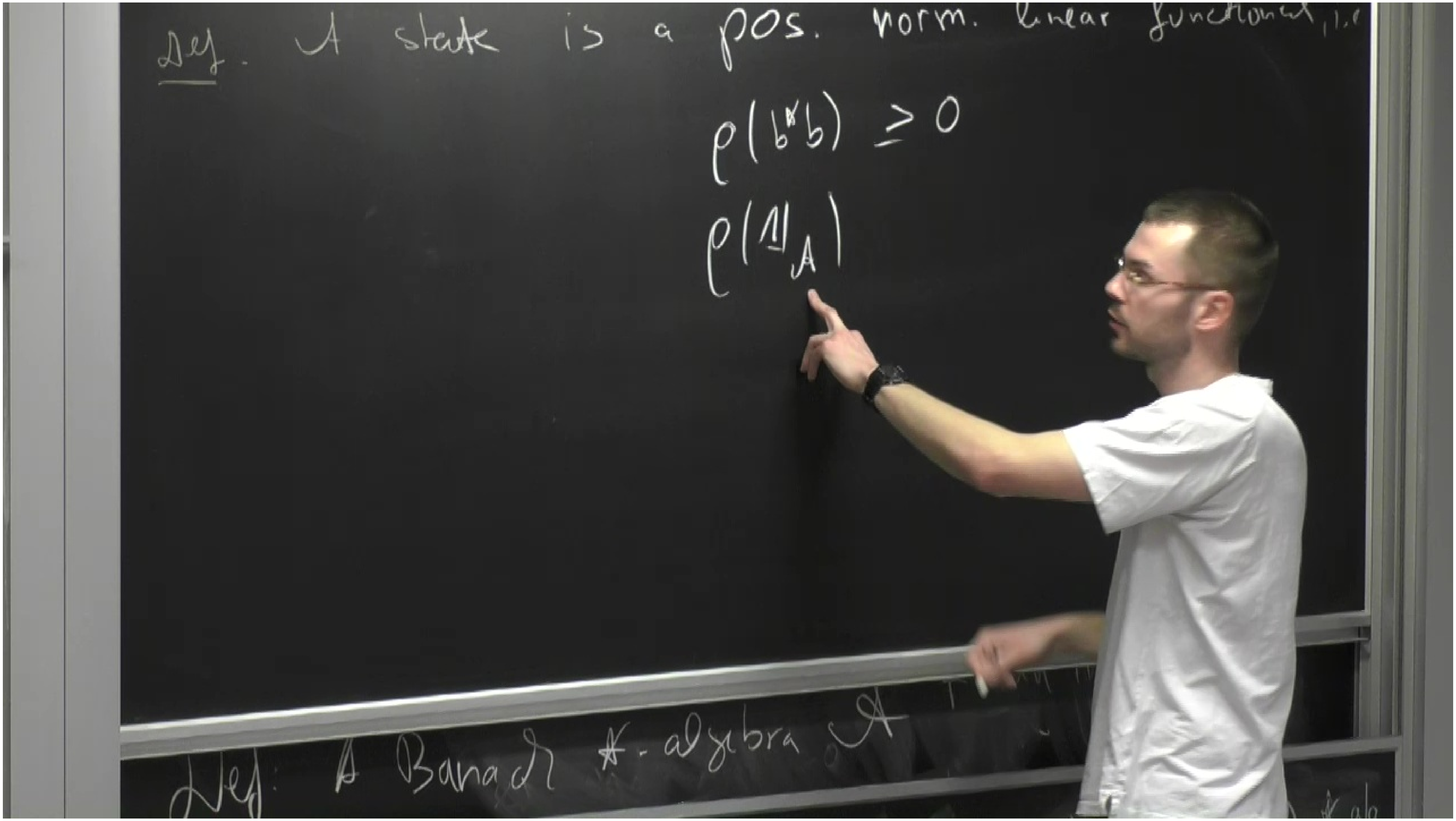
$$\|a\|_{\mathcal{A}} \geq \|\pi(a)\|_{B(\mathcal{H})}$$

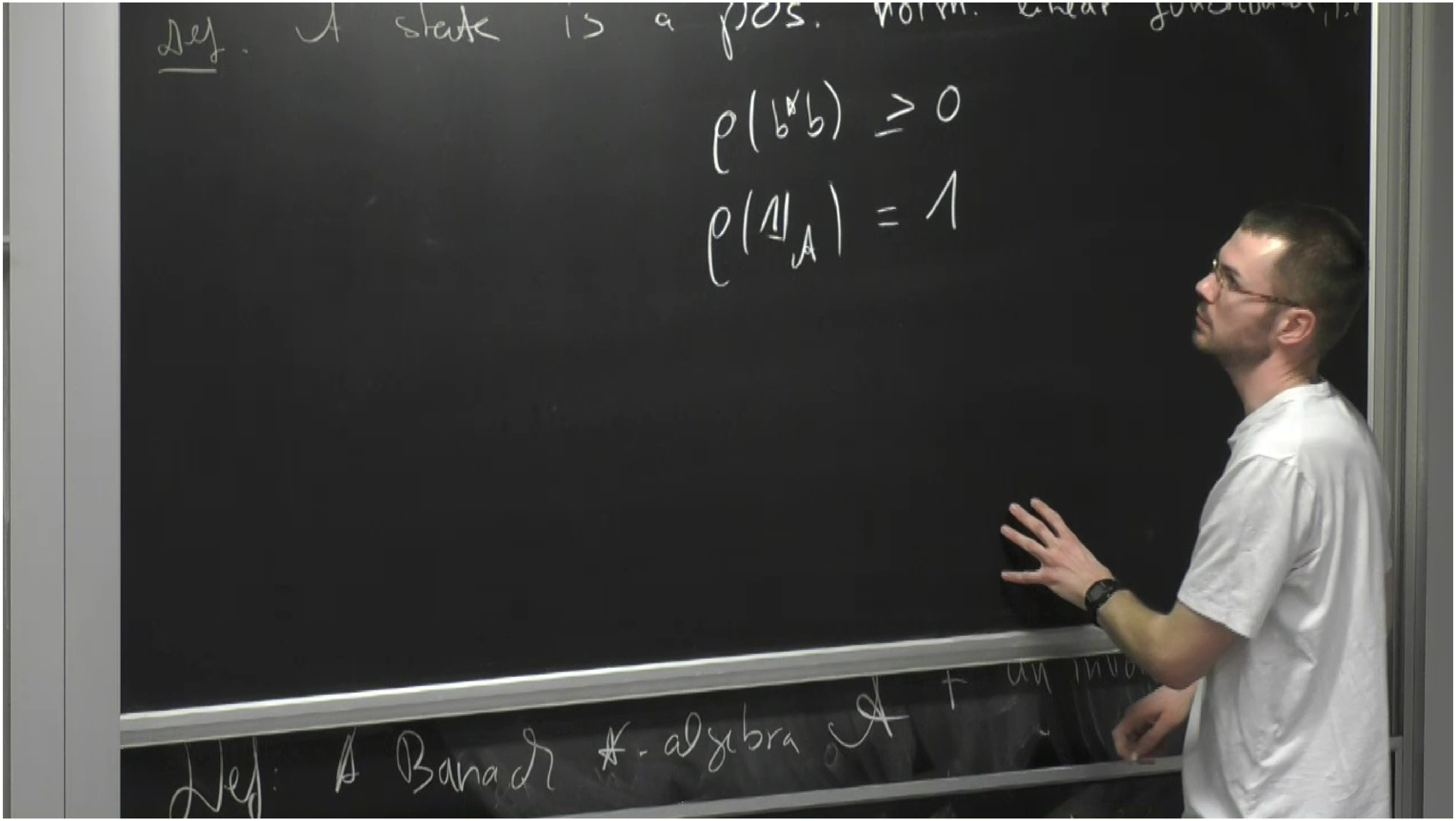
$$\|a\|_{\mathcal{A}} \geq \|\pi(a)\|_{\mathcal{B}(\mathcal{H})}$$

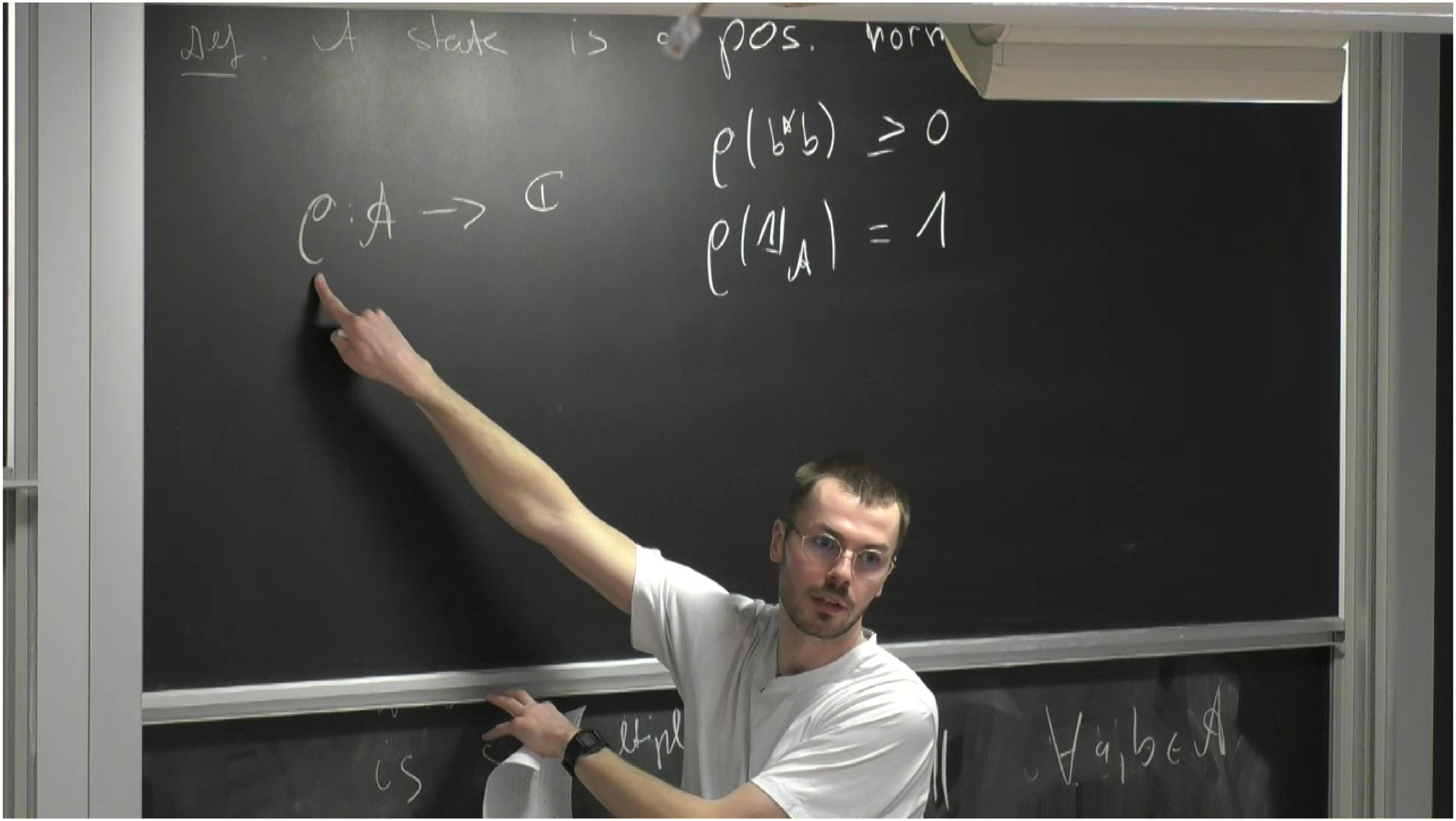
Def. π is cyclic, iff $\exists\, \Omega \in \mathcal{H}$ s.t.

$$\overline{\pi(\mathcal{A})\Omega} = \mathcal{H}$$


Def: An abstract C^*-algebra $\mathcal{A}$ is a Banach-*alg.
s.t. $\|a^*a\| = \|a\|^2 \quad \forall a \in \mathcal{A}$
Def. An $a \in \mathcal{A}$ is positive, if $\exists b$ s.t. $a = b^*b$
$\mathcal{L}(\mathcal{H})$


Def. A state is a pos. norm. linear functional, i.e.
ρ(b*b) ≥ 0
ρ(1_A)
Def: A Banach *-algebra A


Def. A state is a pos.
ρ(b*b) ≥ 0
ρ(1_A) = 1
Def: A Banach *-algebra A + an


Def. A state is a pos. norm
$\rho(b^*b) \geq 0$
$\rho : \mathcal{A} \to \mathbb{C}$
$\rho(1_{\mathcal{A}}) = 1$
is
$\forall a, b \in \mathcal{A}$

$S(\mathcal{A}) \subseteq \mathcal{A}^*$

$\{ f \in \mathcal{A}^* : \sup_{u \in \mathcal{U}} | f(u) | \leq 1 \}$

Def: A Banach Algebra $\mathcal{A}$ is a complex alg... which is a Banach space under norm which ...

$\rho(\mathbb{1}_{\mathcal{A}}) = \ldots$

$S(\mathcal{A}) \subseteq \mathcal{A}^*$

$\{ f \in \mathcal{A}^* : \sup_{u \in \mathcal{U}} | f(u) | \leq 1 \}$

Weak-$*$ compact

$\cap \ \{ f \in \mathcal{A}^* : f(\mathbb{1}_{\mathcal{A}}) = 1 \}$

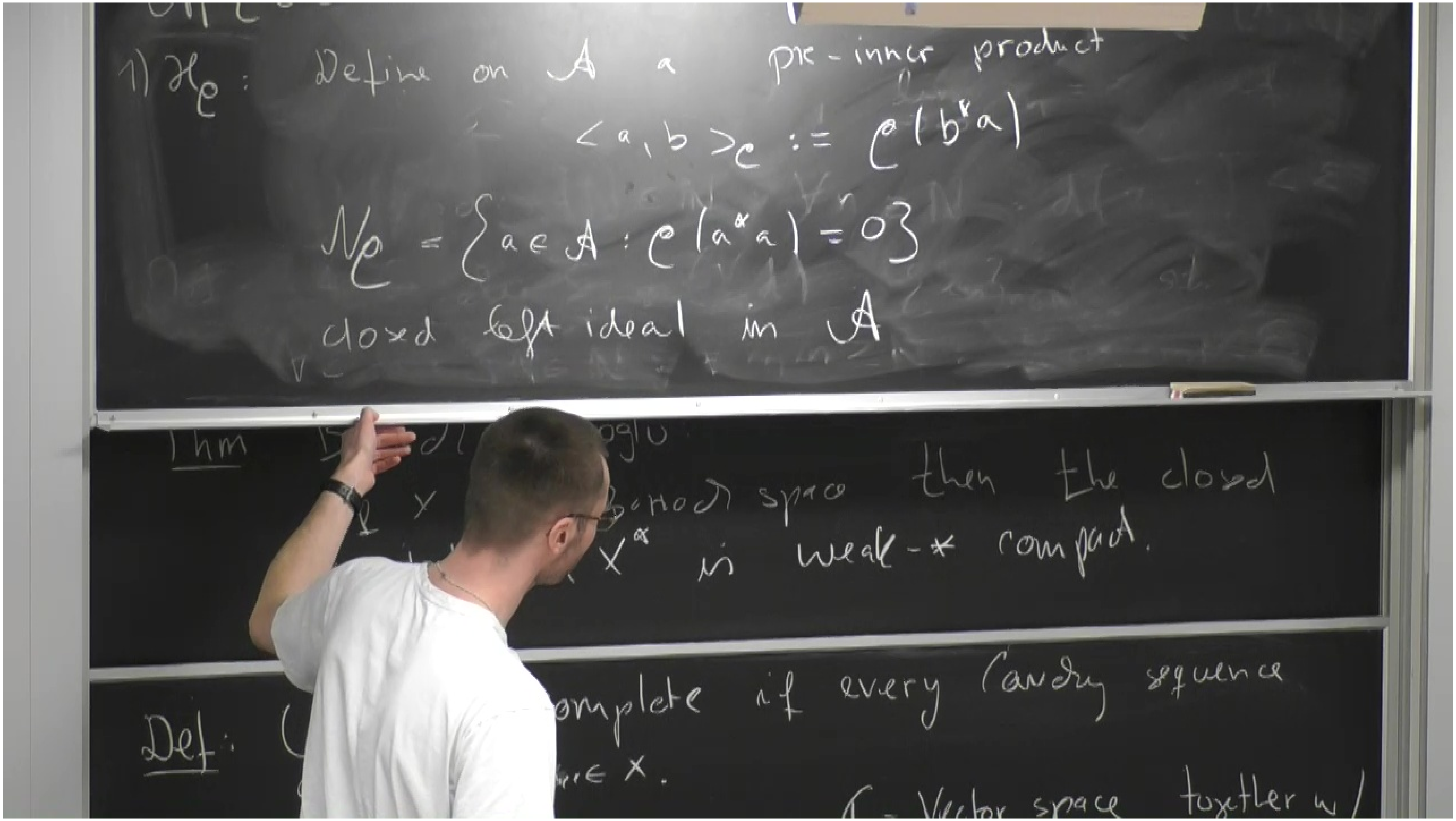
Def: A **Banach Algebra** $\mathcal{A}$ is a complex algebra which is a Banach space under norm which

C^*-algebra $\mathcal{A}$, $\mathcal{H}$, $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$

a <u>vector state</u> $\phi_\Omega \in S(\mathcal{A})$ to a unit

vector $\Omega \in \mathcal{H}$ is given by

$$\phi_\Omega : \mathcal{A} \longrightarrow \mathbb{C}$$
$$a \longmapsto \langle \pi(a)\Omega, \Omega \rangle_{\mathcal{H}}$$



1) $\mathcal{H}_\varphi$: Define on $\mathcal{A}$ a pre-inner product
$\langle a, b \rangle_\varphi := \varphi(b^* a)$
$N_\varphi = \{ a \in \mathcal{A} : \varphi(a^* a) = 0 \}$
closed left ideal in $\mathcal{A}$
Thm
Banach space then the closed
X^* is weak-* compact.
Def:
omplete if every Cauchy sequence
$\in X$.
Vector space together w/

1) $\mathcal{H}_\varphi$: Define on $\mathcal{A}$ a p

$$\langle a, b \rangle_\varphi := \varphi(b^* a)$$

$$\mathcal{N}_\varphi = \{a \in \mathcal{A} : \varphi(a^* a) = 0\}$$

$\langle v, v \rangle \geq 0$

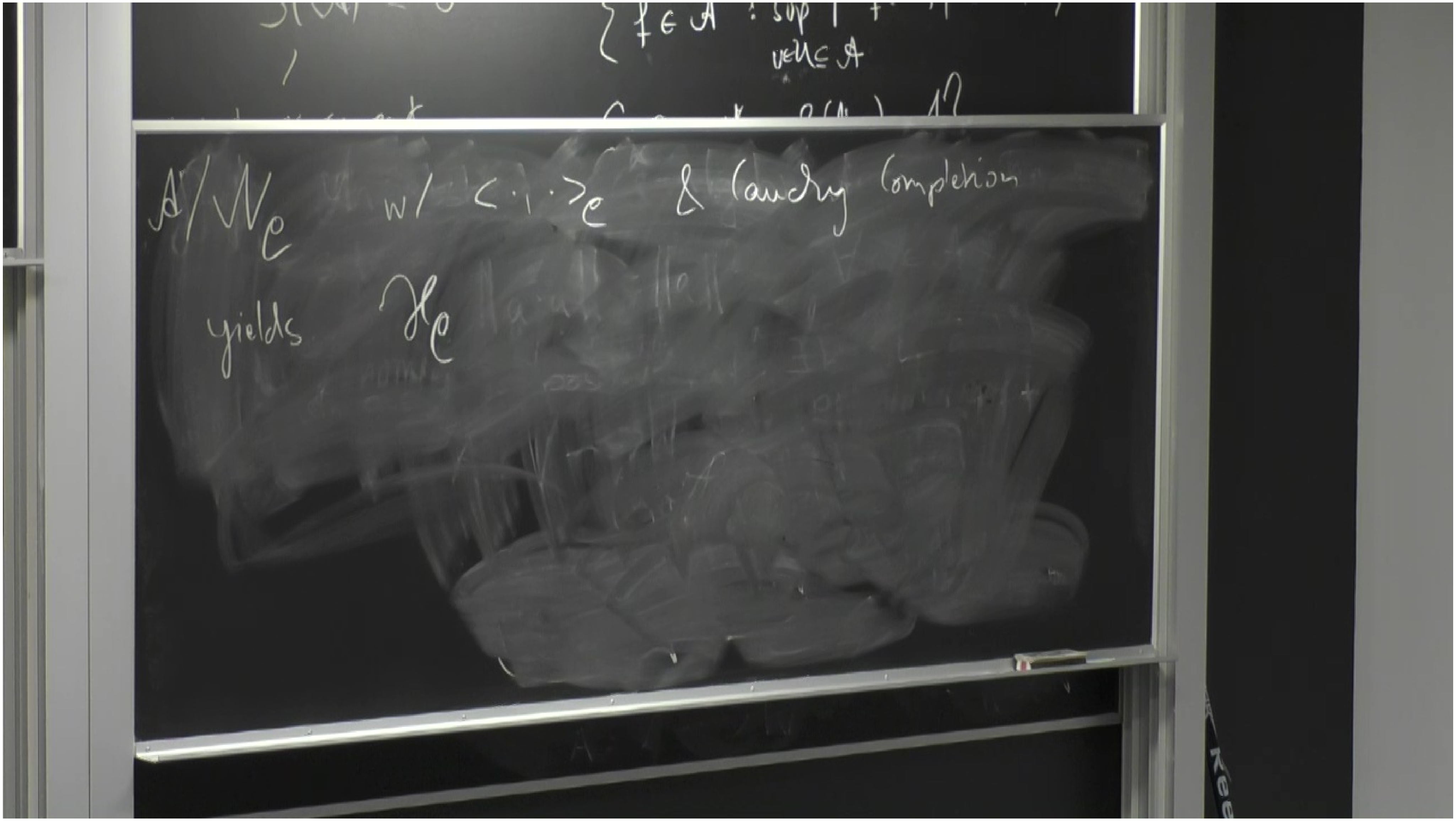
$\langle v, v \rangle = 0 \Leftrightarrow v = 0$

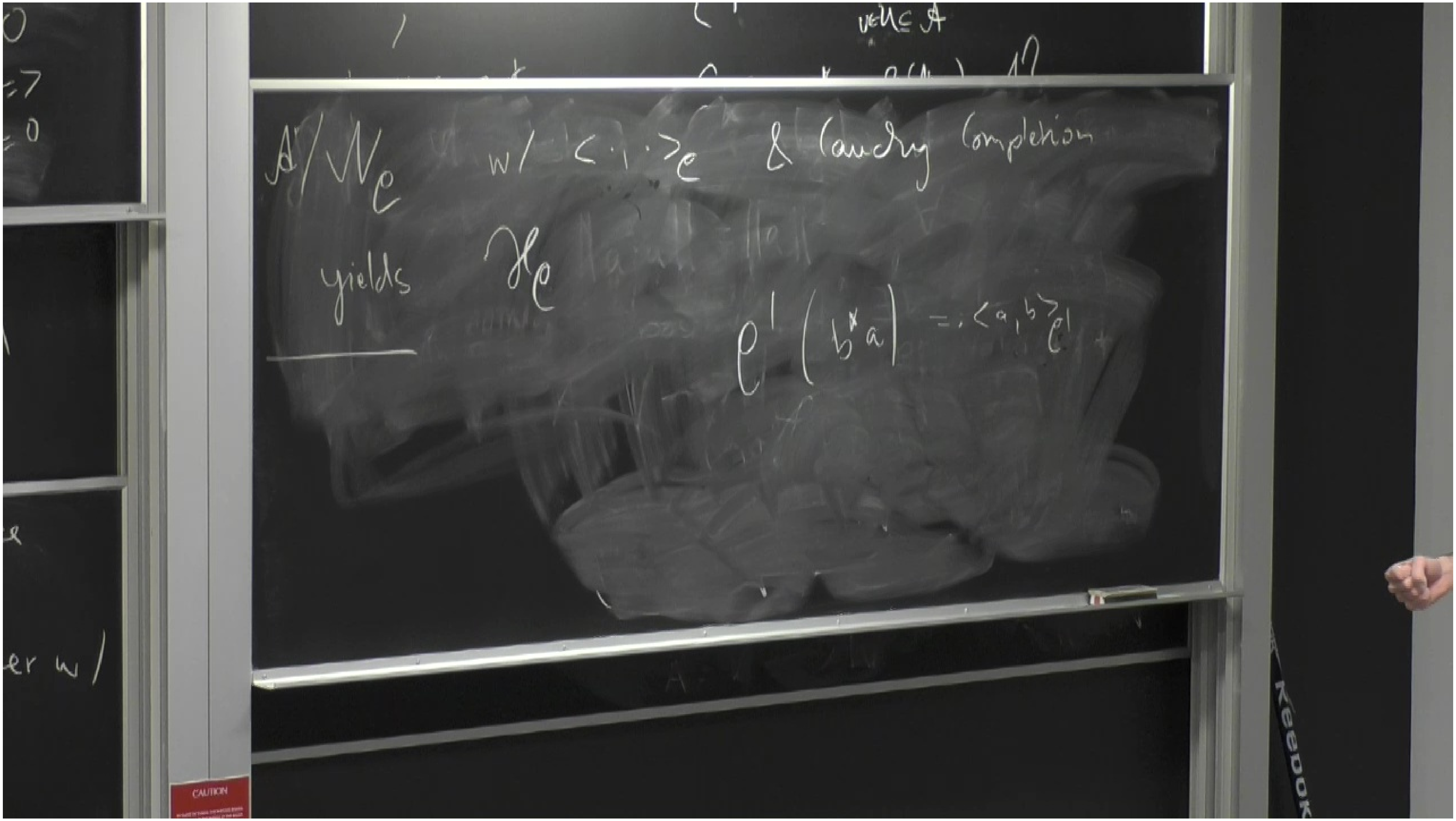
closed left ideal in $\mathcal{A}$

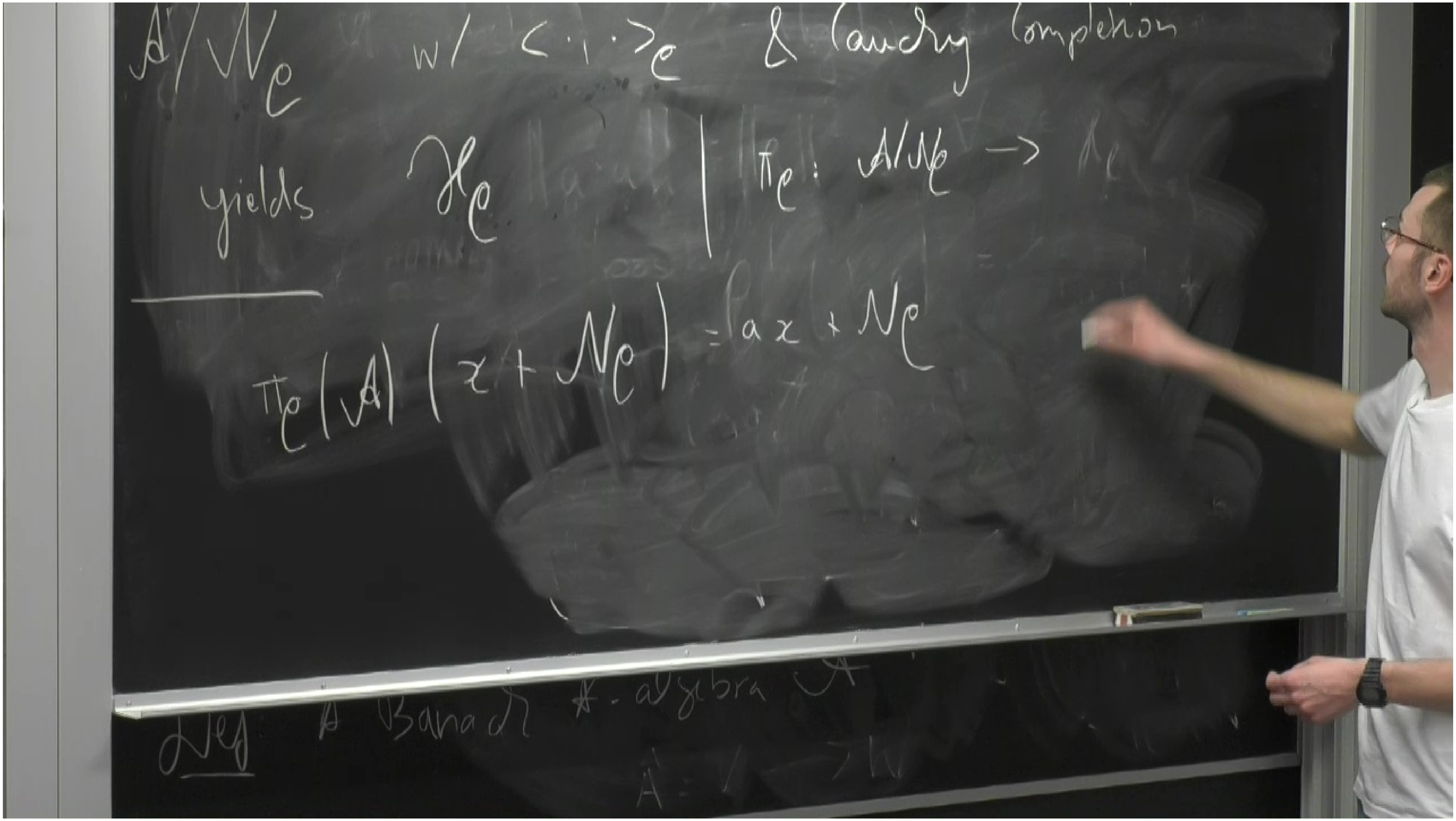
Thm Banach-Alaoglu:

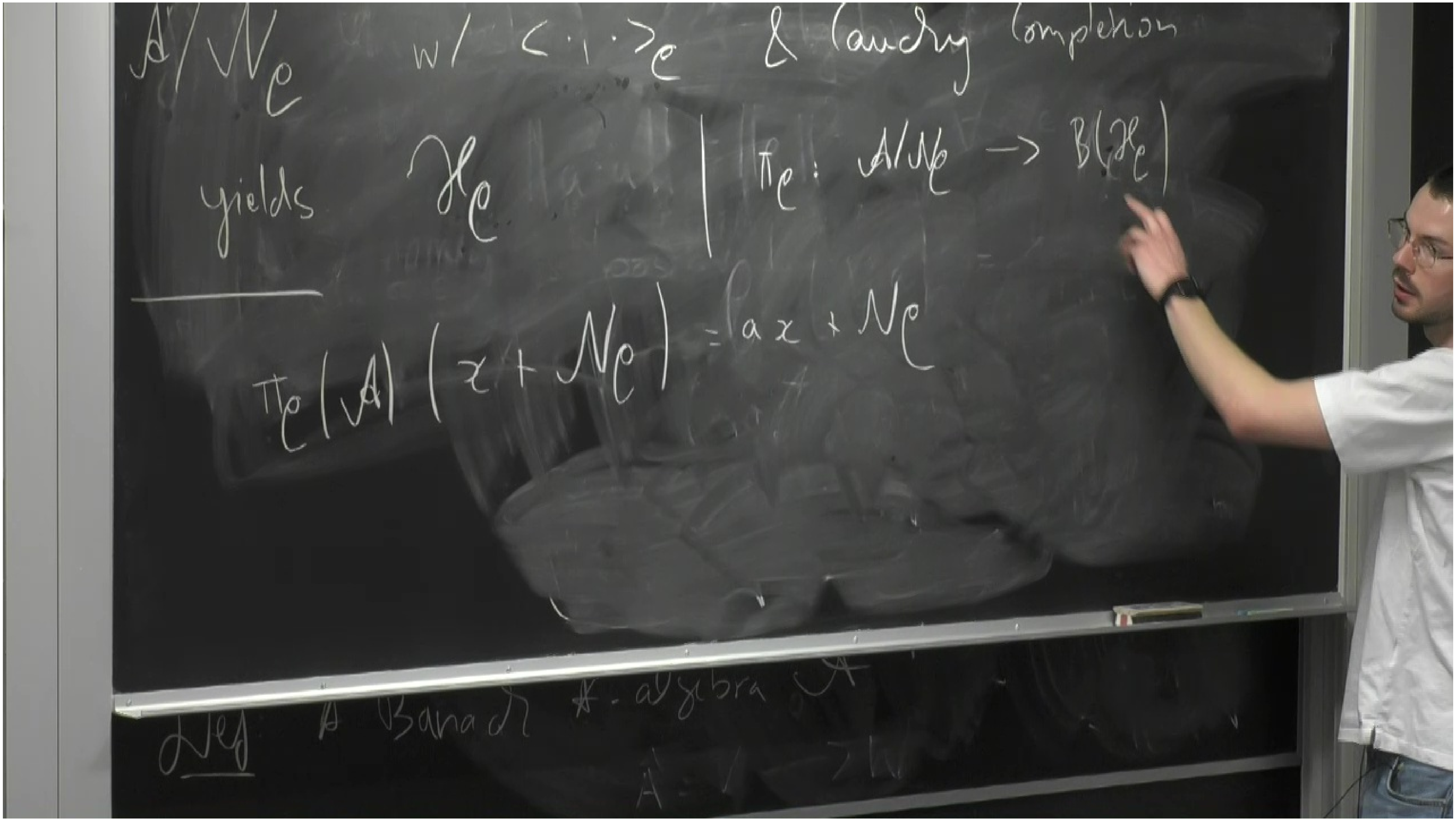
If X is a Banach space then the closed unit ball in X^* is weak-$*$ compact.

Def: (X, d) is complete if every Cauchy sequence


A/N_e w/ <·,·>_e & Cauchy completion
yields H_e


$\mathcal{A}/\mathcal{N}_\rho$ w/ $\langle\cdot,\cdot\rangle_\rho$ & Cauchy completion
yields $\mathcal{H}_\rho$
$\rho'(b^*a) =: \langle a,b\rangle_{\rho'}$


$\mathcal{A}/\mathcal{N}_\mathcal{C}$ w/ $\langle \cdot , \cdot \rangle_\mathcal{C}$ & Cauchy completion
yields $\mathcal{H}_\mathcal{C}$
$\pi_\mathcal{C} : \mathcal{A}/\mathcal{N}_\mathcal{C} \to$
$\pi_\mathcal{C}(A)(x + \mathcal{N}_\mathcal{C}) = ax + \mathcal{N}_\mathcal{C}$
Def A Banach *-algebra $\mathcal{A}$


A/N_ρ w/ <·,·>_ρ & Cauchy completion
yields H_ρ
π_ρ : A/N_ρ → B(H_ρ)
π_ρ(A)(x + N_ρ) = ax + N_ρ
Def A Banach *-algebra A

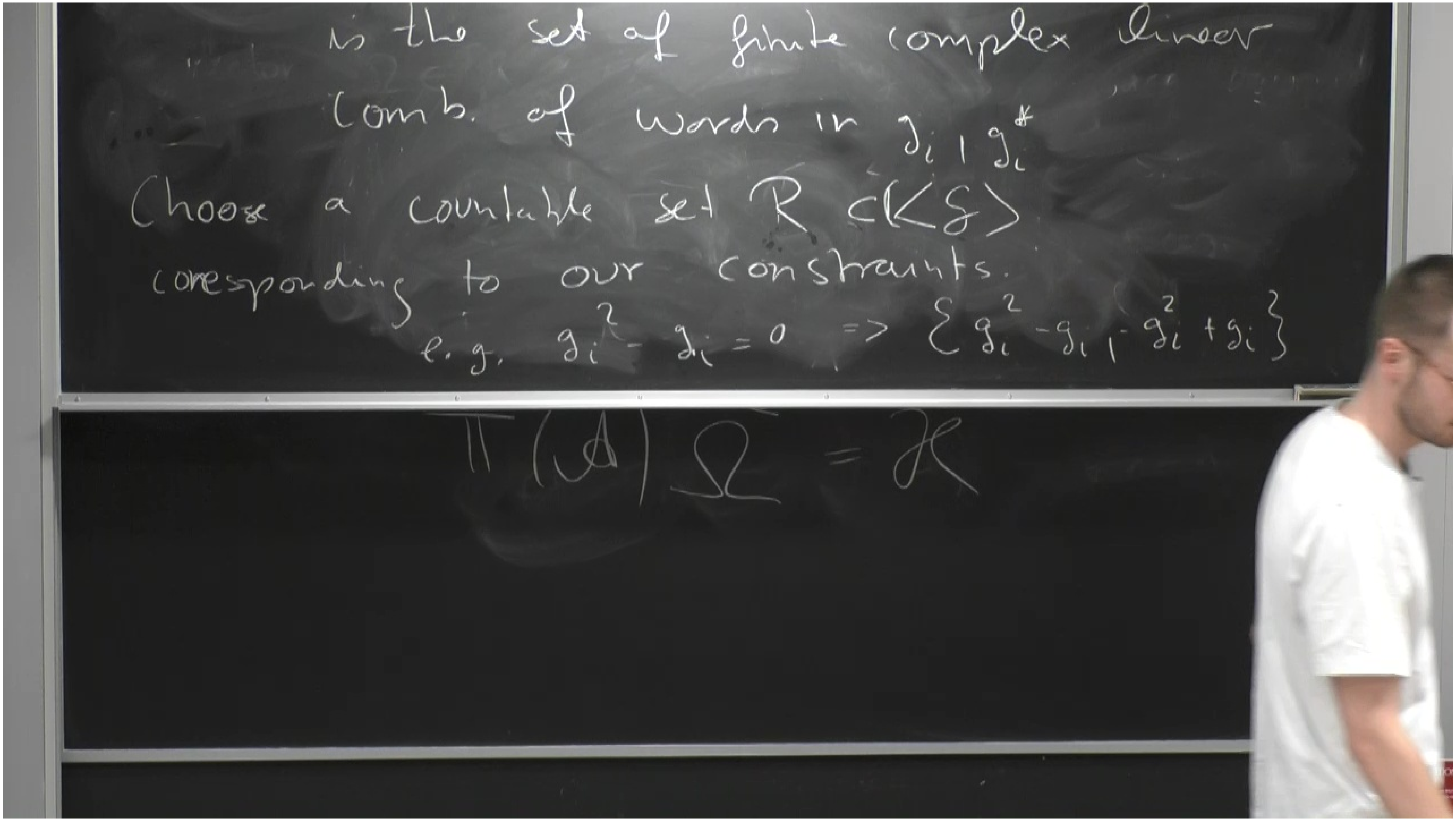
yields $\mathcal{H}_\rho$ | $\pi_\rho : \mathcal{A}/\mathcal{N}_\rho \to B(\mathcal{H}_\rho)$

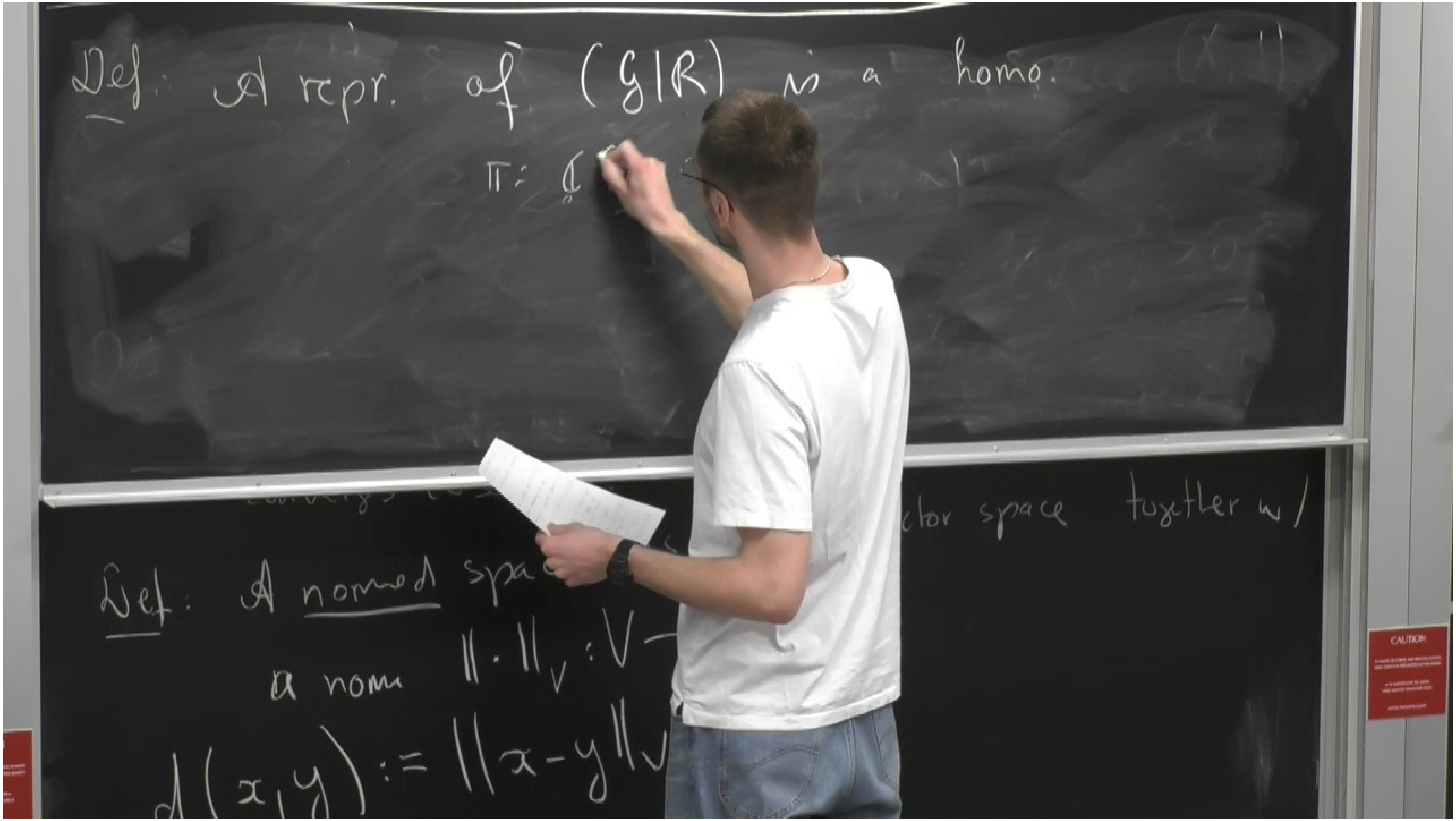
$$\pi_\rho(\mathcal{A})(x + \mathcal{N}_\rho) = ax + \mathcal{N}_\rho$$

GNS cyclic vector is just image of $\mathbb{1}_\mathcal{A}$ in $\mathcal{H}_\rho$

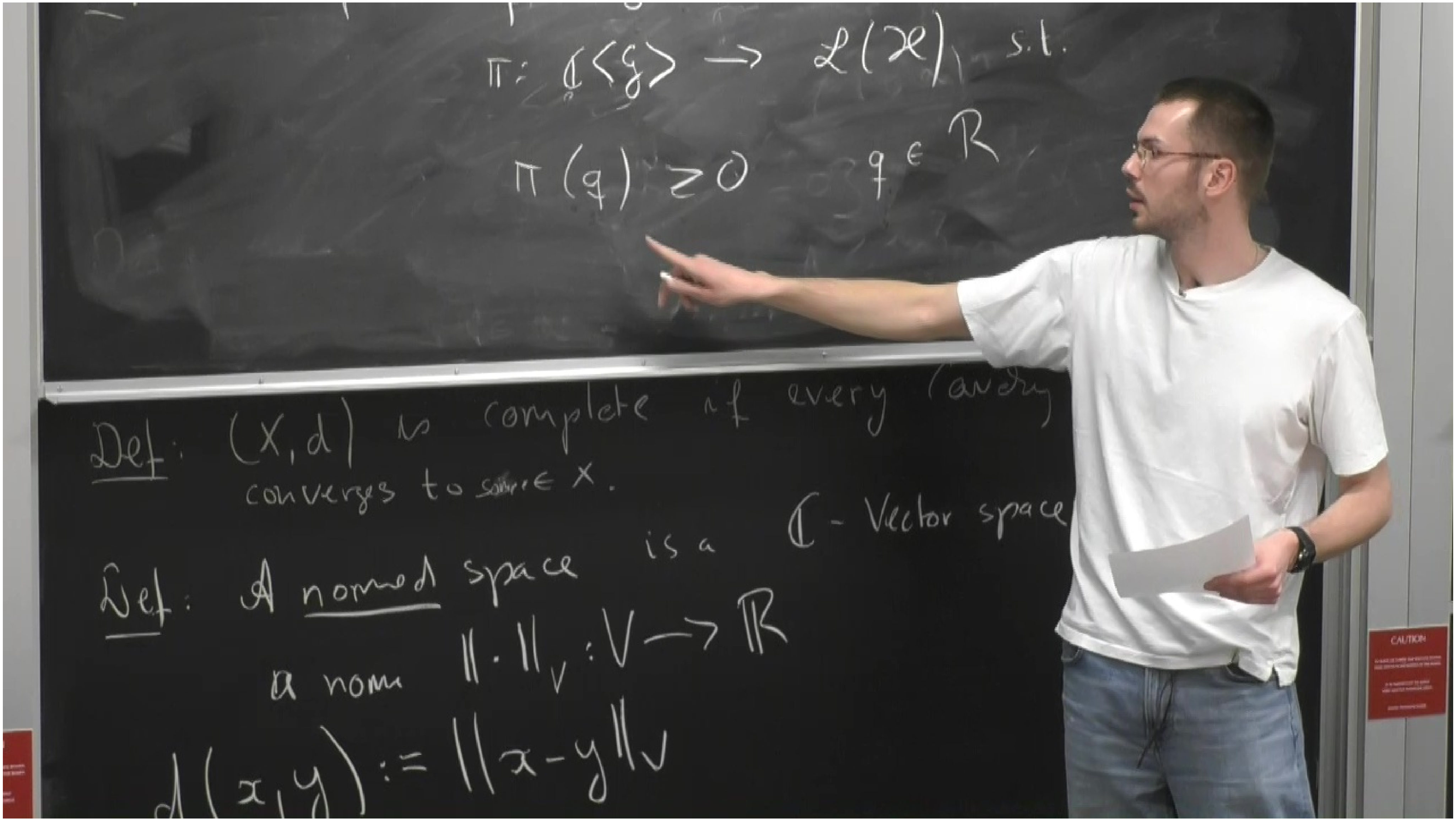
Def: A Banach $*$-algebra $\mathcal{A}$ + an involution.

Def: free complex $*$-algebra $\mathbb{C}\langle \mathcal{G} \rangle$ generated by $\mathcal{G} = \{ g_i \}_{i \in \mathbb{N}}$ is the set of finite complex linear comb. of words in g_i, g_i^*


is the set of finite complex linear
comb. of words in g_i, g_i^*
Choose a countable set $R \subset \mathbb{C}\langle \mathcal{G} \rangle$
corresponding to our constraints.
e.g. $g_i^2 - g_i = 0 \Rightarrow \{g_i^2 - g_i, -g_i^2 + g_i\}$
$\pi(\mathcal{A})\Omega = \mathcal{H}$



Def. A repr. of (G|R) is a homo.
π: G
ctor space together w/
Def: A normed spa
a norm ||·||_V : V
d(x,y) := ||x−y||_V
CAUTION



$\pi: \mathbb{C}\langle g \rangle \to \mathcal{L}(\mathcal{H})$ s.t.
$\pi(q) \geq 0$ $q \in \mathbb{R}$
Def: (X, d) is complete if every Cauchy
converges to some $\in X$.
Def: A normed space is a $\mathbb{C}$ - Vector space
a norm $\|\cdot\|_V : V \to \mathbb{R}$
$d(x,y) := \|x-y\|_V$

$\iff \langle \pi(q)\varphi, \varphi \rangle_{\mathcal{H}} \geq 0 \ \forall \varphi \in \mathcal{H}$

If X is a Banach space then the closed unit ball in X^* is weak-$*$ compact.

Def: (X,d) is complete if every Cauchy sequence converges to some $\in X$.

Def: A normed space is a $\mathbb{C}$-Vector space together w/ a norm $\|\cdot\|_V : V \to \mathbb{R}$

$d(x,y) := \|x-y\|_V$

$$\|x\|_{C^*\langle g\rangle} = \sup\left\{ \|\pi(x)\|_{B(\mathcal{H})} : \pi \text{ is a repr. of } (g \mid R) \right\}$$

Need to ensure $\|x\|_{C^*\langle g\rangle} < \infty$ via R

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution,

$$\|x\|_{\mathbb{C}\langle g\rangle} = \sup\left\{ \|\pi(x)\|_{B(\mathcal{H})} : \pi \text{ is a repr. of } (g|R) \right\}$$

[Need to ensure $\|g_v\|_{\mathbb{C}\langle g\rangle} < \infty$ via R]

$g_v \in g$

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution,

$$\|x\|_{\mathbb{C}\langle g\rangle} = \sup\{ \|\pi(x)\|_{B(\mathcal{H})} \ldots (g|R)\}$$

[Need to ensure $\|g_i\|_{\mathbb{C}\langle g\rangle} < \infty$ via R]

$g_i \in g$

Cauchy Compl. of $\mathbb{C}\langle g\rangle$ w.r.t. $\|\cdot\|_{\mathbb{C}\langle g\rangle}$

$\longrightarrow$ Universal C^*-Algebra: $C^*(G|R)$

$\|a \cdot b\| \le \|a\|\|b\| \quad \forall a, b \in \mathcal{A}$

Def: A Banach $*$-algebra $\mathcal{A}$ + an involution.

<u>Thm</u> Banach-Alaoglu:

then the closed

<u>NPO</u>:

$$p^* = \min_{\rho \in S(C^*(G|R))} \rho(p)$$

NPO from NPA

$$p^* = \min_{\pi, \Omega} \langle \pi(p)\Omega, \Omega \rangle$$

s.t. $\pi(q) \geq 0 \quad q \in R$