Title: Wall-crossing structures and Chern-Simons theory.

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Abstract: In 2008 jointly with Maxim Kontsevich we introduced the notion of stability data on graded Lie algebras. In the case of the Lie algebra of vector fields on a symplectic torus it underlies the wall-crossing formulas for Donaldson-Thomas invariants of 3-dimensional Calabi-Yau categories. In 2013 we introduced the notion of wall-crossing structure, which is a locally-constant sheaf of stability data. Wall-crossing structures naturally appear in complex integrable systems, Homological Mirror Symmetry and many other topics not necessarily related to Donaldson-Thomas theory. Recently, in 2020 we introduced a sublass of analytic wall-crossing structures. We formulated a general conjecture that analytic wall-crossing structure gives rise to resurgent (i.e. Borel resummable) series.

Many wall-crossing structures have geometric origin, and moreover they naturally appear in our Holomorphic Floer Theory program. Aim of my talk is to discuss wall-crossing structures associated with a pair of holomorphic Lagrangian submanifolds of a complex symplectic manifold (in most cases it will be the cotangent bundle). These wall-crossing structures underly Cecotti-Vafa wall-crossing formulas, and as such they appear naturally in the study of exponential integrals in finite and infinite dimensions. I am going to explain our conjectural approach to Chern-Simons theory which is based on the idea of wall-crossing structure. In some aspects this approach is related to the work of Witten on analytic continuation of Chern-Simons theory.

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Introduction

Let $X, dim_{\mathbb{C}}X = n$ be a Kähler manifold endowed with a holomorphic Morse function f ("superpotential"). Denote its critical set by $Crit(f) = \{x_1, ..., x_k\}$, and the corresponding set of critical values by $S := Critval(f) = \{z_1, ..., z_k\}, f(x_i) = z_i$. Let us assume that all z_i's are distinct. Critical points x_i generate the Morse complex of (X, f). Generically there is no interaction between critical points, since $ind(x_i) = n, 1 \leq i \leq k$. Morse differential becomes non-trivial for the function $f/\hbar$ and "Stokes rays" $Arg(\hbar) = Arg(z_i - z_j) \subset \mathbb{C}^*_\hbar$. Similar story for the Floer complex of the pair of holomorphic Lagrangian submanifolds $L_0 = X, L_1 = graph(df)$ in a complex symplectic manifold $(T^*X, \omega^{2,0})$ (generically: no pseudo-holomorphic discs).

Idea of the approach from the perspective of Holomorphic Floer Theory

Main aim of my talk is to explain a conjectural approach to the resurgence of perturbative series in the complexified Chern-Simons theory (i.e. we replace the compact gauge group by the corresponding complex one, like e.g. $SU(2) \to SL(2,\mathbb{C})$). It is based on an analytic WCS defined in terms of a certain pairs of complex Lagrangian subvarieties. In this way the subject of my talk can be thought of as a particular application of our program "Holomorphic Floer Theory" which studies Floer-theoretical problems in the framework of complex symplectic manifolds.

Rough idea of the approach: critical points of the CS functional (i.e. flat connections) correspond to intersection points of the complex Lagrangian subvarieties. Borel transforms of the local perturbative expansions at critical points can be combined in a unique multivalued analytic function provided we know how the critical points "interact". In the case of rigid flat connections the interaction is encoded in the "Stokes indices", which is the properly defined number of gradient lines of $Re(e^{-i\theta}CS)$ between pairs of critical points and approapriate θ. This idea is reminiscent to the one in the "algebra of the infrared" story (for the latter see the foundational paper by Gaiotto-Moore-Witten, for a mathematical treatment of their work see my papers written jointly with Kapranov, Kontsevich, Soukhanov).

The underlying "Chern-Simons wall-crossing structure" can be described in different ways. One way (will not be discussed today) uses our theory of quantum wave functions. Quantum wave functions are cyclic vectors of holonomic DQ-modules associated to the complex Lagrangian subvarieties. In a sense this is the *"B-side of HFT"*. Another one (which will be discussed today) is based on a conjectural Hodge structure of countable rank. In a sense this is the *"A-side of HFT"*. In general the relation between A and B sides of HFT is *not* a Homological Mirror Symmetry but rather a version of the Riemann-Hilbert correspondence.

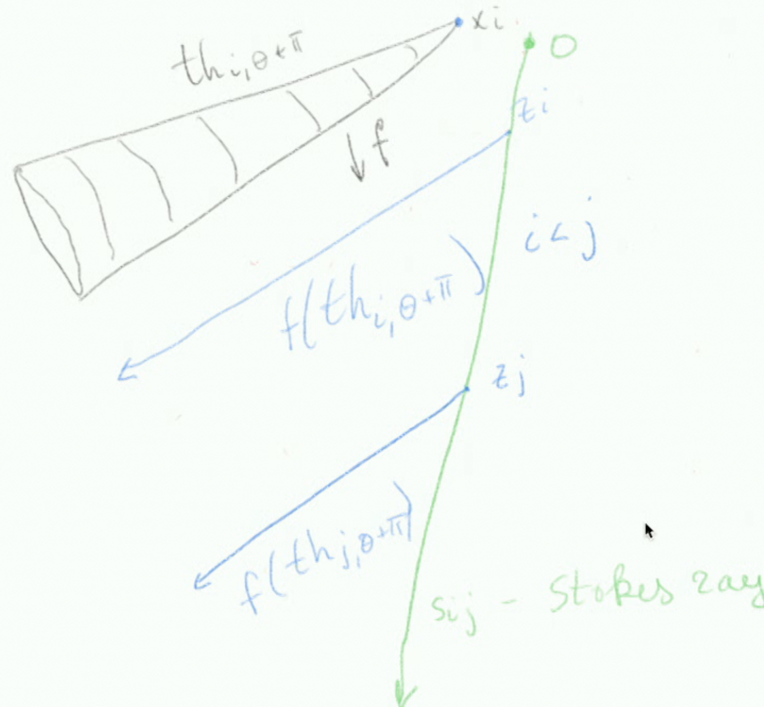
Global and local Fukaya-Seidel categories

Wall-crossing structures I am going to talk about today underly the so-called $2d$ wall-crossing formulas (a.k.a. as Cecotti-Vafa or Picard-Lefschetz WCF). These formulas are corollaries of the *isomorphism of local and global Betti cohomology*. I start by recalling the story in the framework of Landau-Ginzburg model.
Assume for simplicity that f is Morse (this condition can and will be relaxed). Recall that under some mild conditions one can describe the global Fukaya-Seidel category $FS(X, f)$ in terms of the local Fukaya-Seidel categories $FS(U_\varepsilon(z_i), f), 1 \leq i \leq k$, where $U_\varepsilon(z_i)$ is a small tubular neighborhood of the critical locus $f^{-1}(z_i)$ of the critical value z_i.

Thimbles

In the Morse case global (resp. local) FS-categories are generated by global (resp. local) thimbles.
For a given $\theta = Arg(\hbar) \in \mathbb{R}/2\pi\mathbb{Z} = S^1_\theta$ a thimble $th_{z_i,\theta+\pi}$ is defined as the union of gradient lines (for the Kähler metric) of the function $Re(e^{-i\theta}f)$ which originate at the critical point $x_i \in X$. Then $f(th_{z_i,\theta+\pi})$ is a ray $Arg(z) = \theta + \pi$ emanating from the critical value $z_i \in S$.

Global thimbles as generators of the global FS category



Betti local-to-global isomorphism

Homology classes of thimbles $th_{z_i,\theta+\pi}, 1 \leq i \leq k$ generate the lattice of relative homology $\Gamma_\theta = H_n(X, f^{-1}(z), \mathbb{Z})/tors \simeq \mathbb{Z}^k$. Here $|z| \gg 1, Arg(z) = \theta + \pi$. Variation of θ gives a local system of lattices on S^1_θ with the fiber $\Gamma_\theta = H_n(X, f^{-1}(\infty), \mathbb{Z})/tors$. We can replace $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ by $\hbar \in \mathbb{C}^*, Arg(\hbar) = \theta + \pi$ and get a local system of lattices on $\mathbb{C}^*_\hbar$.
This local system of global relative homology groups $H_n^{Betti,\theta+\pi}(X, f)$ can be identified with the direct sum over all z_i of similarly defined local relative homology groups generated by local thimbles. We will often use the cohomology groups instead. The direct sum $H^\bullet_{Betti,loc,\theta+\pi}(X, f)$ of the local cohomology groups over all z_i is called Betti local cohomology of (X, f) in the direction $\theta + \pi$. Then at the level of cohomology the equivalence of global and local FS-categories amounts to the Betti local-to-global isomorphism. Taking two rays with slopes $\theta_\pm = Arg(z_i - z_j) \pm \varepsilon$ one obtains the Stokes isomorphism between Betti local cohomology at z_i and z_j.

Betti local cohomology in terms of the sheaf of vanishing cycles

Assume X is *affine algebraic, Kähler*, $dim_{\mathbb{C}} X = n$, $f : X \to \mathbb{C}$ is *regular* Morse function. Then the local Betti cohomology can be defined via the cohomology with coefficients in the sheaf of vanishing cycles:

$$H^{\bullet}_{Betti,loc,\hbar}(X, f) = \oplus_{z \in \{z_1,\dots,z_k\}} H^{\bullet}((f/\hbar)^{-1}(z/\hbar), \varphi_{\frac{f-z}{\hbar}}(\underline{\mathbb{Z}}_X)).$$

This formula holds without the assumption that f is Morse.

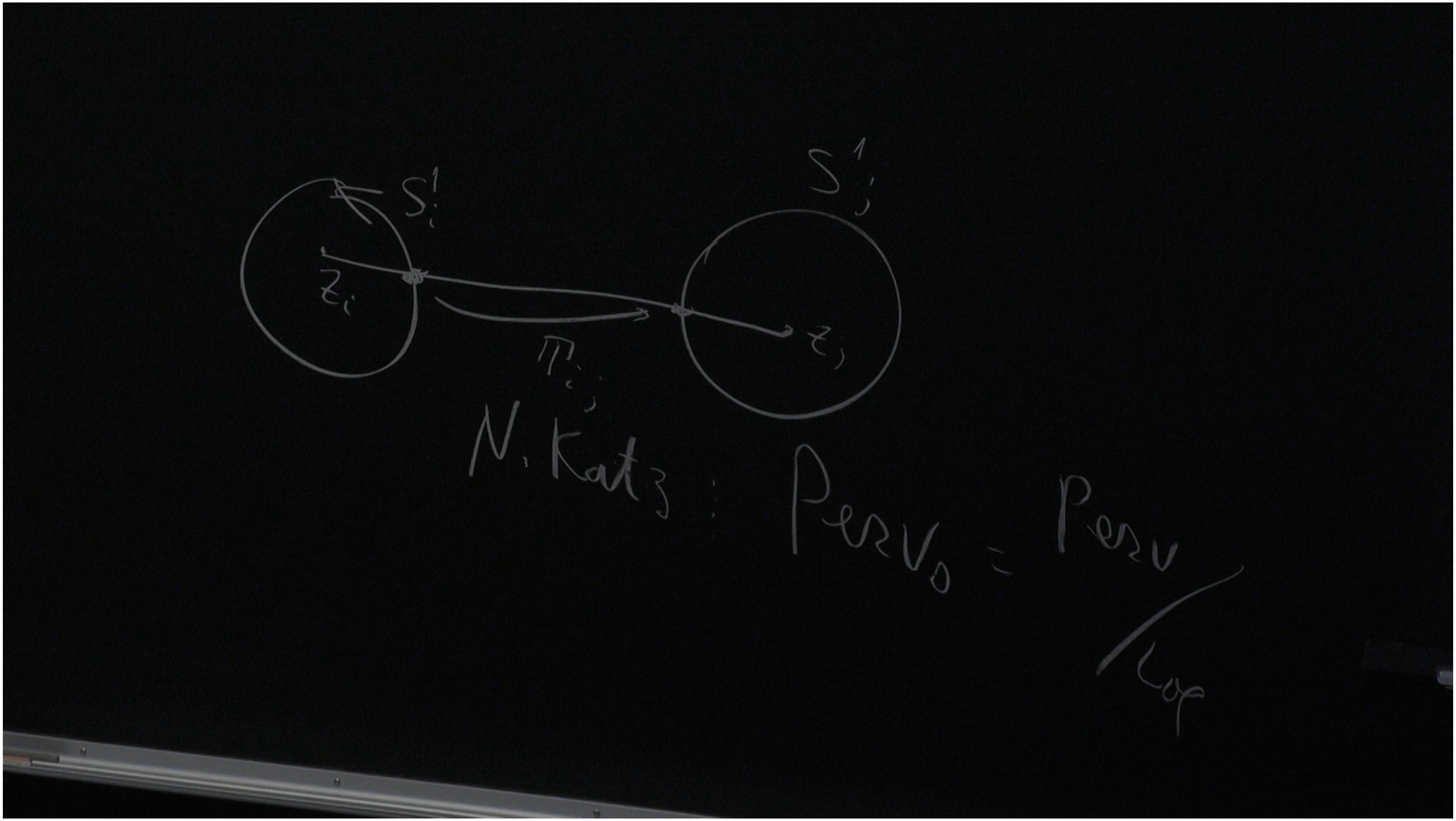
Betti data

Both sides of the Betti local-to-global isomorphism can be described in many equivalent ways. I will call either of those descriptions Betti data and list them below without comments:

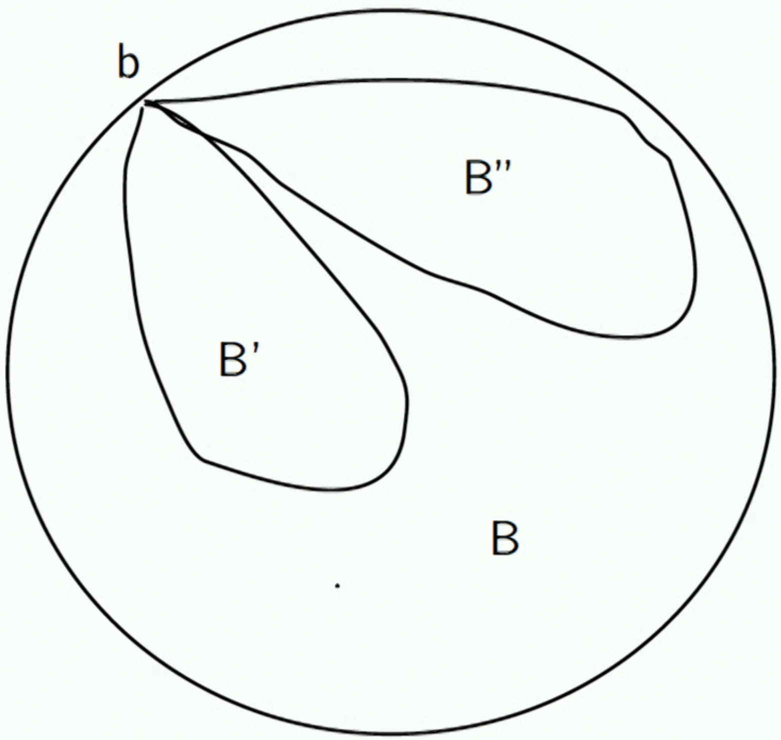
a) For each critical value z_i a local system E_i of abelian groups on S^1_θ and for any Stokes ray $\theta_{ij} = Arg(z_i - z_j), i \neq j$ a morphism of fibers $T_{ij} = T_{\theta_{ij}} : E_{i,\theta} \to E_{j,\theta}$, also invertible $T_{ii} : E_{i,\theta} \to E_{i,\theta}$.

b) Constructible sheaves $\mathcal{F}$ on $\mathbb{C}$ with the set of singularities $\{z_i\}$ such $\mathbf{R}\Gamma(\mathbb{C}, \mathcal{F}) = 0$.

c) Perverse sheaves $\mathcal{G}^\bullet$ on $\mathbb{C}$ with the set of singularities $\{z_i\}$ such that $\mathbf{R}\Gamma(\mathbb{C}, \mathcal{G}^\bullet) = 0$.


S_i^1
S_j^1
z_i
z_j
π_ij
N. Katz : Perv_0 = Perv / Loc

The following three descriptions are equivalent to a)-c), but will not be used today.
d) Deligne-Malgrange-Stokes sheaves of exponential type (a.k.a. Betti part of the non-commutative Hodge structure in the sense of Katzarkov-Kontsevich-Pantev).
e) Constructible sheaves on $\mathbb{C}_\hbar$ with the singular support which belongs to the union of $\mathbb{C}_\hbar - \{|\hbar| < \varepsilon\}$ and conormal bundles to certain closed real analytic curves about z_i.
f) Collections of vector spaces $V(B, b)$, where B is a disc $b \in \partial B$, such that $z_i \notin \partial B$, and subject to the "direct sum property": $V(B) = V(B') \oplus V(B'')$ if B' and B'' form a bouquet which is a retraction of B as on the following figure:


b
B''
B'
B

Exponential integrals

Roughly speaking the exponential integral associated with the pair (X, f) and a choice of the holomorphic volume form *vol* on X corresponds to the Fourier transform of the perverse sheaf $\mathcal{F}$. The description a) of $\mathcal{F}$ corresponds to the Betti local-to-global isomorphism. This isomorphism is responsible for the wall-crossing formulas (and wall-crossing structure) for exponential integrals. I review the wall-crossing formulas below in the Morse case. Later I will discuss a generalization to the non-Morse and the infinite-dimensional cases.

Let us fix the top-degree holomorphic form *vol* on X and consider the following collection of exponential integrals for those $\hbar \in \mathbb{C}^*$ which do not belong to the Stokes rays $s_{ij} := \{\hbar | Arg(\hbar) = Arg(z_i - z_j), i \neq j\}$:

$$I_i(\hbar) = \int_{th_{z_i,\theta+\pi}} e^{f/\hbar} vol.$$

Assume that the set of critical values $S = \{z_1, ..., z_k\}$ is in generic position in the sense that no straight line contains three points from S. Then a Stokes ray contains two different critical values, say z_i, z_j which can be ordered $z_i < z_j$ by their proximity to the vertex.

Wall-crossing formulas

Picard-Lefschetz formulas imply the following fact.

Lemma

If in the $\hbar$-plane we cross the Stokes ray $s_{ij} = s_{\theta_{ij}}$ containing critical values $z_i, z_j, z_i < z_j, \theta_{ij} = Arg(z_i - z_j)$, then the integral $I_i(\hbar)$ changes such as follows:

$$I_i(\hbar) \mapsto I_i(\hbar) + n_{ij} I_j(\hbar),$$

where $n_{ij} \in \mathbb{Z}$ is the number of gradient trajectories of the function $Re(e^{i(Arg(z_i - z_j))} f)$ joining critical points x_i and x_j.

Let us modify the exponential integrals. This will be useful in the infinite-dimensional case as well.

$$I_i^{mod}(\hbar) := \left(\frac{1}{2\pi\hbar}\right)^{n/2} e^{-z_i/\hbar} I_i(\hbar).$$

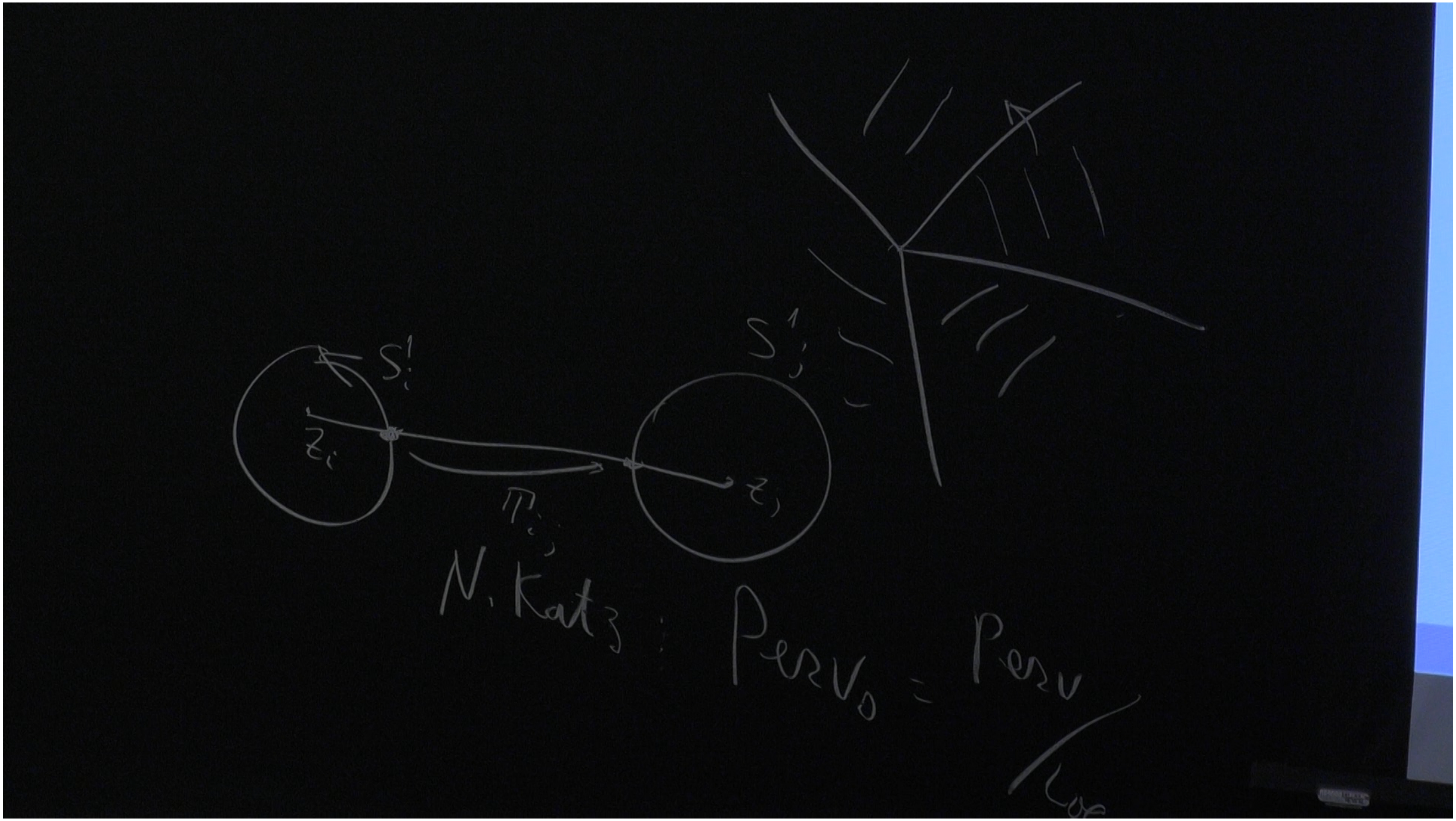
Then as $\hbar \to 0$ the stationary phase expansion gives a formal series

$$I_i^{mod}(\hbar) = c_{i,0} + c_{i,1}\hbar + ... \in \mathbb{C}[[\hbar]],$$

where $c_{i,0} \neq 0$. The jump of the modified exponential integral when we cross the Stokes ray s_{ij} is given by $\Delta(I_i^{mod}(\hbar)) = n_{ij} I_j^{mod}(\hbar) e^{-(z_i - z_j)/\hbar}$.

RH problem

Wall-crossing formulas give rise to a Riemann-Hilbert problem which is in turn gives rise to a holomorphic vector bundle obtained by gluing along Stokes rays trivial bundles in sectors bounded by Stokes rays. Explicitly, the vector $\overline{I}^{mod}(\hbar) = (I_1^{mod}(\hbar), ..., I_k^{mod}(\hbar)), k = |S|$ satisfies the Riemann-Hilbert problem on $\mathbb{C}$ with known jumps across the Stokes rays and known asymptotic expansion as $\hbar \to 0$. In this way we obtain the data consisting of a holomorphic vector bundle on $\mathbb{C}_\hbar$ and its section. These data encode analytic and resurgence properties of any exponential integral of the type $I_C(\hbar) = \int_C e^{f/\hbar} vol$ as $\hbar \to 0$.



S^1_i
z_i
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z_j
N. Katz : $\mathrm{Perv}_0 = \mathrm{Perv}$

Same data can be derived from the pair $(X, graph(df))$ of Lagrangian submanifolds of $(T^*X, \omega = Re(\omega^{2,0}/\hbar))$. Then n_{ij} appears as the number of pseudo-holomorphic discs with boundary on $X \cup graph(df)$ for $\hbar \in \mathbb{R}_{>0} \cdot (z_i - z_j)$.
Exponential integral can be treated as exponential period representing the pairing of the (twisted) global de Rham and global Betti cohomology. From the point of view of HFT this pairing is an incarnation of an equivalence of (some) Fukaya category of T^*X and the (some) category of holonomic DQ-modules on T^*X. This equivalence is an example of our generalized RH-correspondence.

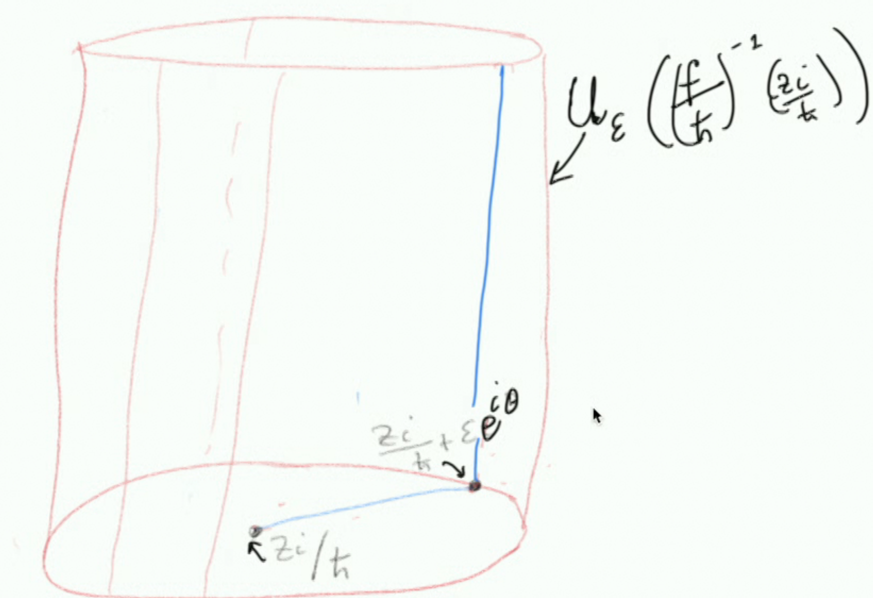
Wall-crossing structure for exponential integrals

In general WCS is a locally constant sheaf of stability data on a graded Lie algebra. In the case of exponential integrals the WCS is described by the following data:
i) local system of lattices $\Gamma_\hbar = \Gamma_{Arg(\hbar)} \simeq \mathbb{Z}^k$ on $\mathbb{C}^*_\hbar$,
ii) local system of central charges $Z_\hbar(e_i) = z_i/\hbar, 1 \leq i \leq k$ for the standard basis $e_i, 1 \leq i \leq k$ of $\mathbb{Z}^k$.
iii) Local system Γ-graded Lie algebras $\mathfrak{g}_\hbar = H^\bullet_{Betti,loc,\hbar}(X, f) \otimes \mathbb{C}$.
The RHS can be written as

$$\oplus_{i,j} Hom(H^\bullet((f/\hbar)^{-1}(D_\varepsilon(z_i/\hbar)), (f/\hbar)^{-1}(\frac{z_i}{\hbar} - \varepsilon), \mathbb{C}), H^\bullet((f/\hbar)^{-1}(D_\varepsilon(z_j/\hbar)), (f/\hbar)^{-1}(\frac{z_j}{\hbar} - \varepsilon), \mathbb{C}))$$

where $D_\varepsilon(p)$ is a small disc centerd at p. In algebra-geometric framework it is better to use $H^\bullet(Crit(f), \varphi_{f/\hbar}(\mathbb{Z}))$, where $\varphi_{f/\hbar}$ is the functor of the sheaf of vanishing cycles of $f/\hbar$.
iv) Final piece of WCS consists of Stokes automorphisms which arise from the Betti local-to-global isomorphisms. In the Morse case Stokes isomorphisms are derived from the collection of Stokes indices n_{ij}.

Local relative cohomology



Pairs of complex Lagrangian subvarieties related to Chern-Simons theory

1) Let M^3 be a compact oriented 3-manifold. Let G_c be a compact group and G be its complexification. Representing M^3 via a surgery along a knot $K \subset M^3$ one gets a complex symplectic manifold (more precisely, symplectic stack) of flat G-connections on the boundary of a small neighborhood of K. One has two complex Lagrangian subvarieties L_{in} and L_{out} consisting of those connections which can be extended inside (resp. outside) of a small neighborhood of K. The intersection $L_{in} \cap L_{out}$ can be identified with the set of flat G-connections on M^3. It is the same as the set of critical points of the (complexified) multivalued Chern-Simons functional

$$CS(A) = \int_{M^3} Tr(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A).$$

To avoid stacks one can work with the infinite-dimensional *manifold* $\mathcal{A}_{\mathbb{C}}^{fr}$ of G-connections trivialized at $x_0 \in M^3$. Then CS becomes a *multivalued* holomorphic function on $\mathcal{A}_{\mathbb{C}}^{fr}$. We can either consider it as a function on the universal abelian covering $\widehat{\mathcal{A}}_{\mathbb{C}}^{fr}$ or work with the holomorphic closed 1-form $\alpha_{CS} = dCS$.

K_2-Lagrangian subvarieties

The pair below gives in the end another approach to the same WCS. I am not going to discuss it today.

2) Another example related to the Chern-Simons theory and to the quantum dilogarithm

$$L_0 := \{(q_1, p_1, q_2, p_2, \ldots, q_n, p_n) \in \mathbb{C}^{2n} | e^{q_i} + e^{p_i} = 1, i = 1, \ldots, n\} \subset (\mathbb{C}^*)^{2n}$$

which is a complex Lagrangian submanifold of the manifold $(\mathbb{C}^*)^{2n}_{x_1,\ldots,x_n,y_1,\ldots,y_n}$ endowed with the standard symplectic form $\omega^{2,0} = \sum_{1\le i\le n} \frac{dx_i}{x_i} \wedge \frac{dy_i}{y_i}$. Here $x_i = e^{q_i}, y_i = e^{p_i}, 1 \le i \le n$.
Fix an integer symmetric matrix $(a_{ij})_{1\le i,j\le n}$ and consider the abelian subgroup in $\mathbb{Z}^n$.

$$P = \{(q_1, p_1, ..., q_n, p_n) | p_j = \sum_{1\le i\le n} a_{ij} q_i\}, 1 \le j \le n.$$

Finally $L_1 := L_P \subset (\mathbb{C}^*)^{2n}$ is the corresponding Lagrangian torus.

When quantizing the CS theory one should give a meaning to the Feynman integral over the totally real subvariety $\mathcal{A}_c^{fr} \subset \mathcal{A}_{\mathbb{C}}^{fr}$ of G_c-connections on principal G_c-*bundles* trivialized (framed) at the point x_0

$$I(k) = \int_{\mathcal{A}_c^{fr}} e^{kCS(A)/2\pi i} \mathcal{D}A.$$

Here $k \in \mathbb{Z}_{>0}$ is the level of the CS theory. If understood properly, this partition function should be given a mathematical meaning in terms of a $3d$ TQFT. This TQFT is defined in terms of the representation theory of quantum groups at roots of 1 (works of Reshetikhin,Turaev, Viro, Witten and others).

I would like to approach this ill-defined exponential integral perturbatively as a function $I = I(\hbar)$ of $\hbar = 2\pi i/k, \hbar \to 0$ by analogy with finite-dimensional exponential integrals. Then the level k does not have to be integer, and the group G_c can be replaced by G.

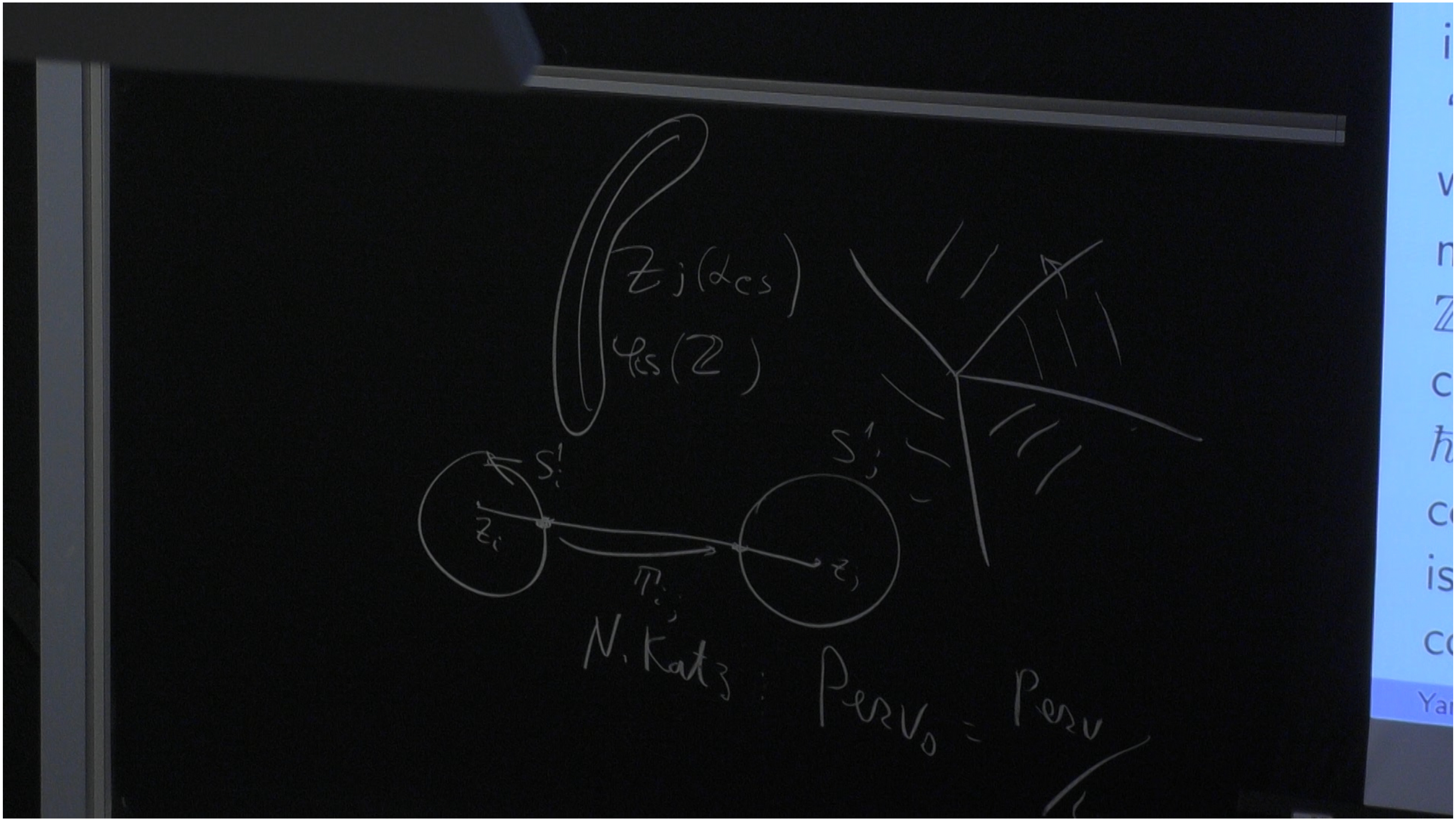
Remarks

In order to explain our approach I am going to assume that the *CS functional considered as a holomorphic function on the abelian universal covering* $\widehat{\mathcal{A}}_{\mathbb{C}}^{fr} \to \mathbb{C}$ *gives rise to a locally trivial infinite-dimensional fiber bundle outside of the set of critical values*.
Under some choices (which include the so-called *orientation data*) one can define the *sheaf of vanishing cycles* $\varphi_{CS}(\mathbb{Z})$ (maybe this is already a theorem). Sometimes I will abuse the notation and treat *CS* as an antiderivative of the 1-form α_{CS} in a small neighborhood of the zero locus $\mathcal{Z}(\alpha_{CS})$ with fixed value on each connected component. Then the sheaf of vanishing cycles is supported on the zero locus.

Let $\mathcal{Z}(\alpha_{CS}) = \sqcup_{j\in J}\mathcal{Z}_j(\alpha_{CS})$ be the decomposition of the zero locus into the finite union of connected components, i.e. $J = \pi_0(\mathcal{Z}(\alpha_{CS}))$.
Physics predicts existence of a $\mathbb{Z}$-equivariant linear map

$$R_j : H^0_c(\mathcal{Z}_j(\alpha_{CS}), \varphi_{CS}(\mathbb{Z})) \to \overline{\mathbb{C}((\hbar))}[\log \hbar],$$

which should be thought of as the formal expansion in $\hbar \to 0$ of the ill-defined "local Feynman integral" $\int_{\gamma_j} e^{\frac{CS}{\hbar}} vol, \hbar = 2\pi i/k$. Here γ_j should be thought of (via Poincaré duality) as a "middle-dimensional integration cycle", which coincides with the local thimble in the case when $\mathcal{Z}_j(\alpha_{CS})$ is an isolated simple zero. Informally *vol* is a "complexification of the Feynman measure" restricted to γ_j. The algebraic closure $\overline{\mathbb{C}((\hbar))} = \cup_{N\geq 1}\mathbb{C}((\hbar^{1/N}))$.
$\mathbb{Z}$-action on the LHS comes from the standard monodromy action on the sheaf of vanishing cycles, while the $\mathbb{Z}$-action on the RHS is given by $\hbar^{1/N} \mapsto e^{2\pi i/N}\hbar^{1/N}, \log\hbar \mapsto \log\hbar + 2\pi i, N \geq 1$. The degree of the compactly supported cohomology group (it is dual to the space of integration cycles) is chosen in such a way that it is compatible with the middle perversity. Hence the middle-dimensional integration cycle corresponds to degree zero.



$Z_j(\alpha_{cs})$
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S^1_i
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N. Katz : $\mathrm{Perv}_0 = \mathrm{Perv}$

One can treat R_j as a black box, without trying to give it a mathematical meaning. With Maxim we have another approach to CS theory via our theory of quantum wave functions. In that approach R_j can be defined rigorously, but then one has to show that the rigorous definition agrees with physics. In a special case when $\mathcal{Z}_j(\alpha_{CS})$ is a G-orbit of a rigid non-trivial flat irreducible connection R_j is given by the sum running over 3-valent graphs of expressions obtained by the standard Feynman rules. All summands are convergent integrals, and the sum does not depend on a choice of propagator.
In a sense all the information about Chern-Simons partition function is encoded in the analog of the above-discussed perverse sheaf $\mathcal{F}$ on $\mathbb{C}$ such that $\mathbf{R}\Gamma(\mathbb{C}, \mathcal{F}) = 0$. But first I am going to discuss a finite-dimensional model, which after generalization to infinite-dimensions will take into account the fact that CS is multivalued.

Finite-dimensional model

Let (X, vol, f) be a triple consisting of an n-dimensional complex manifold endowed with a holomorphic volume form vol and a holomorphic Morse function $f : X \to \mathbb{C}^*$. We assume that critical values of f is a finite set $Critval(f)$, and that f defines a locally trivial fiber bundle over the complement of the set of critical values.

Let $exp(s/2\pi i) : \mathbb{C} \to \mathbb{C}^*$ be the universal abelian covering and (X_1, vol_1, f_1) be the pullback of the above data to the universal $\mathbb{Z}$-covering $X_1 \to X$. Then the set $Critval(f_1))$ of critical values of f_1 consists of finitely many arithmetic series, and f_1 gives rise to a locally trivial fiber bundle outside of this set.

For any $s_1 \in \mathbb{C} - Critval(f_1)$ we have an isomorphism

$$H_n(X_1, f_1^{-1}(s_1), \mathbb{Z}) \simeq \oplus_{z_i \in Critval(f_1)} H_n(f_1^{-1}(D_\varepsilon(z_i)), f_1^{-1}(z_i + \varepsilon e^{\sqrt{-1}Arg(\theta(\gamma(s_1,z_i)))}), \mathbb{Z}),$$

where $D_\varepsilon(p)$ denote a small disc of radius ε with the center at p.

Remarks

This isomorphism depends on a choice of infinite collection of paths $\gamma(s_1, z_i)$ (Gabrielov paths) from s_1 to the critical values z_i such that the paths are disjoint outside of s_1. Generically s_1 does not belong to the countable set of straight lines through different pairs of critical values of f_1. Then there is a canonical choice of Gabrielov paths consisting of straight intervals. In this case $\theta(\gamma(s_1, z_i)) = Arg(s_1 - z_i)$. Let us assume for simplicity that we are in the generic case. The relative homology groups $H_n(X_1, f_1^{-1}(s_1), \mathbb{Z})$ form a local system of infinite rank over $\mathbb{C} - Critval(f_1)$. In the case of Morse critical points the fiber can be identified with $\mathbb{Z}[T^{\pm 1}] \otimes \mathbb{Z}^{Critval(f)}$.

We would like to do generalize this construction to the case of the complexified Chern-Simons theory, with CS functional being an analog of the function f_1 (but now $n = \infty$).
We do not want to develop a semi-infinite differential geometry in order to make the story completely similar to the finite-dimensional one. This is possible to do in case of Morse-Bott critical points, but not in general. We propose to use the cohomology with coefficients in the sheaf of vanishing cycles instead. As I have already mentioned this sheaf might be easier to define rigorously.
Assuming that we replace each summand $H_n(f_1^{-1}(D_\varepsilon(z_j)), f_1^{-1}(z_j + \varepsilon e^{\sqrt{-1}Arg(s_1 - z_j)}), \mathbb{Z})$ by the dual to the abelian group $H^0(\mathcal{Z}_j(\alpha_{CS}), \varphi_{CS \cdot e^{-\sqrt{-1}Arg(s_1 - z_j)}}(\mathbb{Z}))$.

Combining all critical values together we obtain a local system of infinite rank. Its fiber over $s_1 \in \mathbb{C} - Critval(CS)$ can be thought of as the "semi-infinite homology group" $H_\infty(\widehat{\mathcal{A}}^{fr}_{\mathbb{C}}, CS^{-1}(s_1), \mathbb{Z})$. In other words the fiber over generic s_1 is isomorphic to the dual to the product

$$\prod_{z_j \in Critval(CS)} H^0(\mathcal{Z}_j(\alpha_{CS}), \varphi_{CS \cdot e^{-\sqrt{-1}Arg(s_1 - z_j)}}(\mathbb{Z})).$$

We will sometimes use the notation $H_\infty(\widehat{\mathcal{A}}^{fr}_{\mathbb{C}}, CS^{-1}(s_1), \mathbb{Z})$ for this dual cohomology group, but you should remember that we do not have an alternative definition of the semi-infinite homology.

We expect that this local system can be extended to critical values of CS giving rise to a cosheaf of *countable* rank on $\mathbb{C}$. Keeping the notation $\mathcal{F}_{CS}$ for the dual sheaf one can show that $\mathbf{R}\Gamma(\mathbb{C}, \mathcal{F}_{CS}) = 0$ in agreement with the finite-dimensional case (warning: not quite trivial fact because of the failure of the Mittag-Leffler property).

CS Hodge structure

a) By analogy with the finite-dimensional case one can speculate that the abelian group $H_\infty(\widehat{\mathcal{A}}^{fr}_{\mathbb{C}}, CS^{-1}(s_1), \mathbb{Z})$ carries a weight filtration of infinite rank, while the vector space $H_\infty(\widehat{\mathcal{A}}^{fr}_{\mathbb{C}}, CS^{-1}(s_1), \mathbb{Z}) \otimes \mathbb{C}$ caries a Hodge filtration of infinite rank. In this sense one can speak about the Chern-Simons Hodge structure of infinite rank. We expect that the weight filtration is non-trivial (evidence: non-symmetry in the interaction of rigid critical flat connections $\rho \neq 1$ and $\rho = 1$).

b) The sheaf $\mathcal{F}_{CS}$ can be thought of as a "perverse sheaf of infinite rank". Perverse extension to singular points should be possible because the non-trivial blocks of the monodromy matrices have finite size.

c) By analogy with the finite-dimensional case one can hope that this "perverse sheaf of infinite rank" gives a wall-crossing structure. This "CS wall-crossing structure" is expected to be analytic. According to the general resurgence conjecture from our paper with Kontsevich arXiv:2005.10651 analyticity of this WCS should imply resurgence of the local perturbative expansions in the CS theory.

Resurgence of Chern-Simons local perturbative expansions

For each rigid flat connection $\rho \neq 1$ with a trivial stabilizer consider the Borel transform $\mathcal{B}(I_\rho(\hbar))$ of the corresponding (modified) local asymptotic expansion $I_\rho(\hbar) \in \mathbb{C}[[\hbar]]$ of the complexified CS partition function. Since $I_\rho(\hbar)$ is obtained by means of Feynman rules it is well-defined as a series in $\hbar$. Then after analytic continuation $\mathcal{B}(I_\rho(\hbar)) := \mathcal{B}(I_\rho(\hbar))(s)$ has poles at the points $s = s_m^\rho$ which belong to the arithmetic series $z_\rho + (2\pi i)^2 m, m \in \mathbb{Z}$, where z_ρ is the critical value of CS at ρ. Hence the general resurgence conjecture means that: Local perturbative expansion $I_\rho(\hbar) \in \mathbb{C}[[\hbar]]$ at the rigid flat connection $\rho \neq 1$ as above is resurgent.

By analogy with finite-dimensional case we may assume that the integrals over the thimbles in the above Conjecture give sections of the holomorphic bundle of finite rank over a small punctured disc in $\mathbb{C}_\hbar$. The Conjecture guarantees that the LHS is also a holomorphic section of this bundle. One can speculate that the values of this section at the points $\hbar = 2\pi i/k, k = 1, 2, ...$ coincide with the corresponding Reshetikhin-Turaev invariants RT_k.

More explicit conjecture about the integral over the cycle of unitary connections

Let us fix the level $k \in \mathbb{Z}_{\geq 1}$. Then there exist cycles γ_{ρ_j} and $n_{\rho_j} \in \mathbb{Z}$ such that

$$\int_{\mathcal{A}_c^{fr}} e^{\frac{kCS}{2\pi i}} \mathcal{D}A = \sum_{\rho_j,\, s.t.\, |exp(CS(\rho_j)/2\pi i)| \leq 1} n_{\rho_j} \int_{\gamma_{\rho_j}} e^{\frac{kCS}{2\pi i}} \mathcal{D}A,$$

where $n_{\rho_j} \in \mathbb{Z}$ is the virtual number solutions of the (generalized) Kapustin-Witten equation on $M^3 \times [0, \infty)$ with the unitary boundary conditions at $M^3 \times \{0\}$ and flat boundary conditions $\rho_j, |exp(CS(\rho_j)/2\pi i)| \leq 1$ at $M_3 \times \{\infty\}$. Here γ_{ρ_j} denote a cycle (i.e. a representative of an element of $H_\infty(\widehat{\mathcal{A}}_{\mathbb{C}}^{fr}, Im(CS) \lll 0, \mathbb{Z})$) emerging from the critical point ρ_j and satisfying the condition $Re(kCS/2\pi i) < 0$ along the cycle.

There is a finite-dimensional toy-model example illustrating the conjecture. Namely, let $f = z - log(z)$ and $I_k = \int_{|z|=1} e^{kf(z)} \frac{dz}{z}$. Then $I_k = \frac{1}{2\pi i} \int_{|z|=1} z^{-k} e^{kz} \frac{dz}{z} = \frac{k^k}{k!}$. Consider the generating function $N(w) = \sum_{k \geq 1} w^k k^k / k!$. This function has ramifications at the critical values of f i.e. at $0, e^{-1}, \infty$. Setting $0^0 = 1$ we can rewrite $N(w)$ such as follows

$$N(w) = \sum_{k \geq 0} \frac{1}{2\pi i} w^k \int_{|z|=1} (e^z/z)^k \frac{dz}{z} = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{w^{-1} - \frac{e^z}{z}} \frac{dz}{z}.$$

The poles $z_n(w), n \geq 1$ of the denominator $w^{-1} - \frac{e^z}{z}$ form a countable subset of $\mathbb{C}$, and the group $H_1(\mathbb{C}^* - \{z_n(w)\}_{n \geq 1}, \mathbb{Z})$ is generated by the cycle $|z| = 1$ and small circles about the poles. One can show that $\int_{|z - z_n(w)| \ll 1} \frac{1}{w^{-1} - \frac{e^z}{z}} \frac{dz}{z} = 1 + \frac{1}{z_n(w)}$. The generic fiber of the homology bundle carries two permutations: one is a permutation of two elements, while the other pne is an infinite cyclic shift. Making the analogy with the CS functional we see that the cycle about 0 corresponds to the trivial local system $\rho = 1$, while other cycles give residues and correspond to rigid local systems $\rho \neq 1$.