

Title: Horizon entropy and the Einstein equation - Lecture 20230221

Speakers: Ted Jacobson

Collection: Horizon entropy and the Einstein equation

Date: February 21, 2023 - 10:00 AM

URL: <https://pirsa.org/23020042>

Abstract:

<https://pitp.zoom.us/j/96212372067?pwd=dWVaUFFFFc3c5NTlVTDFHOGhCV2pXdz09>

Course purpose : (a) review & critique two derivations of
Einstein eq'n based on assumptions abt.
horizon entropy.

(b) introduce some useful tools & facts abt. GR.

$$C=1$$

1st law of BH mechanics

surface grav $\rightarrow \kappa$ area mass (energy) angular velocity of horizon
 $\frac{\kappa}{8\pi G}$ δA $=$ $\delta M - \Omega \delta J$ (stationary companion) version

T_H δS

$\frac{\hbar \kappa}{2\pi}$ $\delta \left(\frac{A}{4\hbar G} \right)$

S_{BH}

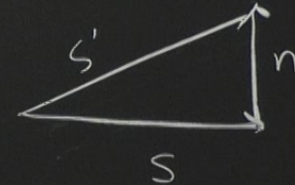
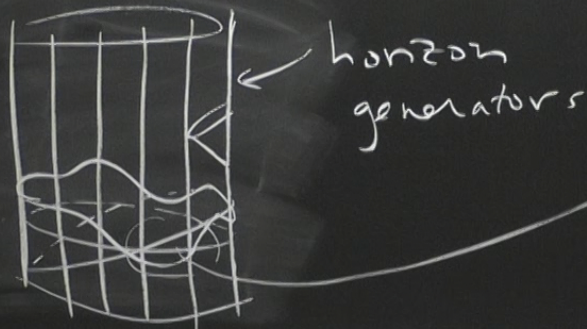
Hawking temperature.

$$T_H = \frac{\hbar \kappa}{2\pi}$$

horizon entropy

(b) introduce some useful tools & facts abt GR.

$$C=1$$



$$S'S' = (S+n) \cdot (S+n) = S \cdot S + 2S \cdot n$$

$$\frac{\hbar c}{2\pi}$$

$$\frac{A}{4\hbar G}$$

$$S_{BH}$$

$$l_H = \frac{\hbar c}{2\pi}$$

$$\lambda' = a\lambda + b,$$

class of affine parameters on null geodesics.



if k^a is null tangent vector to a geodesic,

& $k^b \nabla_b k^a = 0$, then k^a is an affine param tangent.

& $k^a \nabla_a \lambda = 1$ defines affine param. λ



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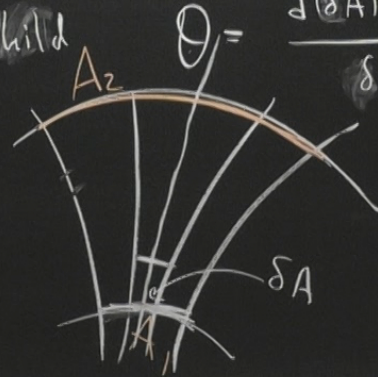
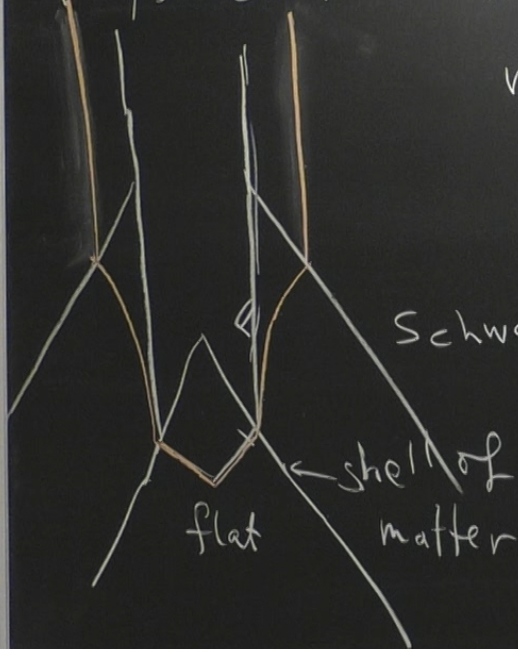
$$k^\alpha = \frac{dx^\alpha}{d\lambda}$$

& $k^a \nabla_a \lambda = 1$ defines affine param. λ

Physical Process version of 1st law of BH mechanics
 want to compute ΔA

introduce expansion Θ of null generators

$$\Theta = \frac{d(\delta A)/d\lambda}{\delta A}, \quad \lambda = \text{affine parameter}$$



Schwarzschild

A_2

δA

CAUTION

CAUTION

$$A_2 - A_1 = \int_1^2 d\lambda \, dA \left(\frac{d(SA)/d\lambda}{SA} \right)^\theta = \int_1^2 dA = A_2 - A_1$$

$$= \int_1^2 d\lambda \, dA \left[\frac{d}{d\lambda} (\lambda \theta) - \lambda \frac{d\theta}{d\lambda} \right]$$

$$= \left[\int dA \, \lambda \theta \right]_1^2 - \int_1^2 d\lambda \, dA \, \lambda \frac{d\theta}{d\lambda}$$

CAUTION

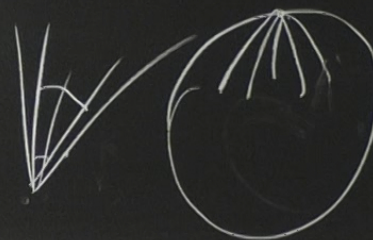
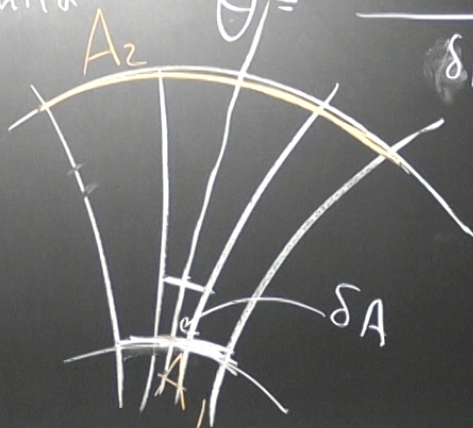
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Schwarzschild

$$\Theta = \frac{d(SA)/d\lambda}{SA}$$

λ = affine parameter

← shell of matter
flat



$$A_2 = \int_1^2 d\lambda \, dA \left(\frac{d(SA)/d\lambda}{SA} \right) = \int_1^2 dA = A_2 - A_1$$

$$\frac{d\theta}{d\lambda} = -\frac{1}{2}\theta^2 - \underset{\substack{\uparrow \\ \text{shear}}}{\sigma_{ab}\sigma^{ab}} - R_{ab}k^ak^b$$

(hypersurface
orthogonal case—
otherwise also
a twist² term)

$$A_2 - A_1 = \int_1^2 d\lambda \, dA \left(\frac{d(SA)/d\lambda}{SA} \right)^\theta = \int_1^2 dA = A_2 - A_1$$

$$= \int_1^2 d\lambda \, dA \left[\frac{d}{d\lambda}(\lambda\theta) - \lambda \frac{d\theta}{d\lambda} \right]$$

$$= \left[\int dA \, \lambda \theta \right]_1^2 - \int_1^2 d\lambda \, dA \, \lambda \left(\frac{d\theta}{d\lambda} \right) = \left[\right]_1^2 + \int_1^2 d\lambda \, dA \, \lambda \left[\frac{1}{2} \theta^2 + \sigma_{ab} \theta^{ab} + R_{ab} k^a k^b \right]$$

$$= \int_1 d\lambda dA \left[\frac{d}{d\lambda} (\lambda \Theta) - \lambda \frac{d\Theta}{d\lambda} \right]$$

$$= \left[\int dA \lambda \Theta \right]_1^2 - \int_1^2 d\lambda dA \lambda \left(\frac{d\Theta}{d\lambda} \right) = \left[\right]_1^2 + \int_1^2 d\lambda dA \lambda \left[\frac{1}{2} \dot{\theta}^2 + \sigma_{ab} \dot{\theta}^{ab} + R_{ab} k^a k^b \right]$$

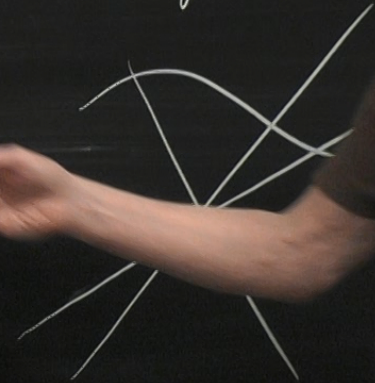
Physical Process version of 1st law of BH mechanics
want to compute ΔA

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$k^a \nabla_a \lambda = 1$ affine

if χ^a is the killing vector,

$$\chi^a \nabla_a v = 1, \quad v = \text{"Killing time"}$$



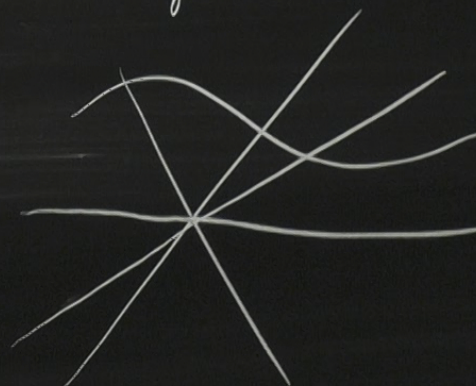
$$K^\alpha = \frac{dX^\alpha}{d\lambda}$$

& $K^a \nabla_a \lambda = 1$ defines affine param. λ

if χ^a is the Killing vector,

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$$\mapsto \chi^\alpha = \frac{dX^\alpha}{dv}$$



if χ^α is the killing vector,

$$\chi^\alpha \nabla_\alpha v = 1, \quad v = \text{"Killing time"}$$

$$\rightarrow \chi^\alpha = \frac{dx^\alpha}{dv} = \frac{d\lambda}{dv} \frac{dx^\alpha}{d\lambda} = \left(\frac{d\lambda}{dv} \right) K^\alpha;$$

$$\lambda = a e^{kv} + b$$

$$\lambda' = a\lambda + b, \quad a, b \text{ constant.}$$

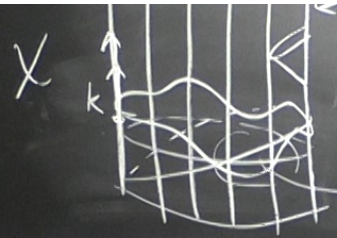
("Raychaudhuri eqn")

$$R_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2} T g_{ab} \right) \quad \text{Surface gravity } \kappa$$
$$\kappa = \lim_{\rightarrow \mathcal{H}} |d|X||$$

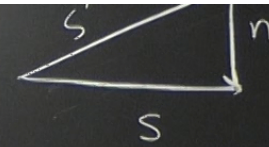
if X^a is the killing vector,

τ = "Killing time"

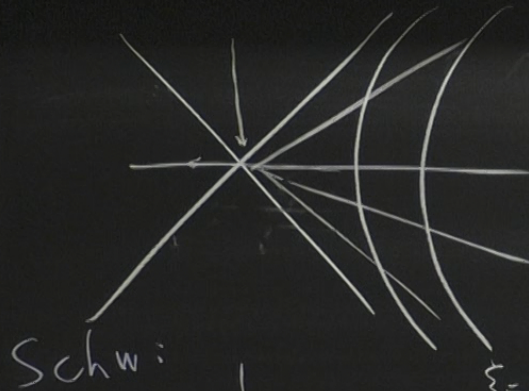
$$C=1$$



generators



$$S'S' = (S+n) \cdot (S+n) = S \cdot S + 2S \cdot n$$



$$\kappa = \frac{1}{4GM}$$

$$ds^2 = \xi^2 d\eta^2 - d\xi^2$$

(hyperbolic polar coordinates)

$$X = \frac{\partial}{\partial \eta} \quad \text{"boost killing vector"}$$

$$\xi = \text{const}$$

$$|X| = \xi, \quad d|X| = d\xi, \quad |d|X|| = 1$$

$$\kappa = \lim_{\lambda \rightarrow \infty} \frac{d|\chi|}{d\lambda}$$

if χ^α is the killing vector, $\frac{d\lambda}{dv} = \kappa(\lambda - \lambda_0)$

$$\chi^\alpha \nabla_\alpha v = 1, \quad v = \text{"Killing time"}$$

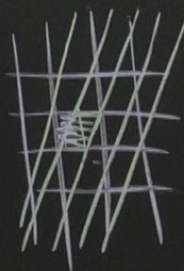
$$\rightarrow \chi^\alpha = \frac{dx^\alpha}{dv} = \frac{d\lambda}{dv} \frac{dx^\alpha}{d\lambda} = \left(\frac{d\lambda}{dv} \right) \chi^\alpha; \quad \lambda = a e^{\kappa v} + b$$

$$\lambda' = a\lambda + b, \quad a, b \text{ constant.}$$

$$= \dots + \int d\lambda dA \lambda 8\pi G T_{ab} k^a k^b$$

$$\chi^a = \kappa \lambda k^a \Big| = \dots + \frac{8\pi G}{\kappa} \int_{d\Sigma^b} d\lambda dA k^b T_{ab} \chi^a$$

$$dA \cancel{dx} \frac{dx^a}{dx}$$



Killing energy current \cdot Killing η

$$\nabla_a (T^{ab} \chi_b) = (\nabla_a T^{ab}) \chi_b + T^{ab} \nabla_a \chi_b$$

Physical Process version of 1st law of BH mechanics
want to compute ΔA

CAUTION
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WHEN IT IS BEING USED BY THE STUDENTS

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$$\Delta A = \left[\int dA \lambda \theta \right]^2 + \frac{8\pi G}{\kappa} \Delta E_{\chi}$$

Killing energy flux into
BH

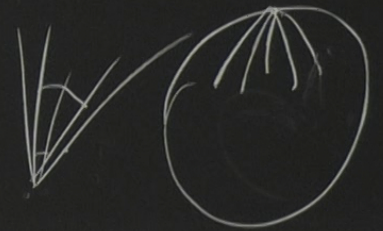
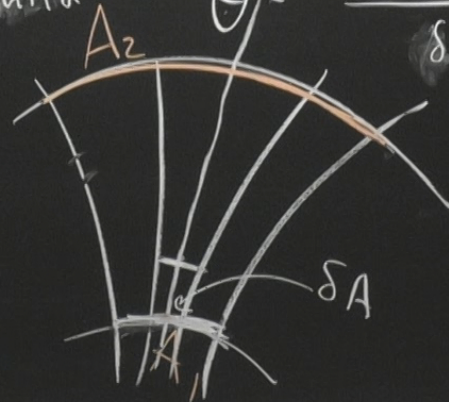
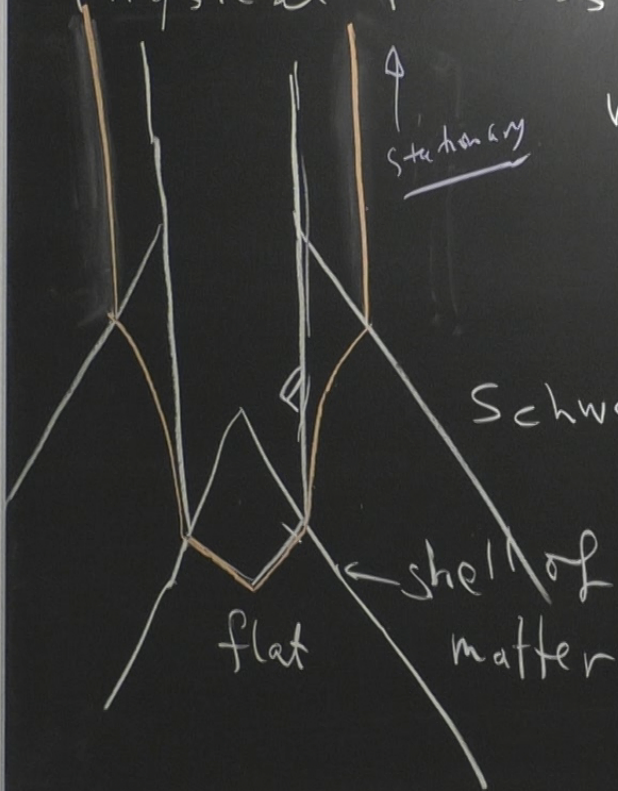
$\leftrightarrow \Delta M$

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(stationary companion version)

$$T_H \delta S$$

$$\frac{\hbar \kappa}{2\pi} \delta \left(\frac{A}{4\hbar G} \right)$$

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Hawking temperature.

$$T_H = \frac{\hbar \kappa}{2\pi}$$

Course purpose: (a) review & critique two derivations of Einstein eq'n based on assumptions abt.

if final state is

stationary, $\partial_t \rightarrow 0$

$$\leftrightarrow \Delta M$$

$$\lambda = \frac{1}{a} e^{k\psi}, \text{ so } \lambda \rightarrow 0 \text{ as } \psi \rightarrow "-\infty"$$

$$\frac{\kappa}{8\pi G} \Delta A = \Delta E_\chi; \quad \chi = \partial_t + \frac{\Omega}{\lambda} \partial_\phi$$

$$\chi^\alpha \nabla_\alpha \psi = 1, \quad \psi = \text{Killing time}$$

$$\hookrightarrow \chi^\alpha = \frac{dx^\alpha}{d\psi} = \frac{d\lambda}{d\psi} \frac{dx^\alpha}{d\lambda} = \left(\frac{d\lambda}{d\psi} \right) k^\alpha$$

$$\lambda = a e^{k\psi} + b$$

$$\Delta A = \left[\int dA \chi \theta \right]_1^2 + \frac{8\pi G}{\kappa} \Delta E_\chi \quad \text{Killing energy flux into BH}$$

If final state is

stationary, $\theta_2 \rightarrow 0$

$$\boxed{\begin{aligned} p \cdot \chi &= p \cdot (\partial_t + \Omega \partial_\phi) \\ &= \delta E - \Omega \delta J \end{aligned}} \quad \longleftrightarrow \quad \Delta M$$

$$\lambda = \frac{1}{a} e^{\kappa v}$$

so $\lambda \rightarrow 0$ as $v \rightarrow -\infty$

$$\begin{aligned} L &= -p \cdot \partial_\phi \\ E &= p \cdot \partial_t \end{aligned}$$

$$\frac{\kappa}{8\pi G} \Delta A = \Delta E_\chi; \quad \chi = \partial_t + \Omega \partial_\phi$$