Title: The structure of translational tilings

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Abstract: Translational tiling is a covering of a space (e.g., Euclidean space) using translated copies of a building block, called a "tile", without any positive measure overlaps. What are the possible ways that a space can be tiled?
One of the most well known conjectures in this area is the periodic tiling conjecture. It asserts that any tile of Euclidean space can tile the space periodically. This conjecture was posed 35 years ago and has been intensively studied over the years. In a joint work with Terence Tao, we disprove the periodic tiling conjecture in high dimensions. In the talk, I will motivate this result and discuss our proof.

The structure of translational tilings

Rachel Greenfeld

Institute for Advanced Study

Joint work with Terence Tao

Colloquium

Perimeter Institute, February 2023

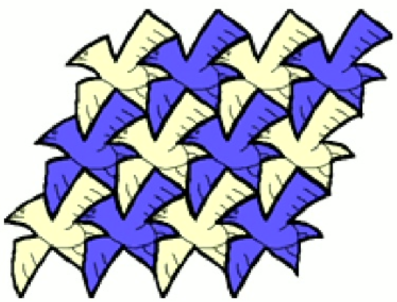
Tiling $\mathbb{R}^d$ by translations

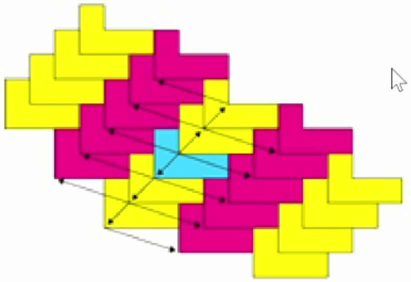
A bounded, measurable set $\Omega \subset \mathbb{R}^d$ **tiles the space by translations** if there exists a countable set $T \subset \mathbb{R}^d$ such that the family

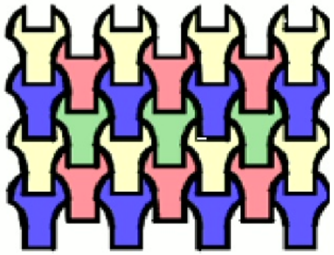
$$\{\Omega + t\}_{t \in T}$$

constitutes a partition of $\mathbb{R}^d$ (up to measure zero).

In which case we say that T **is a tiling of** $\mathbb{R}^d$ **by** Ω.







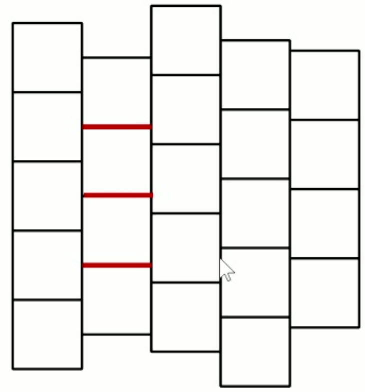
These tiling sets are lattices, i.e. subgroups of $\mathbb{R}^d$ with compact quotient.

Tiling the plane by a rectangle

Examples

Let $\Omega = Q_d$ be a d-dimensional cube.

Then $\mathrm{Tile}(Q_d; \mathbb{R}^d)$ gets extremely complicated as d increases.



Does any cube tiling contain two cubes that share a common facet?

Keller's conjecture (1930): The answer is positive.

Keller's conjecture (1930)

Any tiling by the cube Q_d contains two cubes sharing a common facet.

Theorem (Perron, 1940)

If there is a counterexample to Keller's conjecture in $\mathbb{R}^d$, then there is also a counterexample tiling T having two properties:

- T is $2\mathbb{Z}^d$-periodic.
- $T \subset 2^{-d}\mathbb{Z}^d$.

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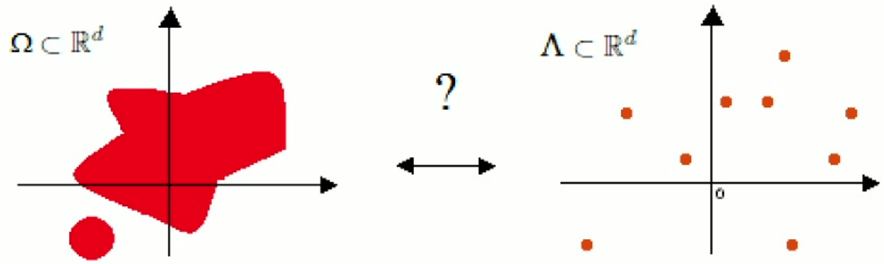
- T is $2\mathbb{Z}^d$-periodic.
- $T \subset 2^{-d}\mathbb{Z}^d$.

For each d, the d-dimensional question is **decidable**.

There is an algorithm which yields a solution in a finite number of steps.

Spectral sets

Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set.



If there exists a frequency set $\Lambda \subset \mathbb{R}^d$ such that the system

$$E(\Lambda) = \left\{ e^{2\pi i \langle \lambda, x \rangle} \right\}_{\lambda \in \Lambda}$$

constitutes an orthogonal basis for $L^2(\Omega)$, then the set Ω is called **spectral** and Λ is a **spectrum** for Ω.

Fuglede's conjecture (1974)

A bounded measurable set $\Omega \subset \mathbb{R}^d$ is spectral if and only if $\mathrm{Tile}(\Omega; \mathbb{R}^d)$ is not empty.

- **Fuglede (1974):** When Λ is a lattice:
  $\Lambda \in \mathrm{Tile}(\Omega; \mathbb{R}^d)$ if and only if Λ^* is a spectrum for Ω.
- **Fuglede (1974):** Fuglede's conjecture holds for **balls** and **triangles**.
- **Łaba (2000):** Fuglede's conjecture holds for **unions of two intervals** in $\mathbb{R}$.

Spectral sets

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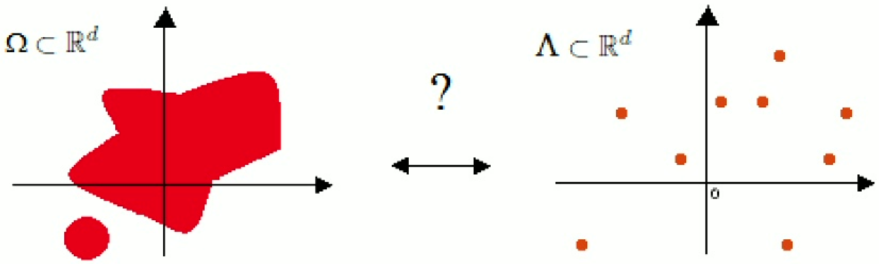

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Fuglede's conjecture (1974)

A bounded measurable set $\Omega \subset \mathbb{R}^d$ is spectral if and only if $\mathrm{Tile}(\Omega;\mathbb{R}^d)$ is not empty.

- **Tao (2004):** There are **spectral sets** in $\mathbb{R}^d$, $d \geq 5$ which **do not tile**.
- **Subsequently (Kolountzakis–Matolcsi (2006), others):** Both directions of the conjecture were **disproved** in dimensions $d \geq 3$.

All the known counterexamples to Fuglede's conjecture originate in finite abelian groups.

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All the known counterexamples to Fuglede's conjecture originate in finite abelian groups.

- Both directions of the conjecture are open in dimensions $d \leq 2$.

The structure of tilings

Let Ω be a bounded measurable set in $\mathbb{R}^d$.

What does the space $\mathrm{Tile}(\Omega;\mathbb{R}^d)$ look like?

The structure of tilings

Let Ω be a bounded measurable set in $\mathbb{R}^d$.

What does the space $\mathrm{Tile}(\Omega;\mathbb{R}^d)$ look like?

Periodic tiling conjecture
(Grünbaum–Shephard, 1987; Lagarias–Wang Y., 1996):

If Ω tiles $\mathbb{R}^d$ by translations, then $\mathrm{Tile}(\Omega;\mathbb{R}^d)$ contains a **periodic** set.

Ω must admit a tiling which is invariant under translations by some lattice.

Results

Let Ω be a bounded measurable set in $\mathbb{R}^d$.

The periodic tiling conjecture

If Ω tiles $\mathbb{R}^d$ by translations, then it admits a **periodic** tiling.

- **Venkov (1954), McMullen (1980):** The periodic tiling conjecture holds for **convex domains**. (Geometry)

Results

Let Ω be a bounded measurable set in $\mathbb{R}^d$.

The periodic tiling conjecture

If Ω tiles $\mathbb{R}^d$ by translations, then it admits a **periodic** tiling.

- **Venkov (1954), McMullen (1980):** The periodic tiling conjecture holds for **convex domains**. (Geometry)
- **Lagarias–Wang (1996):** **Every** tiling in $\mathbb{R}$ is periodic. (Fourier analysis)
- **Beauquier–Nivat (1990), Kenyon (1992):** The periodic tiling conjecture holds for **topological disks in** $\mathbb{R}^2$. (Geometry, topology)

The failure of PTC in high dimensions

Theorem (G.–Tao, 2022):

For sufficiently large d, the periodic tiling conjecture **fails** in $\mathbb{R}^d$.

Our threshold for "sufficiently large" is explicitly computable, but enormous. It remains open whether the periodic tiling conjecture could still be true in $\mathbb{R}^2$.

Our construction originates in a discrete setup.

Examples

For any singleton $F = \{x\} \subset \mathbb{Z}^d$,

$$\mathbb{Z}^d \oplus \{x\} = \mathbb{Z}^d.$$

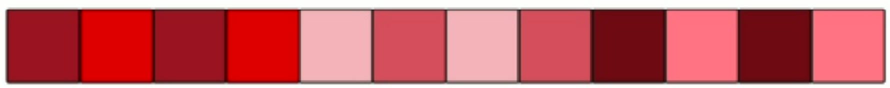
In fact, this is the **only** tiling be $F = \{x\}$, i.e.,

$$\mathrm{Tile}(\{x\}; \mathbb{Z}^d) = \{\mathbb{Z}^d\}.$$

Examples

For $F = \{0,2\} \subset \mathbb{Z}$,

$$(4\mathbb{Z} + \{0,1\}) \oplus \{0,2\} = \mathbb{Z}.$$



Discrete periodic tiling conjecture

Let F be a finite subset of $\mathbb{Z}^d$.

Discrete periodic tiling conjecture:

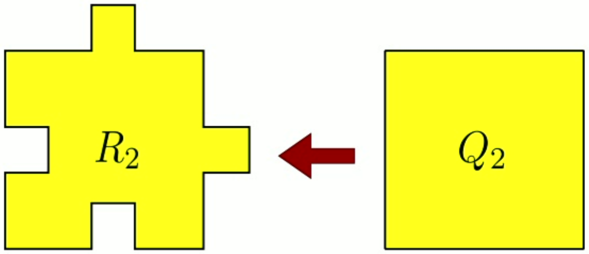
If F is a tile of $\mathbb{Z}^d$ then $\mathrm{Tile}(F;\mathbb{Z}^d)$ contains a **periodic** set.

Theorem (G.–Tao, 2022)

If the discrete periodic tiling conjecture fails in $\mathbb{Z}^d$, then the continuous periodic tiling conjecture fails in $\mathbb{R}^d$.

Let Q_d be a d-dimensional unit cube.

- If $A \in \mathrm{Tile}(F; \mathbb{Z}^d)$ then $A \in \mathrm{Tile}(F + Q_d; \mathbb{R}^d)$.
- Replace Q_d with a "rigid" fundamental domain of $\mathbb{R}^d/\mathbb{Z}^d$, R_d.



- If $A \in \mathrm{Tile}(F + R_d; \mathbb{R}^d)$ then $A + t \in \mathrm{Tile}(F; \mathbb{Z}^d)$ for some $t \in \mathbb{R}^d$.

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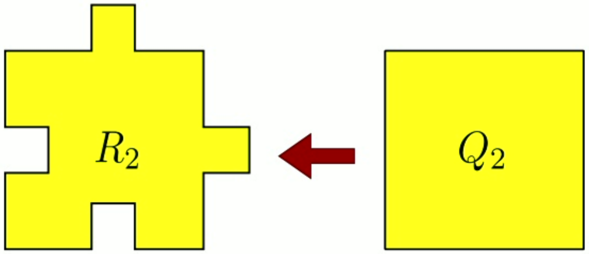

- If $A \in \mathrm{Tile}(F + R_d; \mathbb{R}^d)$ then $A + t \in \mathrm{Tile}(F; \mathbb{Z}^d)$ for some $t \in \mathbb{R}^d$.

Theorem (Meyerovitch–Sanadhya–Solomon, 2022)

The periodic tiling conjecture in $\mathbb{Z}^d$ implies the periodic tiling conjecture in $\mathbb{Z}^d/\Lambda$ for any subgroup Λ of $\mathbb{Z}^d$.

Decidability of tiling

Let F be a finite subset of $\mathbb{Z}^d$.

Can we **decide** whether F is a tile of $\mathbb{Z}^d$ in **finite** time?

Is there any algorithm which yields a solution in a finite number of steps?

H. Wang (1961)

The question: "can F tile $\mathbb{Z}^d$ by translations?" is **decidable** if any tile of $\mathbb{Z}^d$ admits a **periodic** tiling.

Available tools

- Geometry.
- Combinatorics. Coloring. Graph theory.
- Fourier analysis. $A \oplus F = \mathbb{Z}^d \Leftrightarrow \mathbb{1}_A * \mathbb{1}_F = 1 \Leftrightarrow \widehat{\mathbb{1}}_A \cdot \widehat{\mathbb{1}}_F = \delta_0$.
  Study the distribution $\widehat{\mathbb{1}}_A$ in $\mathcal{Z}(\widehat{\mathbb{1}}_F) \cup \{0\}$, where $\mathcal{Z}(\widehat{\mathbb{1}}_F)$ denotes the zero set of the trigonometric polynomial $\widehat{\mathbb{1}}_F$.
- Ergodic theory. $\mathrm{Tile}(F; \mathbb{Z}^d)$ can be interpreted as an ergodic point process on $\mathbb{Z}^d$.
- Spectral analysis. Study the spectral measure of the latter process.
- Number theory.
- Probabilistic methods.
- Model theory. Logical decidability.
- Computability theory. Algorithmic decidability.

Results (partial list)

Let F be a finite subset of $\mathbb{Z}^d$.

Discrete periodic tiling conjecture

If F is a tile of $\mathbb{Z}^d$ then $\mathrm{Tile}(F;\mathbb{Z}^d)$ contains a **periodic** set.

- **Newman (1977)**: **Any** set in $\mathrm{Tile}(F;\mathbb{Z})$ is periodic.
  (Pigeonholing)
- **Bhattacharya (2019); G.–Tao (2020)**: If $\mathrm{Tile}(F;\mathbb{Z}^2)$ is nonempty, then it contains a periodic set.
  (Ergodic theory, spectral analysis; combinatorics)
- **Szegedy (1998)**: The conjecture holds when $|F|$ is a prime and also when $|F| = 4$.
  (Combinatorics, number theory)

A counterexample to PTC

Theorem (G.–Tao, 2022):

For sufficiently large d, the discrete periodic tiling conjecture **fails** in $\mathbb{Z}^d$ and the continuous periodic tiling conjecture **fails** in $\mathbb{R}^d$.

In fact:

There exists a finite abelian group G_0 and a finite set F in $\mathbb{Z}^2 \times G_0$ such that F tiles $\mathbb{Z}^2 \times G_0$, but does **not** admit any periodic tiling.

Sudoku puzzle

9	4	2	1	6	3	8	5	7
5	3	6	2	8	7	9	4	1
8	7	1	9	5	4	2	3	6
3	2	7	8	1	9	4	6	5
1	5	4	3	2	6	7	9	8
6	9	8	7	4	5	1	2	3
2	6	5	4	7	1	3	8	9
7	8	9	6	3	2	5	1	4
4	1	3	5	9	8	6	7	2

A solution to a **standard Sudoku puzzle** consists of a function from a 9×9 board to a set of digits $\{1, \ldots, 9\}$, such that the restriction of the function to any row, column, or 3×3 block is a permutation.

Our construction: "q-adic Sudoku"

Let $q = 2^s$ be a sufficiently large power of 2.

Step 1: We introduce the notion of a **"q-adic Sudoku solution"**: a function from a board

$$\{1, \ldots, q^2\} \times \mathbb{Z}$$

to $\{1, \ldots, q-1\}$, which has a certain structure on any line of integer slope. We show that these solutions are non-periodic.

$\vdots$

3	2	1	3	3	2	1	2	3	2	1	1	3	2	1	2
2	1	3	3	2	1	2	3	2	1	1	3	2	1	1	3
1	3	3	2	1	2	3	2	1	1	3	2	1	1	3	2
3	3	2	1	2	3	2	1	1	3	2	1	3	3	2	1
3	2	1	2	3	2	1	1	3	2	1	2	3	2	1	3
2	1	2	3	2	1	1	3	2	1	1	3	2	1	3	3

$\vdots$

Encoding "q-adic Sudoku"

Let $q = 2^s$ be a sufficiently large power of 2.

Step 1: We introduce an **aperiodic** **"q-adic Sudoku puzzle"**: any solution is a "q-adic" structured function from a board

$$\{1, \ldots, q^2\} \times \mathbb{Z}$$

to $\{1, \ldots, q-1\}$.

Step 2: We **encode** the **q-adic Sudoku puzzle** as a **tiling equation** in $\mathbb{Z}^2 \times G_0$.

Step 1: Constructing "q-adic Sudoku"

Let $q = 2^s$ be a power of 2.

The function $f_q \colon \mathbb{Z} \to (\mathbb{Z}/q\mathbb{Z}) \setminus \{0\}$:

$$f_q(n) := \frac{n}{q^{\nu_q(n)}} \pmod q, \quad f_q(0) := 1$$

exhibits a q-**adic structure**. ($\nu_q(n)$ is the number of times q divides n)

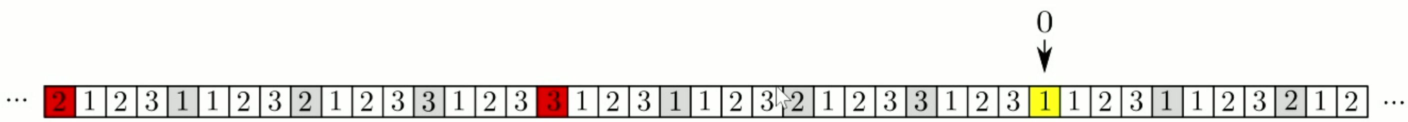

Figure: The function f_4. Gray cells indicate $\nu_4(n) = 1$ and red $\nu_4(n) = 2$.

Step 1: Constructing "q-adic Sudoku"

Let $\mathcal{S}_q$ denote the class of functions

$$g\colon \{1,\ldots,q^2\} \to (\mathbb{Z}/q\mathbb{Z}) \setminus \{0\}$$

such that there exist $a, b \in \mathbb{Z}$ such that

$$g(n) = f_q(an+b)$$

for all $n \in \{1,\ldots,q^2\}$.

2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

Figure: $a = 0$ and $b = 2$. No gray cells.

A $\mathcal{S}_q$-**Sudoku solution** is a function

$$F \colon \{1, \dots, q^2\} \times \mathbb{Z} \to (\mathbb{Z}/q\mathbb{Z}) \setminus \{0\}$$

with the property that for every slope $j \in \mathbb{Z}$ and intercept $i \in \mathbb{Z}$, the function

$$n \mapsto F(n, jn + i), \quad n = 1, \dots, q^2$$

lies in the class $\mathcal{S}_q$.

$\vdots$

2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

$\vdots$

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$\vdots$

3	2	1	3	3	2	1	2	3	2	1	1	3	2	1	2
2	1	3	3	2	1	2	3	2	1	1	3	2	1	1	3
1	3	3	2	1	2	3	2	1	1	3	2	1	1	3	2
3	3	2	1	2	3	2	1	1	3	2	1	3	3	2	1
3	2	1	2	3	2	1	1	3	2	1	2	3	2	1	3
2	1	2	3	2	1	1	3	2	1	1	3	2	1	3	3

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lies in the class $\mathcal{S}_q$.

$$\vdots$$

2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2

$$\vdots$$

A **Good $\mathcal{S}_q$-Sudoku solution** is a Sudoku solution

$$F\colon \{1,\dots,q^2\}\times\mathbb{Z}\to(\mathbb{Z}/q\mathbb{Z})\setminus\{0\}$$

such that for each $n\in\{1,\dots,q^2\}$ the column $m\mapsto F(n,m)$ of F satisfies

$$F(n,m)=\sigma_n(m \pmod q)\ \text{ for every } m\in\mathbb{Z} \text{ with } \sigma_n(m \pmod q)\neq 0$$

for some **permutation** $\sigma_n\colon \mathbb{Z}/q\mathbb{Z}\to\mathbb{Z}/q\mathbb{Z}$.

$\vdots$

3	2	1	3	3	2	1	2	3	2	1	1	3	2	1	2
2	1	3	3	2	1	2	3	2	1	1	3	2	1	1	3
1	3	3	2	1	2	3	2	1	1	3	2	1	1	3	2
3	3	2	1	2	3	2	1	1	3	2	1	3	3	2	1
3	2	1	2	3	2	1	1	3	2	1	2	3	2	1	3
2	1	2	3	2	1	1	3	2	1	1	3	2	1	3	3

$\vdots$

Step 2: Tiling encoding

Tiling language: we translate certain sentences into **tiling equations**.

Example: Let G_0 be a finite abelian group.

We want to say: A `is a graph of some function` $f : \mathbb{Z}^2 \to G_0$.

Step 2: Tiling encoding

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Example: Let G_0 be a finite abelian group.

We want to say: `A is a graph of some function` $f : \mathbb{Z}^2 \to G_0$.

But instead, we can say: $A \in \mathrm{Tile}(\{0\} \times G_0; \mathbb{Z}^2 \times G_0)$.

Indeed, $A \oplus (\{0\} \times G_0) = \mathbb{Z}^2 \times G_0$ if and only if for every $x \in \mathbb{Z}^2$

$$A_x \oplus G_0 = G_0$$

where A_x denotes the x'th slice $A_x := \{g \in G_0 \colon (x, g) \in A\}$ of A.

This means that A_x contains **one** point. In other words,

$A = \{(x, f(x)) \colon x \in \mathbb{Z}^2\}$ is a graph of some function $f : \mathbb{Z}^2 \to G_0$.

Step 2: Tiling encoding

Example: Let $N \geq 2$ and $G_0 = \mathbb{Z}/N\mathbb{Z}$.

We want to say: A is a graph of a function $f : \mathbb{Z} \to G_0$ with

$$f(x) = \sigma(x \pmod N) \text{ for every } x \in \mathbb{Z}$$

for some permutation $\sigma : G_0 \to G_0$.

But instead, we can say:

$$\begin{cases} A \in \mathrm{Tile}(\{0\} \times G_0; \mathbb{Z} \times G_0) \\ A \in \mathrm{Tile}(\{0, 1, \ldots, N-1\} \times \{0\}; \mathbb{Z} \times G_0) \end{cases}$$

The first equations encodes $A = \{(x, f(x)) : x \in \mathbb{Z}\}$ for some function $f : \mathbb{Z} \to G_0$. Then, the second equations encodes

$$\{f(x), f(x+1), \ldots, f(x+N-1)\} = \mathbb{Z}/N\mathbb{Z} \text{ for every } x \in \mathbb{Z}.$$

Theorem (G.–Tao, 2022):

Any system of M tiling equations in a finitely generated abelian group G

$$\begin{cases} X \oplus F^{(1)} = G \\ X \oplus F^{(2)} = G \\ \quad\vdots \\ X \oplus F^{(M)} = G \end{cases}$$

can be encoded as a **single** tiling equation in $G \times \mathbb{Z}/N\mathbb{Z}$:

$$X \oplus \tilde{F} = G \times \mathbb{Z}/N\mathbb{Z},$$

where $N = N(M)$, $E_1 \uplus \cdots \uplus E_M$ is a "rigid" partition of $\mathbb{Z}/N\mathbb{Z}$ and

$$\tilde{F} = \biguplus_{m=1}^{M} F^{(m)} \times E_m.$$

With our **tiling language** we encode the sentences:

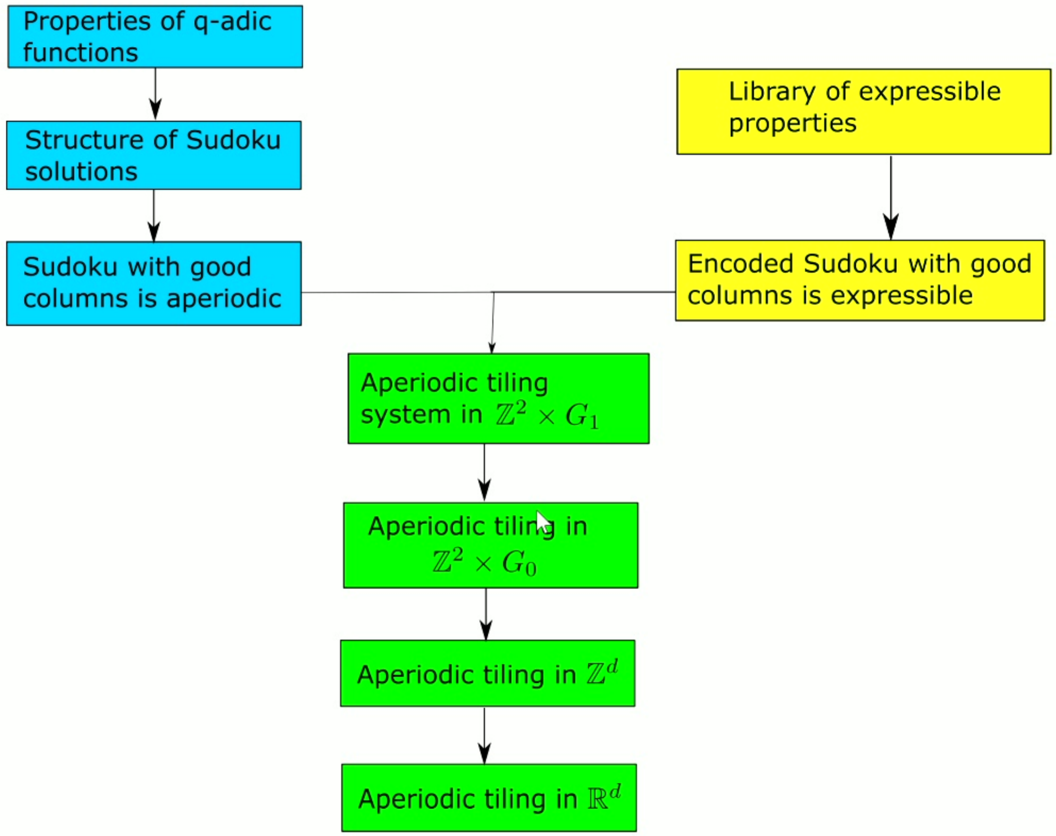
- For any $(i,j) \in \mathbb{Z}^2$, the function
$$n \mapsto g_n(jn+i), \quad n \in \{1, \dots, q^2\}$$
is in $\mathcal{S}_q$.
- For each $n \in \{1, \dots, q^2\}$ there is a permutation $\sigma_n \colon \mathbb{Z}/q\mathbb{Z} \to \mathbb{Z}/q\mathbb{Z}$ such that
$$g_n(m) = \sigma_n(m \pmod{q}) \text{ whenever } \sigma_n(m \pmod{q}) \neq 0.$$

as a **system of tiling equations with a single tile** in $\mathbb{Z}^2 \times G_0$, for some finite abelian group G_0.

We translate the latter **system** of tiling equations into a **single tiling equation with a single tile** in $\mathbb{Z}^2 \times G_0 \times \mathbb{Z}/N\mathbb{Z}$.

This equation encodes a **good $\mathcal{S}_q$-Sudoku puzzle**, thus, is **aperiodic**.

Our proof



Properties of q-adic functions
Structure of Sudoku solutions
Sudoku with good columns is aperiodic
Library of expressible properties
Encoded Sudoku with good columns is expressible
Aperiodic tiling system in $\mathbb{Z}^2 \times G_1$
Aperiodic tiling in $\mathbb{Z}^2 \times G_0$
Aperiodic tiling in $\mathbb{Z}^d$
Aperiodic tiling in $\mathbb{R}^d$

The failure of the PTC

Theorem (G.–Tao, 2022):

- There exists a finite abelian group G_0 such that the **periodic tiling conjecture fails** in $\mathbb{Z}^2 \times G_0$.
- There exists $d_0 > 2$ such that the **periodic tiling conjecture fails** in $\mathbb{Z}^d$ for every $d \geq d_0$.
- There exists $d_0 > 2$ such that the **periodic tiling conjecture fails** in $\mathbb{R}^d$ for every $d \geq d_0$.

What's next?

- What is the **largest** d for which the periodic tiling conjecture tiling conjecture holds in $\mathbb{Z}^d$ (or $\mathbb{R}^d$)?
- Let $d \geq 1$. We saw that the PTC in $\mathbb{R}^d$ implies the PTC in $\mathbb{Z}^d$.

  Does **discrete** PTC in $\mathbb{Z}^d \implies$ **continuous** PTC in $\mathbb{R}^d$?

  The answer is positive when $d = 1$ (Lagarias–Wang, 1996),

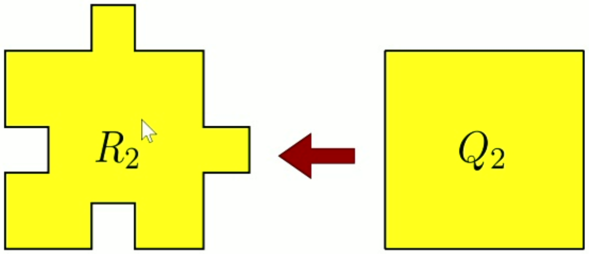
  but for $d \geq 2$, the question is still open.
- Is the tiling problem **decidable** in $\mathbb{Z}^d$ for all $d \geq 1$?
- Is the periodic tiling conjecture true for **connected** tiles?
- Is there any **aperiodic tile** which is **spectral**?
- Could our new tools be adapted to other long standing tiling-related problems?

  E.g., **Nivat's conjecture** and the **Coven–Meyerowitz conjecture**.

Thank you!

Let Q_d be a d-dimensional unit cube.

- If $A \in \mathrm{Tile}(F; \mathbb{Z}^d)$ then $A \in \mathrm{Tile}(F + Q_d; \mathbb{R}^d)$.
- Replace Q_d with a "rigid" fundamental domain of $\mathbb{R}^d/\mathbb{Z}^d$, R_d.



- If $A \in \mathrm{Tile}(F + R_d; \mathbb{R}^d)$ then $A + t \in \mathrm{Tile}(F; \mathbb{Z}^d)$ for some $t \in \mathbb{R}^d$.

Step 1: Constructing "q-adic Sudoku"

Let $q = 2^s$ be a power of 2.

The function $f_q \colon \mathbb{Z} \to (\mathbb{Z}/q\mathbb{Z}) \setminus \{0\}$:

$$f_q(n) := \frac{n}{q^{\nu_q(n)}} \pmod{q}, \quad f_q(0) := 1$$

exhibits a q-**adic structure**. ($\nu_q(n)$ is the number of times q divides n)

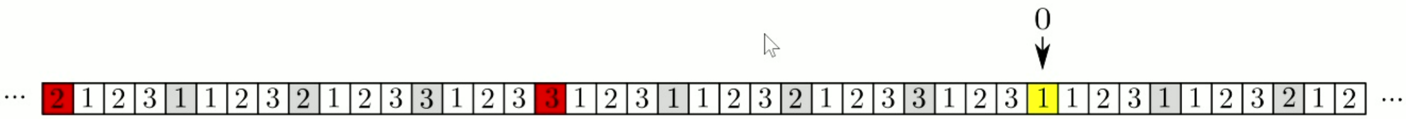


Figure: The function f_4. Gray cells indicate $\nu_4(n) = 1$ and red $\nu_4(n) = 2$.

q-adic structure is **non-periodic** (aka Besicovitch almost periodic).