Title: Introduction to Colloquium

Speakers: Aldo Riello

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Tiling, periodicity, and decidability

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Tiling by translations

Let $G = (G, +)$ be an abelian group (for instance, $\mathbb{R}^d$, $\mathbb{Z}^d$).

A bounded set $F \subset G$ of positive measure **tiles the space by translations** if there exists a set $A \subset G$ such that the family

$$\{\Omega + a\}_{a \in A}$$

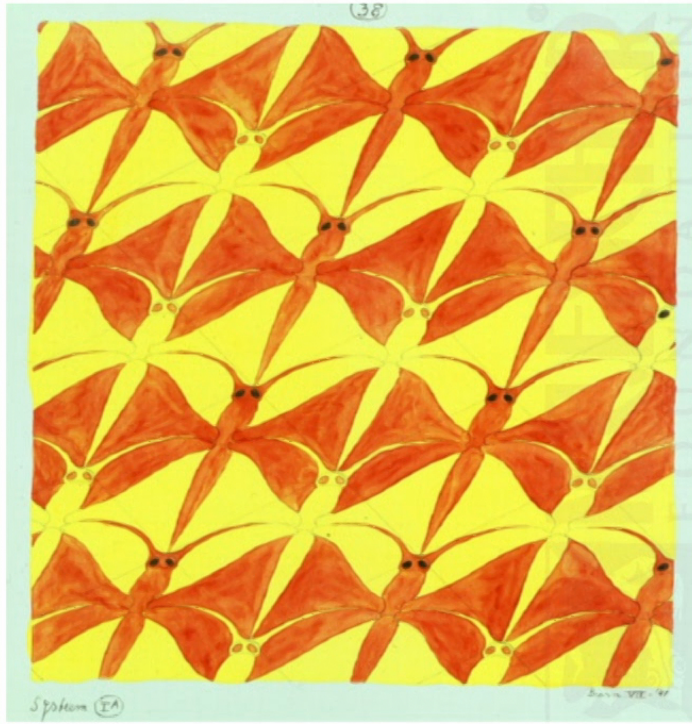
constitutes a partition of G (up to measure zero). In other words,

$$\mathbb{1}_A * \mathbb{1}_F(x) = \sum_{f \in F} \mathbb{1}_A(x - f) =_{\text{a.e.}} 1.$$

We say that **F is a tile of G and A is a tiling of G by F**, and write:

$$A \oplus F = G.$$

Translational tilings



Tiling $A \oplus F = \mathbb{R}^2$ of the plane by translations of a butterfly F.
[Escher, 1941]

Translational tilings

Tiling $A \oplus F = \mathbb{R}^2$ of the plane by translations of a rectangle F.

Periodic tiling

Suppose that A is a tiling of G by F, i.e.,

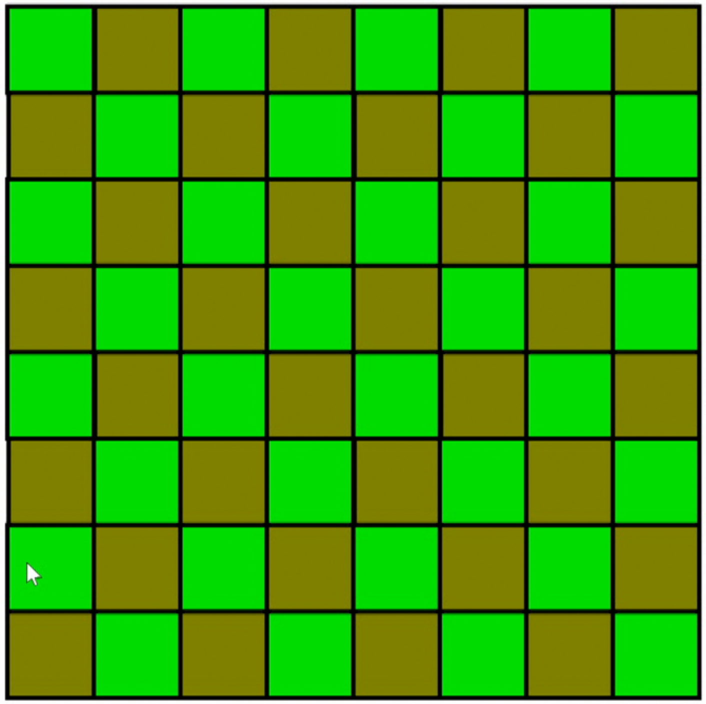
$$A \oplus F =_{\text{a.e.}} G.$$

We say that the tiling A is **periodic** if there is a **lattice** $\Lambda \subset G$ (i.e., a discrete subgroup whose quotient G/Λ is compact) such that

$$A + \lambda = A \quad \text{for every} \quad \lambda \in \Lambda.$$

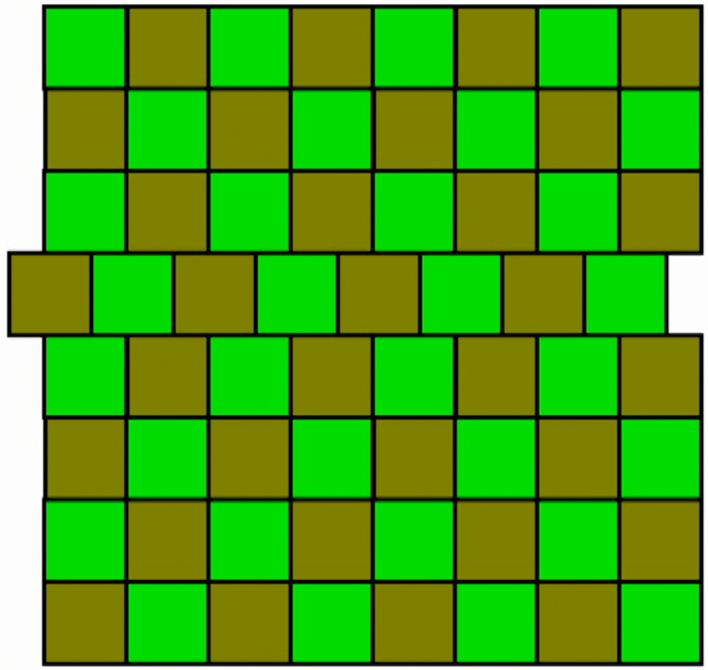
In other words A is invariant under translations by Λ.

Periodic tiling



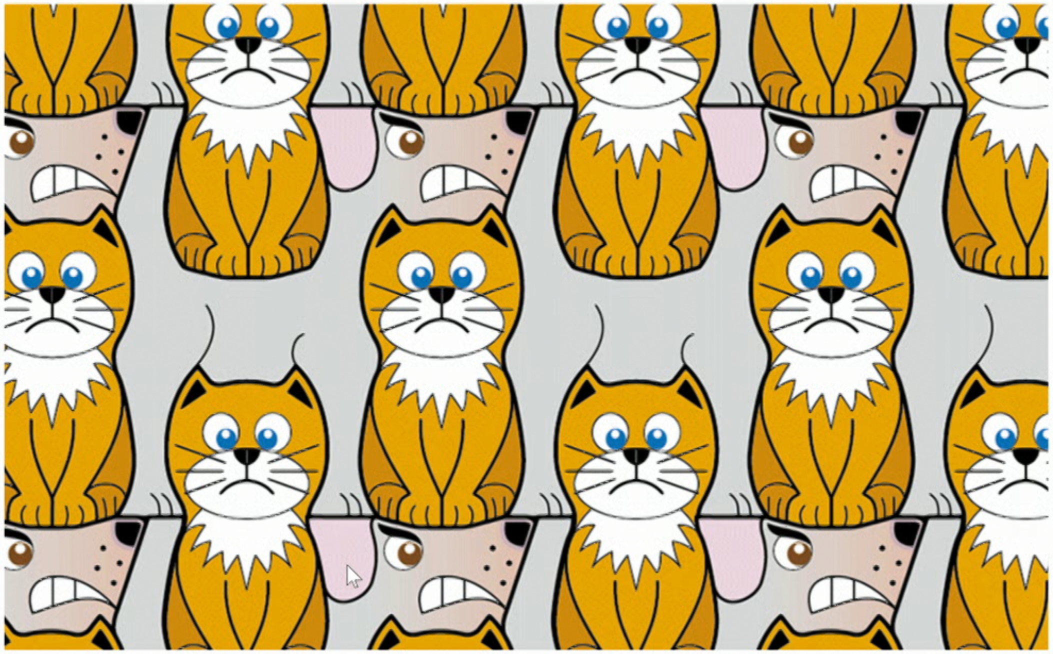
Periodic tiling $\mathbb{Z}^d \oplus F = \mathbb{R}^2$ of the plane by translations of a unit square F.

Non-periodic tiling



Non periodic tiling $A \oplus F = \mathbb{R}^2$ of the plane by translations of a square F.

Tiling by translations of multiple tiles



A periodic tiling $(A_1 \oplus F_1) \uplus (A_2 \oplus F_2) = \mathbb{R}^2$ of the plane by a cat F_1 and a dog F_2. [Nicolas, 1999]

Tiling by translations of multiple tiles

A periodic tiling $(A_1 \oplus F_1) \uplus (A_2 \oplus F_2) = \mathbb{R}^2$ of the plane by a seahorse F_1 and a rotated seahorse F_2. [Escher 1938]

Tiling by translations of multiple tiles

A periodic tiling $(A_1 \oplus F_1) \uplus (A_2 \oplus F_2) \uplus (A_3 \oplus F_3) \uplus (A_4 \oplus F_4) = \mathbb{R}^2$ of the plane by four rotations F_1, F_2, F_3, F_4 of a lizard. [Escher 1937]

Vague questions

Let F be a bounded subset of G with positive measure.

- Can we **decide** whether F is a **tile** of G?

Questions

Let's make it more precise...

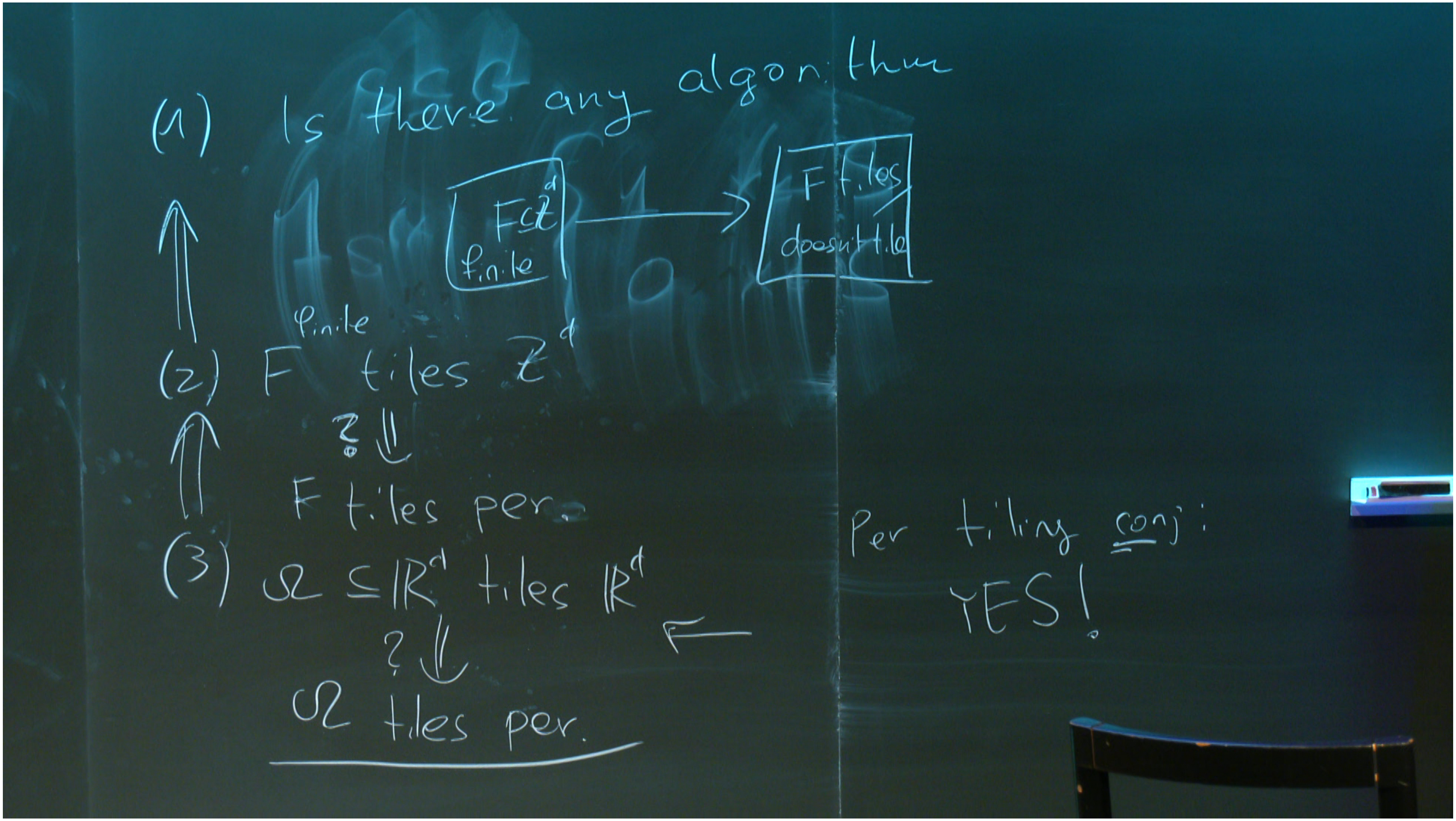
(1) Is there an **algorithm** which computes in a **finitely many steps** upon any given subset of $\mathbb{Z}^d$ whether it tiles $\mathbb{Z}^d$ by translations?

(2) Suppose that a finite $F \subset \mathbb{Z}^d$ tiles $\mathbb{Z}^d$ by translations.

Does it admit a **periodic** tiling?

(3) Suppose that a bounded set $\Omega \subset \mathbb{R}^d$ tiles $\mathbb{R}^d$ by translations.

Does it admit a **periodic** tiling?


(1) Is there any algorithm
$F \subset \mathbb{Z}^d$ finite
F tiles / doesn't tile
(2) F finite tiles $\mathbb{Z}^d$
? ⇓
F tiles per.
(3) $\Omega \subseteq \mathbb{R}^d$ tiles $\mathbb{R}^d$
? ⇓
Ω tiles per.
Per tiling conj:
YES!

Conjecture

Let Ω be a bounded measurable set in $\mathbb{R}^d$.

Periodic tiling conjecture
(Grünbaum–Shephard, 1987; Lagarias–Wang Y., 1996):

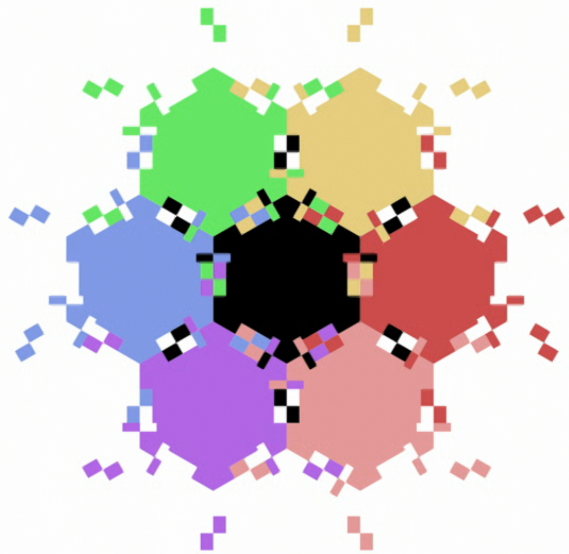
If Ω tiles $\mathbb{R}^d$ by translations, then there is a **periodic** tiling of $\mathbb{R}^d$ by Ω.

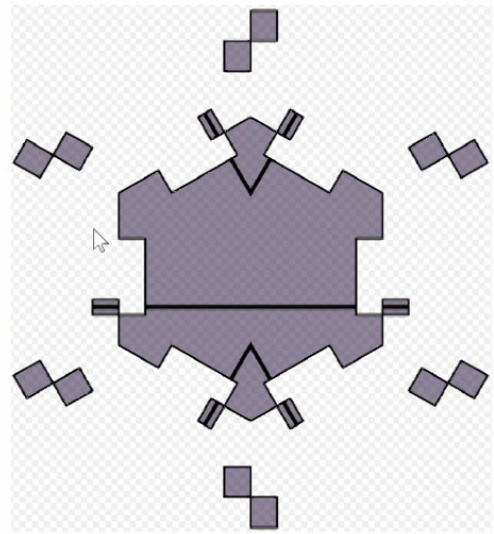
Informally: "Any tiling can be repaired to be periodic."

If the conjecture **holds**, it implies a positive answer to **all** our questions!

Are there any **aperiodic** tilings?

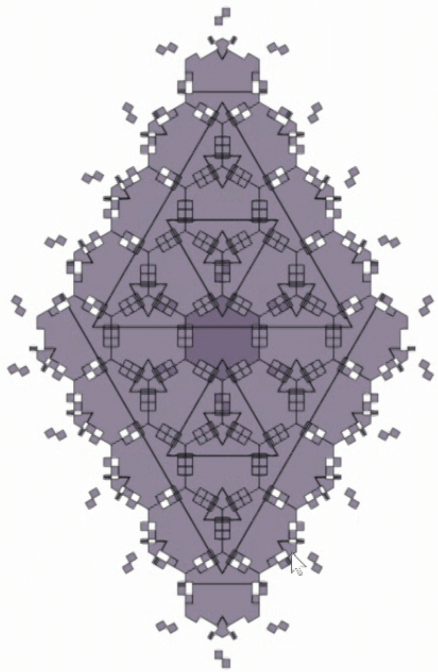
Aperiodic planar tilings





An aperiodic tiling of the plane by **translations, rotations and reflections** of one tile - the Socolar–Taylor tile. [Socolar–Taylor 2011]

Aperiodic planar tilings



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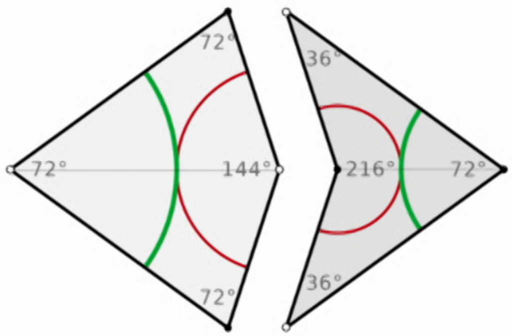
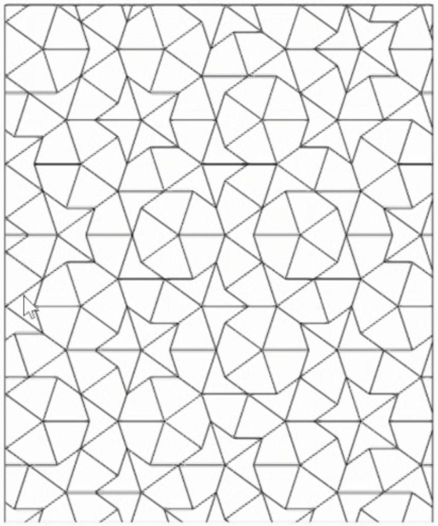
Aperiodic planar tilings





Penrose P2 tiling (aka kite and dart tiling)

An aperiodic tiling of the plane by translations of 20 **tiles**.

Aperiodic planar tilings

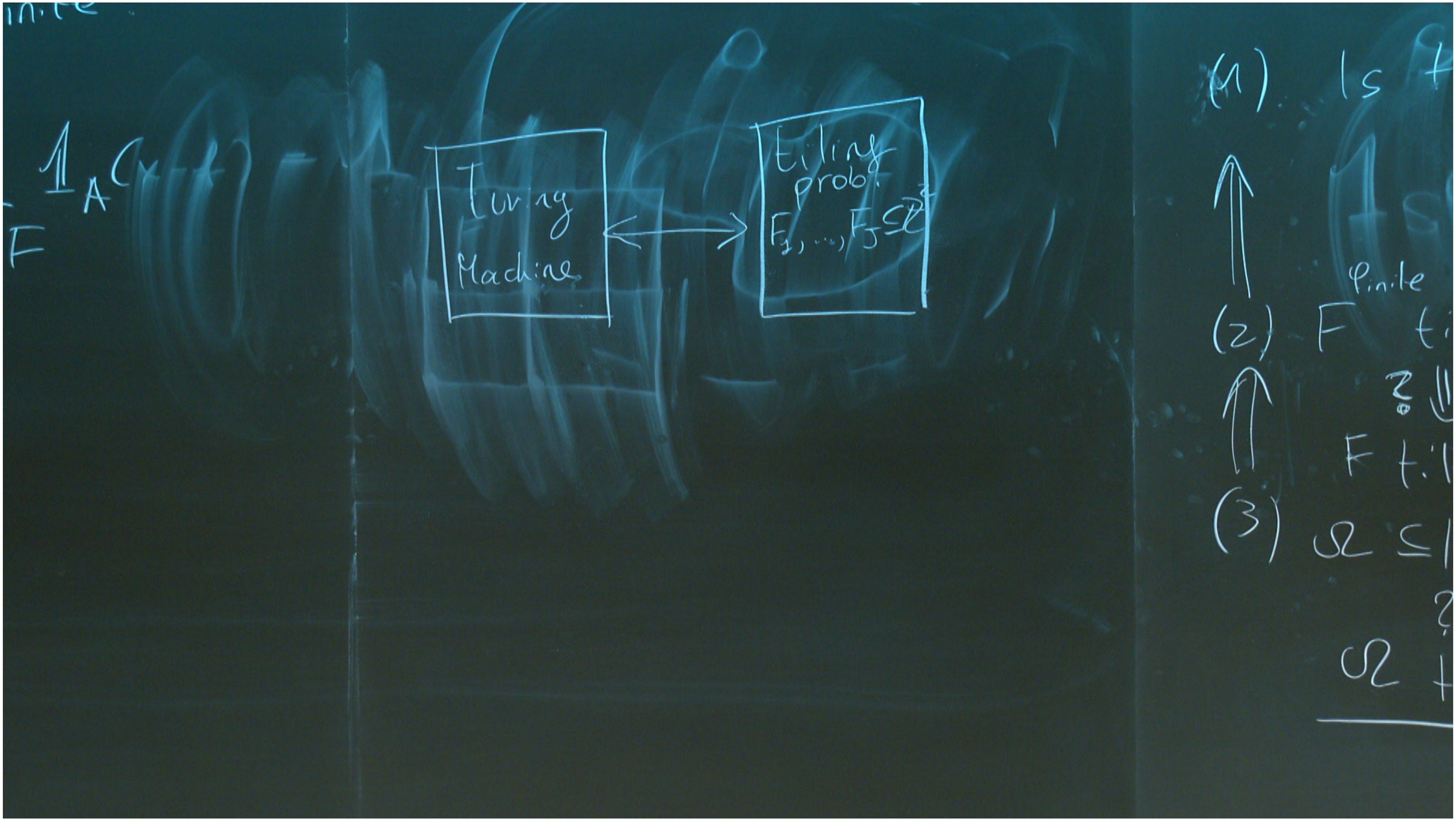
Penrose P3 tiling (aka Rhombus tiling)

An aperiodic tiling of the plane by translations of 10 **tiles**.

Undecidability with multiple tiles

Theorem (Berger, 1966):

Any problem of testing whether a Turing machine eventually halts ("halting problem") can be translated into a translational tiling problem with multiple tiles in $\mathbb{Z}^2$.


Turing
Machine
tiling
prob.
(1) Is
finite
(2) F
(3)

Undecidability with multiple tiles

Theorem (Berger, 1966):

Any problem of testing whether a Turing machine eventually halts ("halting problem") can be translated into a translational tiling problem with multiple tiles in $\mathbb{Z}^2$.

The undecidability of the halting problem then implies the **undecidability of tilings with multiple tiles in** $\mathbb{Z}^2$.

In addition, Berger constructed an **aperiodic** tiling with $20,426$ tiles.

Constructions for $G = \mathbb{Z}^2$ (partial list)

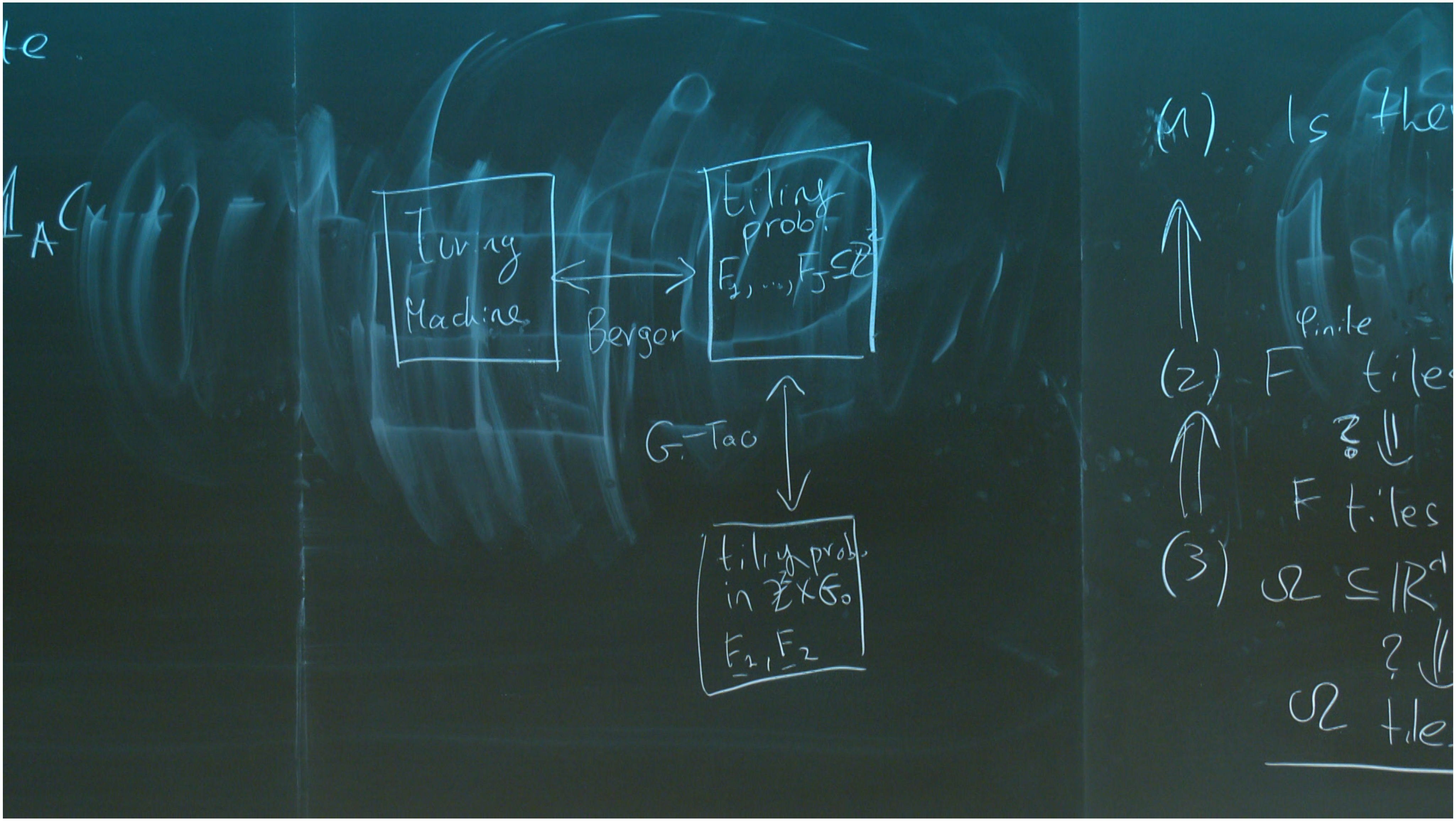
J	Author (Year)	Type
20426	Berger (1966)	aperiodic, undecidable
104	Robinson (1967)	aperiodic
104	Ollinger (2008)	aperiodic, undecidable
103	Berger (1964)	aperiodic
86	Knuth (1968)	aperiodic
56	Robinson (1971)	aperiodic
52	Robinson (1980)	aperiodic
40	Lauchli (1966)	aperiodic
32	Robinson (1975)	aperiodic
24	Robinson (1977)	aperiodic
16	Ammann (1978)	aperiodic
14	Kari (1996)	aperiodic
13	Culik (1996)	aperiodic
11	Ollinger (2009)	undecidable
11	Jeandel–Rao (2020)	aperiodic
8	Ammann et al. (1992)	aperiodic
8	Goodman Strauss (1996)	aperiodic

Undecidability with two tiles

Theorem (G.–Tao, 2021):

There exists an **undecidable** tiling problem involving **two tiles**.

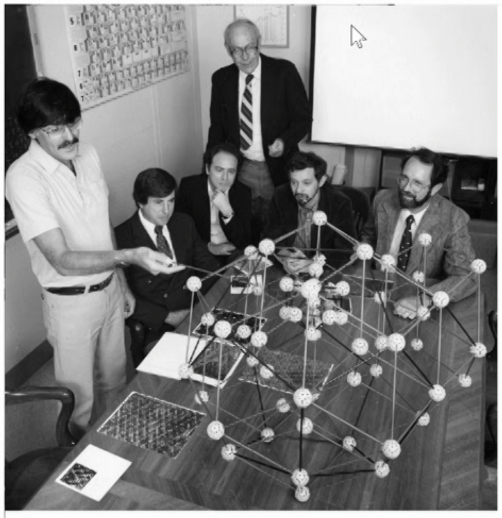
More precisely: There is no algorithm that decides upon any input of finite abelian group G_0 and finite sets $F_1, F_2 \subset \mathbb{Z}^2 \times G_0$, whether $\mathrm{Tile}(F_1, F_2; \mathbb{Z}^2 \times G_0)$ is empty or not.


Turing Machine
Berger
tiling prob.
F1, ..., FJ ⊆ Z2
G.-Tao
tiling prob. in Z×G0
F1, F2
(1) Is the
(2) F tiles
finite
F tiles
(3) Ω ⊆ Rd
Ω tile

Natural quasicrystals

Physical solids whose atoms' arrangement is aperiodic.

Quasicrystals do exist in nature! (1982, Dan Shechtman)



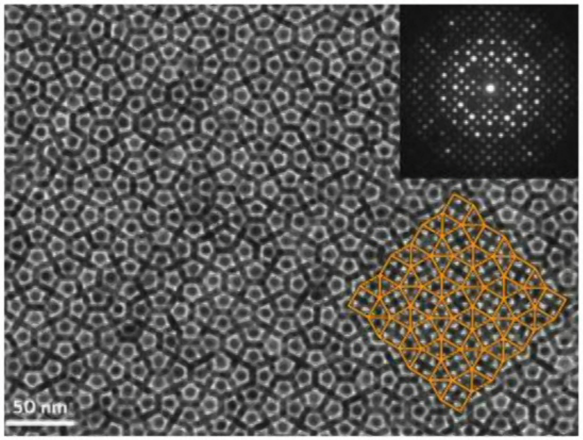


Image credit: University of Pennsylvania; H. Mark Helfer/NIST.

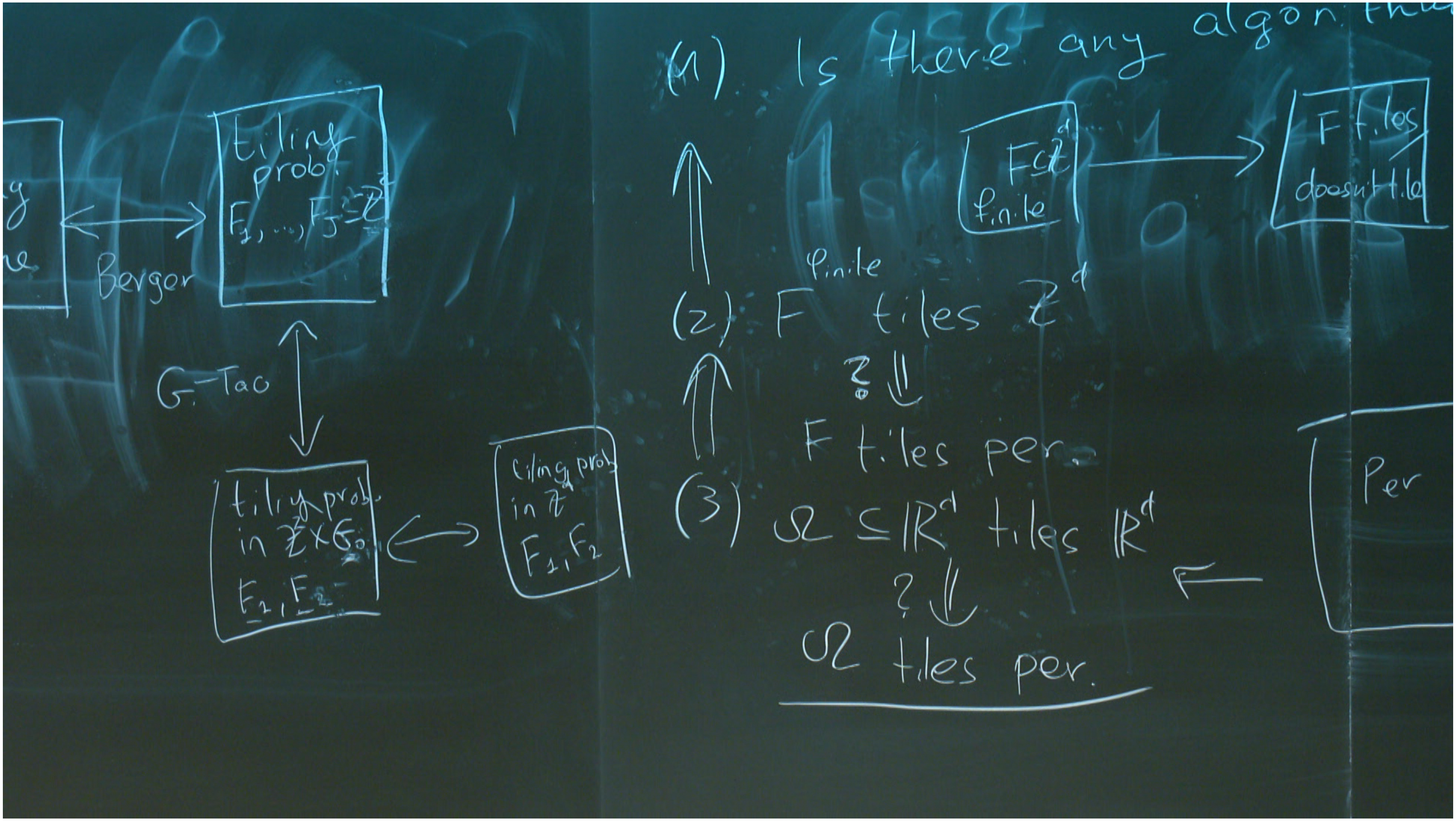
Conjecture

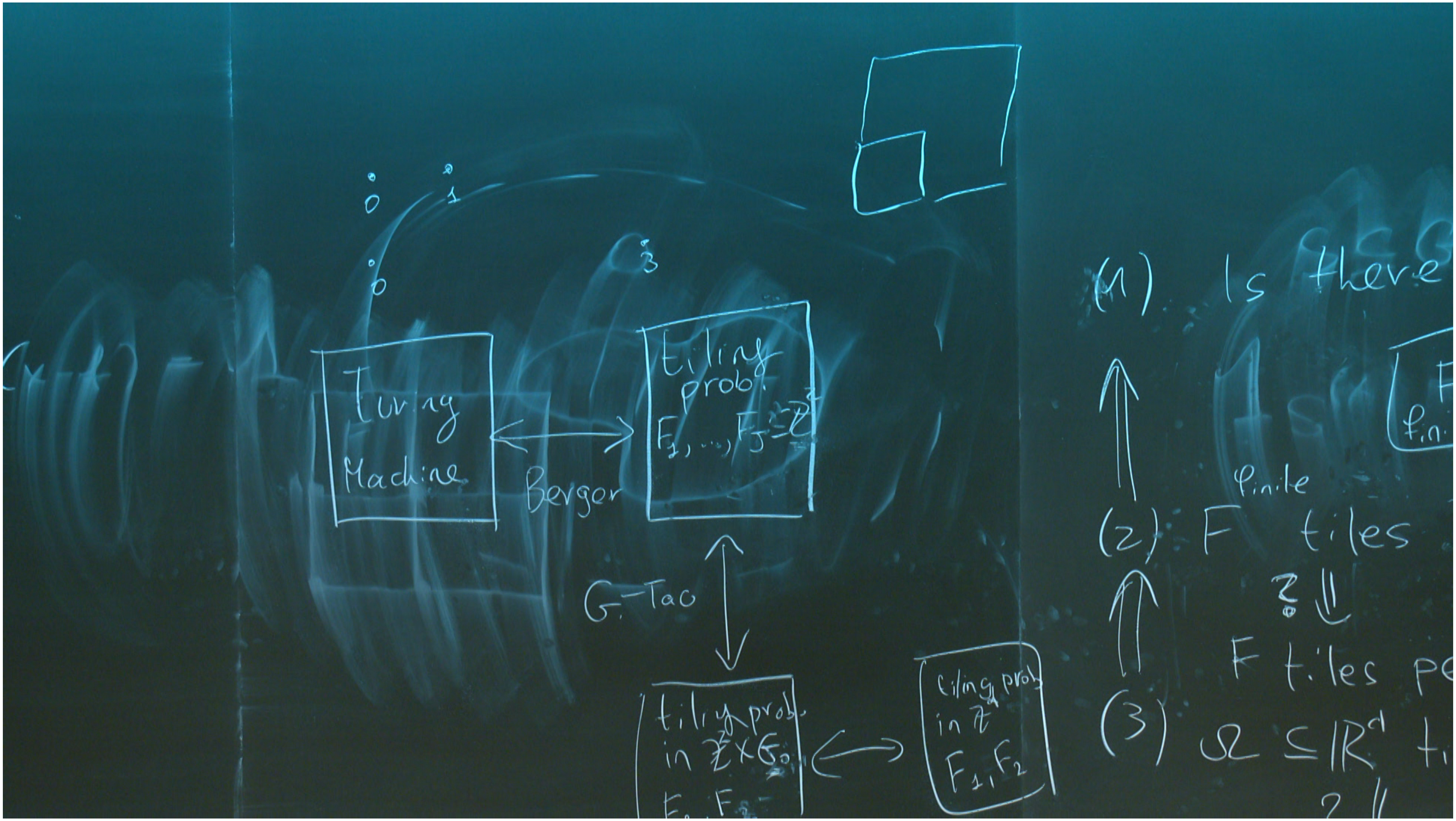
Let Ω be a bounded measurable set in $\mathbb{R}^d$.

Periodic tiling conjecture
(Grünbaum–Shephard, 1987; Lagarias–Wang Y., 1996):

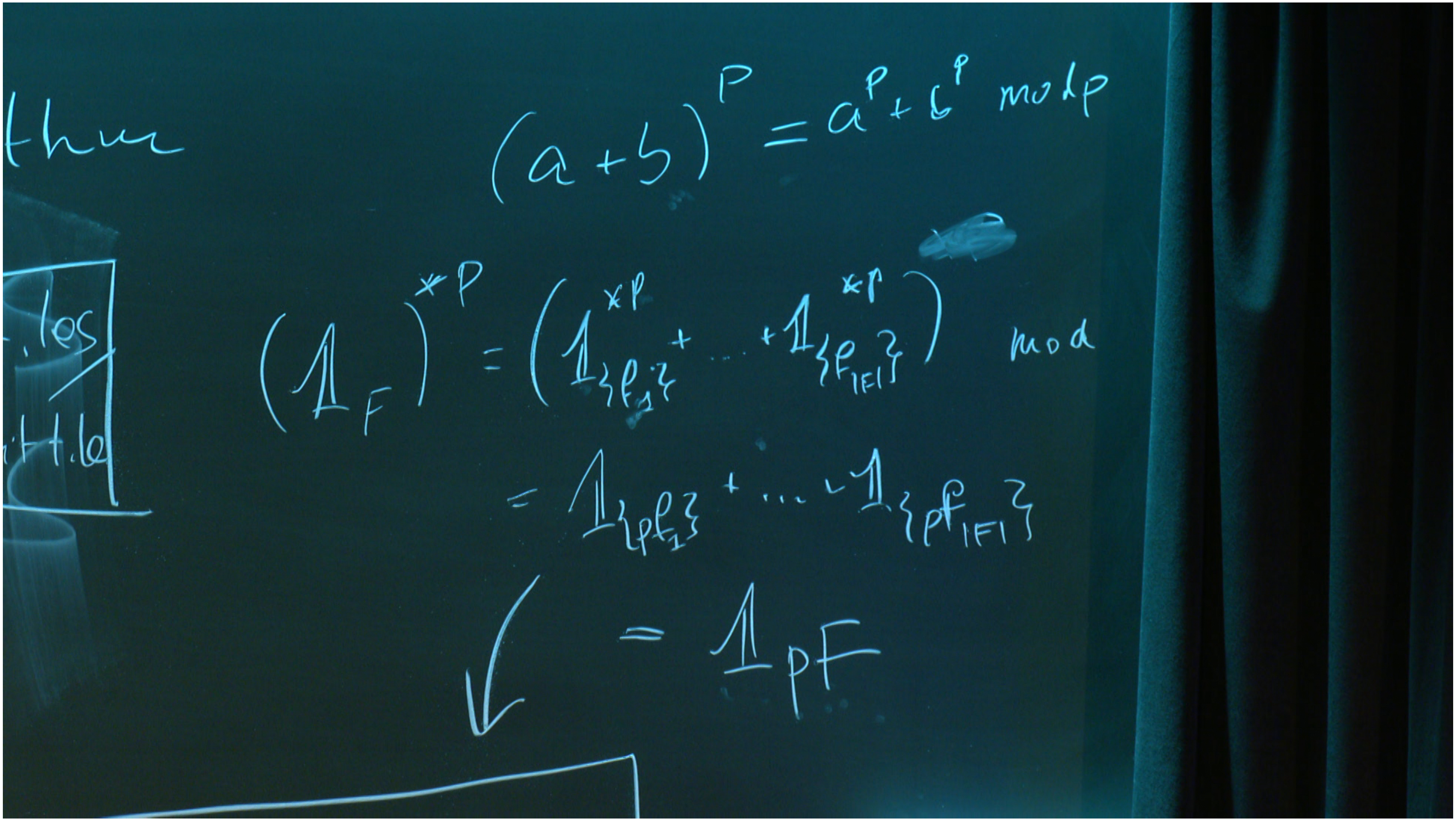
If Ω tiles $\mathbb{R}^d$ by translations, then there is a **periodic** tiling of $\mathbb{R}^d$ by Ω.

Are there any **aperiodic** **translational** tilings by a **single** tile?


(1) Is there any algorithm
tiling prob.
Berger
G.-Tao
tiling prob.
tiling prob
F finite
F tiles
doesn't tile
finite
(2) F tiles Z^d
F tiles per.
(3) Ω ⊆ R^d tiles R^d
Ω tiles per.
Per



Turing Machine
Berger
tiling prob.
G.-Tao
tiling prob. in Z^2 x G_0
tiling prob in Z^d F_1, F_2
(1) Is there
finite
(2) F tiles
F tiles pe
(3) Ω ⊆ R^d ti


$(a+b)^p = a^p + b^p \mod p$
$(\mathbb{1}_F)^{*p} = (\mathbb{1}_{\{f_1\}}^{*p} + \dots + \mathbb{1}_{\{f_{|F|}\}}^{*p}) \mod$
$= \mathbb{1}_{\{pf_1\}} + \dots + \mathbb{1}_{\{pf_{|F|}\}}$
$= \mathbb{1}_{pF}$

$$F \subseteq \mathbb{Z}^d \text{ finite.}$$

$$\mathbb{1}_A * \mathbb{1}_F(x) := \sum_{f \in F} \mathbb{1}_A(\ldots$$

$$\mathbb{1}_F * (\mathbb{1}_{pF} * \mathbb{1}_A) \geq 1 * \mathbb{1}_F$$

$$= 1 * \mathbb{1}_{pF} = |pF| = |F|$$

$$F \oplus A = \mathbb{Z}^d$$

$$\Updownarrow$$

$$\mathbb{1}_F * \mathbb{1}_A = \mathbb{1}$$

$F \subseteq \mathbb{Z}^d$ finite.

$$\mathbb{1}_A * \mathbb{1}_F(x) := \sum_{f \in F} \mathbb{1}_A(x-f)$$

$$\mathbb{1}_F * (\mathbb{1}_{pF} * \mathbb{1}_A) \overset{=}{\geq} 1 * \mathbb{1}_F$$

$$= 1 * \mathbb{1}_{pF} = |pF| = |F|$$

Dilation lemma (Tijdeman 1995; Szegedy 1998; Horak–Kim 2016; Bhattacharya 2019; Kari–Szabados 2020; G.–Tao 2020)

Let F be a finite subset of $\mathbb{Z}^d$ of cardinality $|F|$. Suppose that $A \oplus F = \mathbb{Z}^d$. Then

$$A \oplus rF = \mathbb{Z}^d$$

for every $r \in \mathbb{Z}$ which is co-prime to $|F|$.

Proof: $\mathbb{1}_F * \mathbb{1}_A = 1$

$\Rightarrow (\mathbb{1}_F)^{*p} * \mathbb{1}_A = |F|^{p-1} \mod p$

For any **prime** p which does not divide $|F|$:
$(\mathbb{1}_F)^{*p} = \mathbb{1}_{pF} \mod p$ and: $|F|^{p-1} = 1 \mod p$

$\Rightarrow \mathbb{1}_{pF} * \mathbb{1}_A = 1 \mod p$

$\Rightarrow \mathbb{1}_{pF} * \mathbb{1}_A \geq 1$

$\Rightarrow |F| = \mathbb{1}_F * \mathbb{1}_{pF} * \mathbb{1}_A \geq |F| \Rightarrow \mathbb{1}_{pF} * \mathbb{1}_A = 1.$

Conjecture

Let F be a bounded measurable set in an abelian group G.

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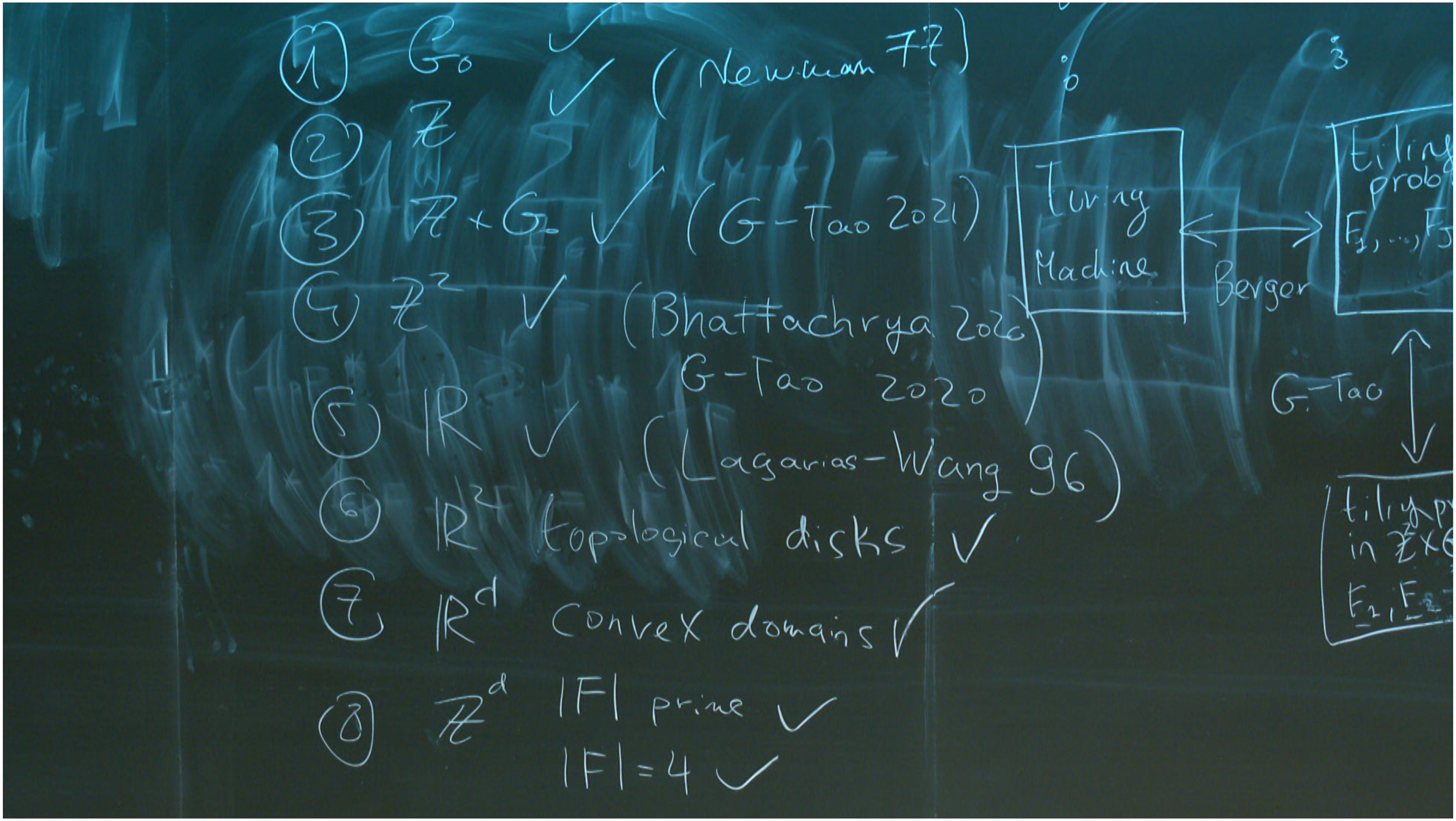

(1) G0 ✓ (Newman 77)
(2) Z ✓
(3) Z × G0 ✓ (G-Tao 2021)
(4) Z^2 ✓ (Bhattachrya 2020
G-Tao 2020)
(5) R ✓ (Lagarias-Wang 96)
(6) R^2 topological disks ✓
(7) R^d convex domains ✓
(8) Z^d |F| prime ✓
|F|=4 ✓
Turing Machine
Berger
G-Tao

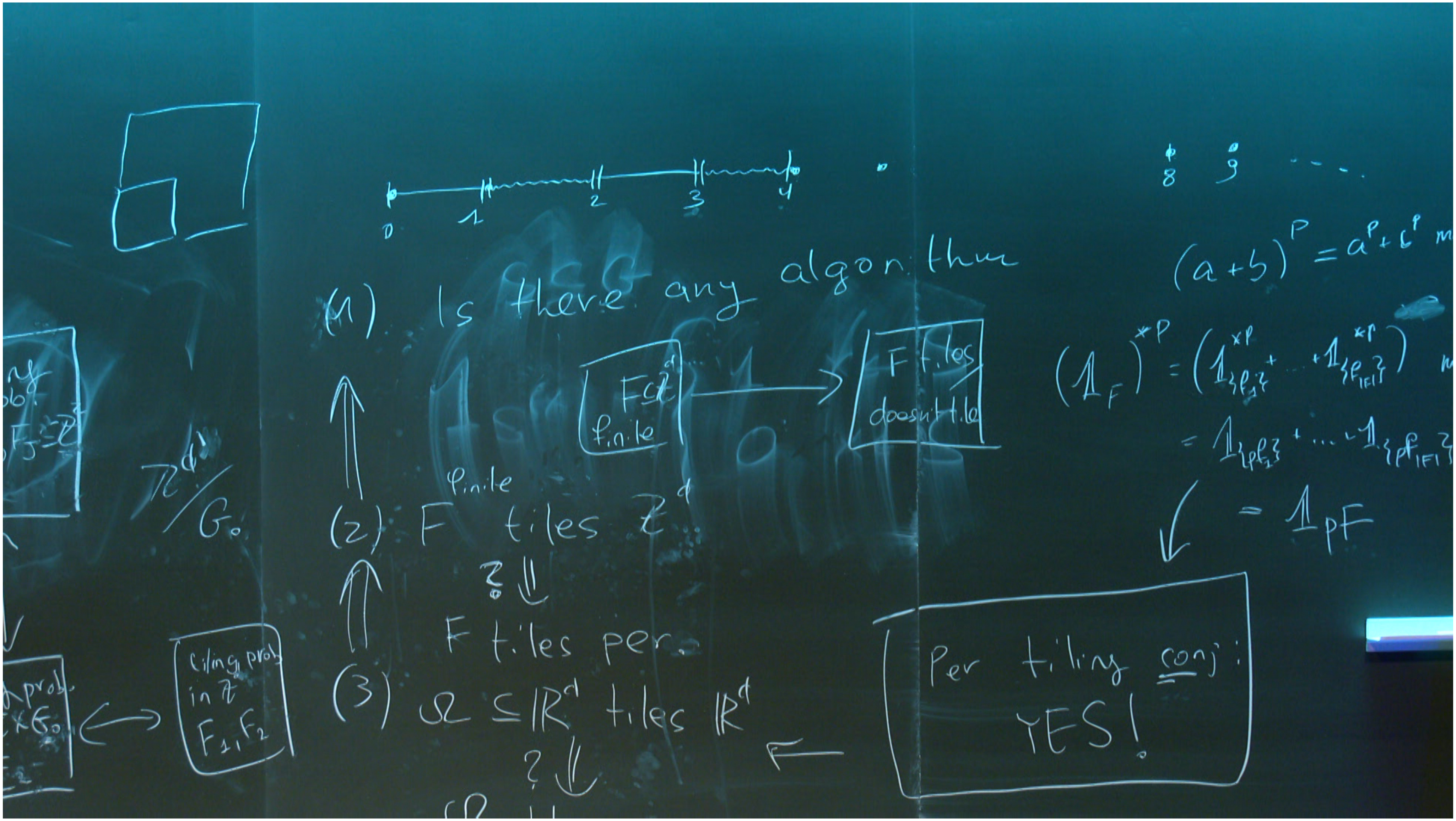
A counterexample to PTC

Theorem (G.–Tao, 2022):

For sufficiently large d, the discrete periodic tiling conjecture **fails** in $\mathbb{Z}^d$ and the continuous periodic tiling conjecture **fails** in $\mathbb{R}^d$.

In fact:

There exists a finite abelian group G_0 and a finite set F in $\mathbb{Z}^2 \times G_0$ such that F tiles $\mathbb{Z}^2 \times G_0$, but does **not** admit any periodic tiling.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Per tiling conj:
YES!
tiling prob
(a+b)^p = a^p + b^p