Title: Gravitational Physics Lecture - 230206

Speakers: Ruth Gregory

Collection: Gravitational Physics (2022/2023)

Date: February 06, 2023 - 9:00 AM

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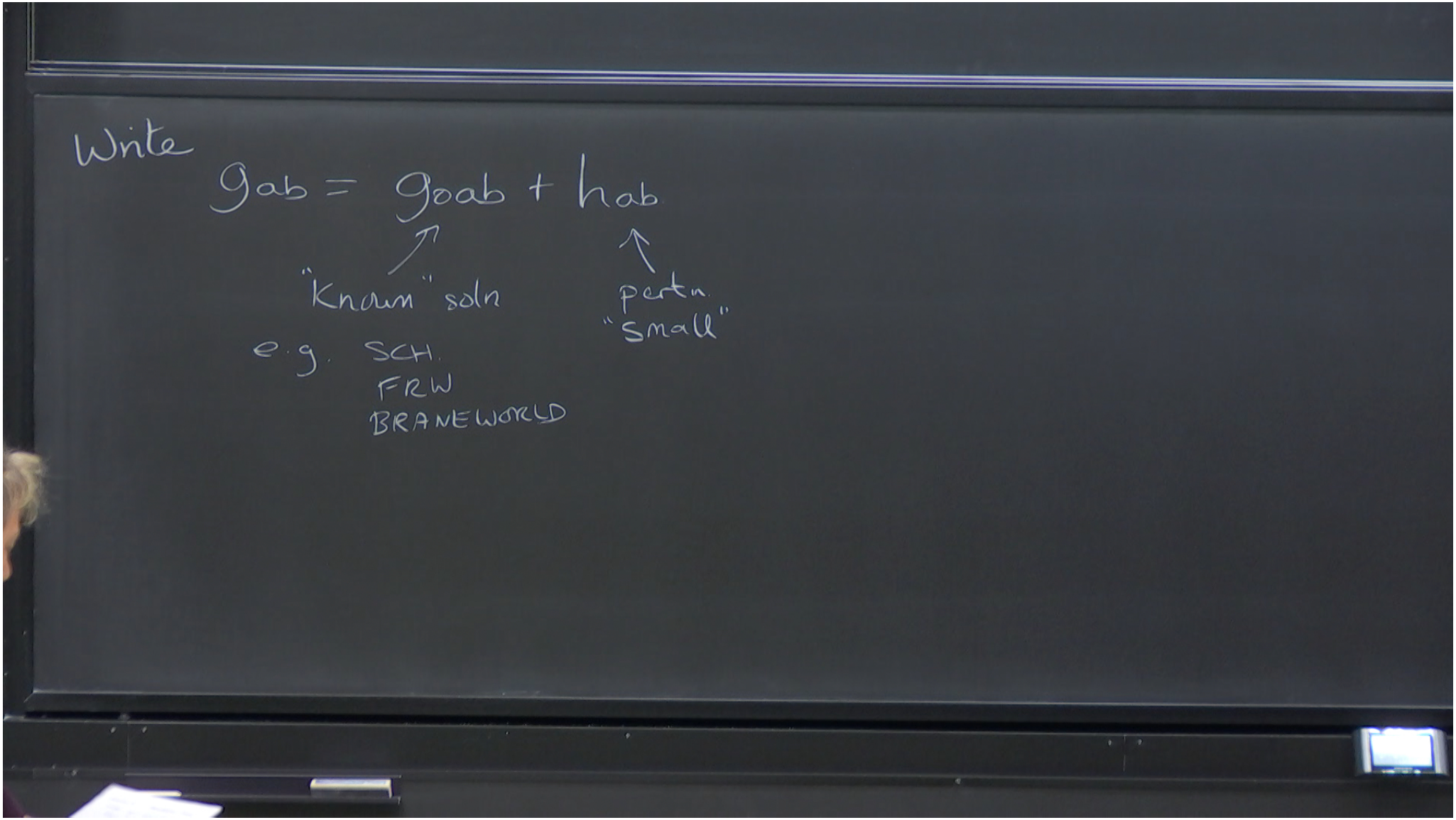
LECTURE 12 Perturbation Theory

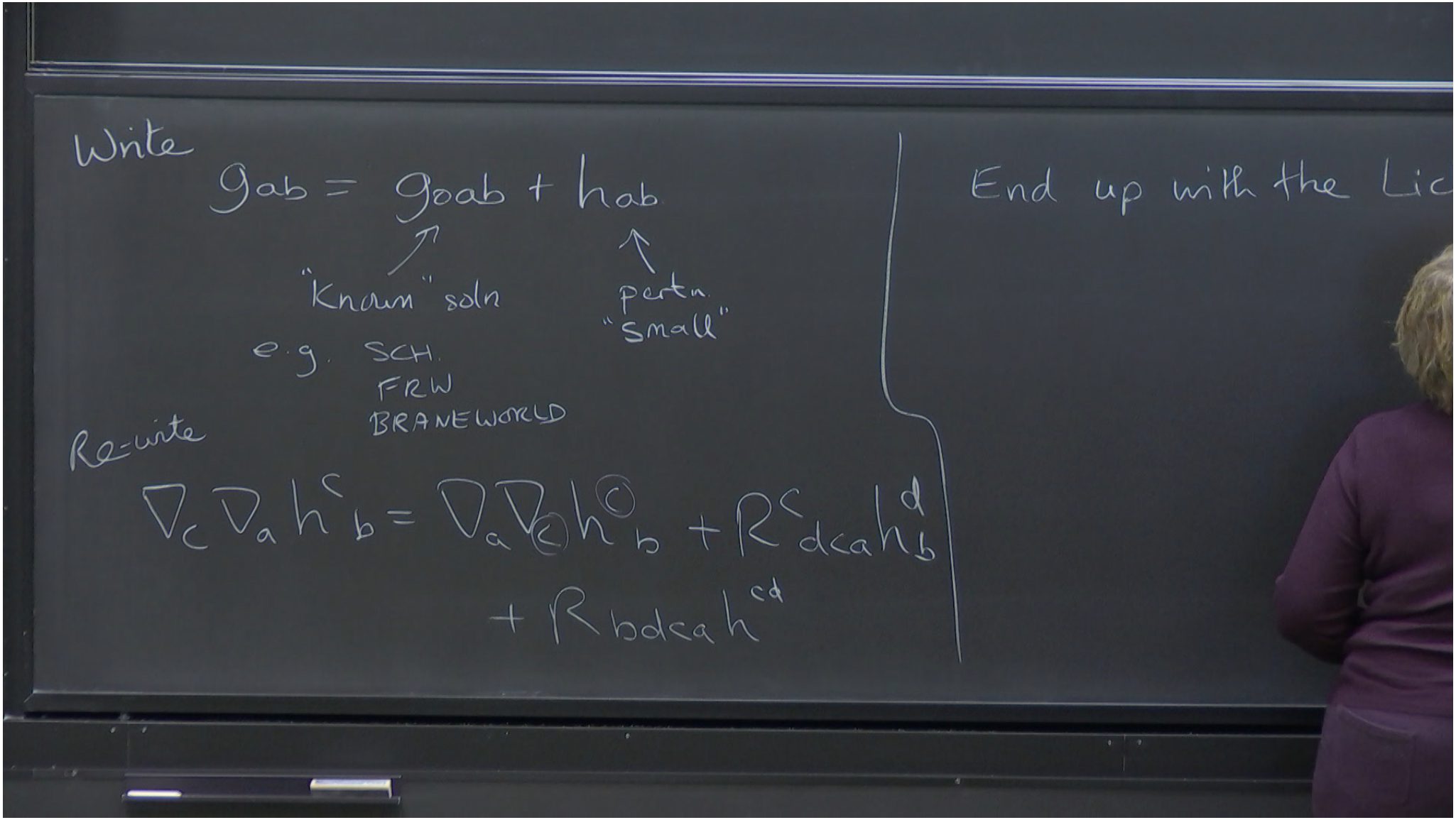
Recall $\delta R_{ab} = \nabla_c \delta \Gamma^c_{ab} - \nabla_b \delta \Gamma^c_{ac}$

$= \frac{1}{2}\left(\nabla_c \nabla_a \delta g^c{}_b + \nabla_c \nabla_b \delta g^c{}_a \right.$

$\left. - \frac{1}{2}\left(\Box \delta g_{ab} - \nabla_a \nabla_b \delta g \right) \right.$

here δg_{ab} not δg^{-1}!


Write
$g_{ab} = g_{0ab} + h_{ab}$
"known" soln
pertn "small"
e.g. SCH.
FRW
BRANEWORLD


Write
$g_{ab} = g_{0ab} + h_{ab}$
"known" soln
pertn. "small"
e.g. SCH.
FRW
BRANEWORLD
Re-write
$\nabla_c \nabla_a h^c{}_b = \nabla_a \nabla_c h^c{}_b + R^c{}_{dca} h^d{}_b$
$+ R_{bdca} h^{cd}$
End up with the Lic

End up with the Lichnerowicz eqn

$$\delta R_{ab} = -\frac{1}{2}\Delta_L h_{ab} \quad (= 0 \text{ in vac})$$

where

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} - 2R_{(a}{}^{c}h_{b)c} - \nabla_{(a}\nabla^{e}\bar{h}_{b)e}$$

(Lich op)

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2}h g_{0\,ab}$$

End up with the Lichnerowicz eqn

$$\delta R_{ab} = -\frac{1}{2}\Delta_L h_{ab} \quad (= 0 \text{ in vac})$$

where

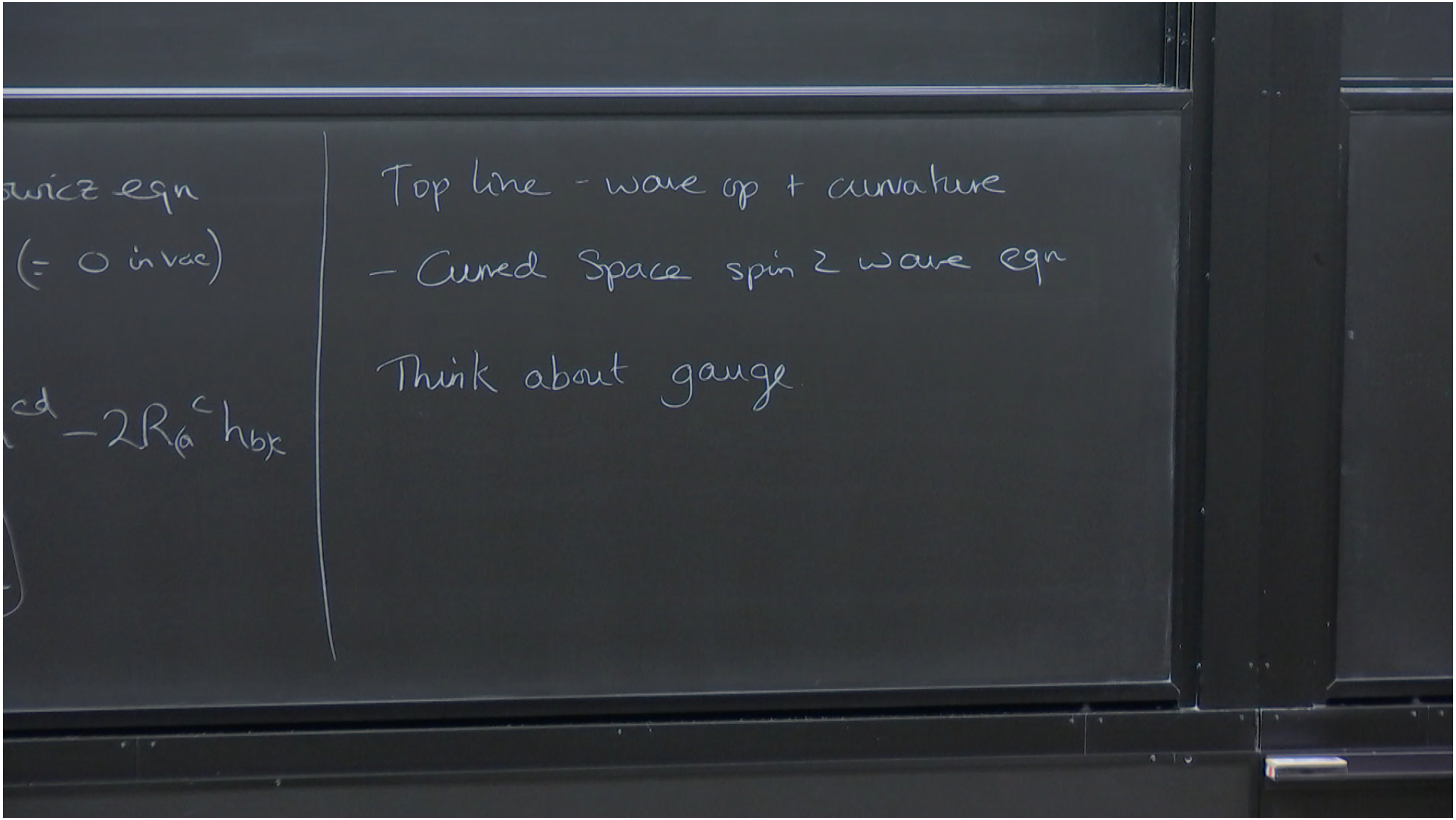
$$\Delta_L h_{ab} = \Box h_{ab} + 2R_{acbd}h^{cd} - 2R_{(a}{}^{c}h_{b)c}$$

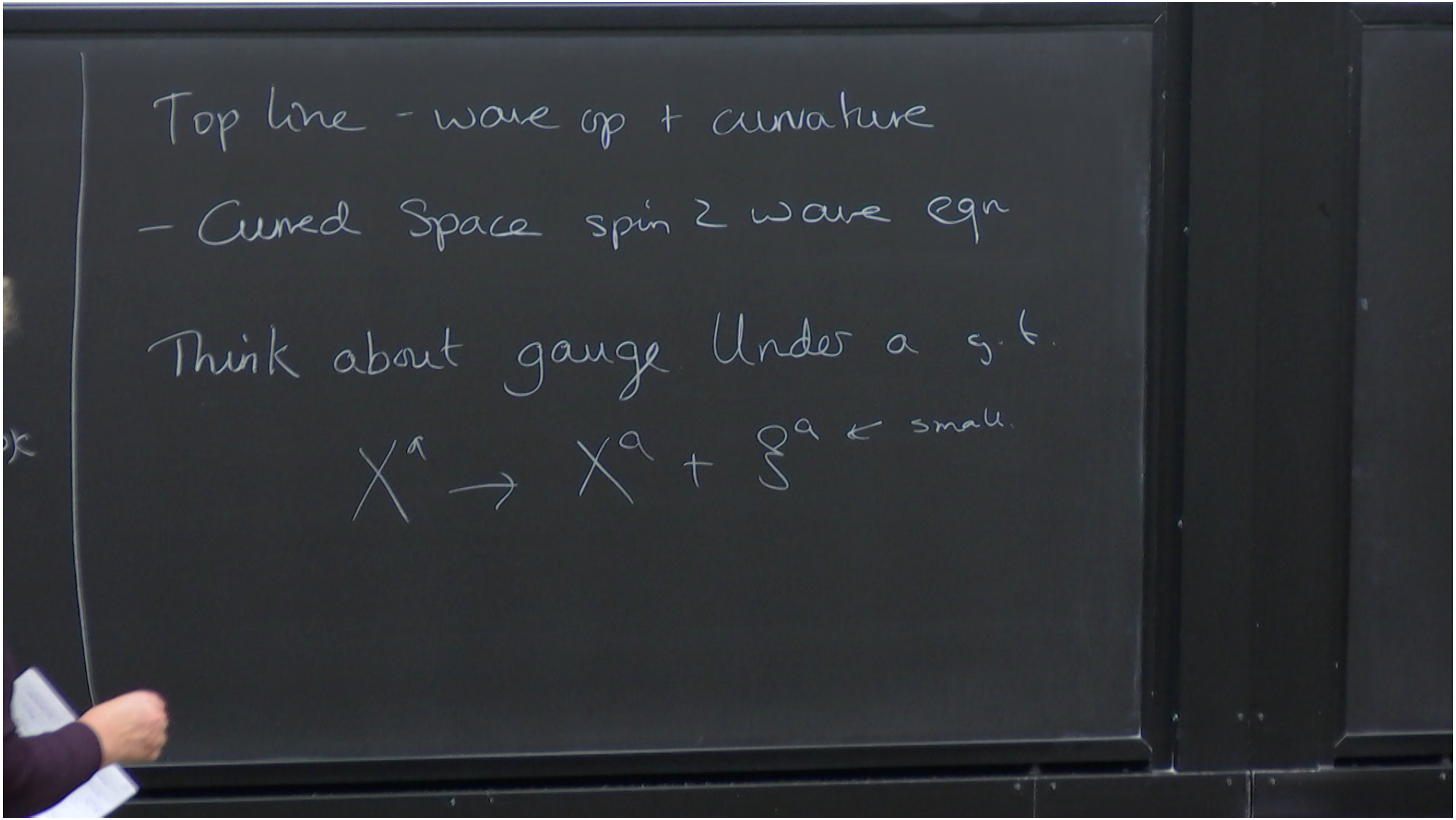
Lich op

$$- \nabla_{(a}\nabla^{e}\bar{h}_{b)e}$$

$$\bar{h}_{ab} = h_{ab} - \frac{1}{2}h g_{0\,ab}$$

$$\cdots {}^{d}{}_{dca}h_{b}$$


wicz eqn
(= 0 in vac)
$^{cd} - 2R_{a}{}^{c} h_{bx}$
Top line - wave op + curvature
- Curved Space spin 2 wave eqn
Think about gauge


Top line - wave op + curvature
- Curved Space spin 2 wave eqn
Think about gauge Under a g.t.
$X^a \to X^a + \xi^a$ ← small

Top line - wave op + curvature

- Curved Space spin 2 wave eqn

Think about gauge Under a g.t.

$$X^a \to X^a + \xi^a \leftarrow \text{small}$$

$$g_{ab} \longmapsto g_{ab} + \mathcal{L}_\xi g_{ab}.$$

Top line - wave op + curvature

- Curved Space spin 2 wave eqn

Think about gauge Under a g.t.

$$X^a \to X^a + \xi^a \leftarrow \text{small}$$

$$g_{ab} \longmapsto g_{ab} + \mathcal{L}_\xi g_{ab}$$

$$= g_{ab} + \nabla_a \xi_b + \nabla_b \xi_a$$

i.e. ξ^a generates an $h_{\xi ab} = \nabla_a \xi_b + \nabla_b \xi_a$

$$\bar{h}_{\xi ab} = 2\nabla_{(a}\xi_{b)} - (\nabla\cdot\xi) g_{0ab}$$

$$\nabla^a \bar{h}_{ab} = \Box \xi_b + \nabla_a \nabla_b \xi^a - \nabla_b \nabla_a \xi^a$$

$$= \Box \xi_b + R^c{}_b \xi_a$$

- well-posed eqn, so can be solved given data.

i.e. if ∇_a

i.e. if $\nabla_a \bar{h}^a{}_b$ is not initially zero

can find a ξ to make

$\nabla^a(\bar{h}_{0\,ab} + \bar{h}_{\xi\,ab}) = 0$.

DE DONDER GAUGE.

Transverse tracefree: $h^a{}_a = 0 = \nabla_a h^a{}_b$.

i.e. if $\nabla_a \bar{h}^a{}_b$ is not initially zero

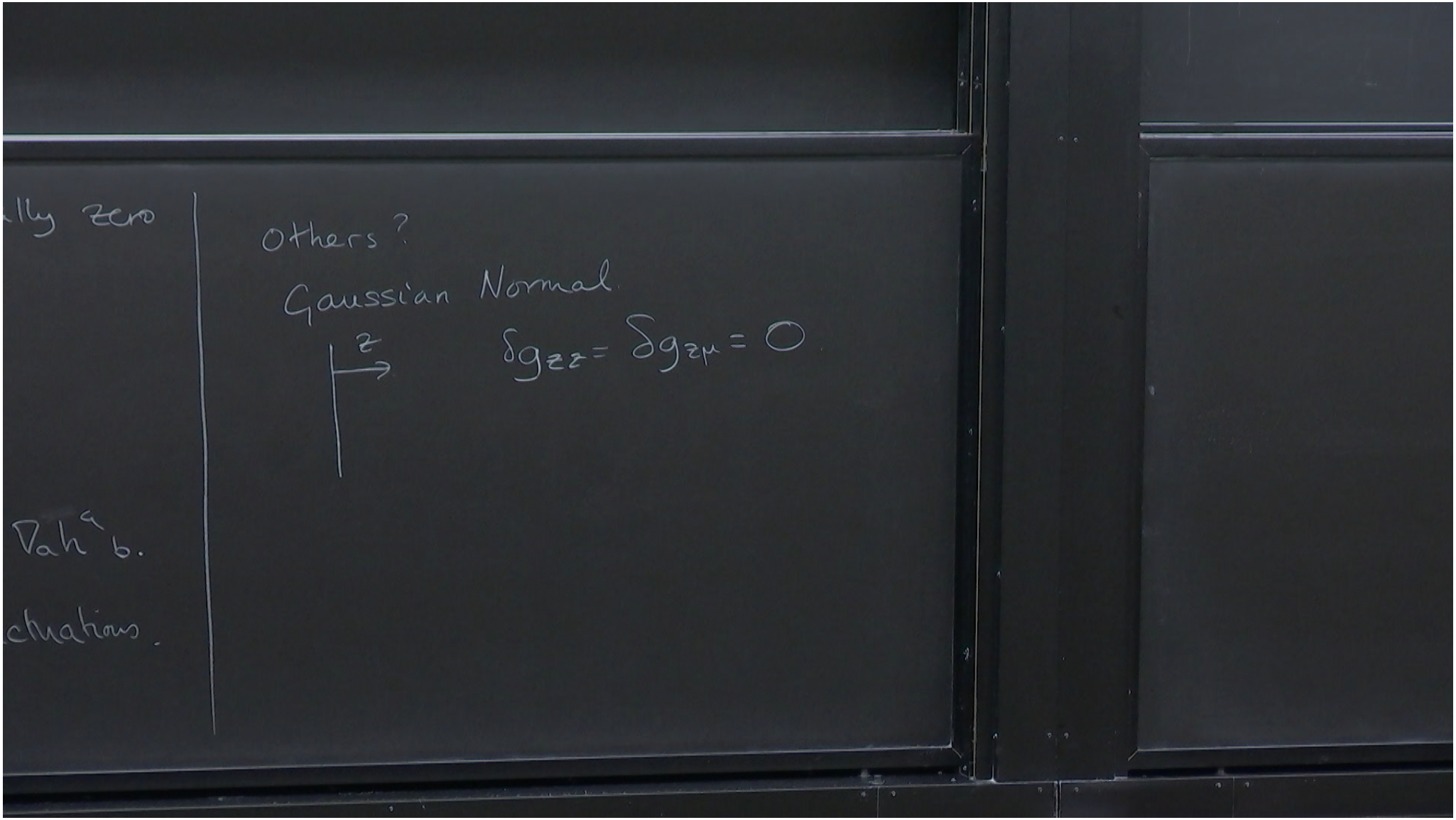
can find a ξ to make

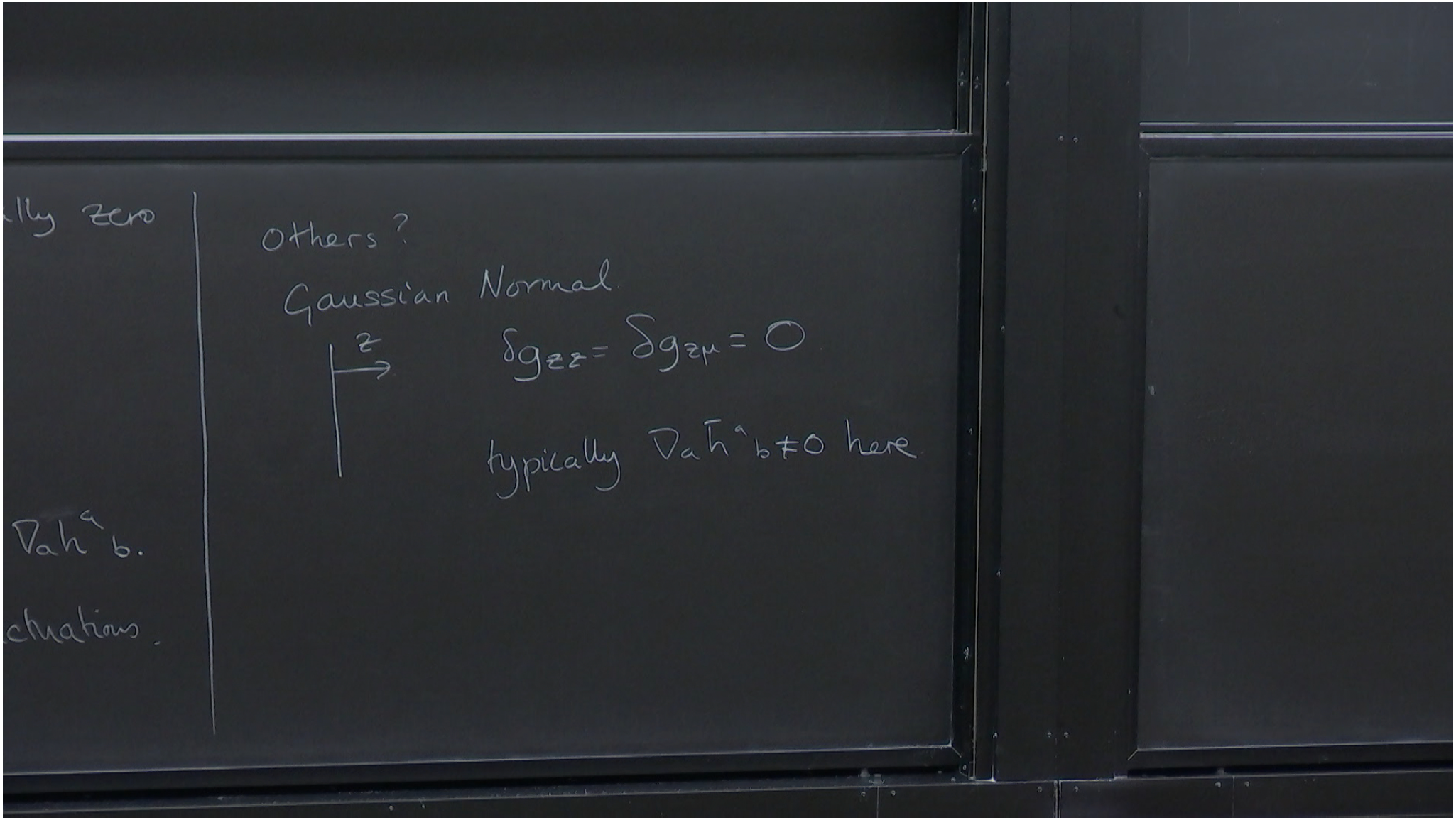
$\bar{\nabla}^a(\bar{h}_{0\,ab} + \bar{h}_{s\,ab}) = 0$.

DE DONDER GAUGE.

Transverse tracefree: $h^a{}_a = 0 = \nabla_a h^a{}_b$.

- typically needs vacuum fluctuations.


lly zero
Others?
Gaussian Normal
z
$\delta g_{zz} = \delta g_{z\mu} = 0$
$\nabla_a h^a{}_b$.
ctuations.


lly zero
$D_a h^a{}_b$.
ctuations.
Others?
Gaussian Normal
z
$\delta g_{zz} = \delta g_{z\mu} = 0$
typically $D_a h^a{}_b \neq 0$ here

lly zero

$D_a h^a{}_b$.

chuations.

Others?

Gaussian Normal.

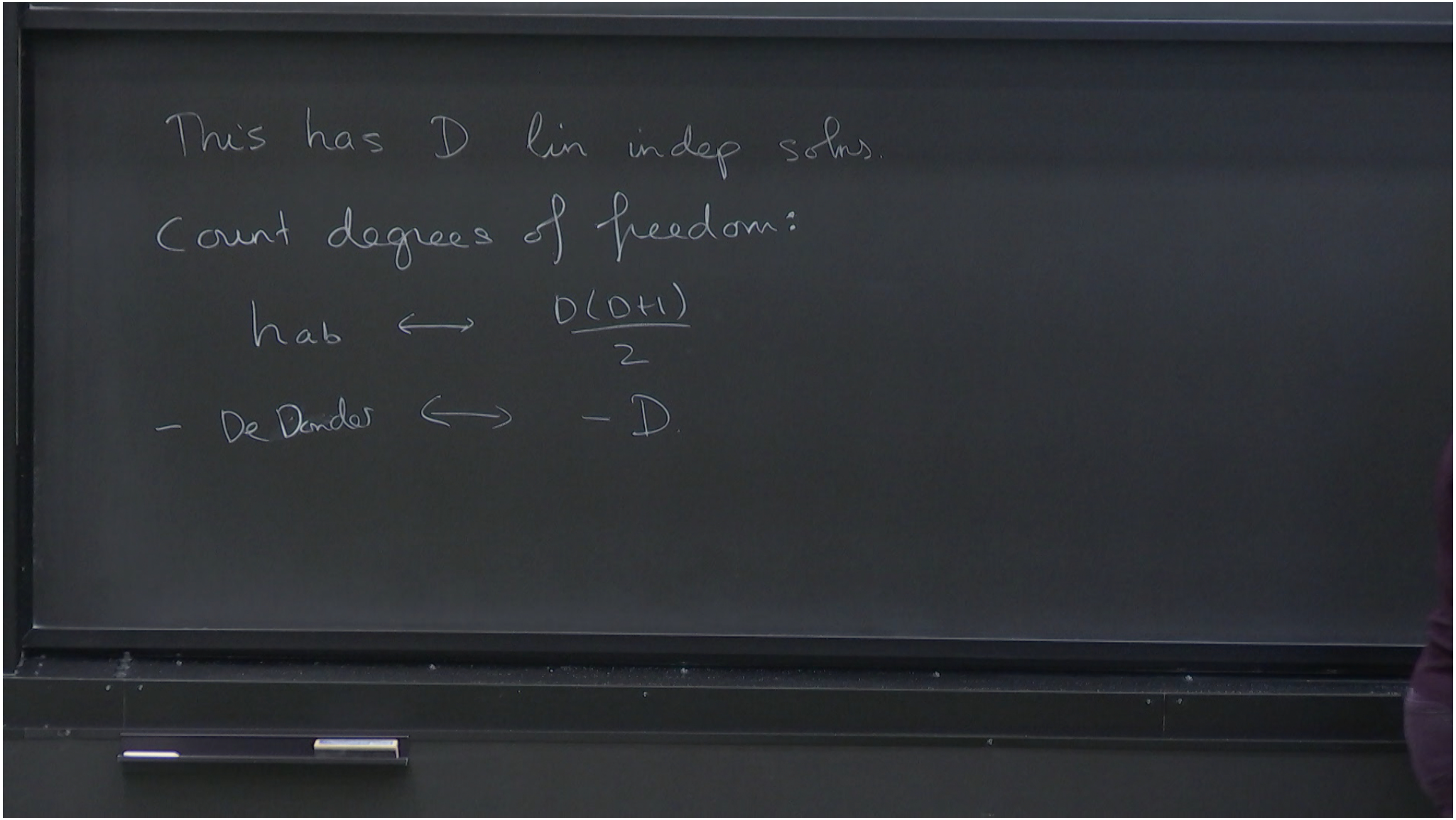
$z \longrightarrow$

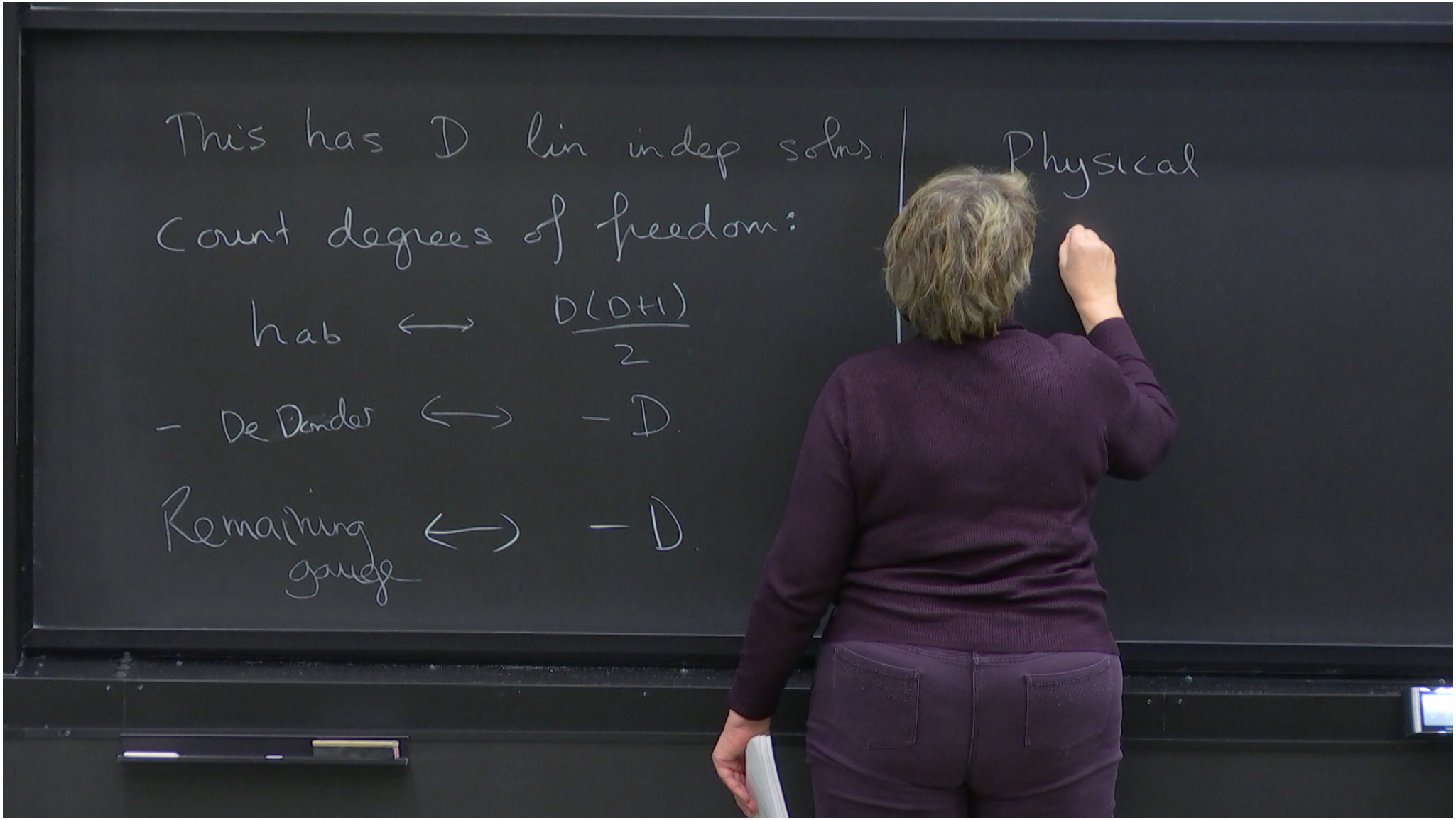
$\delta g_{zz} = \delta g_{z\mu} = 0$

typically $D_a h^a{}_b \neq 0$ here

DD - Remaining gauge freedom?

$X^a \to X^a + \chi^a$ s.t. $D\chi^a + R^a{}_b \chi^b = 0$

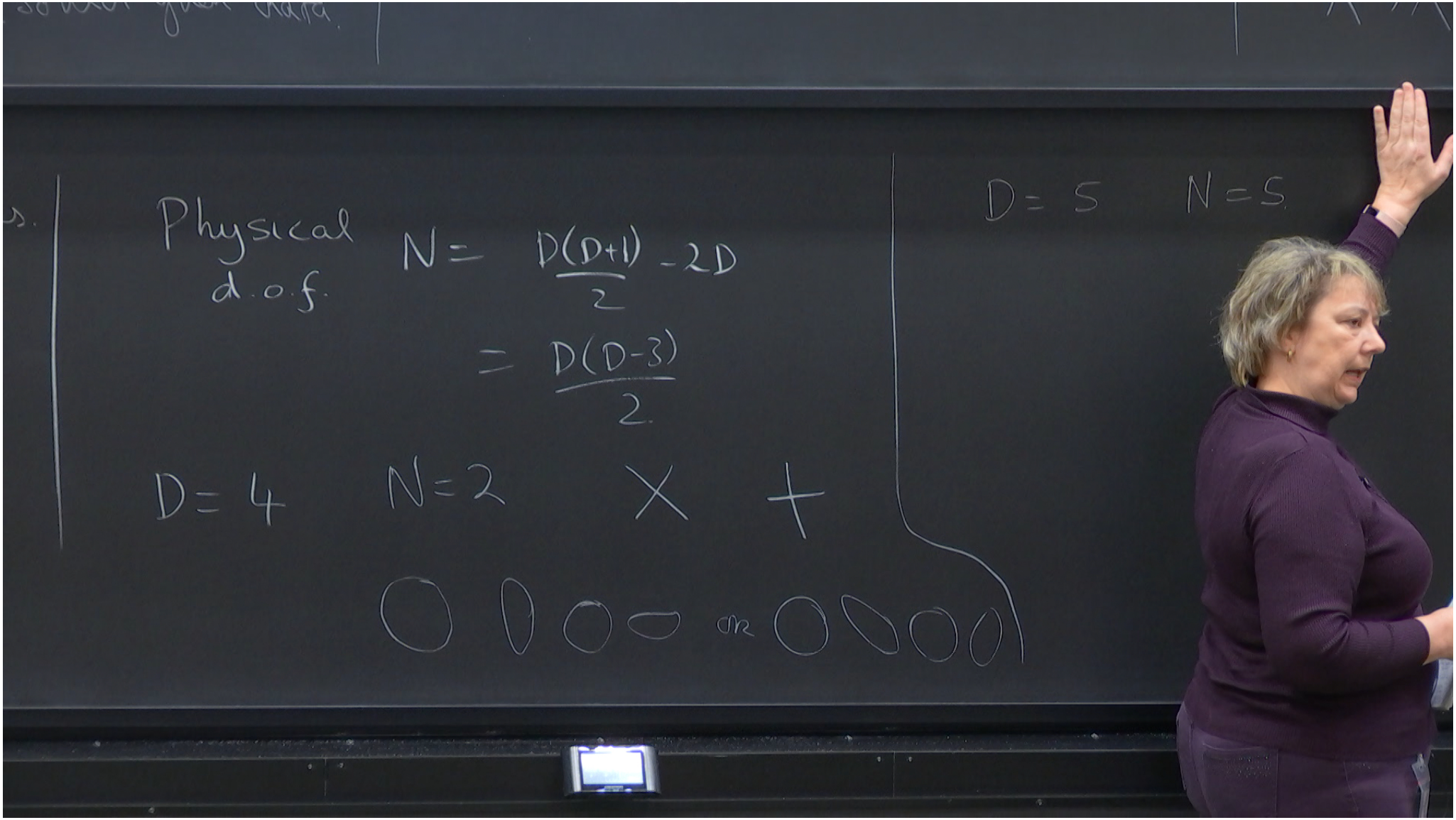

This has D lin indep solns.
Count degrees of freedom:
hab ⟷ D(D+1)/2
- De Donder ⟷ - D.


This has D lin indep solns.
Physical
Count degrees of freedom:
hab ⟷ D(D+1)/2
- De Donder ⟷ - D
Remaining gauge ⟷ - D

Physical d.o.f.

$$N = \frac{D(D+1)}{2} - 2D$$

$$= \frac{D(D-3)}{2}$$


Physical d.o.f.
$N = \frac{D(D+1)}{2} - 2D$
$= \frac{D(D-3)}{2}$
$D = 4 \quad N = 2$
X +
or
$D = 5 \quad N = 5$

$X \to X^a + \chi^a$ s.t. $D\chi^a + R^a{}_b \chi^b = 0$

$D = 5 \quad N = 5$

Black holes Regge-Wheeler (57)

-2D

3)

X +

or

$\chi \to \chi^a + \chi^a$ s.t. $D\chi^a + R^a{}_b \chi^b = 0$

-2D

3)

X +

OR

$D = 5 \quad N = 5$

Black holes Regge-Wheeler (57)

(SCH) — Have ∂_t, 3 angular K.vectors ∂_ϕ

$X \to X^a + \chi^a$ s.t. $\Box \chi^a + R^a{}_b \chi^b = 0$

$D = 5 \quad N = 5$

-2D

3)

X +

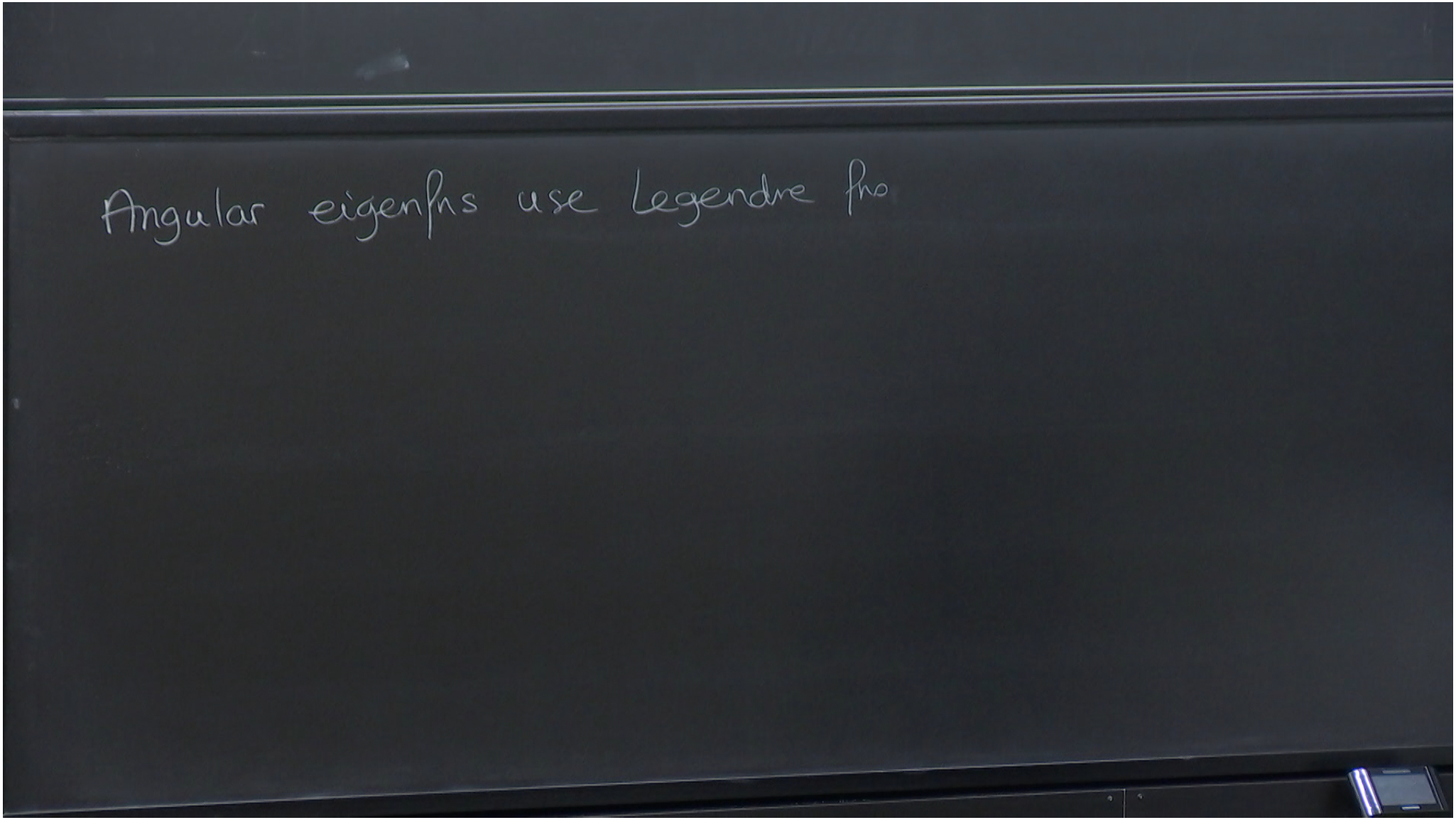
or

Black holes Regge-Wheeler (57)

(SCH) — Have ∂_t, 3 angular K. vectors ∂_ϕ

Symmetry of background allows a decomposition of perturbations.

frequency, $\omega \leftrightarrow \partial_t$ angular mom $\ell \leftrightarrow \{\ell, m\}$

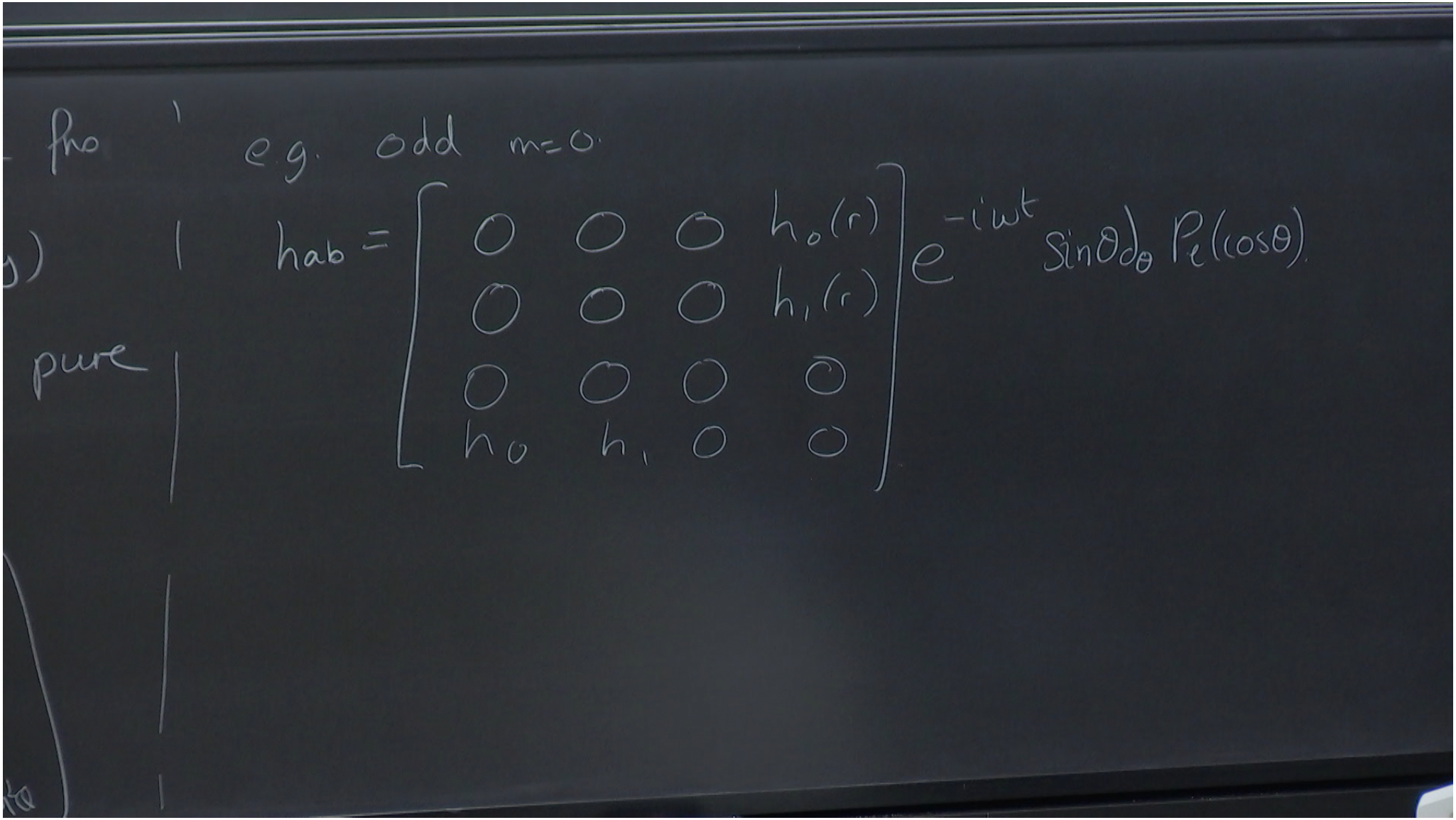

Angular eigenfns use Legendre fns

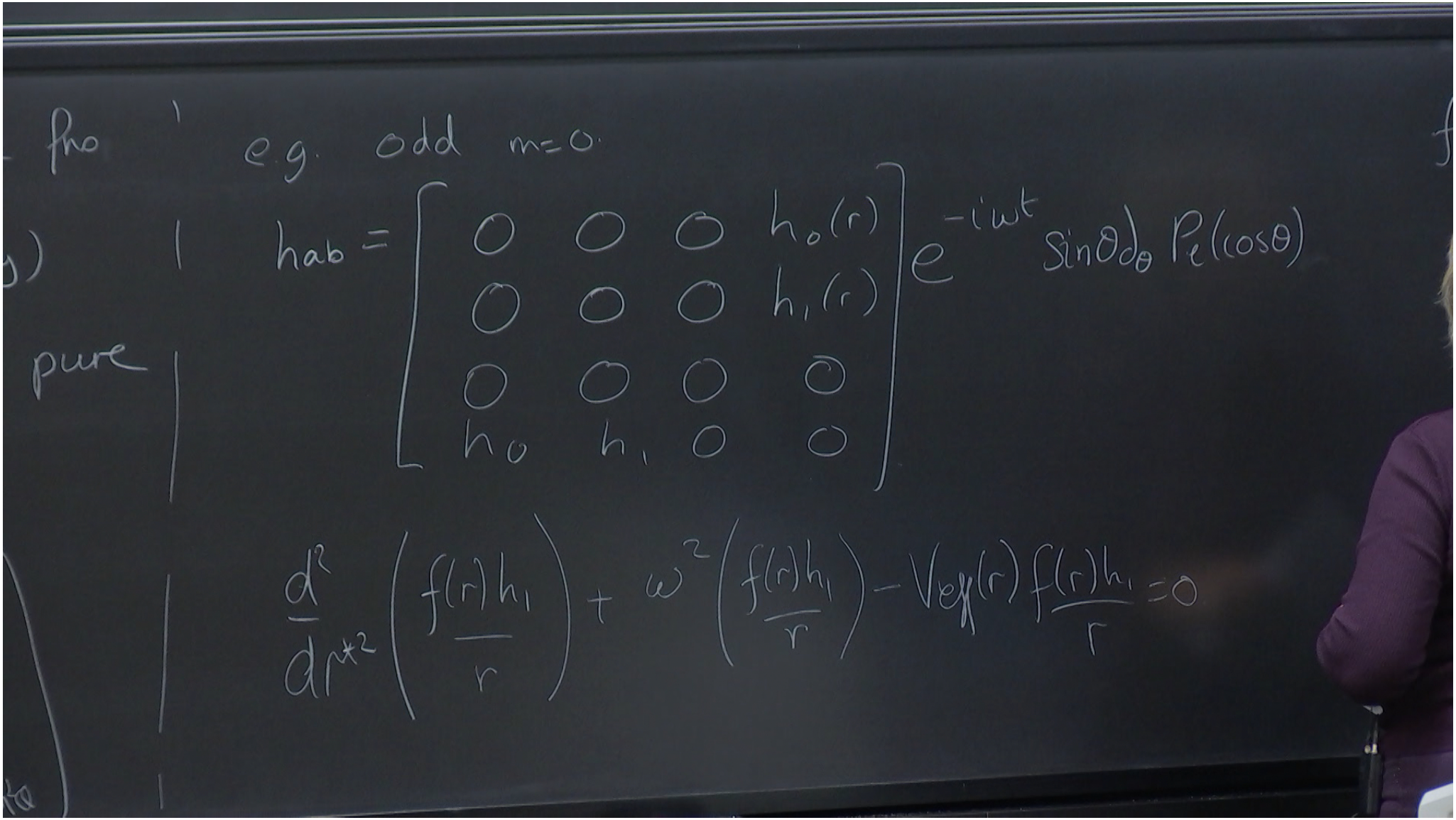
Angular eigenfns use Legendre fns

- Even & odd pfns (parity)

No s-wave ($L=0$) solns to pure 4D Einstein gravity.

$$h_{ab}^{(s)} = e^{-i\omega t} \begin{bmatrix} h_0(r)\, h_1(r) & & \\ h_1 & h_2(r) & \\ & & K(r)\sin^2\theta \end{bmatrix}$$


pho
e.g. odd m=0.
pure
$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} e^{-i\omega t} \sin\theta \partial_\theta P_\ell(\cos\theta)$


Rho
e.g. odd m=0.
pure
$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} e^{-i\omega t} \sin\theta \partial_\theta P_\ell(\cos\theta)$
$\frac{d^2}{dr^{*2}}\left(\frac{f(r)h_1}{r}\right) + \omega^2\left(\frac{f(r)h_1}{r}\right) - V_{eff}(r)\frac{f(r)h_1}{r} = 0$

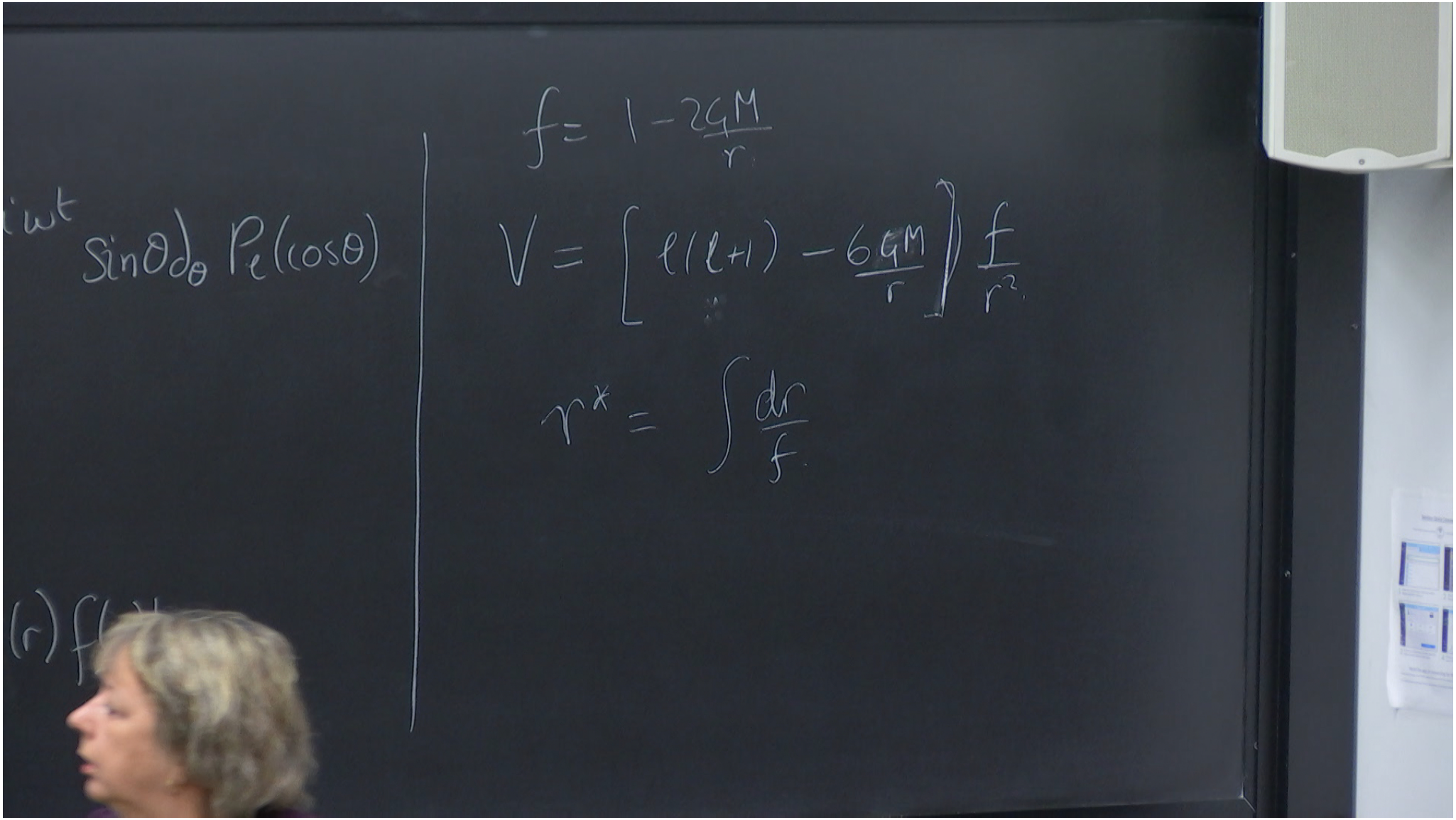
e.g. odd $m=0$.

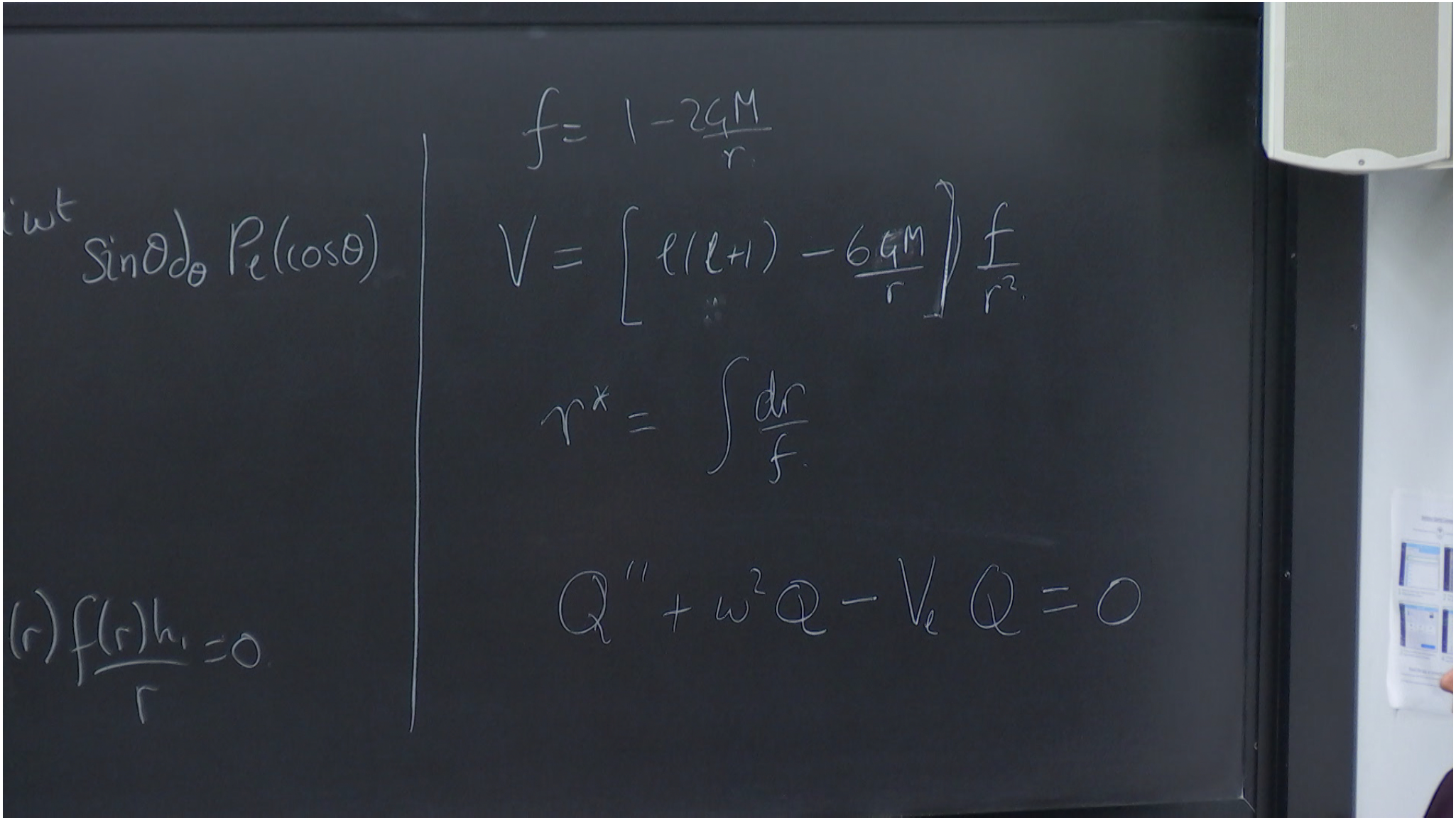
$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} e^{-i\omega t} \sin\theta \partial_\theta P_\ell(\cos\theta)$$

$$f = 1 - \frac{2GM}{r}$$

$$V = \left[\ell(\ell+1) - \frac{6GM}{r} \right) \frac{f}{r^2}$$

$$\frac{d^2}{dr^{*2}} \left(\frac{f(r) h_1}{r} \right) + \omega^2 \left(\frac{f(r) h_1}{r} \right) - V_\ell(r) \frac{f(r) h_1}{r} = 0$$


$f = 1 - \frac{2GM}{r}$
$V = \left[\ell(\ell+1) - \frac{6GM}{r} \right] \frac{f}{r^2}$
$r^* = \int \frac{dr}{f}$
$\sin\theta \partial_\theta P_\ell(\cos\theta)$
$e^{i\omega t}$


$f = 1 - \frac{2GM}{r}$
$V = \left[\ell(\ell+1) - \frac{6GM}{r} \right] \frac{f}{r^2}$
$r^* = \int \frac{dr}{f}$
$Q'' + \omega^2 Q - V_\ell Q = 0$
$\sin\theta \partial_\theta P_\ell(\cos\theta)$
$\frac{f(r) h_1}{r} = 0$

e.g. odd $m=0$.

$$h_{ab} = \begin{bmatrix} 0 & 0 & 0 & h_0(r) \\ 0 & 0 & 0 & h_1(r) \\ 0 & 0 & 0 & 0 \\ h_0 & h_1 & 0 & 0 \end{bmatrix} e^{-i\omega t} \sin\theta \partial_\theta P_\ell(\cos\theta)$$

$$\frac{d^2}{dr^{*2}}\left(\frac{f(r)h_1}{r}\right) + \omega^2\left(\frac{f(r)h_1}{r}\right) - V_\ell(r)\frac{f(r)h_1}{r} = 0$$

Q.

$f =$

$V =$

r^*

Q

eg Black String Instability



SCH X R

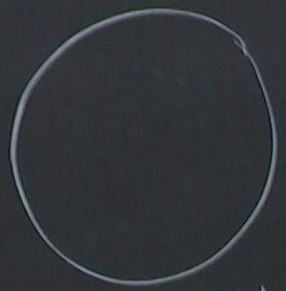


5D BH.

$ds^2_{D} = ($

$\mu = 16$

$(D$

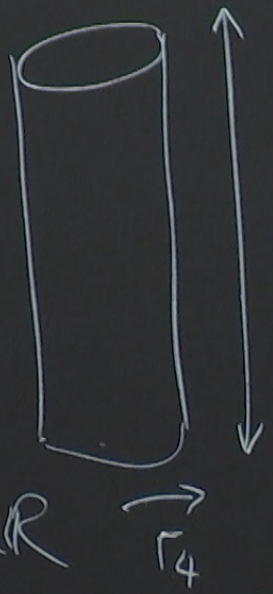


5D BH.

$$ds^2_{D} = \left(1 - \frac{\mu}{r^{D-3}}\right) dt^2 - \frac{dr^2}{1 - \frac{\mu}{r^{D-3}}} - r^2 d\Omega^2_{D-2}$$

$$\mu = \frac{16\pi G M}{(D-2) A_{D-2}}$$

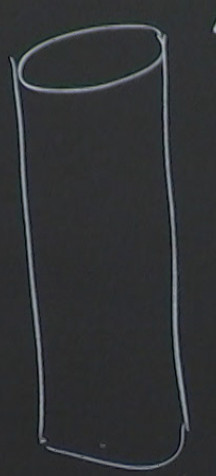
eg Black String Instability



$$\text{Mass} = \frac{r_+ L}{2G_5} = \frac{r_+}{2G_4} = M.$$



eg Black String Instability



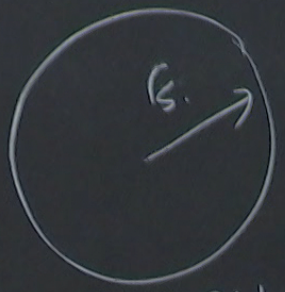
SCHXR $\vec{r_4}$

$$\text{Mass} = \frac{r_+ L}{2G_5} = \frac{r_+}{2G_4} = M.$$

$$S_{BS} = \frac{4\pi r_+^2 L}{4G_5} = \frac{4\pi G_5 M^2}{L}$$

5D BH

$$ds^2_{D} = \left(1 - \frac{M}{r^{D-3}}\right) dt^2 - \frac{dr^2}{1 - \frac{M}{r^{D-3}}} - r^2 d\Omega^2_{D-2}$$



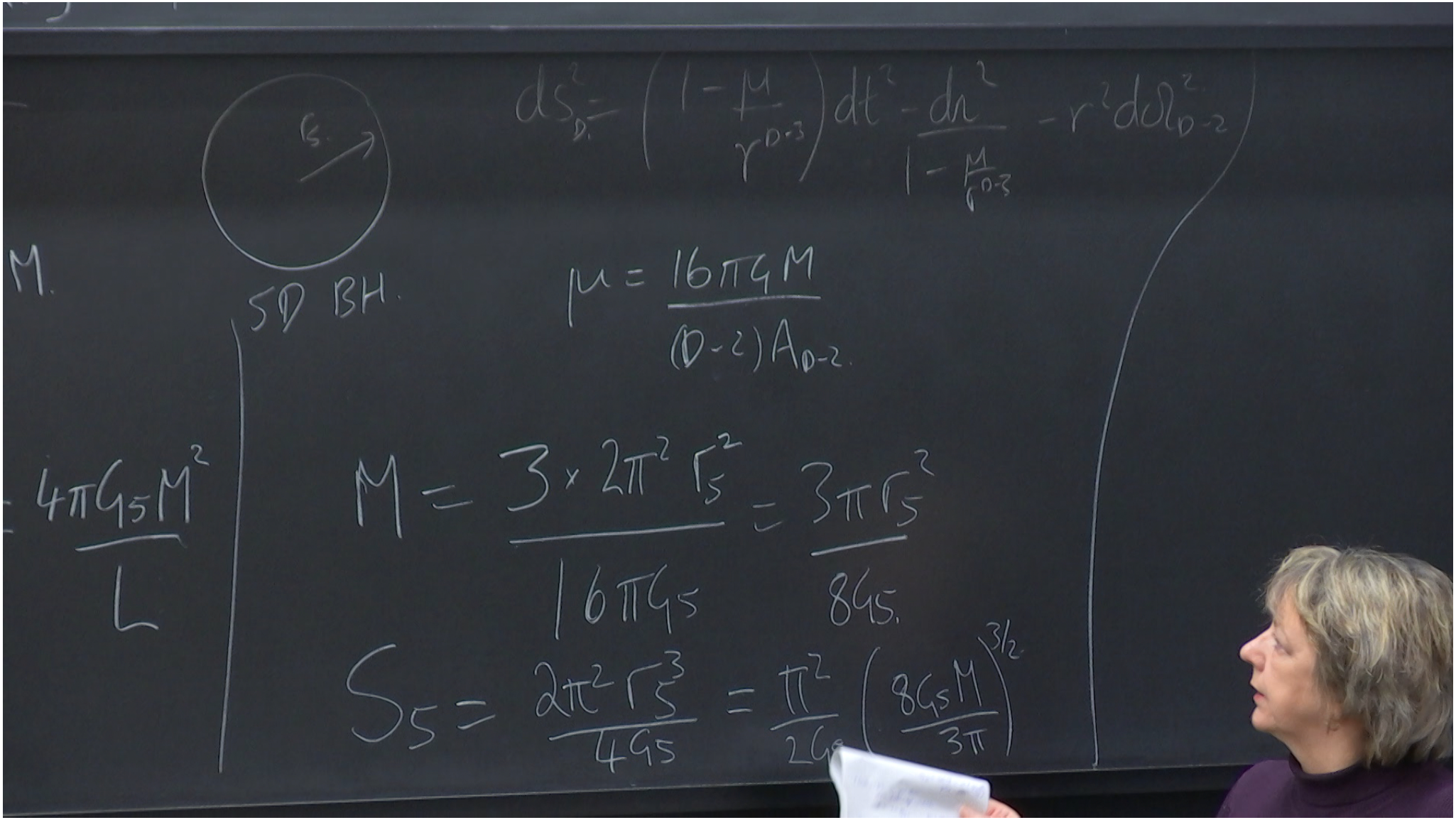
5D BH.

$$\mu = \frac{16\pi G_4 M}{(D-2) A_{D-2}}$$

M.

$$= \frac{4\pi G_5 M^2}{L}$$

$$M = \frac{3 \times 2\pi^2 r_5^2}{16\pi G_5} = \frac{3\pi r_5^2}{8 G_5}$$


$ds^2 = \left(1 - \frac{M}{r^{D-3}}\right) dt^2 - \frac{dr^2}{1 - \frac{M}{r^{D-3}}} - r^2 d\Omega_{D-2}^2$
r_5
5D BH.
M.
$\mu = \frac{16\pi G M}{(D-2) A_{D-2}}$
$= \frac{4\pi G_5 M^2}{L}$
$M = \frac{3 \times 2\pi^2 r_5^2}{16\pi G_5} = \frac{3\pi r_5^2}{8 G_5}$
$S_5 = \frac{2\pi^2 r_5^3}{4 G_5} = \frac{\pi^2}{2G_5}\left(\frac{8 G_5 M}{3\pi}\right)^{3/2}$

$$\frac{dr^2}{\frac{M}{r^{D-3}}} - r^2 d\Omega_{D-2}^2 \Bigg) \quad \text{Then} \quad \frac{S_{BH}}{S_{BS}} = \frac{\pi^2}{2G_S}\left(\frac{8G_S M}{3\pi}\right)^{3/2} \Big/ \frac{4\pi G_S M^2}{L}$$

$$= \sqrt{\frac{8}{27\pi}}\,\frac{L}{\sqrt{G_S M}}$$

$$= \sqrt{\frac{8}{27\pi}}\,\frac{L}{G_4 M}$$

$$\left(\frac{G_S M}{3\pi}\right)^{3/2}$$

$$S_5 = \frac{2\pi^2 r_5^3}{4 G_5} = \frac{\pi^2}{2 G_5} \left(\frac{8 G_5}{} \right.$$

For $L > \frac{27\pi}{16} r_+$

black hole preferred.

$$h_{ab} = e^{\Omega t} e^{i\mu z} \left[\begin{array}{c|c} h_{\mu\nu} & h_{\mu 5} \\ \hline & h_{55} \end{array} \right]$$

$$J_5 = \frac{\partial \mathcal{L}_{IS}}{4 g_5} = \frac{11}{2 g_5}\left(\frac{8 g_5 \Pi}{3\pi}\right)$$

h_{55} is 4D scalar

$$\Delta_L h_{55} = (\Box_4 + \mu^2) h_{55}$$

$$= -f h_{55}'' - 2(r - g_M) h_{55}'$$

$$- \left(\mu^2 + \frac{\Omega^2}{f}\right) h_{55} = 0$$

$h_{\mu 5}^{(r)}$

$h_{55}^{(r)}$

$$J_5 = \frac{d(\Gamma_5)}{4g_5} = \frac{1}{2G_5}\left(\frac{8G_5\Gamma}{3\pi}\right)$$

h_{55} is 4D scalar

$$\Delta_L h_{55} = (\Box_4 + \mu^2) h_{55}$$

$$= -f h_{55}'' - 2(r - G_M) h_{55}'$$

$$- \left(\mu^2 + \frac{\Omega^2}{f}\right) h_{55} = 0$$

$h_{\mu 5}^{(r)}$

$h_{55}^{(r)}$

4D massive wave eqn

$$J_5 = \frac{d(\Gamma_5)}{4G_5} = \frac{\Gamma}{2G_5}\left(\frac{8G_5\Gamma}{3\pi}\right)$$

h_{55} is 4D scalar

$$\Delta_L h_{55} = (\Box_4 + \mu^2) h_{55}$$

$$= -f h_{55}'' - 2(r - 6M) h_{55}'$$

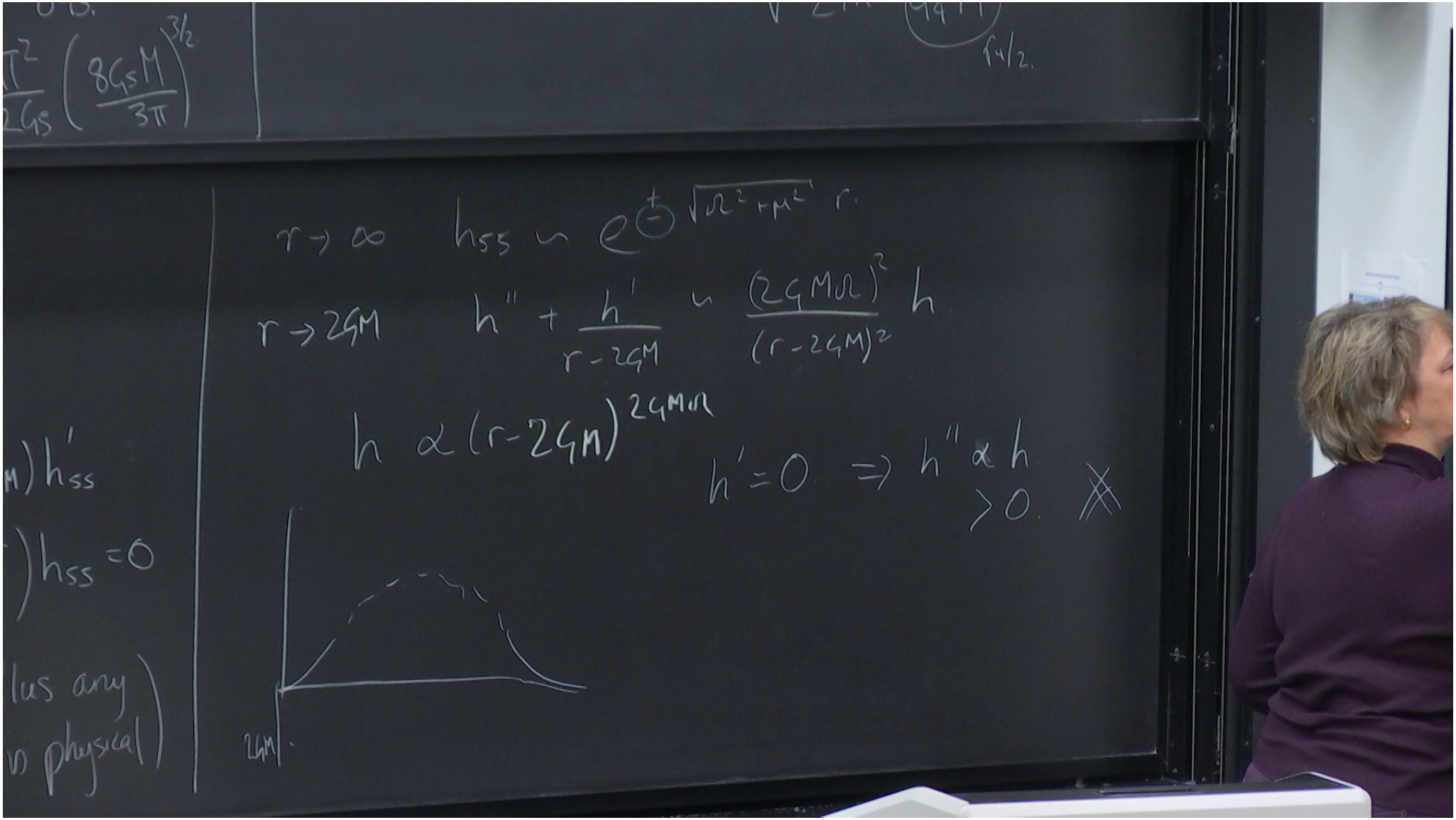
$$- \left(\mu^2 + \frac{\Omega^2}{f}\right) h_{55} = 0$$

4D massive wave eqn (tellus any h is physical)

$h_{\mu 5}^{(r)}$

$h_{55}^{(r)}$

$r \to \infty \quad h_{55} \sim e^{\pm}$


$r \to \infty \quad h_{ss} \sim e^{\pm \sqrt{\omega^2 + M^2}\, r}$
$r \to 2GM \quad h'' + \frac{h'}{r-2GM} \sim \frac{(2GM\omega)^2}{(r-2GM)^2} h$
$h \propto (r-2GM)^{2GM\omega}$
$h' = 0 \Rightarrow h'' \propto h > 0$ ✗
$2GM$

well posed eqn, so can be solved given data.

Eqn has form

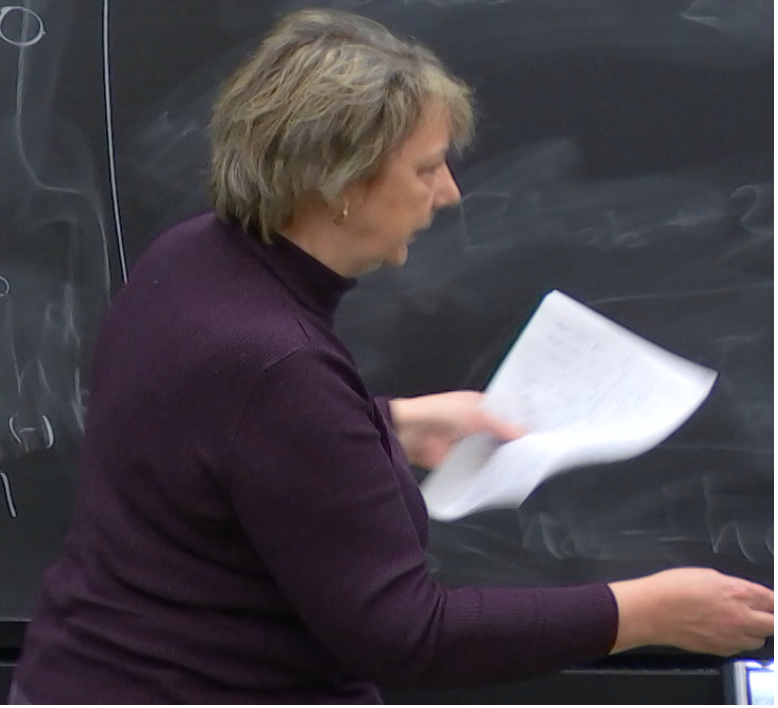
$$\frac{d^2h}{dr^{*2}} - \Omega^2 h - V(r,\mu)h = 0$$

eigenvalue problem for $\Omega(\mu)$.

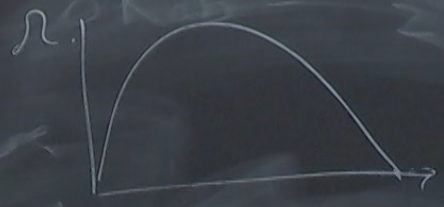
Integrate decaying soln at ∞ to near horizon

$$h = A h^{(+)} + B h^{(-)}$$

Look for zero of



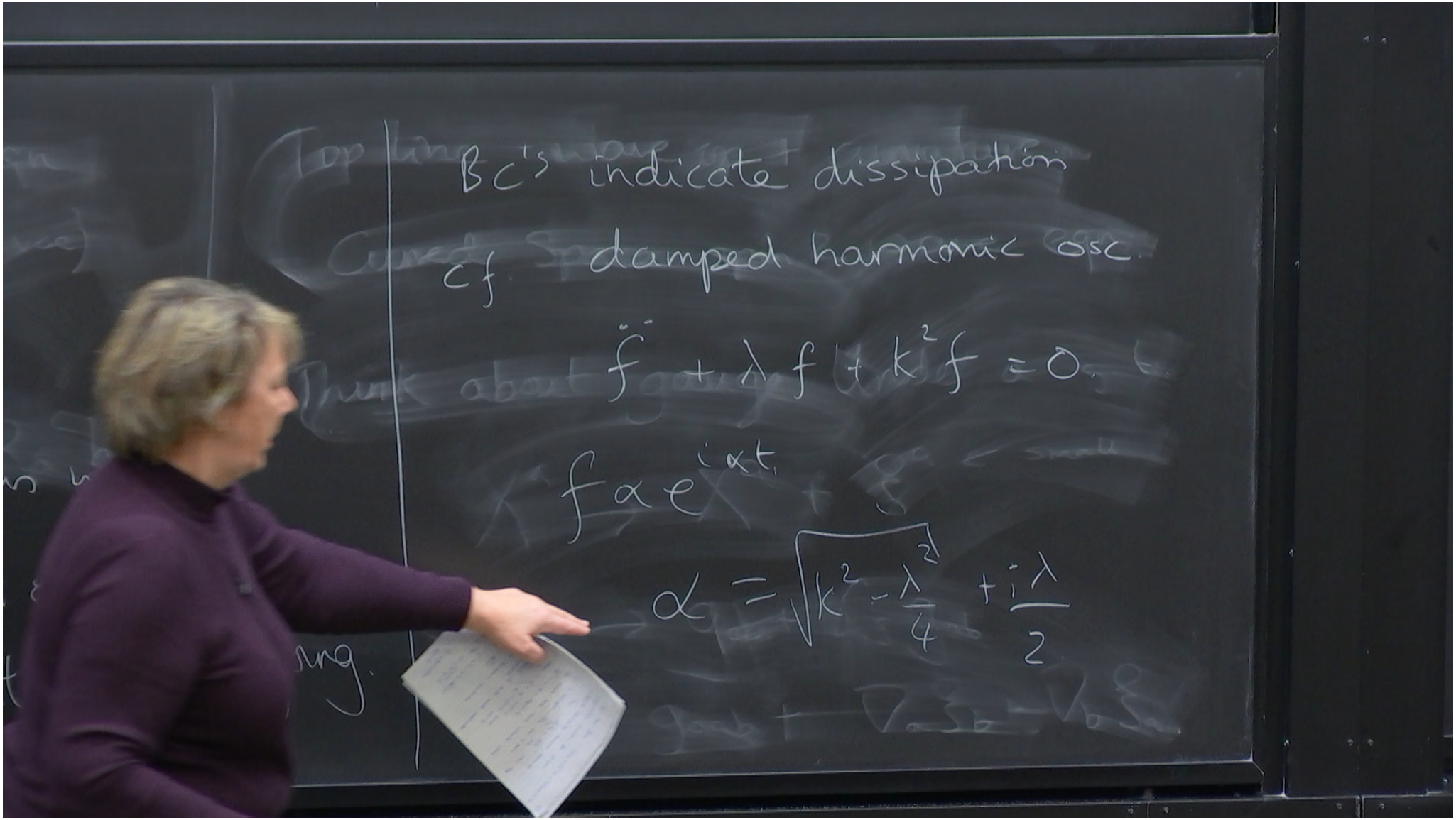
Look for zero of B/A, also in ω

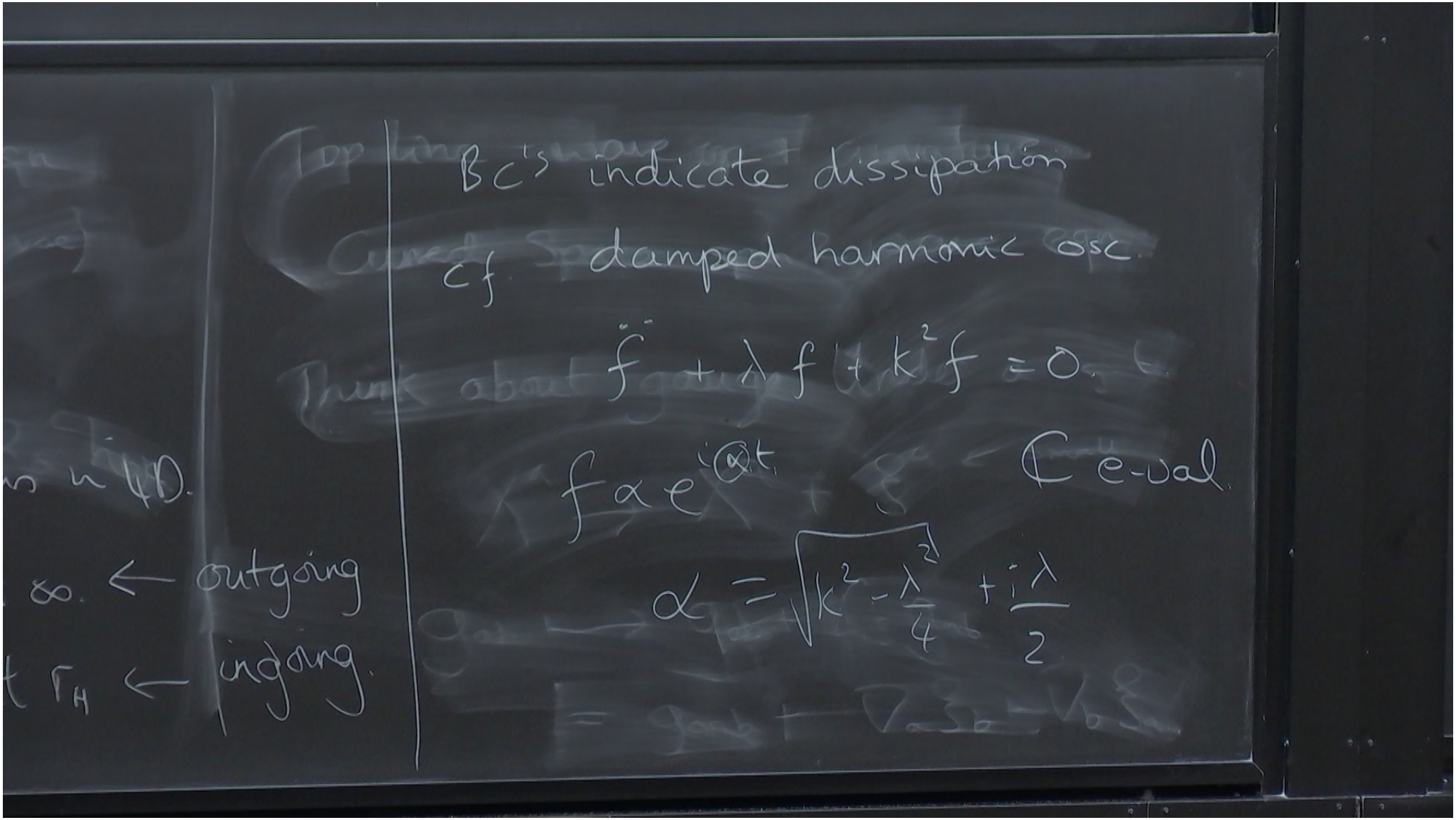


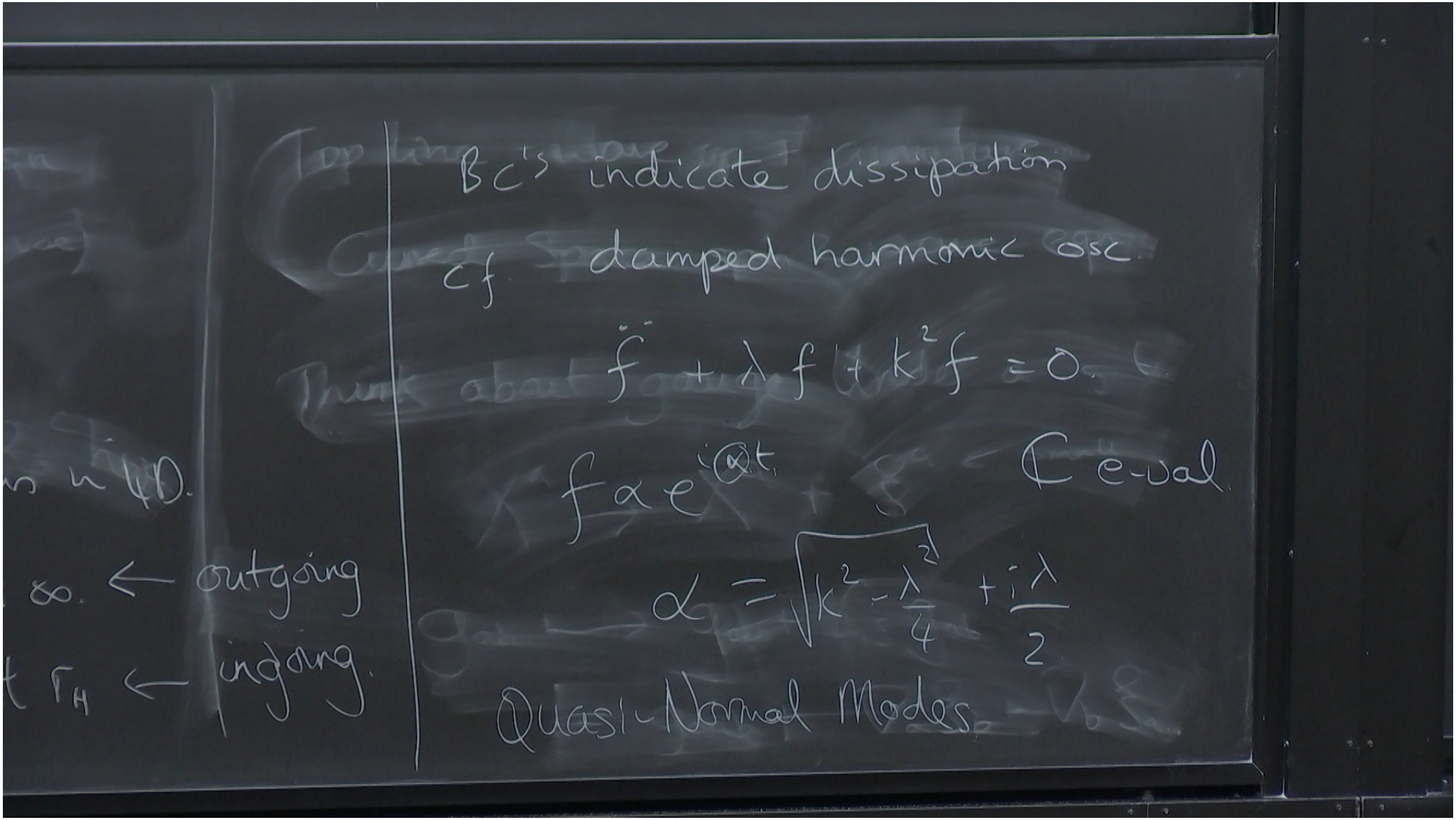
For the black hole perturbations in 4D.

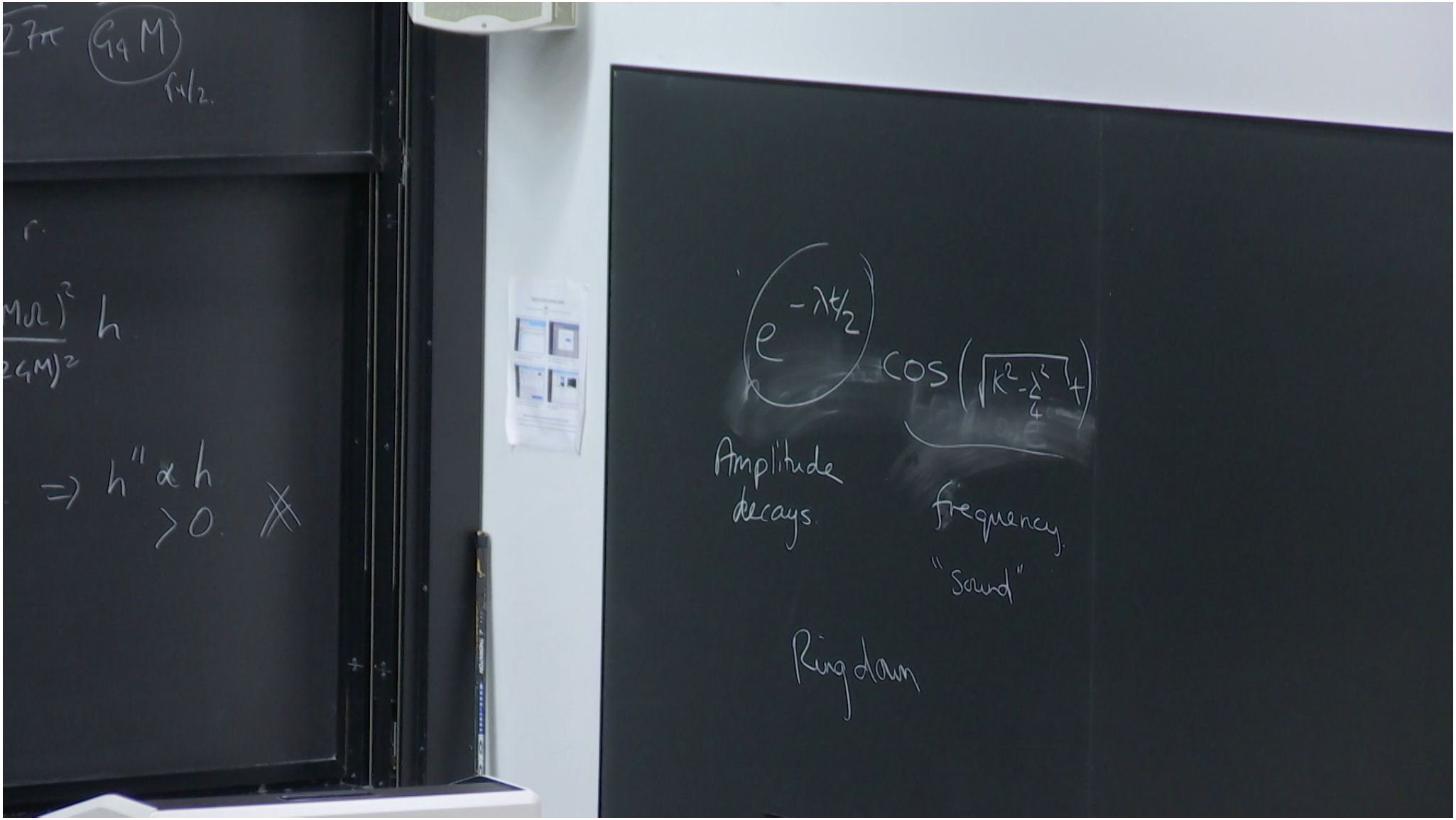
$Q \sim e^{-i\omega(t-r)}$ at ∞. ← outgoing

$e^{-i\omega(t+r^*)}$ at r_H ← ingoing.


BC's indicate dissipation
cf damped harmonic osc
$\ddot{f} + \lambda \dot{f} + k^2 f = 0$
$f \propto e^{i\alpha t}$
$\alpha = \sqrt{k^2 - \frac{\lambda^2}{4}} + i\frac{\lambda}{2}$


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Quasi-Normal Modes
∞ ← outgoing
r_+ ← ingoing


$e^{-\lambda t/2}$
$\cos\left(\sqrt{k^2-\frac{\xi^2}{4}}\,t\right)$
Amplitude decays
frequency
"Sound"
Ringdown