Title: Mathematical Physics Lecture - 230207

Speakers: Kevin Costello

Collection: Mathematical Physics (2022/2023)

Date: February 07, 2023 - 11:00 AM

URL: https://pirsa.org/23020005

2d free scalar

$\mathcal{L}(\varphi) = \int \varphi \Box \varphi$

Important fact

In dimension 2,

the eq^n $\Box \varphi = 0$

is conformally invo

If $g_{\mu\nu}$ is the me

$\Box_g \varphi = \partial_\mu |\det g|^{1/2}$

free scalar

$= \int \varphi \,\Box\, \varphi$

ortant fact

dimension 2,

the eq^n $\Box\varphi = 0$

is conformally invariant.

If $g_{\mu\nu}$ is the metric,

$$\Box_g \varphi = \partial_\mu |\det g|^{1/2} g^{\mu\nu} \partial_\nu \varphi$$

$$g_{\mu\nu} \longrightarrow e^f g_{\mu\nu}$$

$= 0$

y 'invariant'.

he metric,

$\det g|^{1/2} g^{\mu\nu} \partial_\nu \varphi$

$\mu\nu$

$$\det g \longrightarrow e^{2f} \det g \quad (\text{dimension} = 2)$$

$$g^{\mu\nu} \longrightarrow e^{-f} g^{\mu\nu}$$

So if $d = 2$, e^f factors cancel.

Phase space $= \{$ Sol'ns to EOM

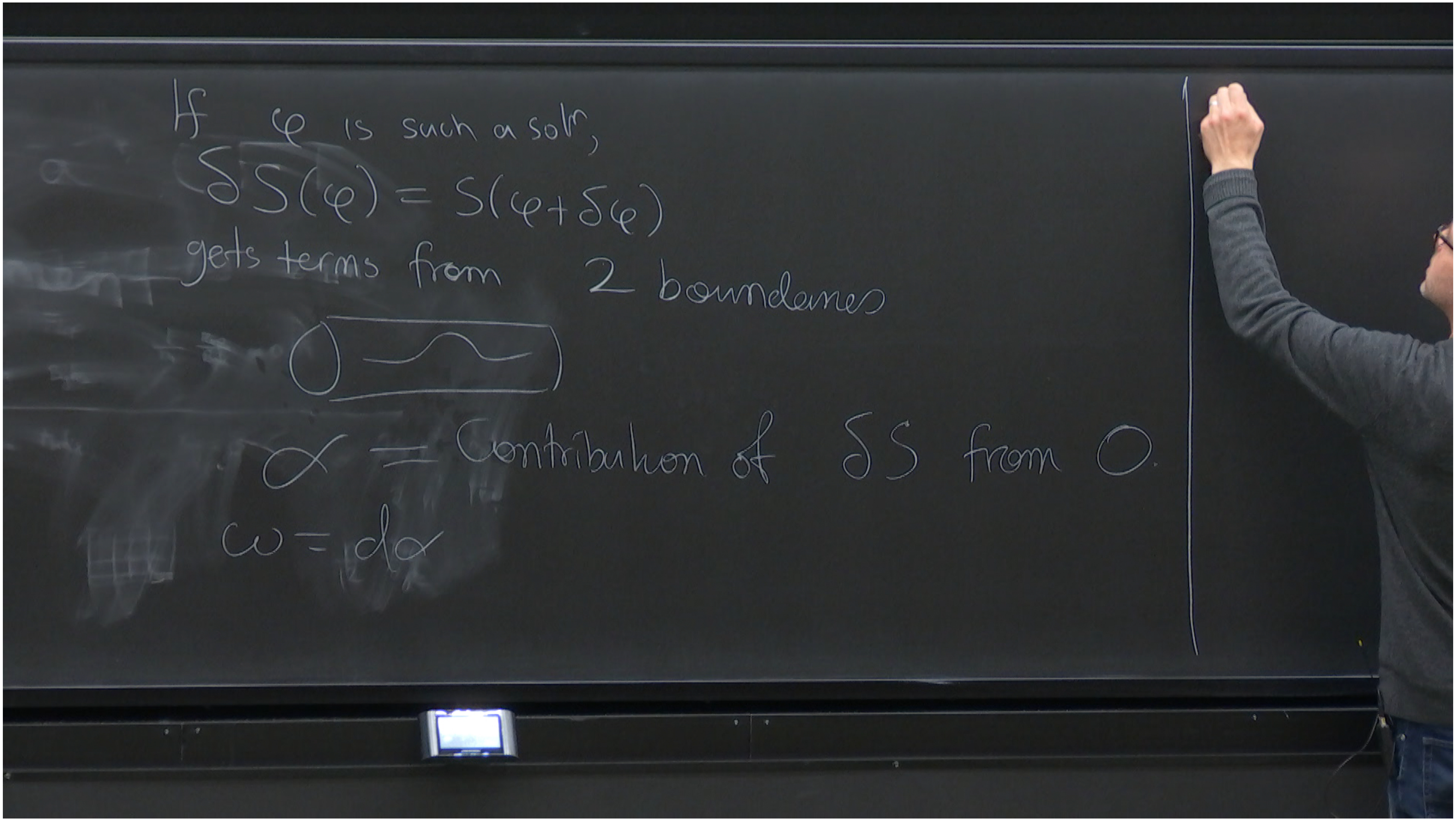
$\varphi(r,\theta)$, defined near $r=1\}$

$q(\theta) = \varphi(1,\theta)$

$p(\theta) = (r\partial_r\varphi)(1,\theta)$

are the coordinates

From before, $H_E = -\int_\theta p(\theta)^2 - \int_\theta q(\theta)\partial_\theta^2 q(\theta)$


If φ is such a soln,
δS(φ) = S(φ+δφ)
gets terms from 2 boundaries
α = Contribution of δS from O.
ω = dα

Why is L Lagrangian?

Recall,

Variational 1-form

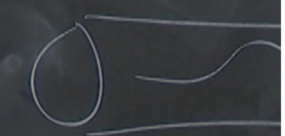
is defined by considering

Solns EOM on $S^1 \times [0,1]$

If φ is such

$\delta S(\varphi) =$

gets terms fro



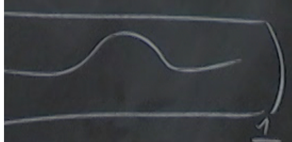
$\alpha =$

$\omega = d\alpha$

such a sol^n, $\alpha_0 - \alpha_1 = dS$

$) = S(\varphi + \delta\varphi)$

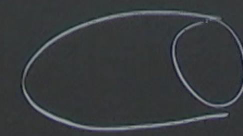
from 2 boundaries



$=$ Contribution of δS from 0.

$d\alpha$

On disc,



$\alpha = dS(\varphi)$

as there is only 1 boundar

$d\alpha = \omega = 0$

on set of sol^{n}s that extend across a disc.

or any manifold M

$= \partial N$

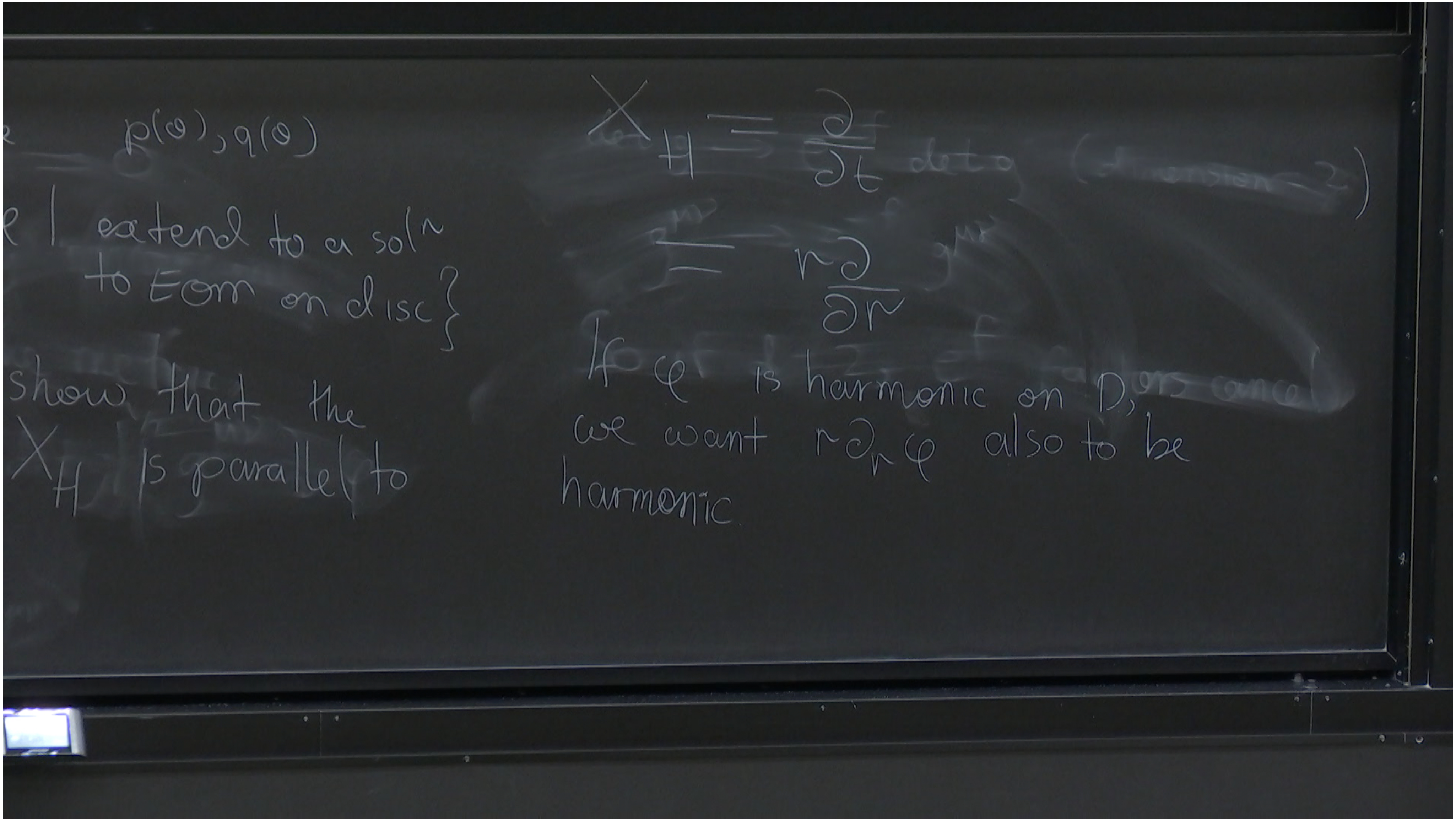
$\{ \text{sol}^{\text{n}}\text{s to EOM on } N \}$

es a state on M.

Phase space $p(\theta), q(\theta)$

$\mathcal{L} = \{ \varphi \mid$ extend to a sol$^{\text{n}}$ to EOM on disc $\}$

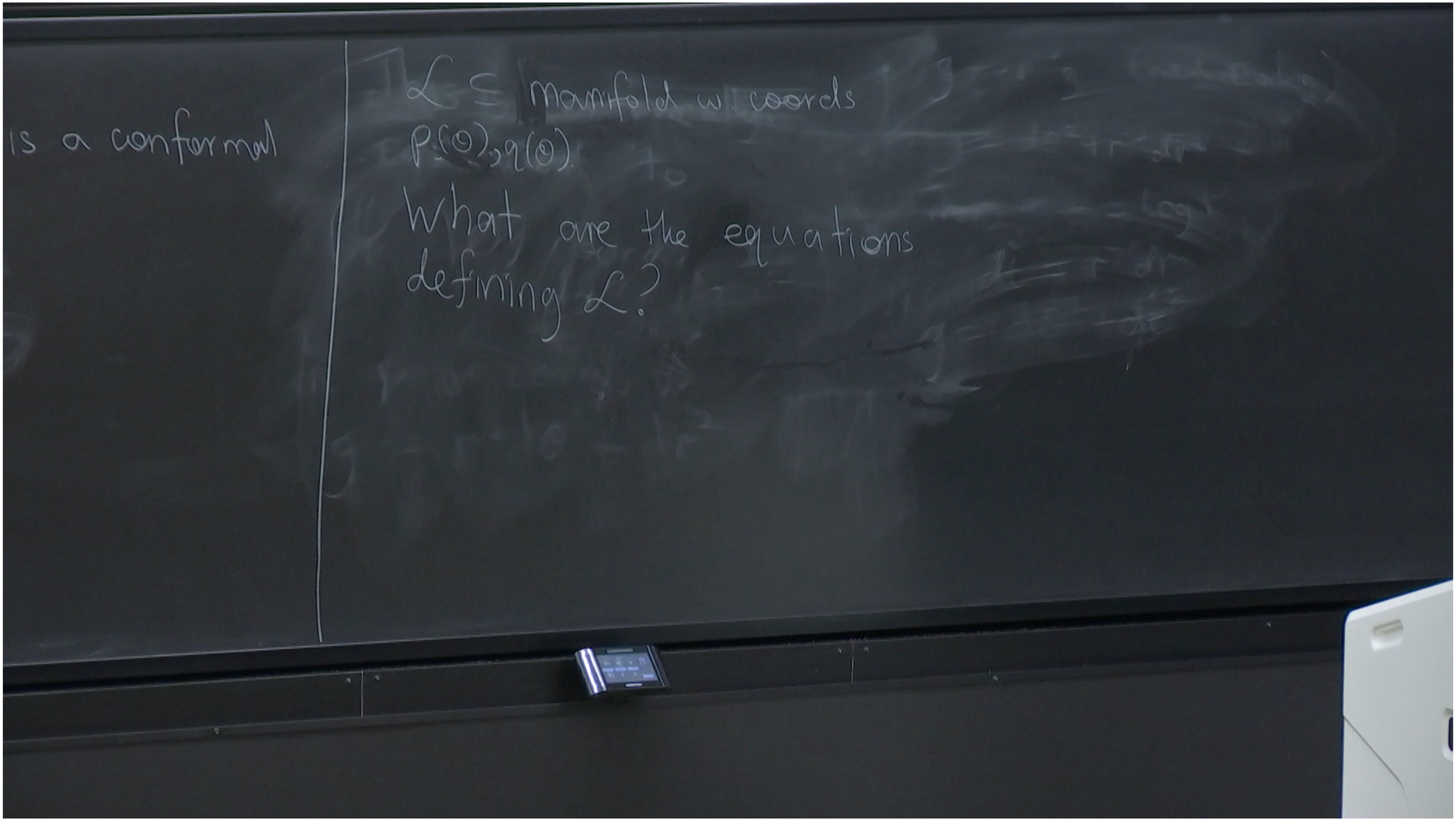
Want to show that the v. field X_H is parallel to $\mathcal{L}$.

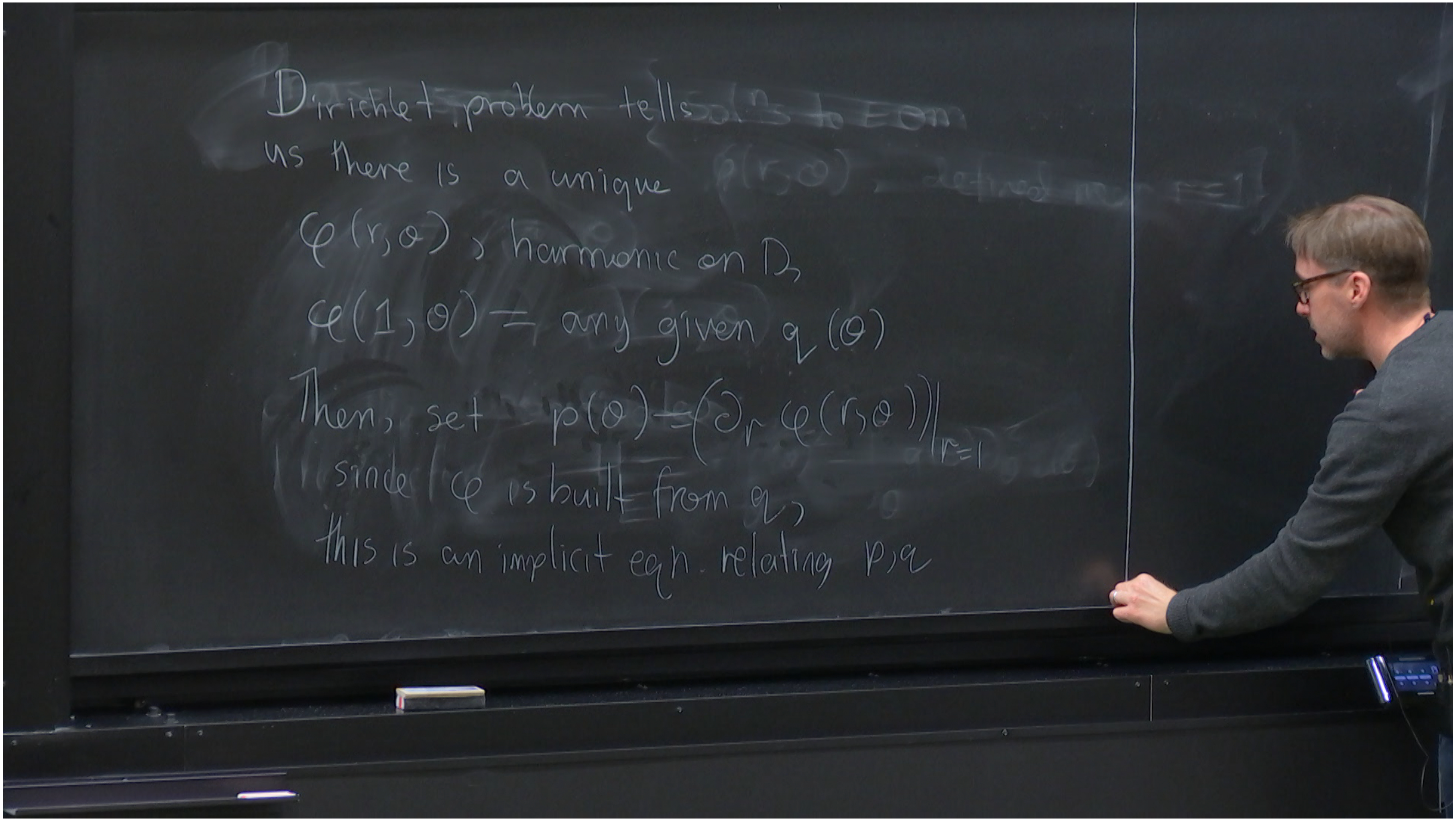

p(θ), q(θ)
extend to a sol^n to EoM on disc?
show that the
X_H is parallel to
X_H = ∂/∂t
= r ∂/∂r
If φ is harmonic on D,
we want r∂_r φ also to be harmonic

This is automatic as $r\partial_r$ is a conformal symmetry. EOM

$$\Delta = \partial_r^2 + r^{-2}\partial_\theta^2$$

$$[r\partial_r, \Delta] = -2\Delta$$


is a conformal
$\mathcal{L} \subseteq$ manifold w/ coords
$p(\theta), q(\theta)$.
What are the equations
defining $\mathcal{L}$?


Dirichlet problem tells
us there is a unique
φ(r,θ), harmonic on D,
φ(1,θ) = any given q(θ)
Then, set p(θ) = (∂r φ(r,θ))|r=1
since φ is built from q,
this is an implicit eqn. relating p,q

To make this explicit,
recall,
any harmonic function
on D is real part
of a holomorphic function
(and conversely).

Write

$$q(\theta) = \sum q_n^+ \cos 2\pi n\theta + q_n^- \sin 2\pi n\theta$$

$$z^n = r^n e^{2\pi i n\theta}$$

$$= r^n(\cos 2\pi n\theta + i \sin 2\pi n\theta)$$

$z = x + iy$

$\ldots\cos 2\pi n\theta + q_n^- \sin 2\pi n\theta$

$\ldots n\theta$

$\ldots \pi n\theta + i \sin 2\pi n\theta)$

If $\mathrm{Re}\, z^n = r^n \cos 2\pi n\theta$

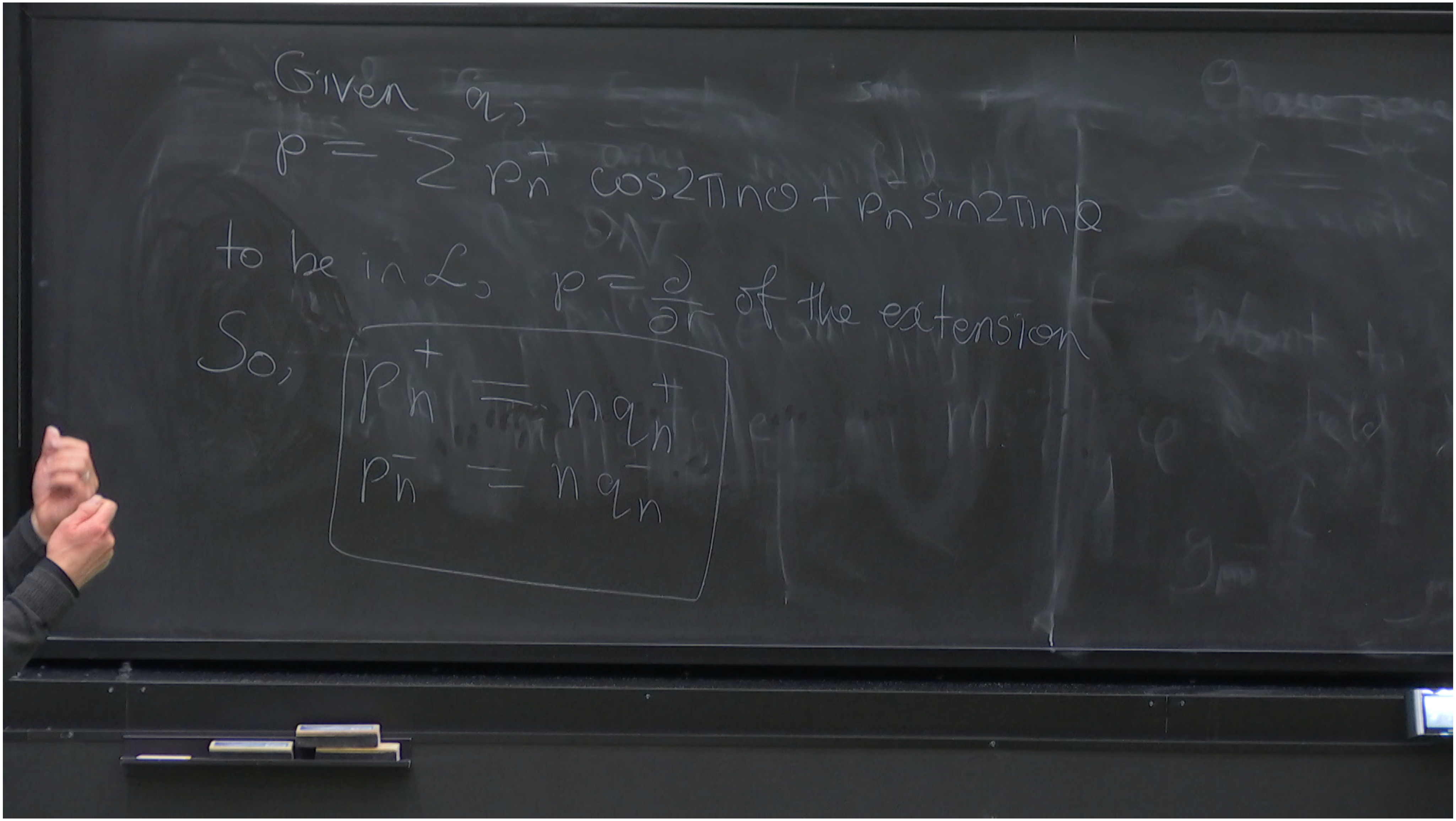
$\mathrm{Im}\, z^n = r^n \sin 2\pi n\theta$

are both harmonic.

So

$$\sum q_n^+ r^n \cos 2\pi n\theta + q_n^- r^n \sin 2\pi n\theta$$

is harmonic on D

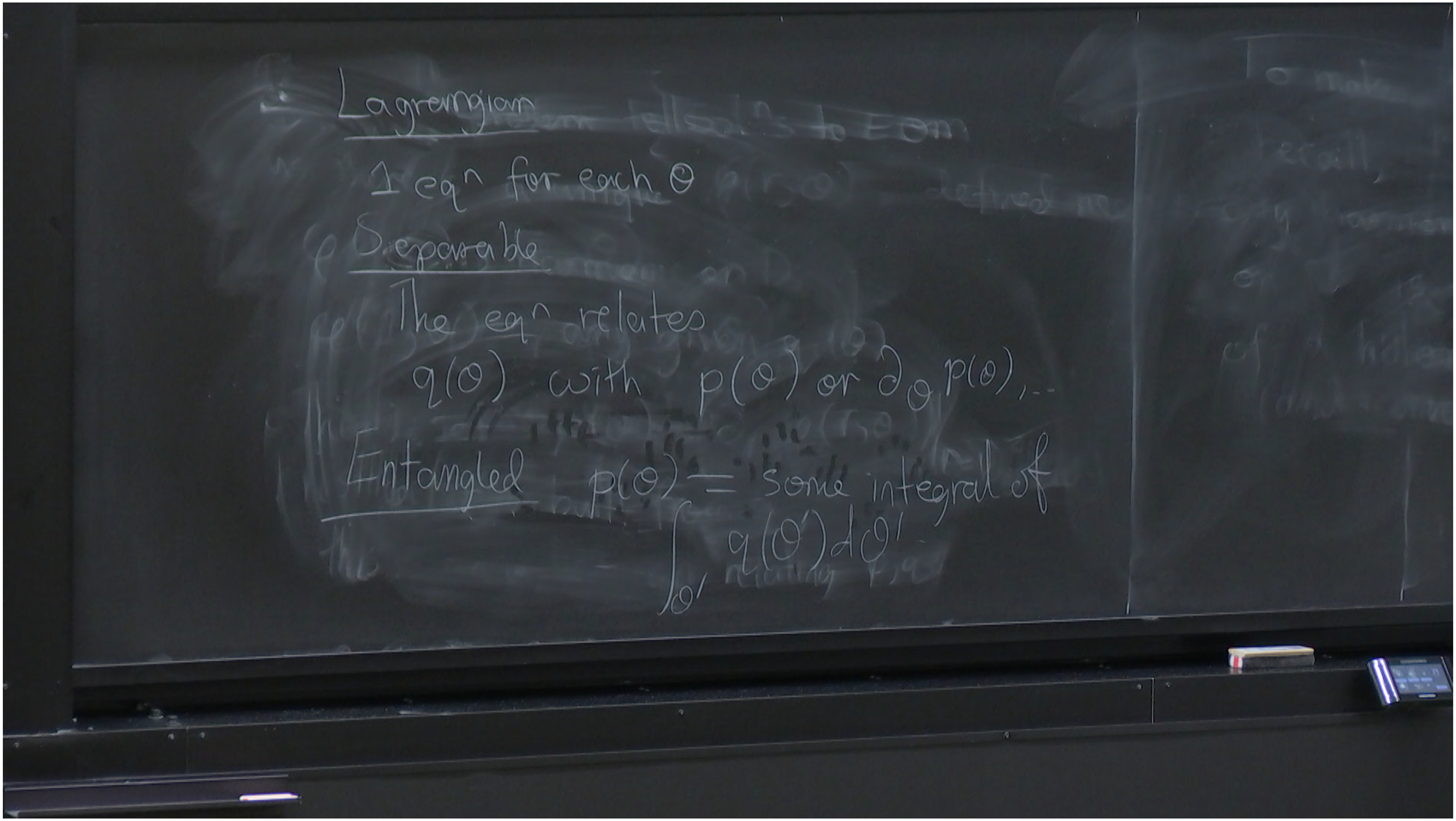

Given a,
$p = \sum p_n^+ \cos 2\pi n\theta + p_n^- \sin 2\pi n\theta$
to be in $\mathcal{L}$, $p = \frac{\partial}{\partial r}$ of the extension
So, $p_n^+ = n a_n^+$
$p_n^- = n a_n^-$

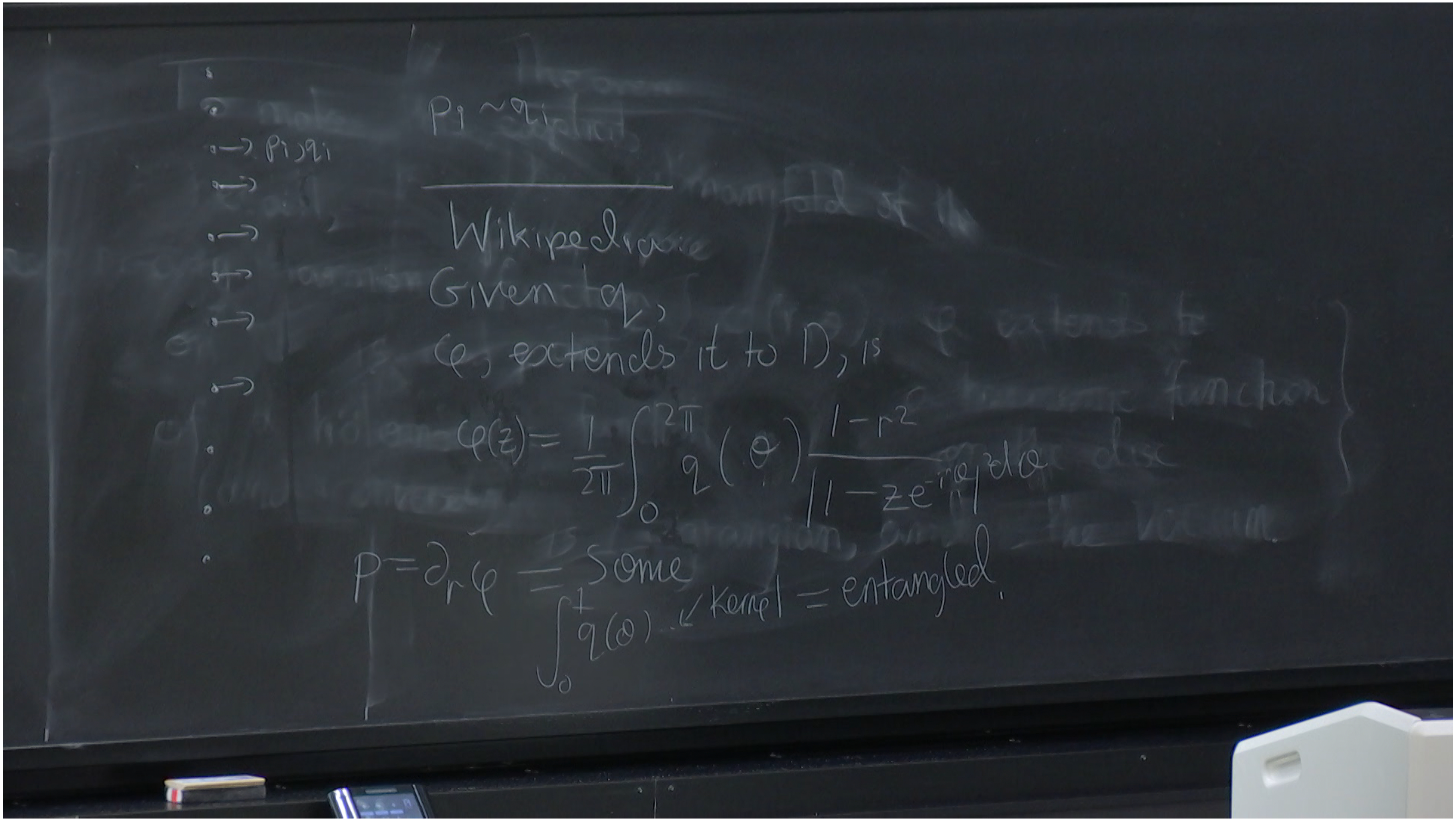
The Hamiltonian is

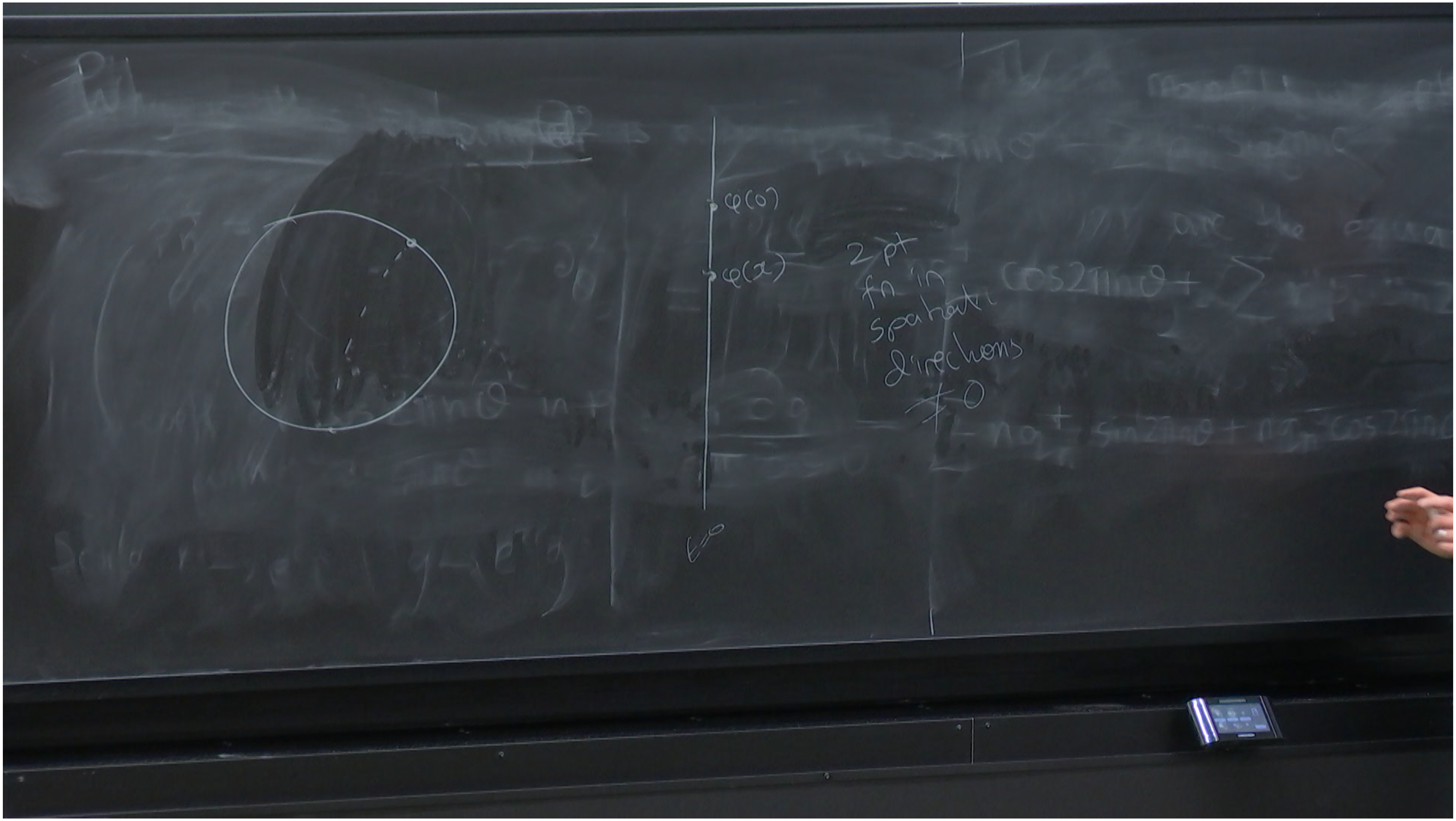
$$= \int p(\theta)^2 - \frac{1}{(4\pi)^2} q(\theta)\,\partial_\theta^2 q(\theta)$$

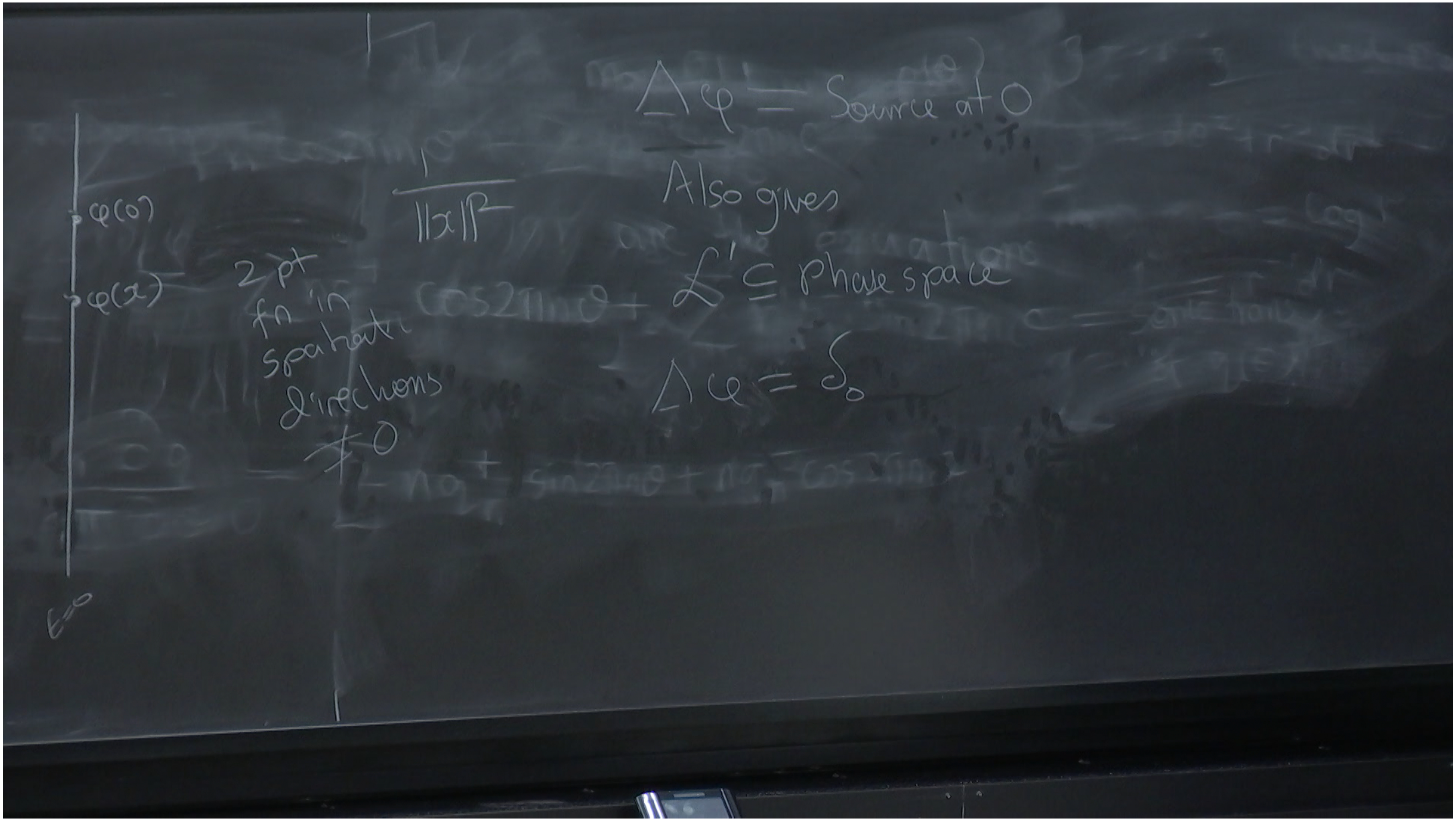
$$= -\sum (p_n^+)^2 - (p_n^-)^2 + n^2 (q_n^+)^2 + n^2 (q_n^-)^2$$

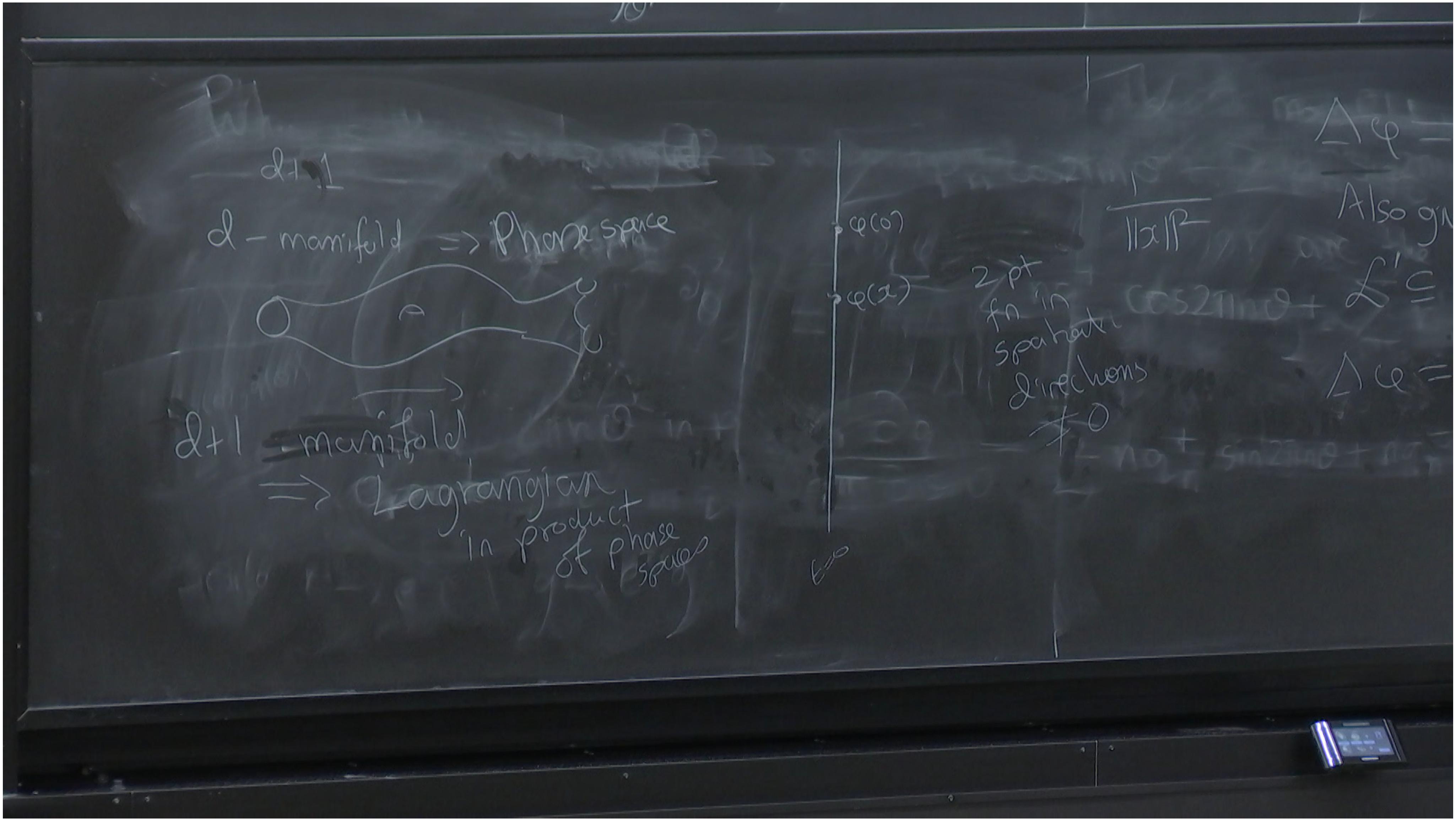
So $H = 0$ on this submanifold!


Lagrangian
1 eqⁿ for each θ
Separable
The eqⁿ relates
q(θ) with p(θ) or ∂θ p(θ), ...
Entangled p(θ) = some integral of
∫θ' q(θ') dθ'


p_i ~ q_i explicit
p_i, q_i
Wikipedia
Given q,
φ, extends it to D, is
φ(z) = 1/2π ∫_0^2π q(θ) (1 − r²)/|1 − ze^{−iθ}|² dθ
p = ∂_r φ = some ∫_0^1 q(θ) ... kernel = entangled.


φ(0)
φ(x)
2 pt
fn in
spatial
directions
≠ 0


Δφ = Source at 0
Also gives
L' ⊆ phase space
Δφ = δ0
1/||x||²
φ(0)
φ(x)
2 pt
fn in
spatial
directions
≠ 0
t=0


d+1
d - manifold => Phase space
d+1 manifold
=> Lagrangian
in product
of phase
spaces
φ(0)
φ(x)
2 pt
fn in
spatial
directions
≠ 0
1/||x||²
cos 2πnθ +
Also