Title: Mathematical Physics Lecture - 230202

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Collection: Mathematical Physics (2022/2023)

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States $\longleftrightarrow$ Lagrangian submanifolds

(M, ω)

$f : M \longrightarrow B$

$\dim B = n = \frac{1}{2} \dim M$

Further, functions on B

Poisson commute

$\mathcal{H} = \{ \mathbb{C}$ valued fns on BS points of $B \}$

$b \in B$, b is Bohr-Sommerfeld

$\delta_b \in$ a state.

Think:

δ_b is associated to the submanifold $f^{-1}(b) \subseteq M$ of dimension n

Def^n

A submanifold $L \subseteq M$ is Lagrangian if

1) $\dim L = \frac{1}{2} \dim M$

2) $\omega|_L = 0$

Equivalently

If, locally L is cut out by $f_1, \ldots, f_n = 0$

2) is equivalent to

$\{f_i, f_j\} = 0$ on L

Equivalently

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2) is equivalent to

$$\{f_i, f_j\} = 0 \text{ on } L$$

δ-fn is a "measure"

$$\int \delta_{x=0} f(x)\, dx = f(0)$$

Equivalently

If, locally L is cut out by $f_1, \dots, f_n = 0$

2) is equivalent to

$$\{f_i, f_j\} = 0 \text{ on } L$$

Eg p_1, p_2, q_1, q_2 $\omega = \sum dq_i \wedge dp_i$

$L = \{p_1 = 0, q_1 = 0\}$ This is not Lagrangian

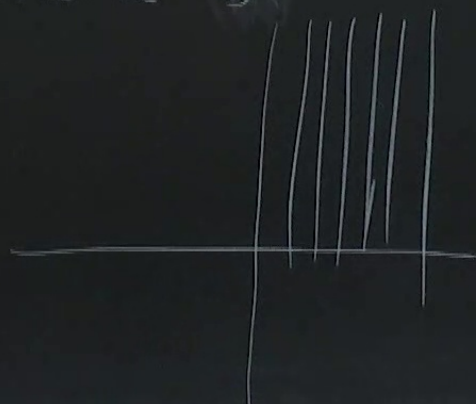
as,

$\{p_1, q_1\} = 1$ is not zero on L

Or: q_2, p_2 are coords on L

$\omega|_L = dq_2 \wedge dp_2$, not zero

If $\pi : M \to B$
is a polarization
$\pi^{-1}(b) \subseteq M$ is Lagrangian
for generic b



Theorem (Darboux-Weinstein)
If L is a Lagrangian,
$\exists$ coords locally
p_i, q_i so
$$\omega = \sum dq_i \wedge dp_i$$
and $L = \{ p_i = 0, i = 1 \ldots n \}$

Quantum State $\overset{\hbar \to 0}{\rightsquigarrow}$ Lagrangian

If $|\psi\rangle$ is a state

$\mathcal{A}$ = def. quantization of $C^\infty(M)$

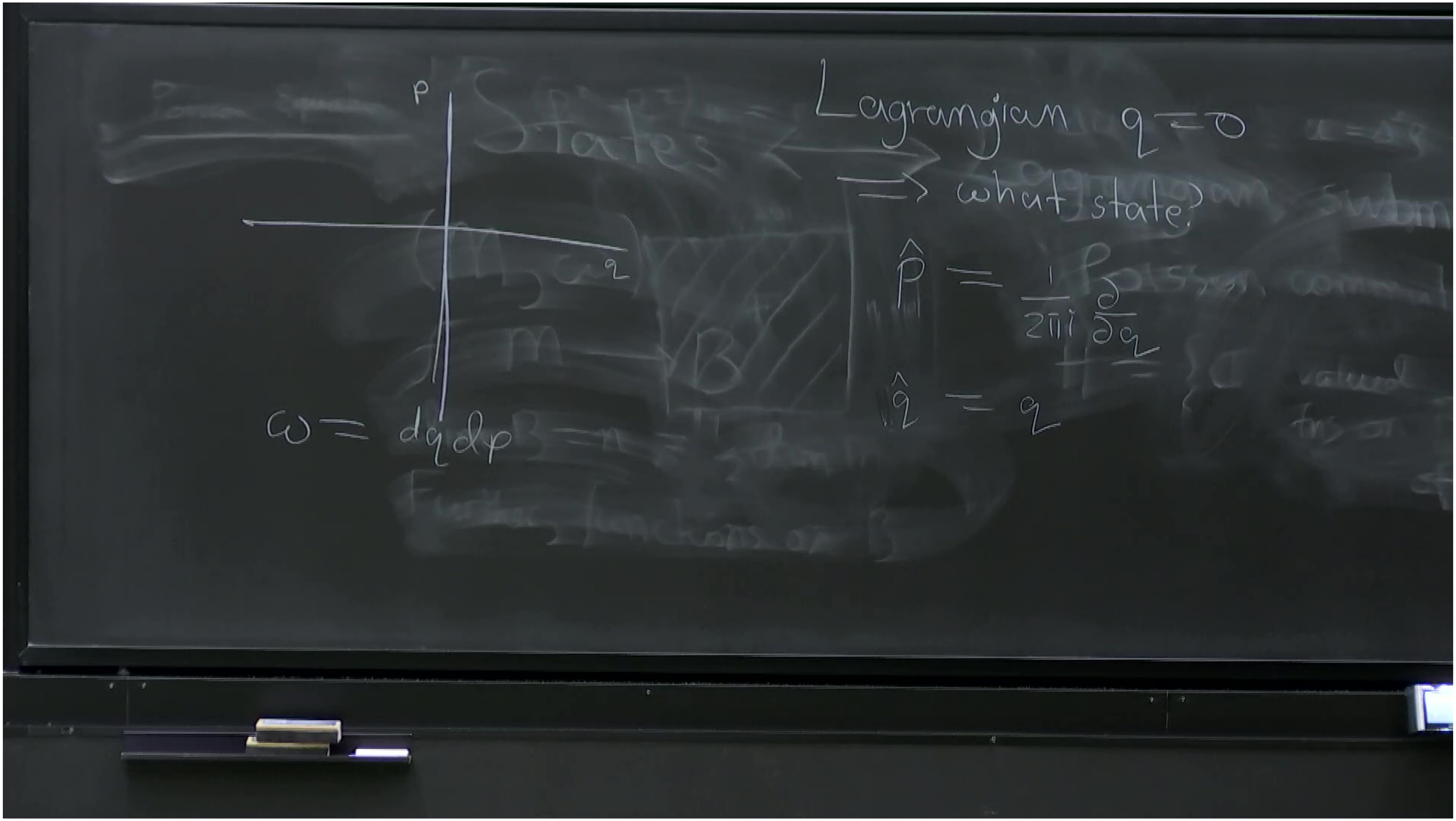
Let $I \subseteq \mathcal{A}$ of operators $\mathcal{O}$ so that

$$\mathcal{O}|\psi\rangle = 0$$

As $\hbar \to 0$ this will become a subset $I_{\hbar=0} \subseteq C^\infty(M)$ of functions which vanish on L.

If $\theta_1, \theta_2 \in I \subseteq A$

$[\theta_1, \theta_2]|\psi\rangle = 0$ as well



States
Lagrangian $q = 0$
$\Longrightarrow$ what state?
$\hat{p} = \frac{1}{2\pi i}\frac{\partial}{\partial q}$
$\hat{q} = q$
$\omega = dq\,dp$
p
q
B

Want $\psi(q)$ ~~to be~~ such that

$$q\,\psi(q) = 0$$

<u>Soln</u> $\psi(q) = \delta_{q=0}$.

Definite position, undetermined momentum

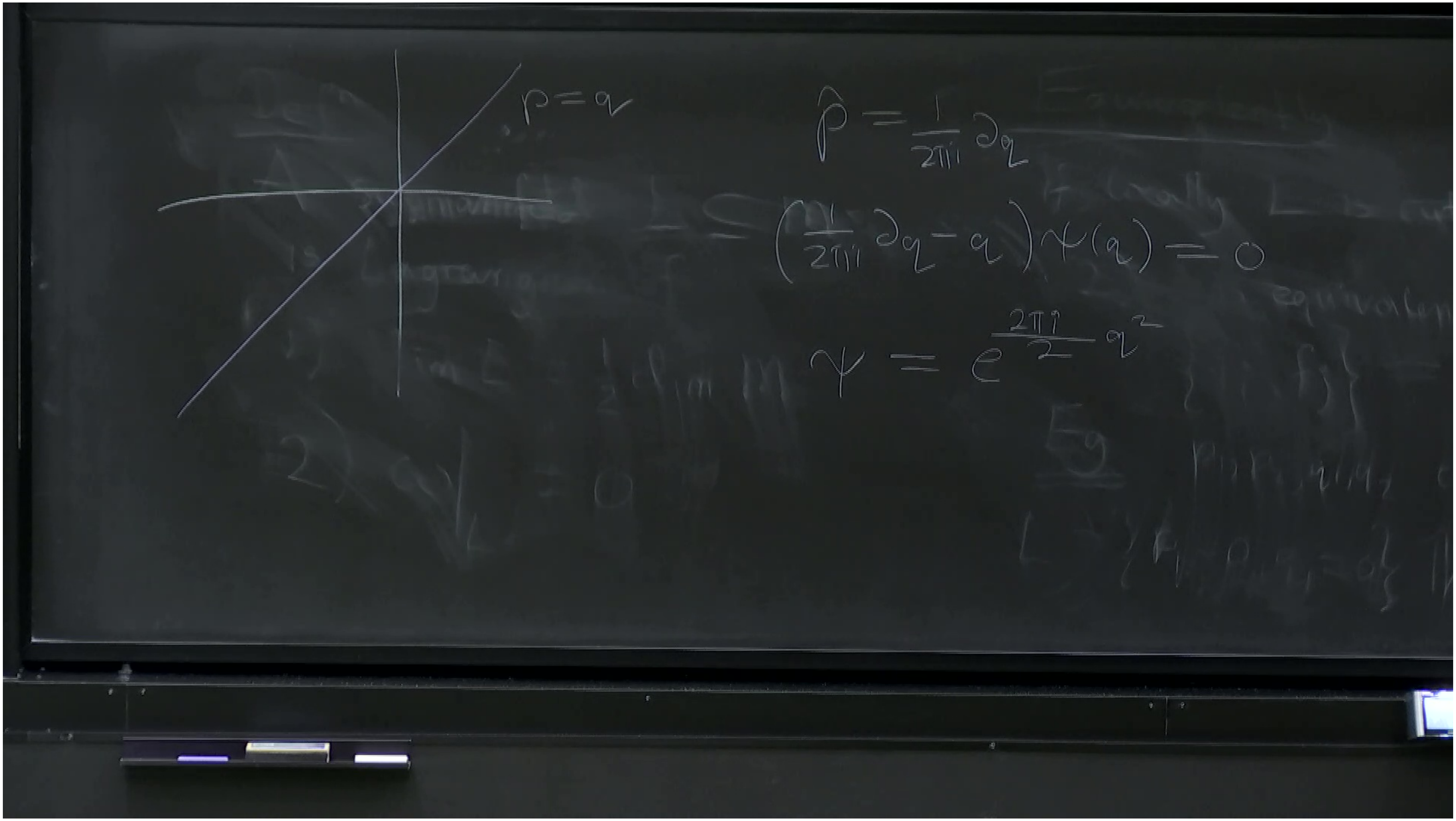
Want $\psi(q)$ to be such that $p = 0$

$q\,\psi(q) = 0$ $\rightsquigarrow$ $\psi = 1$

Definite momentum, undetermined position

<u>Sol^n</u> $\psi(q) = \delta_{q=0}$

Definite position, undetermined momentum


$p = q$
$\hat{p} = \frac{1}{2\pi i}\partial_q$
$\left(\frac{1}{2\pi i}\partial_q - q\right)\psi(q) = 0$
$\psi = e^{\frac{2\pi i}{2} q^2}$

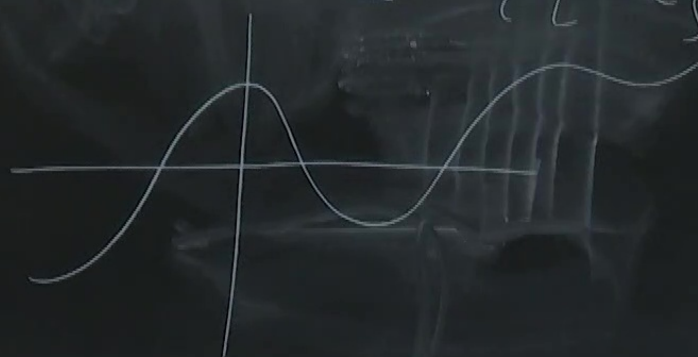
$$\int \delta_{q=0} f(q) = f(0)$$

$$\int \left(\delta_{q=0}\, g(q)\right) f(q) = g(0) f(0)$$

$$g(q)\,\delta_{q=0} = g(0)\,\delta_{q=0}$$

Given $L \subseteq \mathbb{R}^2$, what is $\psi_L(q)$?

Answer: Assume for simplicity that $L \cap \{q = c\}$ is always a point.



Then, L

Then, L = graph of a function

g on $\mathbb{R}$ "$p = g(q)$" defines L

State is $\psi(q)$ satisfies

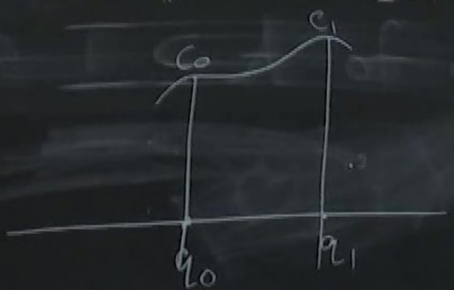
$$\frac{1}{2\pi i} \partial_q \psi = g(q)\psi$$

which we solve by integrating.

Or: $\alpha = 2\pi i q\, dp$

So $\omega = \dfrac{d\alpha}{2\pi i}$

Then, we can compute

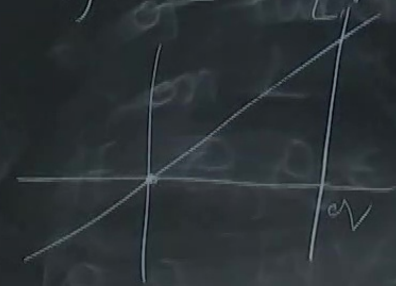


$$\psi(q_1) \quad \psi(q_0)$$

$$= \exp\left(\int_{c_0}^{c_1} \alpha\right)$$

Integral is along the path from c_0 to c_1 in L

If $L = \{p = q\}$

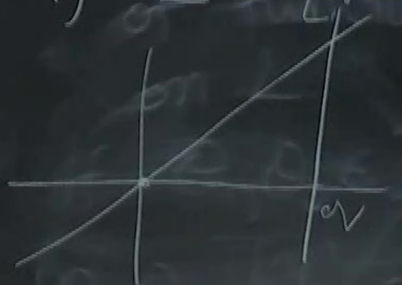


$c_0 = 0$

$c_1 = (p, q)$

Integral is along the path from c_0 to c_1 in L

If $L = \{p = q\}$



$c_0 = 0$

$c_1 = (p, q)$

$\alpha = 2\pi i \, q \, dp$

On L, $dp = dq$

$$\alpha = \frac{2\pi i}{2} dq^2$$

$$\int_0^q \alpha = \frac{2\pi i}{2} q^2$$

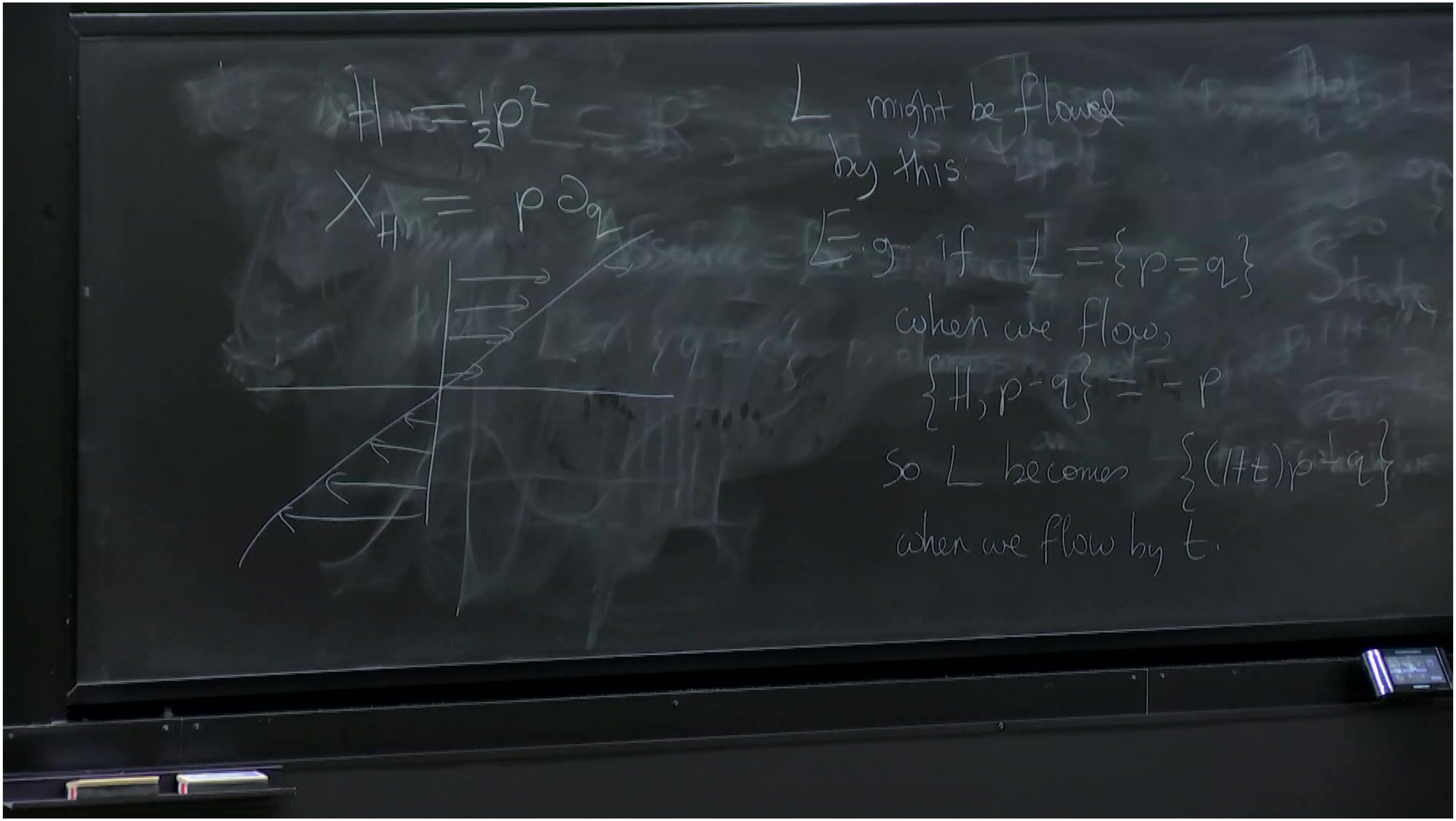
$$\psi = \exp\left(\frac{2\pi i}{2} q^2\right)$$

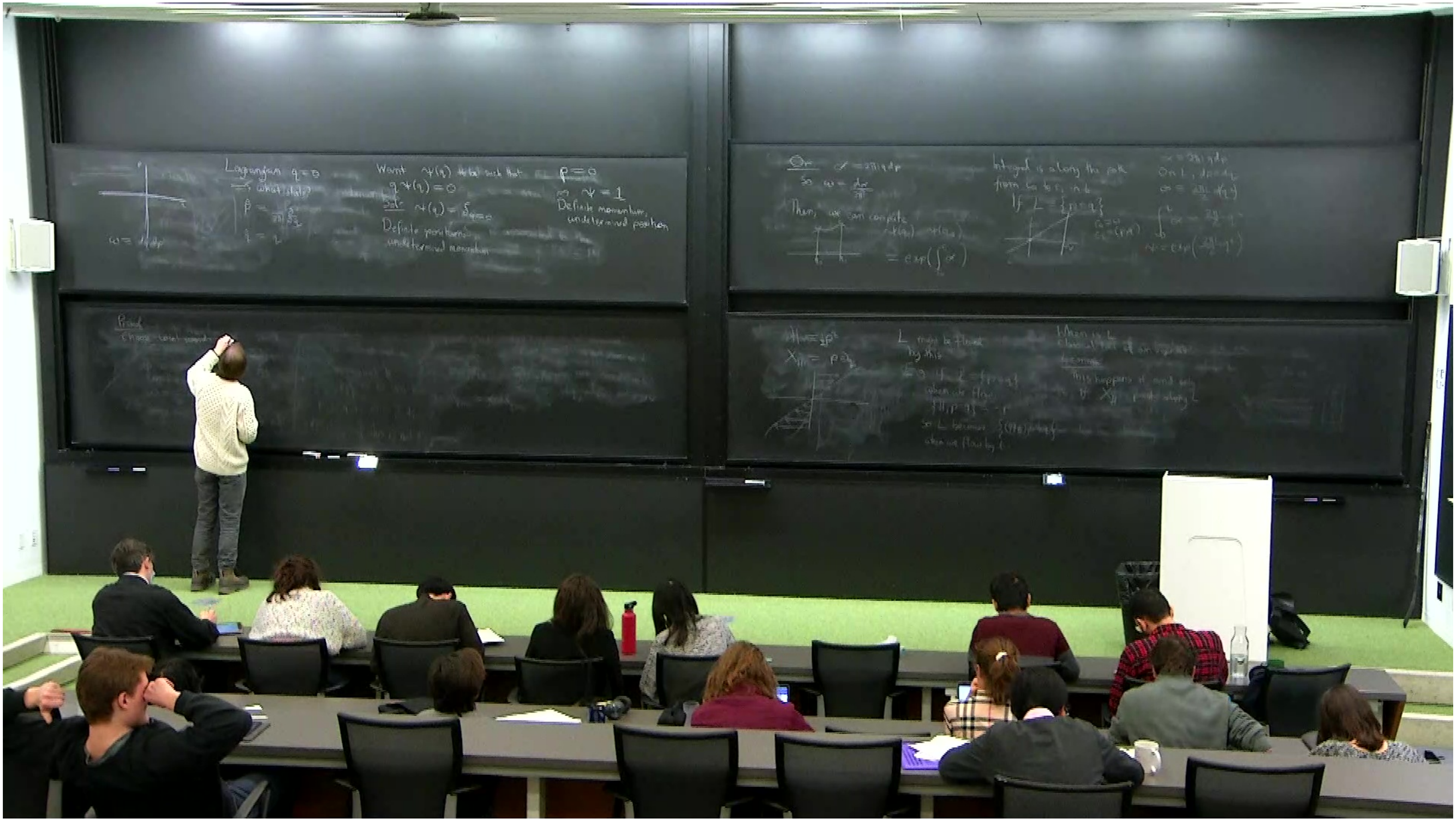
I forgot to mention

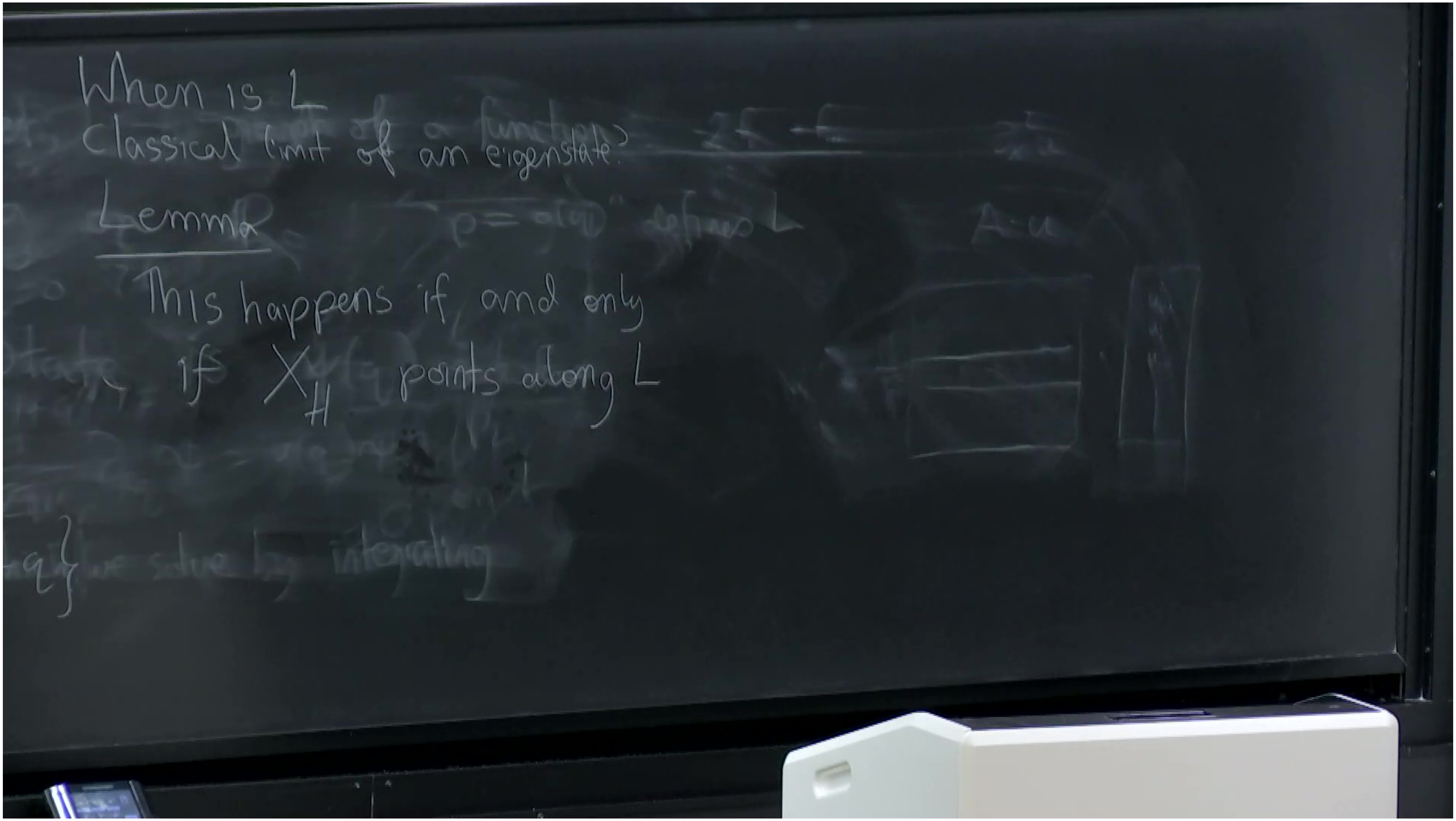
$L \subseteq M$ will correspond to a state only if BS condition is satisfied:

$\omega = \frac{1}{2\pi i} F(A)$ A $u(1)$ gauge field; $e^{\int_\gamma A} = 1$ for all

Loops in L.


$H = \frac{1}{2}p^2$
$X_H = p\,\partial_q$
L might be flowed
by this.
E.g. if $L = \{p = q\}$
when we flow,
$\{H, p - q\} = -p$
So L becomes $\{(1+t)p = q\}$
when we flow by t.



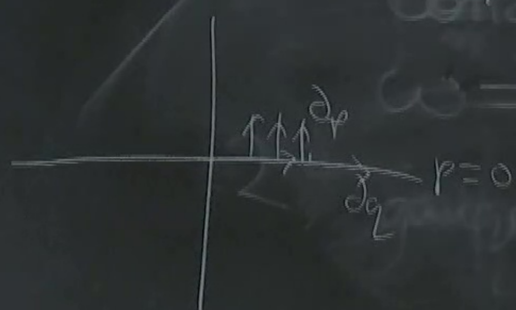

When is L
Classical limit of an eigenstate?
Lemma
This happens if and only
if X_H points along L

Proof:

Choose local coords p_i, q_i

so $L = \{p_i = 0\}$

Then, $$X_H = \sum \underbrace{\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}}_{\text{along } L} - \underbrace{\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}}_{\text{normal to } L}$$



X_H poi

$\Longleftrightarrow$

X_H points along L

$\iff$ $\dfrac{\partial H}{\partial q_i} = 0$ on L

q_i are coords. on L

this means H is constant on L
which is semi-classical version of L being an eigenstate.

$\hat{H}|\psi_L\rangle = c|\psi_L\rangle$

$(\hat{H}-c)|\psi_L\rangle = 0$

In classical limit

X_H points along L

$$\iff \frac{\partial H}{\partial q_i} = 0 \text{ on } L$$

q_i are coords. on L

this means H is constant on L

which is semi-classical version of L being an eigenstate.

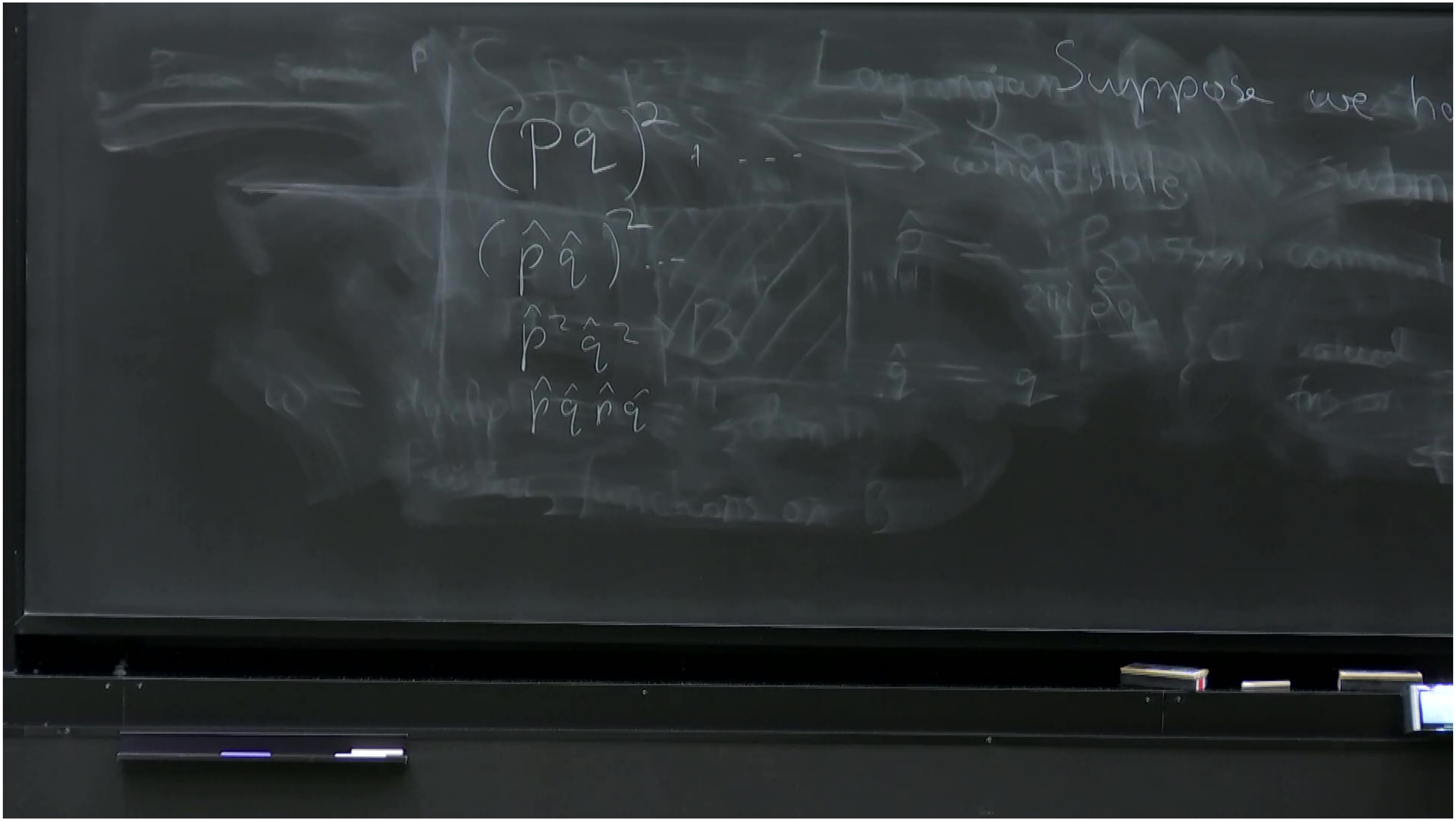
$$\hat{H}|\psi_L\rangle = c|\psi_L\rangle$$

$$(\hat{H}-c)|\psi_L\rangle = 0$$

In classical limit

$H-c$ vanishes on L

$H=c$ on L



Suppose
$(pq)^2, \ldots$
$(\hat{p}\hat{q})^2 \ldots$
$\hat{p}^2\hat{q}^2$
$\hat{p}\hat{q}\hat{p}\hat{q}$

Suppose we have product $M_1 \times M_2$ of 2 systems.

e.g. $M_1 = \mathbb{R}^2$ coords p_1, q_1

$M_2 = \mathbb{R}^2$ " p_2, q_2

A Lagrangian $L \subseteq M_1 \times M_2$ is separable if $L = L_1 \times L_2$

Otherwise, it's entangled.

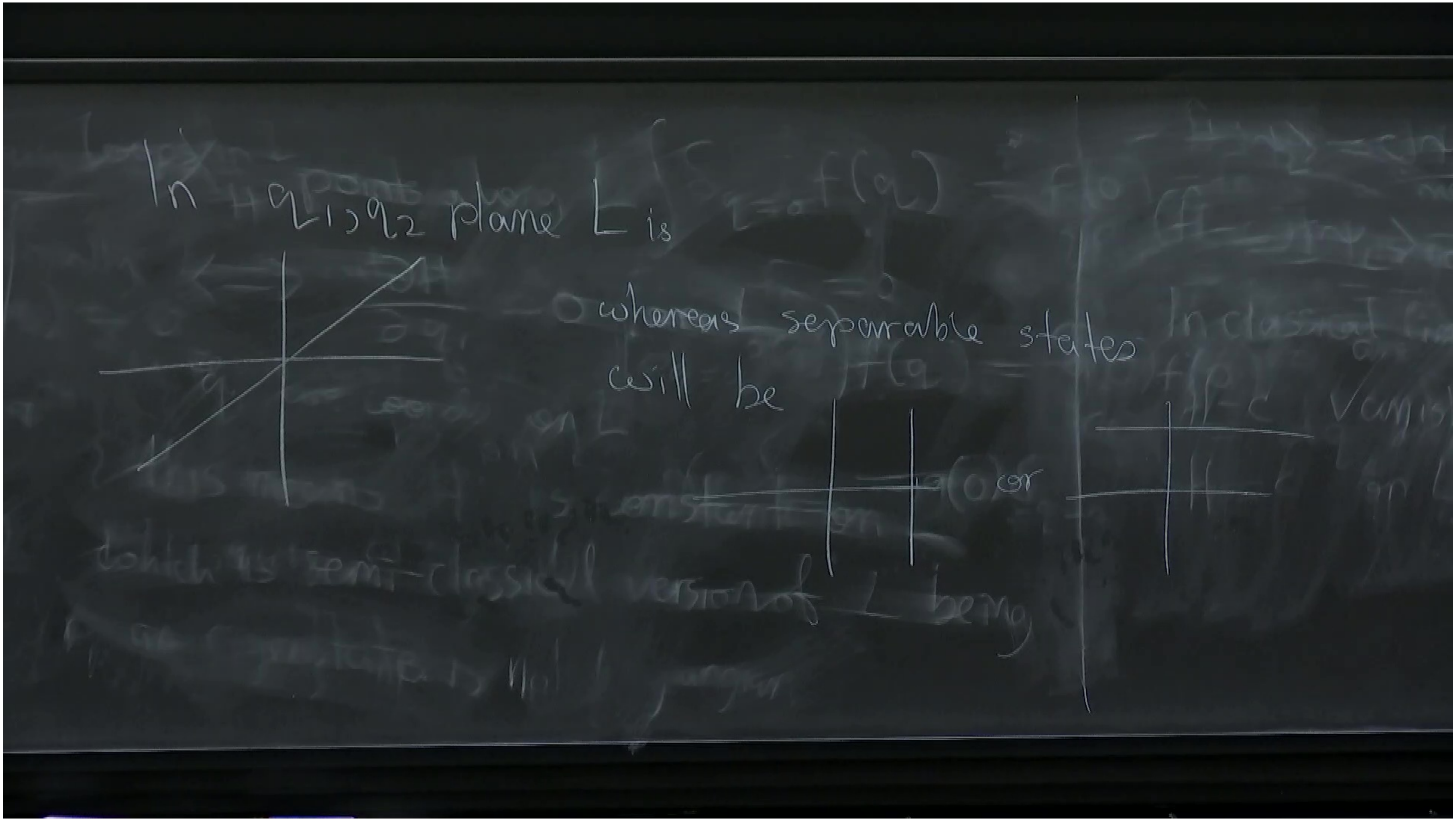
Take

$$L = \left\{ \begin{array}{l} q_1 = q_2 \\ p_1 = -p_2 \end{array} \right\}$$

Since

$$\{ p_1 + p_2 , q_1 - q_2 \} = 0$$

this is Lagrangian.


In q1, q2 plane L is
whereas separable states
will be
or
In classical

The quantum state

$\psi_L(q_1, q_2)$ satisfies

$$q_1 \psi_L = q_2 \psi_L$$

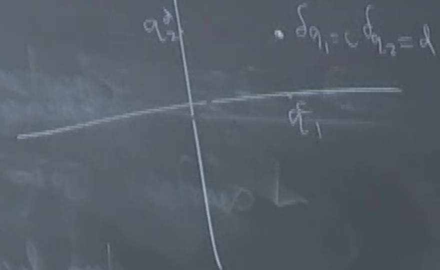
$$\left(\frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2}\right)\psi_L = 0$$

Solution:

$$\psi_L(q_1, q_2) = \delta_{q_1 = q_2}$$

Solution:

$$\Psi_L(q_1,q_2) = \delta_{q_1=q_2}$$

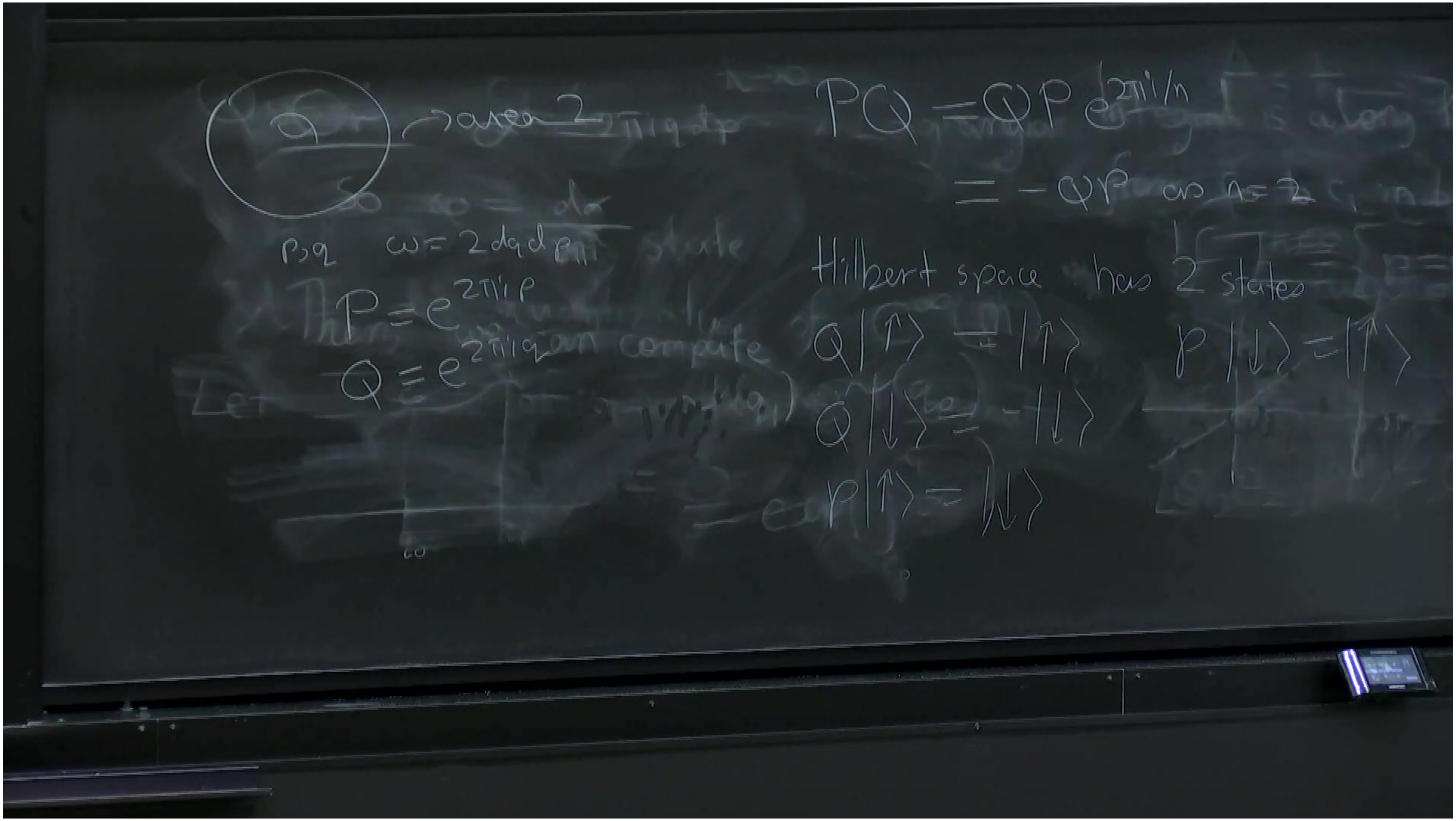


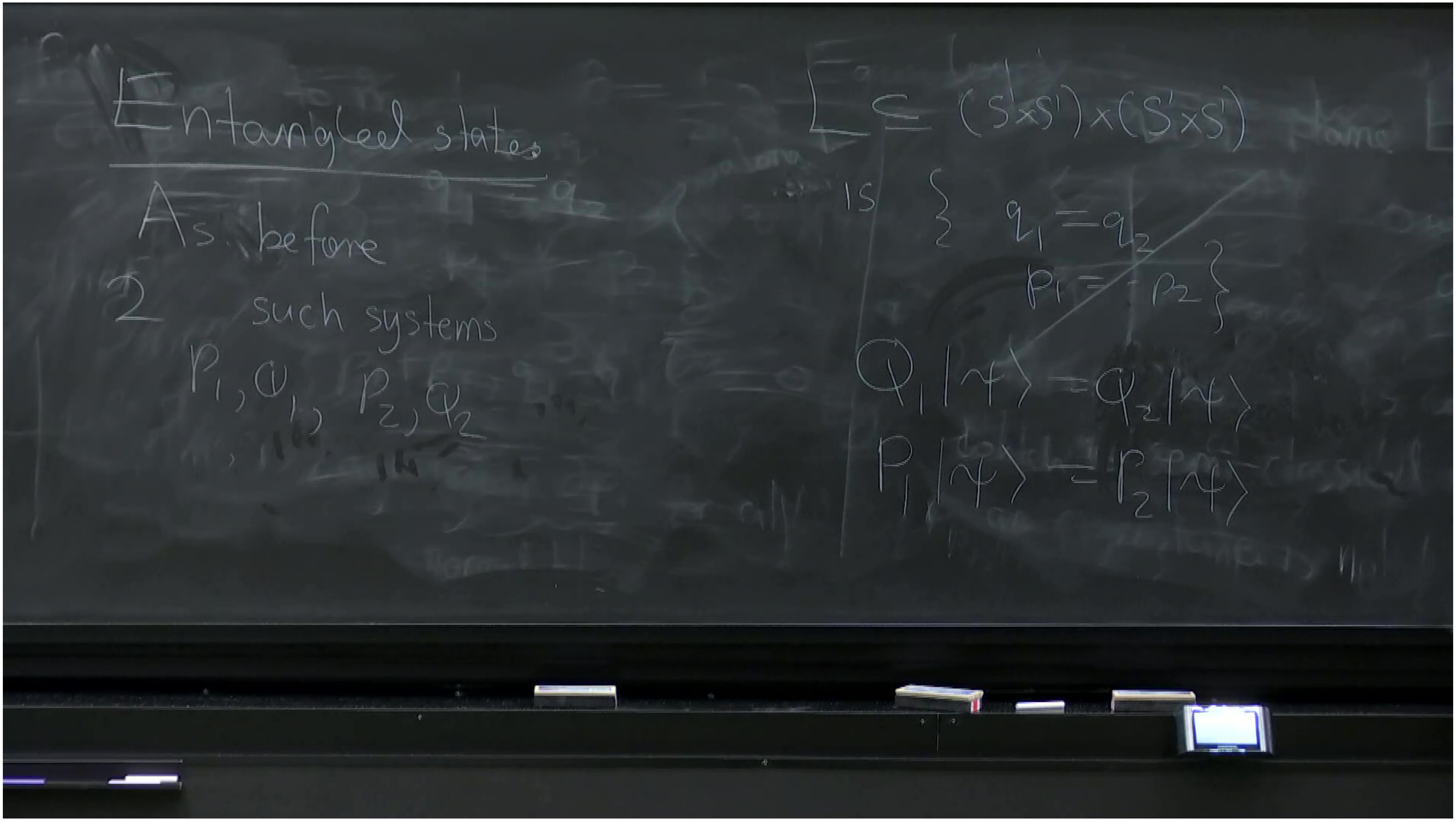
$\delta_{q_1=c}\otimes\delta_{q_2=c}$ are separable states

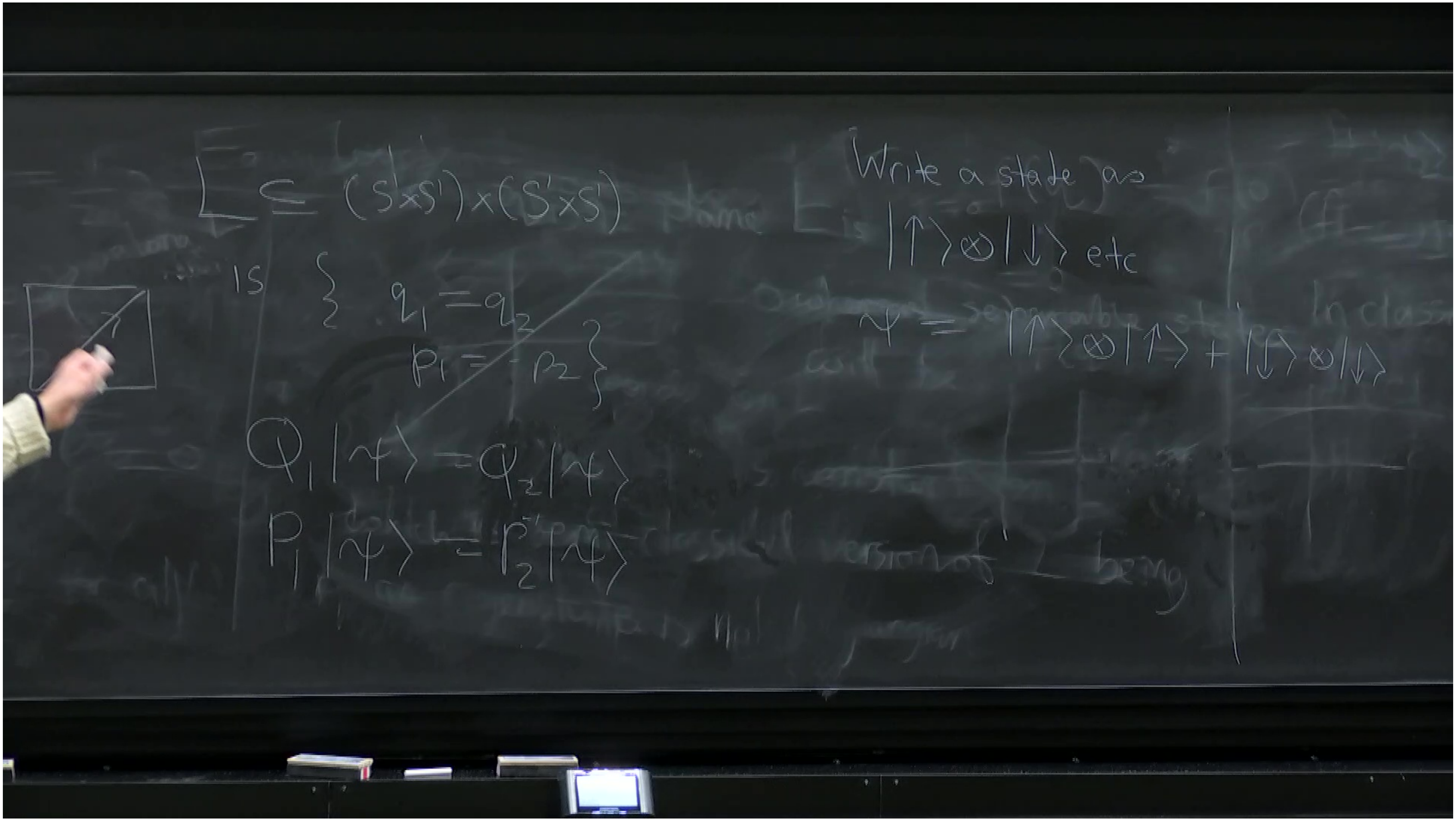
$$\delta_{q_1=q_2} = \int_c \delta_{q_1=c}\otimes\delta_{q_2=c}$$

so it is (very) entangled.

$$\alpha = 2\pi i\, q\, dp$$


area 2
p, q
ω = 2 dq dp
P = e^{2πi p}
Q = e^{2πi q}
PQ = QP e^{2πi/n}
= − QP as n = 2
Hilbert space has 2 states
Q|↑⟩ = |↑⟩
Q|↓⟩ = −|↓⟩
P|↑⟩ = |↓⟩
P|↓⟩ = |↑⟩


Entangled states
As before
2 such systems
P1, Q1, P2, Q2
E ⊆ (S'xS')x(S'xS')
is { q1 = q2
p1 = -p2 }
Q1|ψ⟩ = Q2|ψ⟩
P1|ψ⟩ = -P2|ψ⟩


L ⊆ (S¹×S¹)×(S¹×S¹)
is { q₁ = q₂, p₁ = −p₂ }
Q₁|ψ⟩ = Q₂|ψ⟩
P₁|ψ⟩ = −P₂|ψ⟩
Write a state as
|↑⟩⊗|↓⟩ etc
ψ = |↑⟩⊗|↑⟩ + |↓⟩⊗|↓⟩