Title: Self-Similar Quasicrystals and Hyperbolic Honeycombs

Speakers: Justin Kulp

Series: Quantum Fields and Strings

Date: January 27, 2023 - 11:00 AM

URL: https://pirsa.org/23010112

Abstract: Most people are familiar with periodic tessellations and lattices; from the floor in the PI Bistro to their favourite spin systems. In this talk, I will discuss two less familiar families of tessellations and their applications to high energy physics, condensed matter physics, and mathematics: hyperbolic tessellations and quasicrystals. After introducing the basics of regular hyperbolic lattices, I will survey constructions and surprising properties of quasicrystals (like the Penrose tiling), including their classically forbidden symmetries, long-range order, and self-similar structure. Inspired by the AdS/CFT correspondence, I will describe a mathematical relationship between hyperbolic lattices in (D+1)-dimensions and quasicrystals in D-dimensions, as well as the resolution of a conjecture by Bill Thurston. Based on work to appear with Latham Boyle.

Zoom Link: https://pitp.zoom.us/j/91876287518?pwd=NGNLOXZHa1h1cmdnMzJxQzVMVFJJdz09

Quasicrystal_Presentations
HYPERBOLIC HONEYCOMBS AND
SELF-SIMILAR QUASICRYSTALS
FOR QUANTUM FIELDS & STRINGS SEMINAR
JUSTIN KULP
WITH LATHAM BOYLE
PERIMETER INSTITUTE FOR
THEORETICAL PHYSICS
27/JAN/2023
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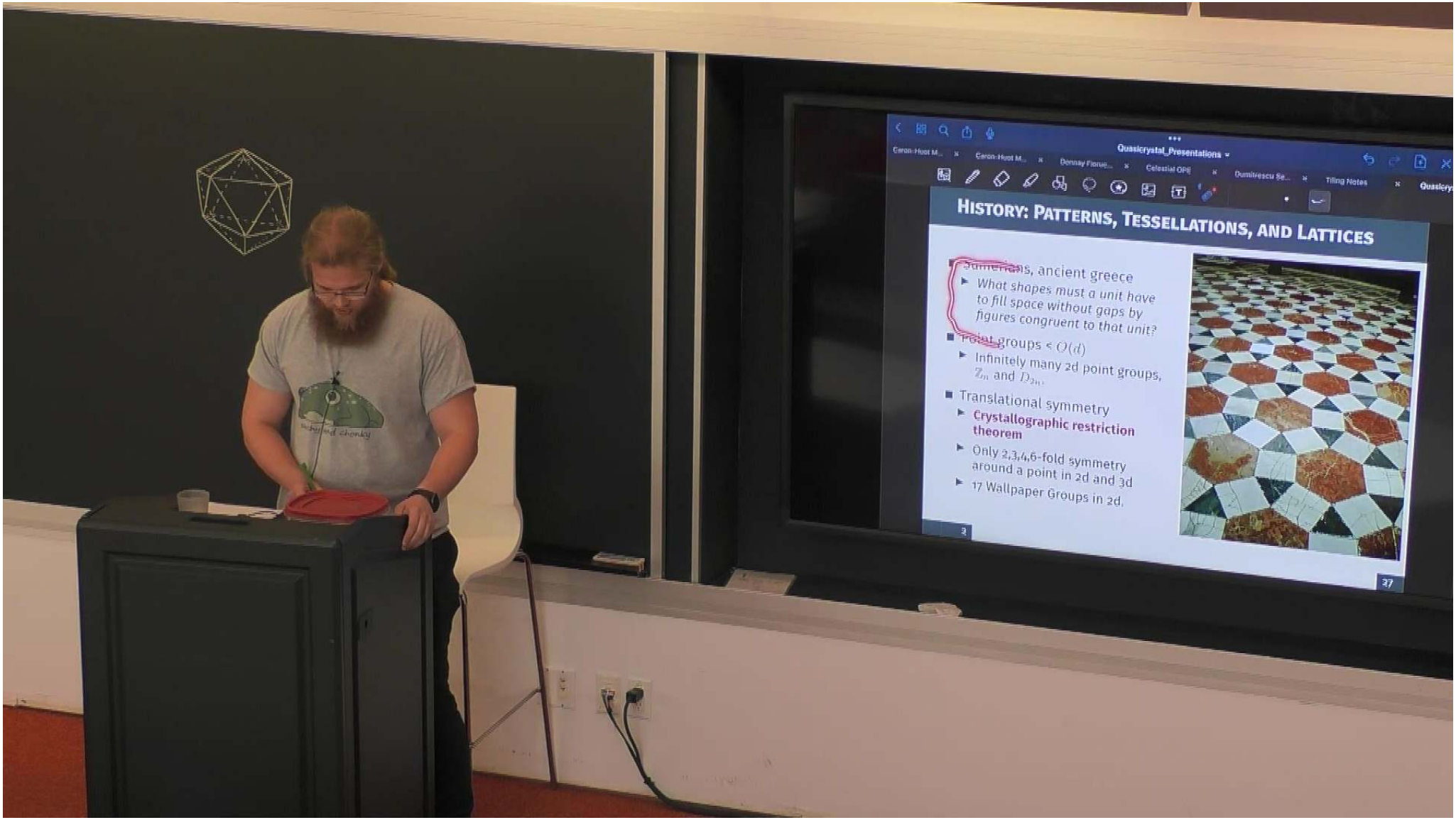
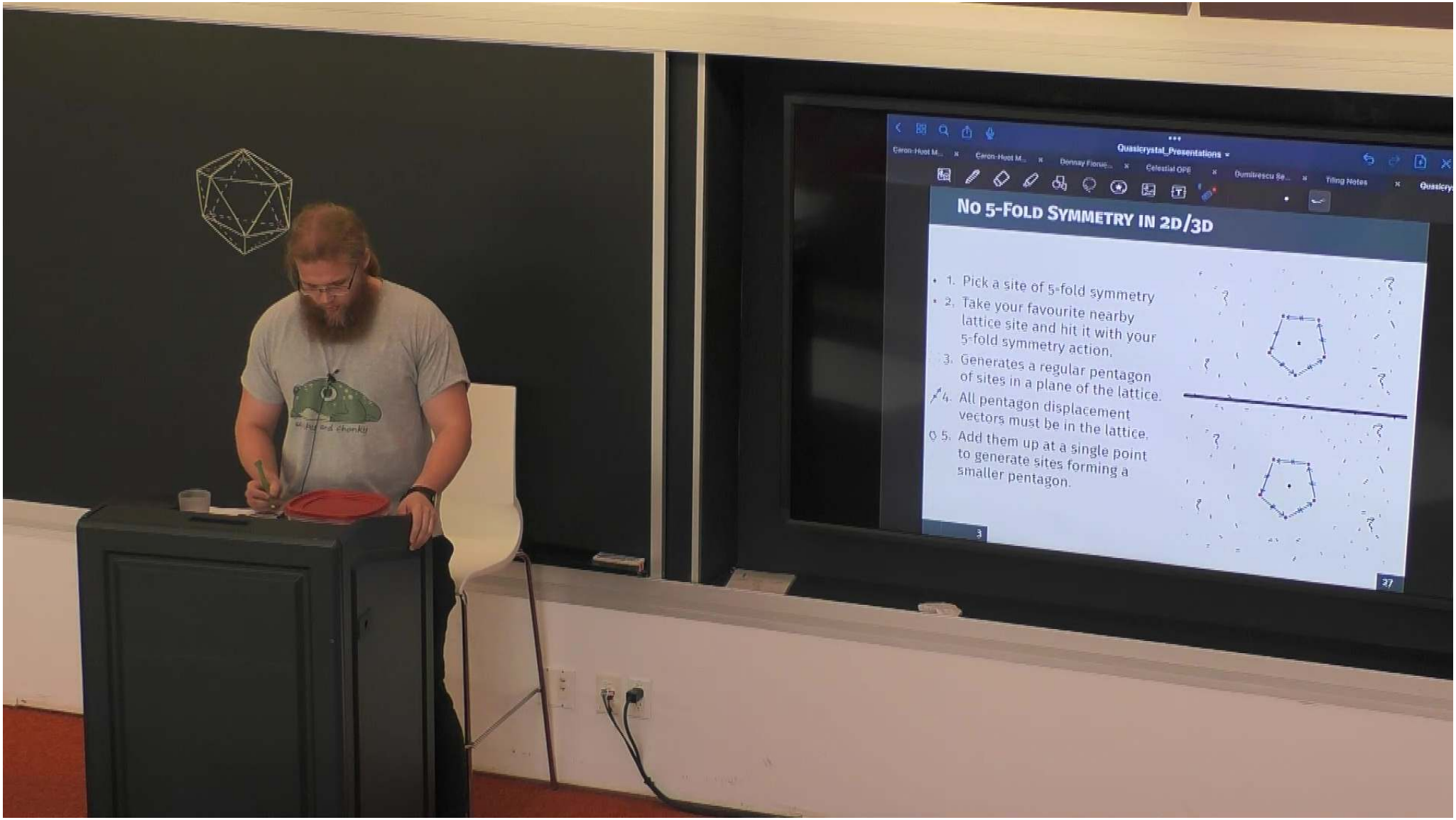
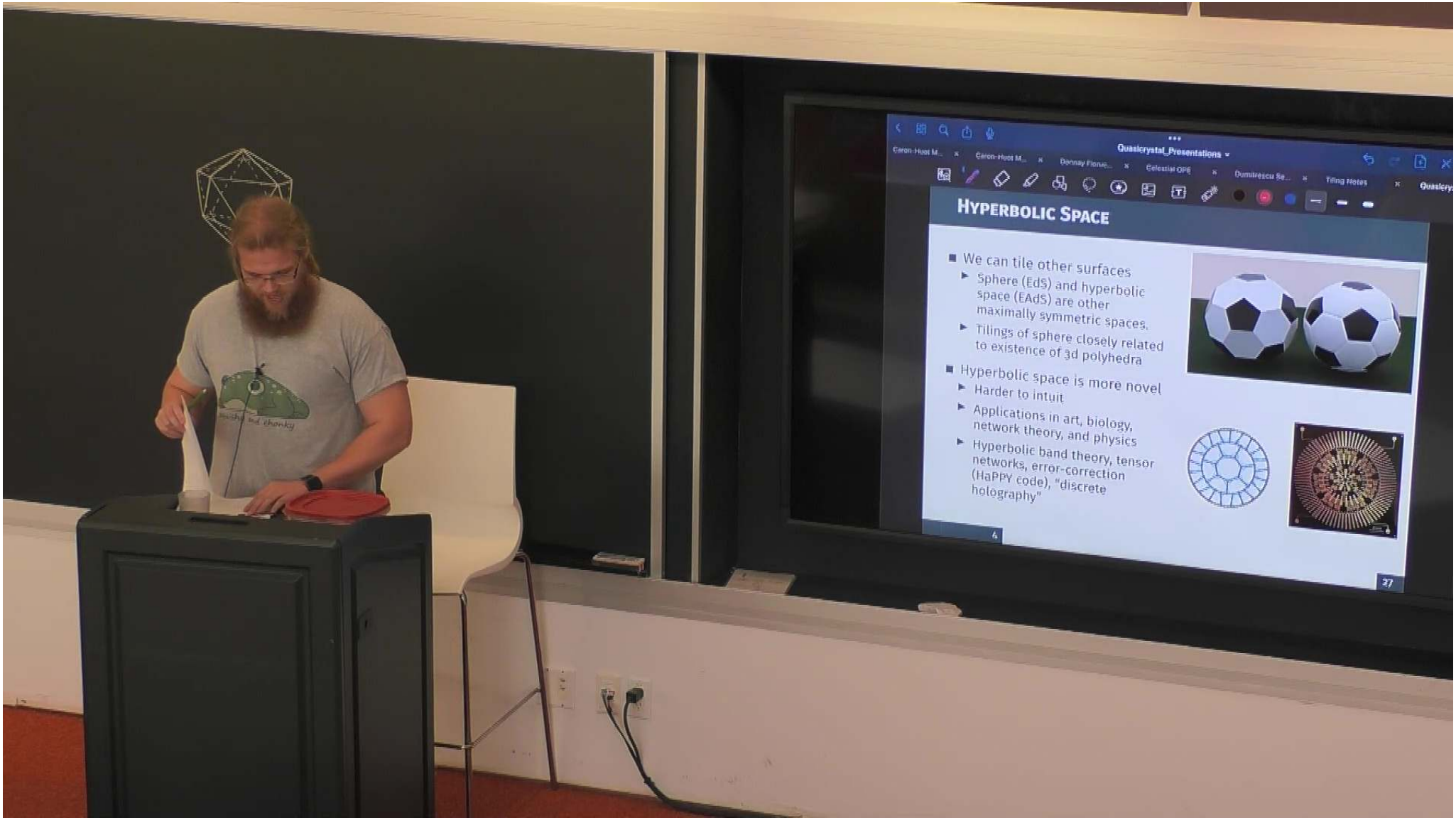
Quasicrystal_Presentations
OUTLINES AND PUNCHLINES
I. Introduction and Motivation.
II. Beyond Euclidean Lattices.
III. Hyperbolic Quasicrystals 2d/1d.
IV. Higher Hyperbolic Quasicrystals.
V. Conclusion.
Three Punchlines
1. Hyperbolic lattices and quasicrystals
2. New quasiperiodic patterns
3. Applications to physics (holography, condensed matter)
27


Quasicrystal_Presentations
Tiling Notes
HISTORY: PATTERNS, TESSELLATIONS, AND LATTICES
s, ancient greece
What shapes must a unit have to fill space without gaps by figures congruent to that unit?
groups < $O(d)$
Infinitely many 2d point groups, $\mathbb{Z}_n$ and D_{2n}.
Translational symmetry
Crystallographic restriction theorem
Only 2,3,4,6-fold symmetry around a point in 2d and 3d
17 Wallpaper Groups in 2d.
3
37


Quasicrystal_Presentations
No 5-Fold Symmetry in 2D/3D
1. Pick a site of 5-fold symmetry
2. Take your favourite nearby lattice site and hit it with your 5-fold symmetry action.
3. Generates a regular pentagon of sites in a plane of the lattice.
4. All pentagon displacement vectors must be in the lattice.
5. Add them up at a single point to generate sites forming a smaller pentagon.
37


Quasicrystal_Presentations
HYPERBOLIC SPACE
We can tile other surfaces
Sphere (EdS) and hyperbolic space (EAdS) are other maximally symmetric spaces.
Tilings of sphere closely related to existence of 3d polyhedra
Hyperbolic space is more novel
Harder to intuit
Applications in art, biology, network theory, and physics
Hyperbolic band theory, tensor networks, error-correction (HaPPY code), "discrete holography"
4
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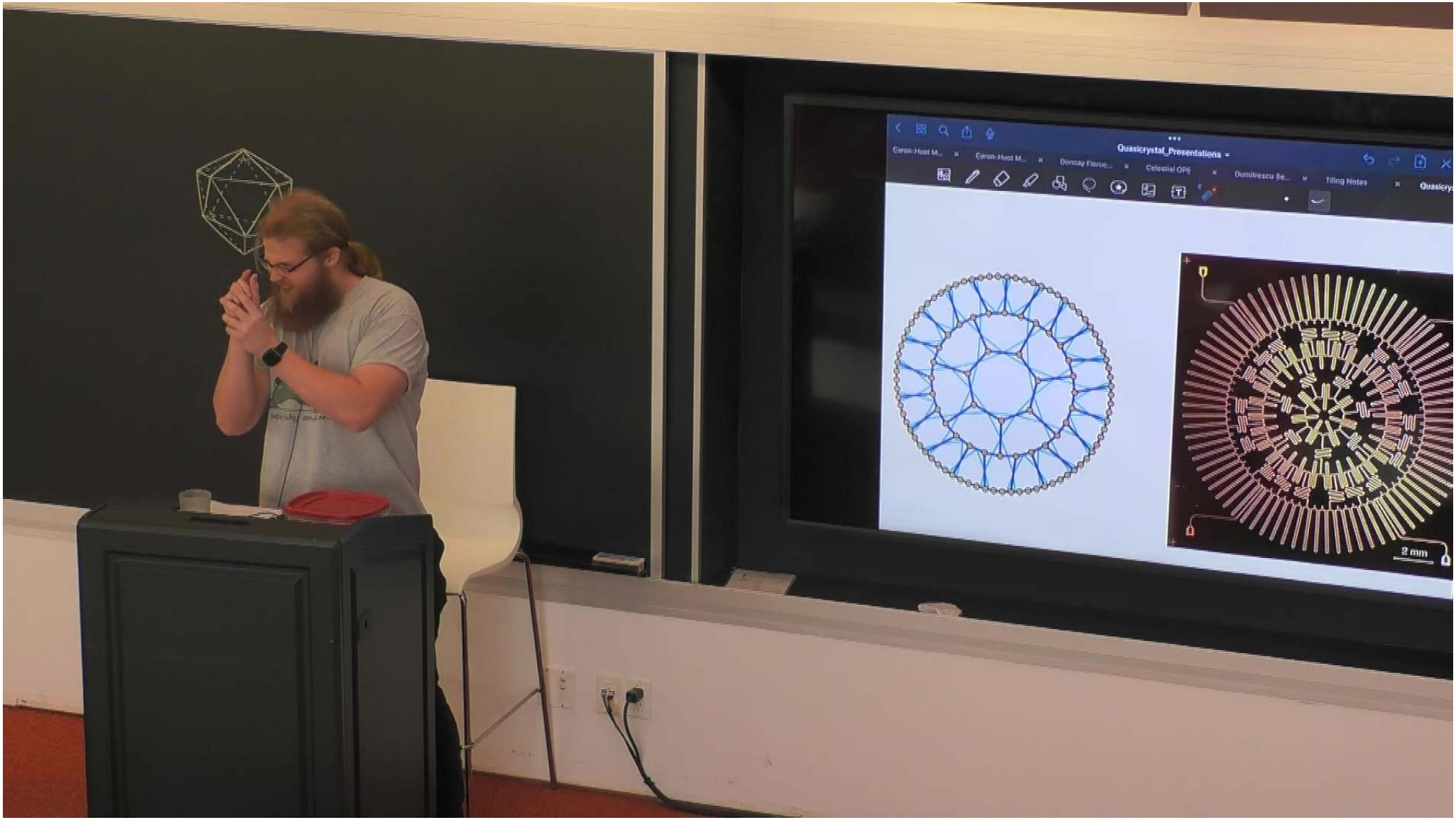
Quasicrystal_Presentations
Tiling Notes
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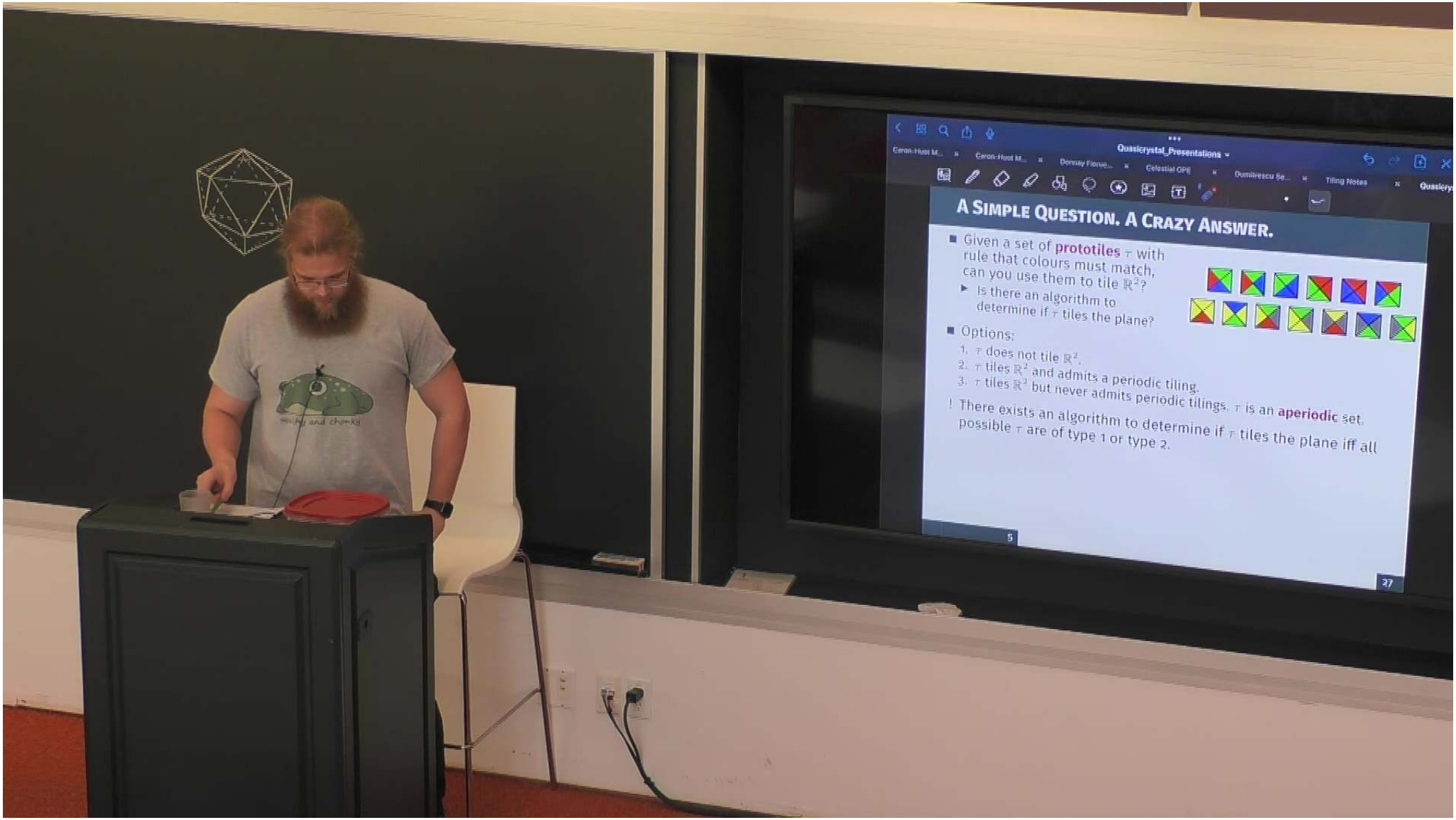
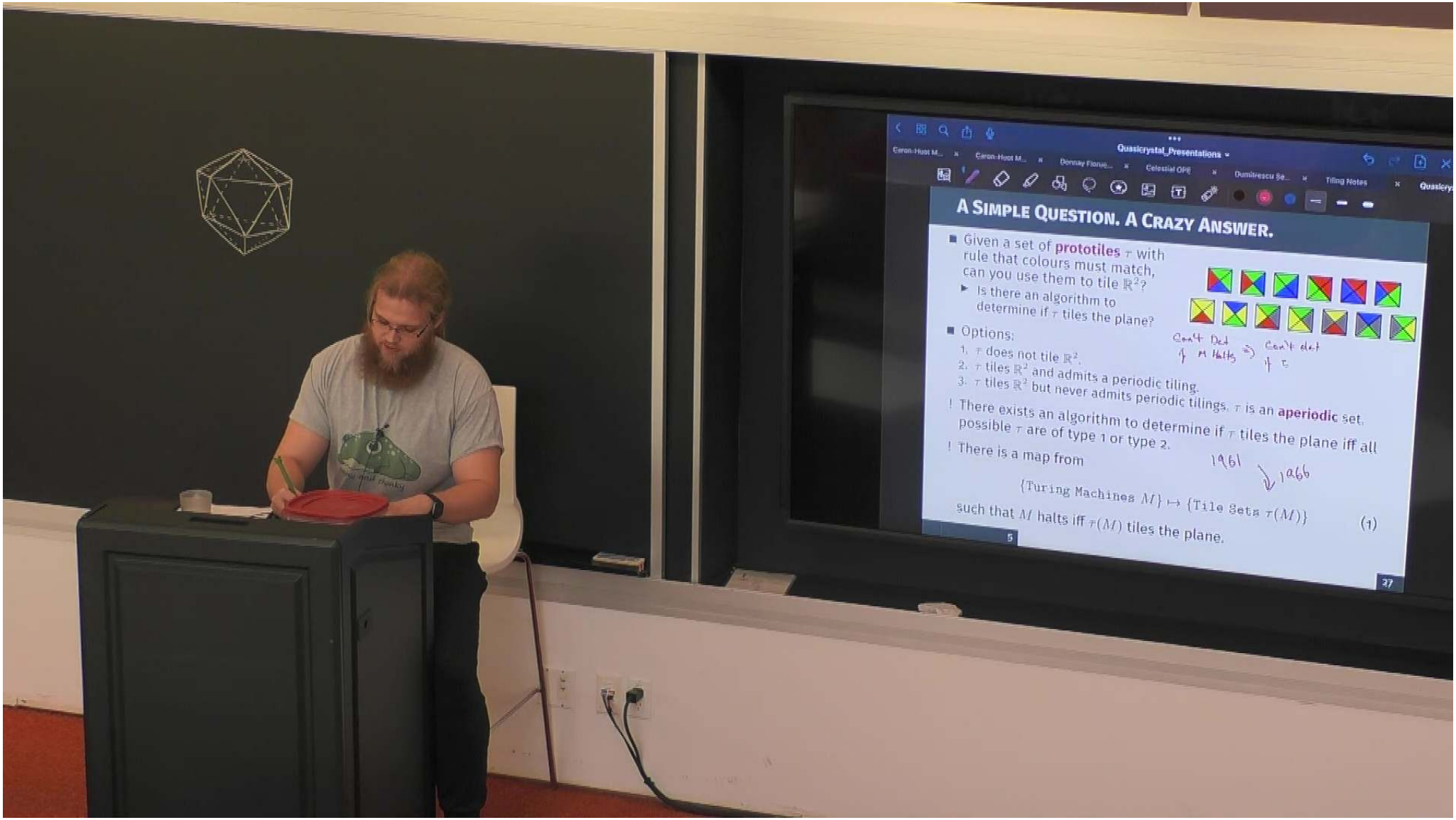

Quasicrystal_Presentations
A Simple Question. A Crazy Answer.
Given a set of prototiles τ with rule that colours must match, can you use them to tile $\mathbb{R}^2$?
Is there an algorithm to determine if τ tiles the plane?
Options:
1. τ does not tile $\mathbb{R}^2$.
2. τ tiles $\mathbb{R}^2$ and admits a periodic tiling.
3. τ tiles $\mathbb{R}^2$ but never admits periodic tilings. τ is an aperiodic set.
! There exists an algorithm to determine if τ tiles the plane iff all possible τ are of type 1 or type 2.
5
37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! There is a map from
$$\{\text{Turing Machines } M\} \mapsto \{\text{Tile Sets } \tau(M)\} \quad (1)$$
such that M halts iff $\tau(M)$ tiles the plane.
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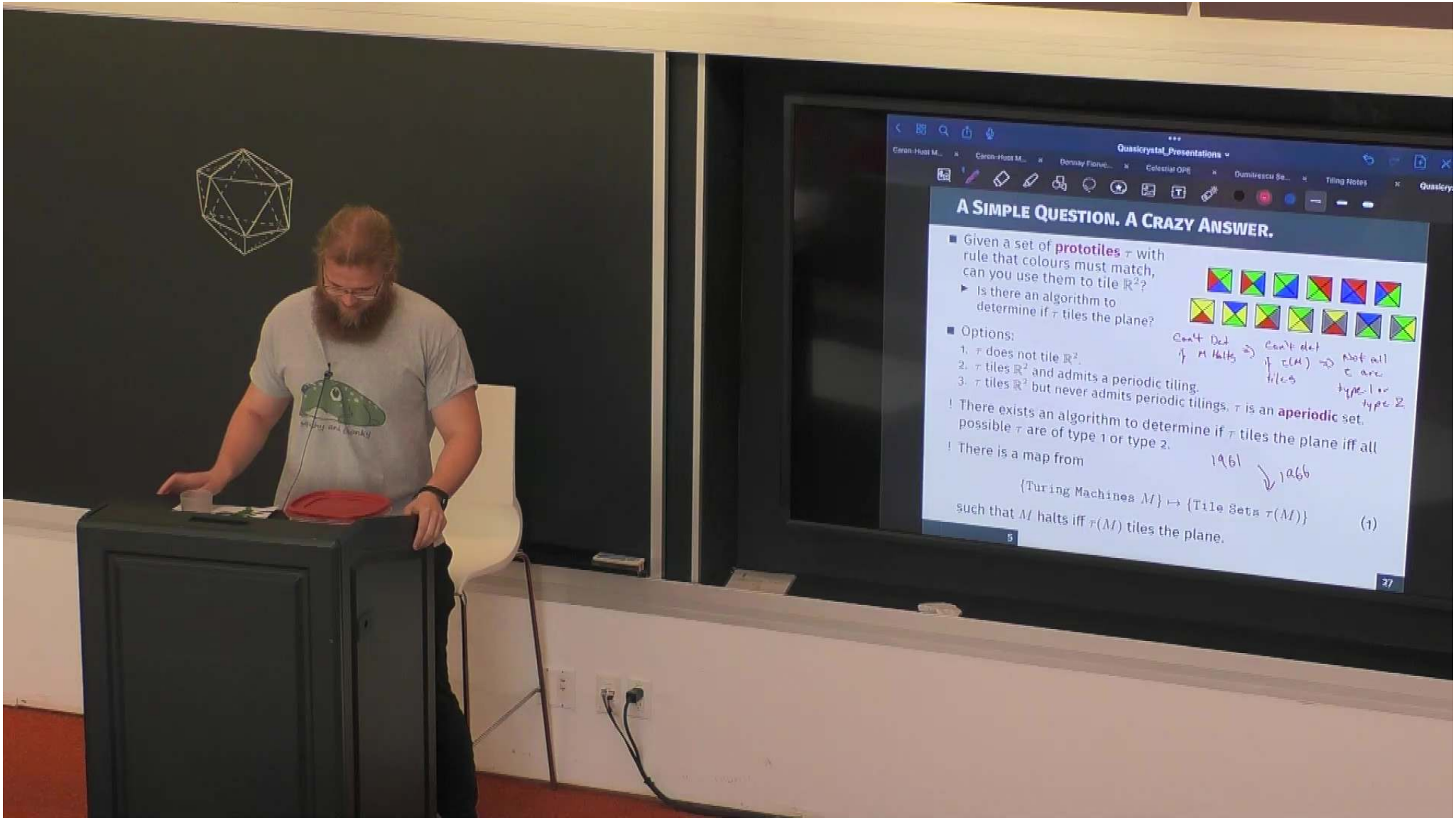
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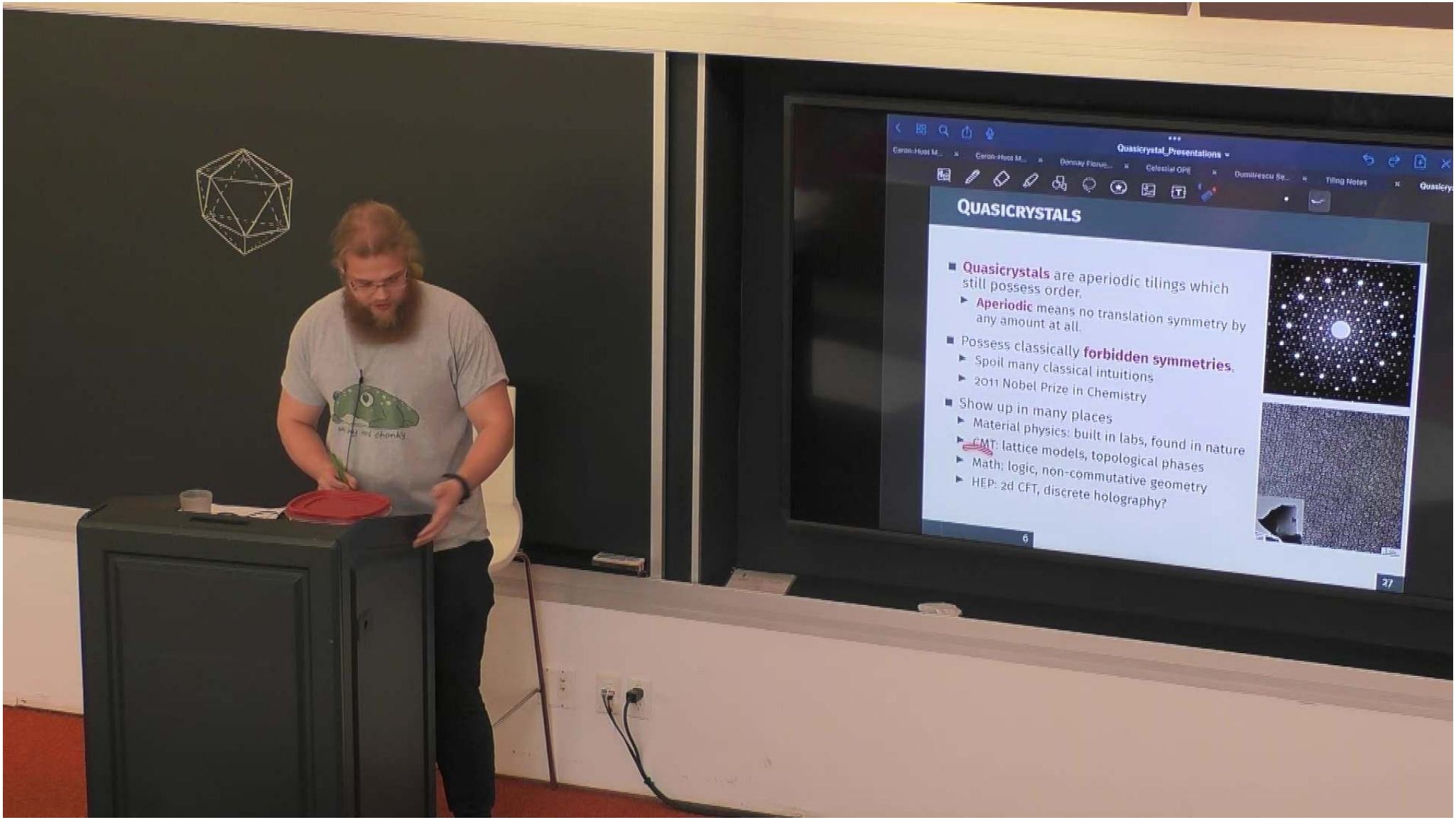
QUASICRYSTALS
Quasicrystals are aperiodic tilings which still possess order.
Aperiodic means no translation symmetry by any amount at all.
Possess classically forbidden symmetries.
Spoil many classical intuitions
2011 Nobel Prize in Chemistry
Show up in many places
Material physics: built in labs, found in nature
CMT: lattice models, topological phases
Math: logic, non-commutative geometry
HEP: 2d CFT, discrete holography?
6
37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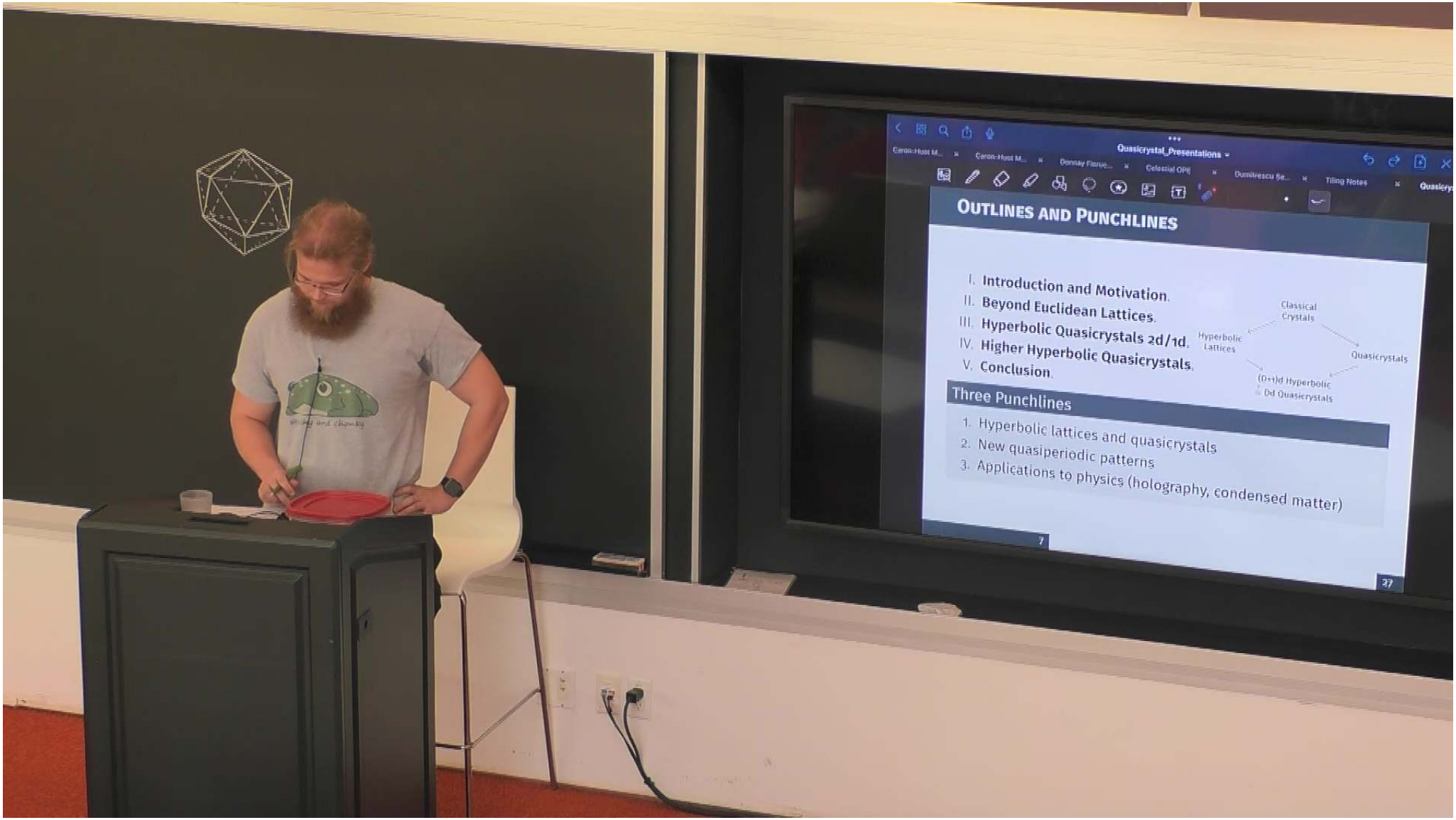
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Classical Crystals
Hyperbolic Lattices
Quasicrystals
(D+1)d Hyperbolic
Dd Quasicrystals
Three Punchlines
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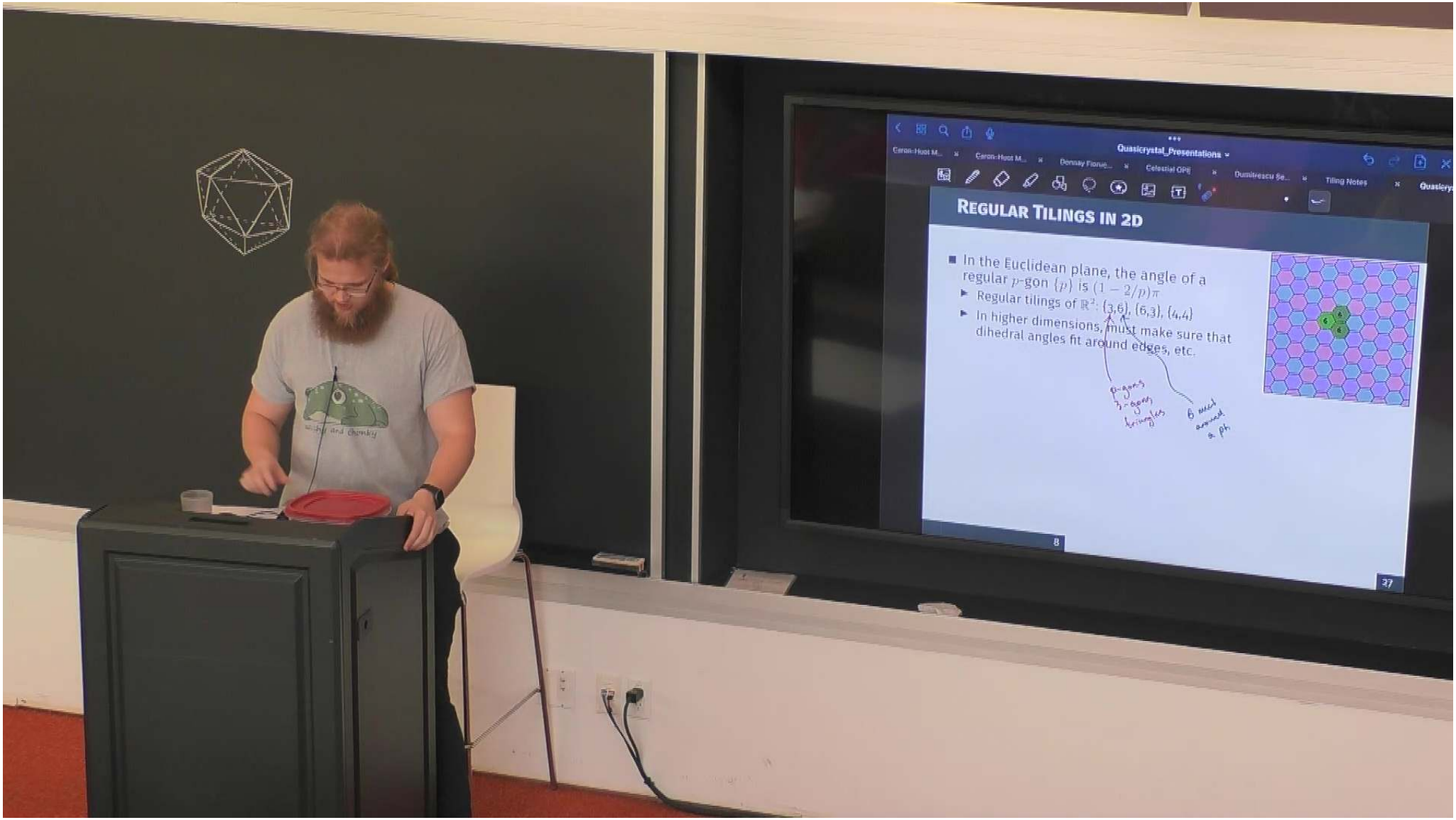
Quasicrystal_Presentations
Tiling Notes
REGULAR TILINGS IN 2D
In the Euclidean plane, the angle of a regular p-gon $\{p\}$ is $(1-2/p)\pi$
Regular tilings of $\mathbb{R}^2$: $\{3,6\}$, $\{6,3\}$, $\{4,4\}$
In higher dimensions, must make sure that dihedral angles fit around edges, etc.
8
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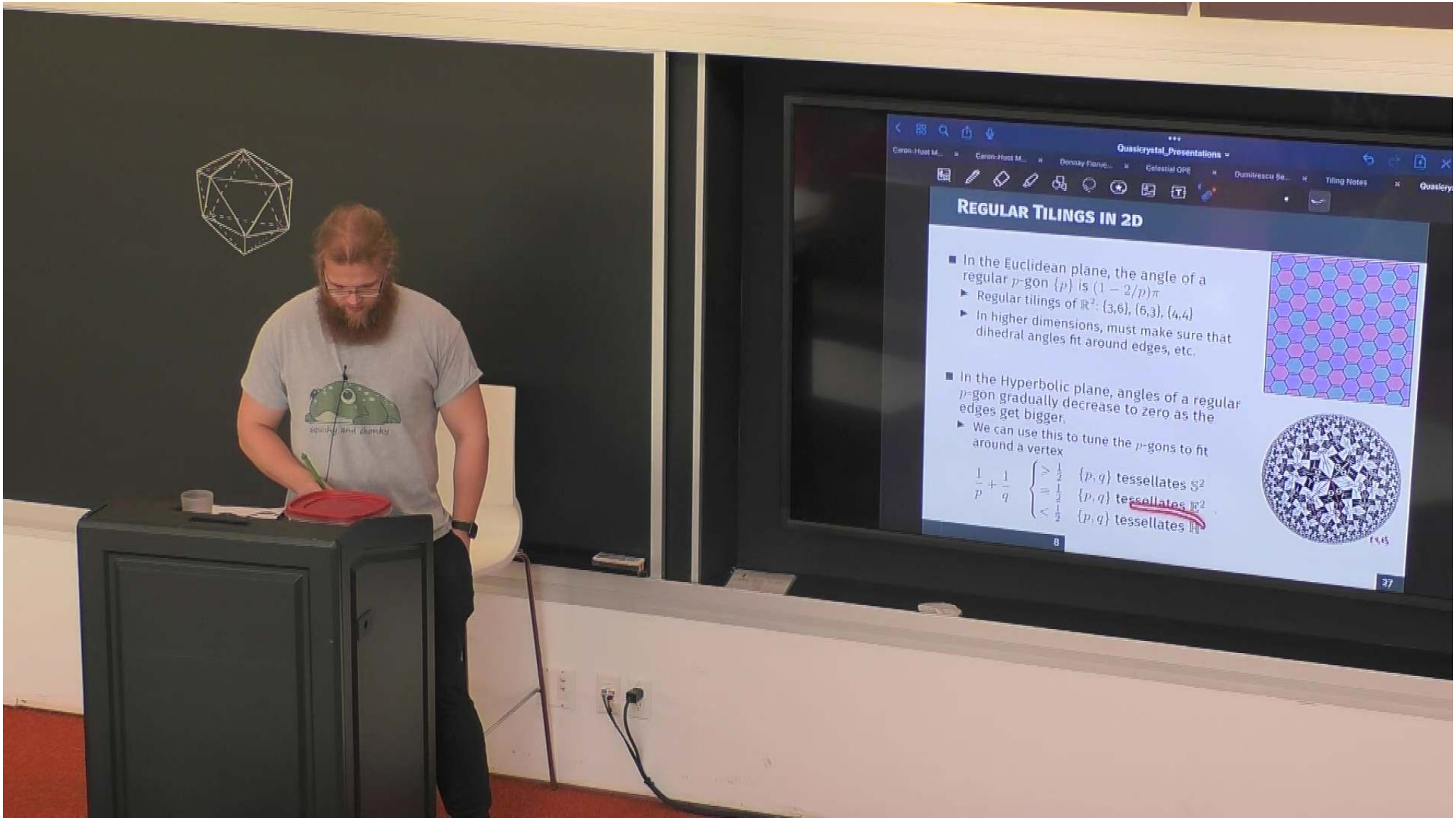
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In higher dimensions, must make sure that dihedral angles fit around edges, etc.
In the Hyperbolic plane, angles of a regular p-gon gradually decrease to zero as the edges get bigger.
We can use this to tune the p-gons to fit around a vertex
$\frac{1}{p}+\frac{1}{q}$ $\begin{cases} > \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{S}^2 \\ = \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{E}^2 \\ < \frac{1}{2} & \{p,q\} \text{ tessellates } \mathbb{H}^2 \end{cases}$
8
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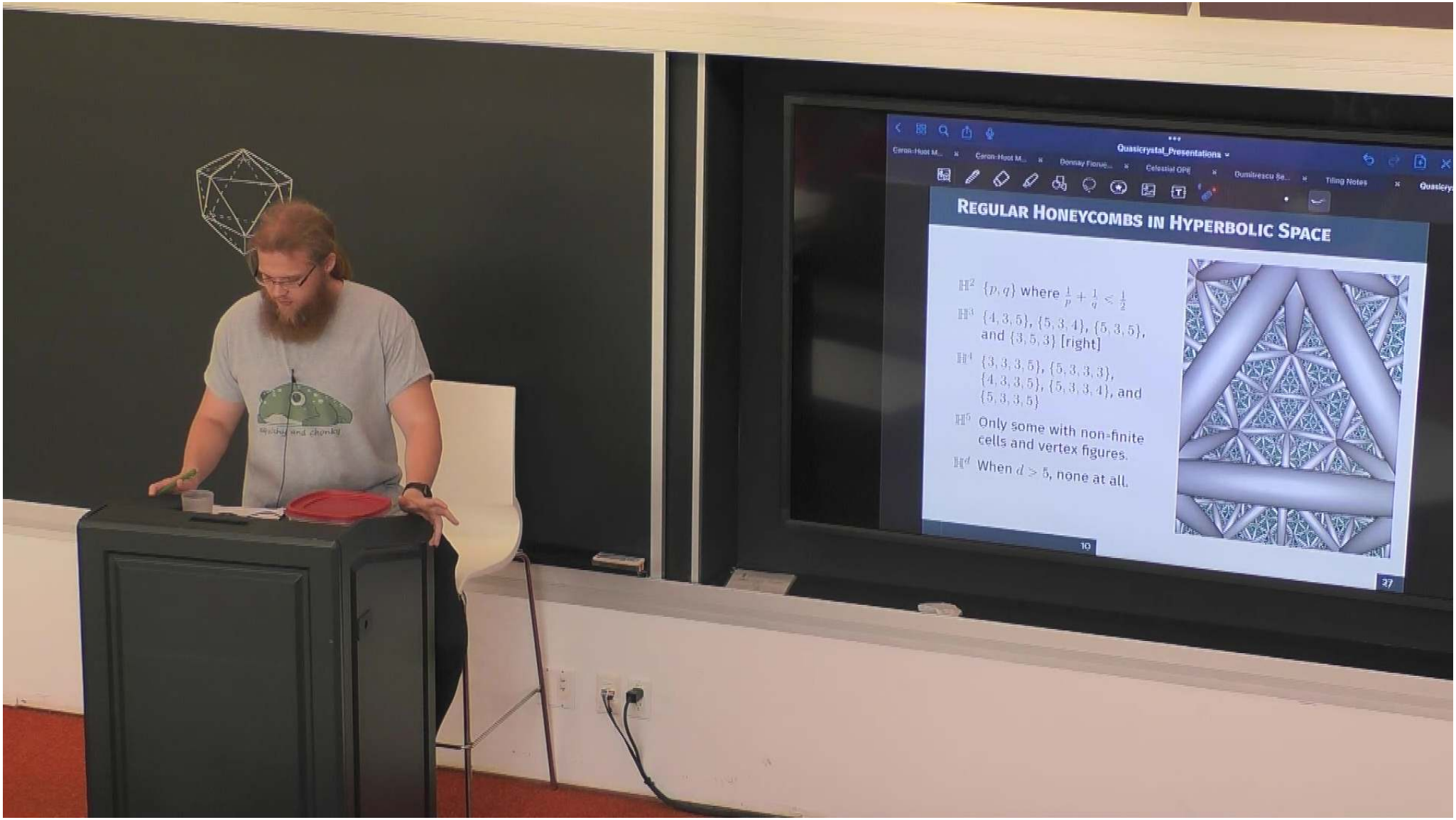
Quasicrystal_Presentations
Tiling Notes
REGULAR HONEYCOMBS IN HYPERBOLIC SPACE
$\mathbb{H}^2$ $\{p, q\}$ where $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$
$\mathbb{H}^3$ $\{4, 3, 5\}$, $\{5, 3, 4\}$, $\{5, 3, 5\}$, and $\{3, 5, 3\}$ [right]
$\mathbb{H}^4$ $\{3, 3, 3, 5\}$, $\{5, 3, 3, 3\}$, $\{4, 3, 3, 5\}$, $\{5, 3, 3, 4\}$, and $\{5, 3, 3, 5\}$
$\mathbb{H}^5$ Only some with non-finite cells and vertex figures.
$\mathbb{H}^d$ When $d > 5$, none at all.
10
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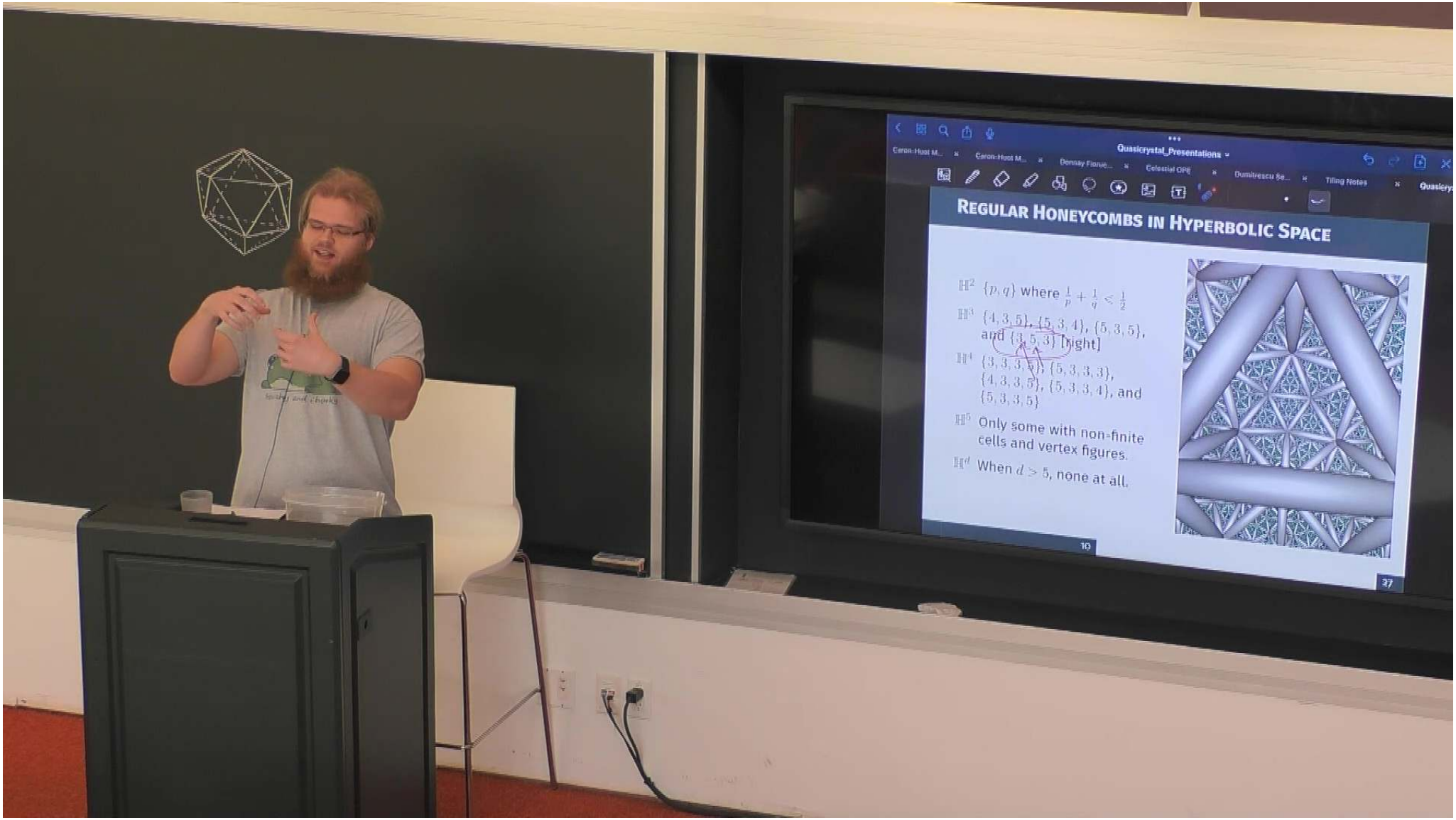
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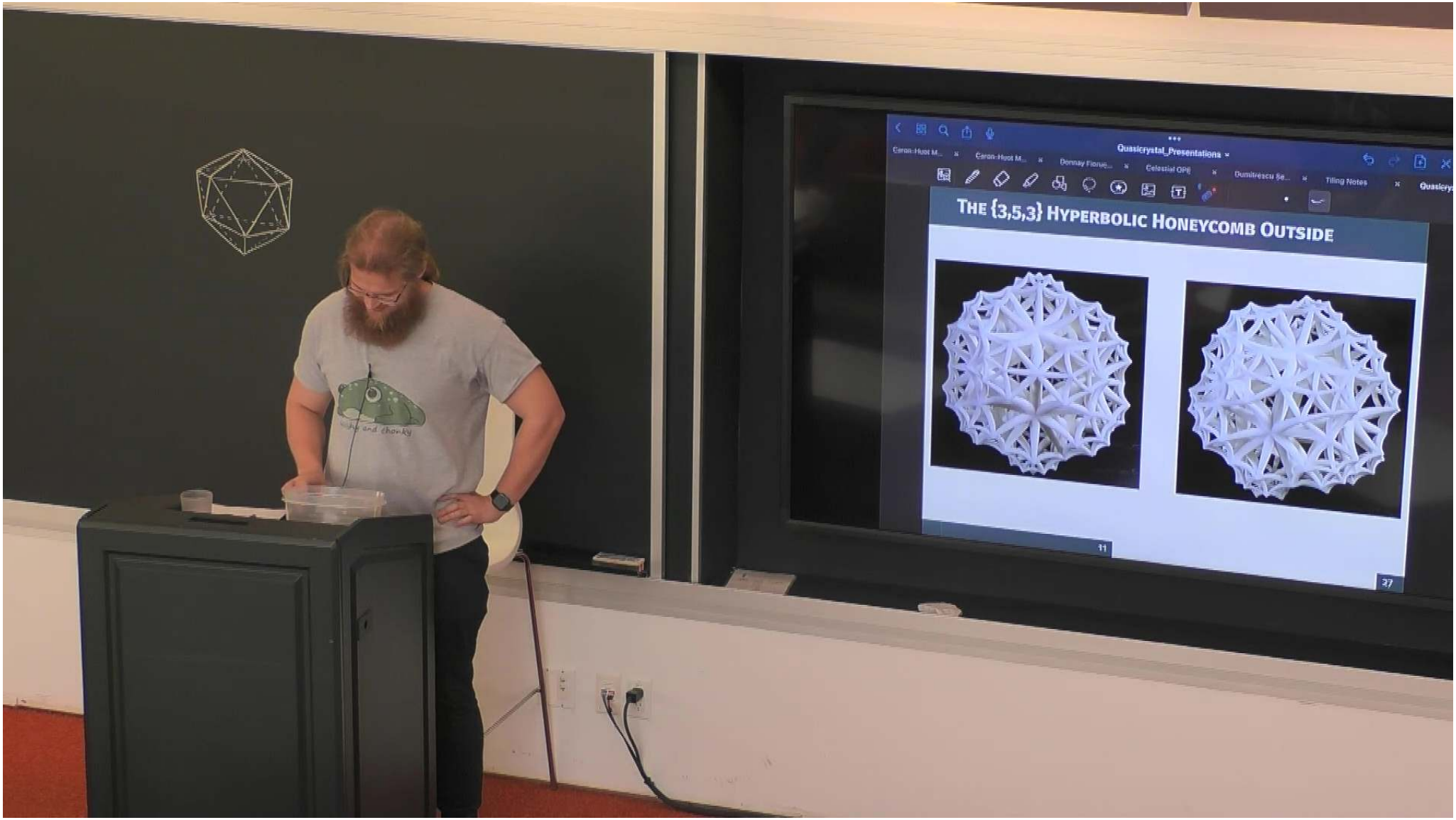
Quasicrystal_Presentations
Tiling Notes
THE {3,5,3} HYPERBOLIC HONEYCOMB OUTSIDE
11
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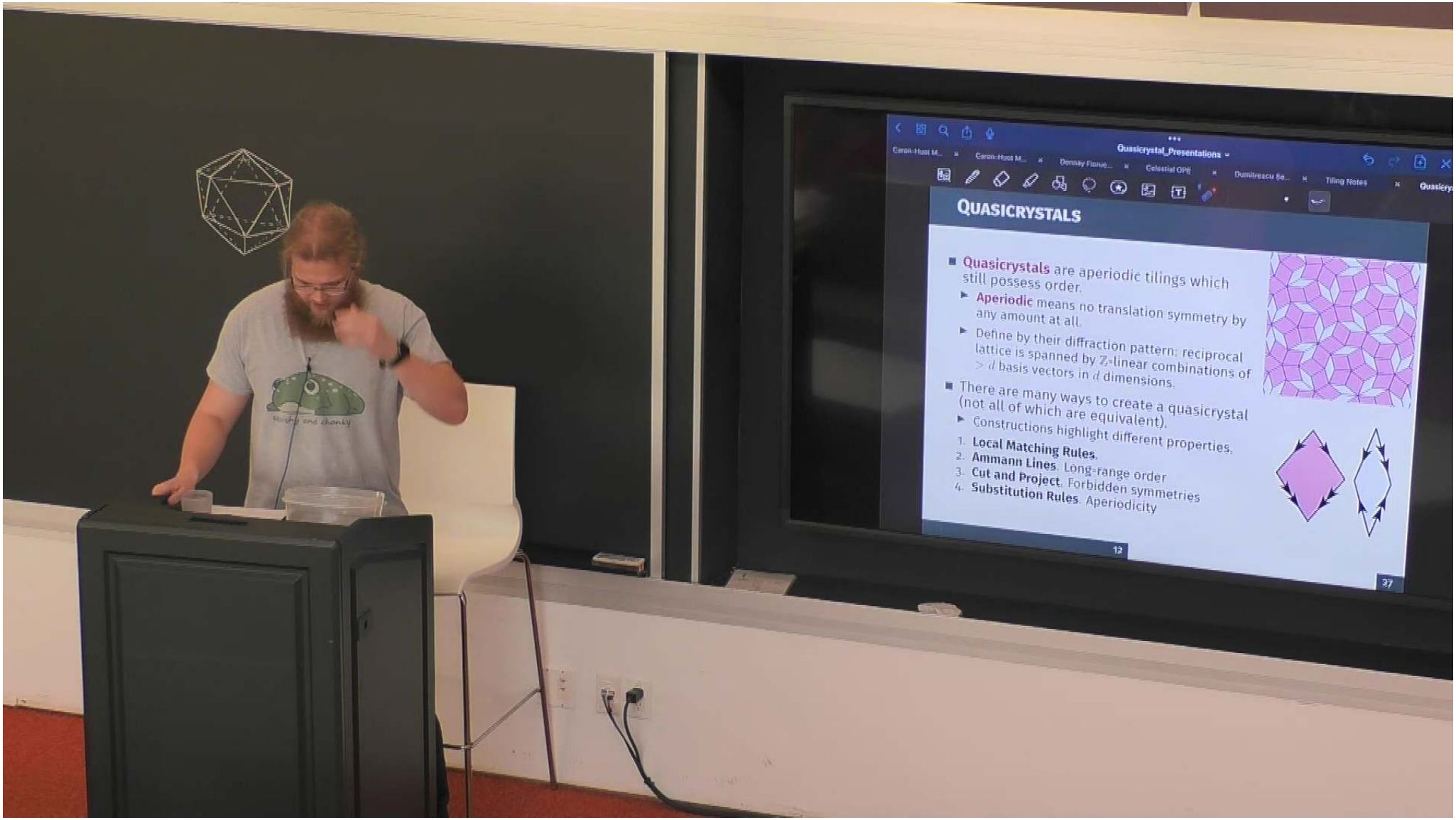
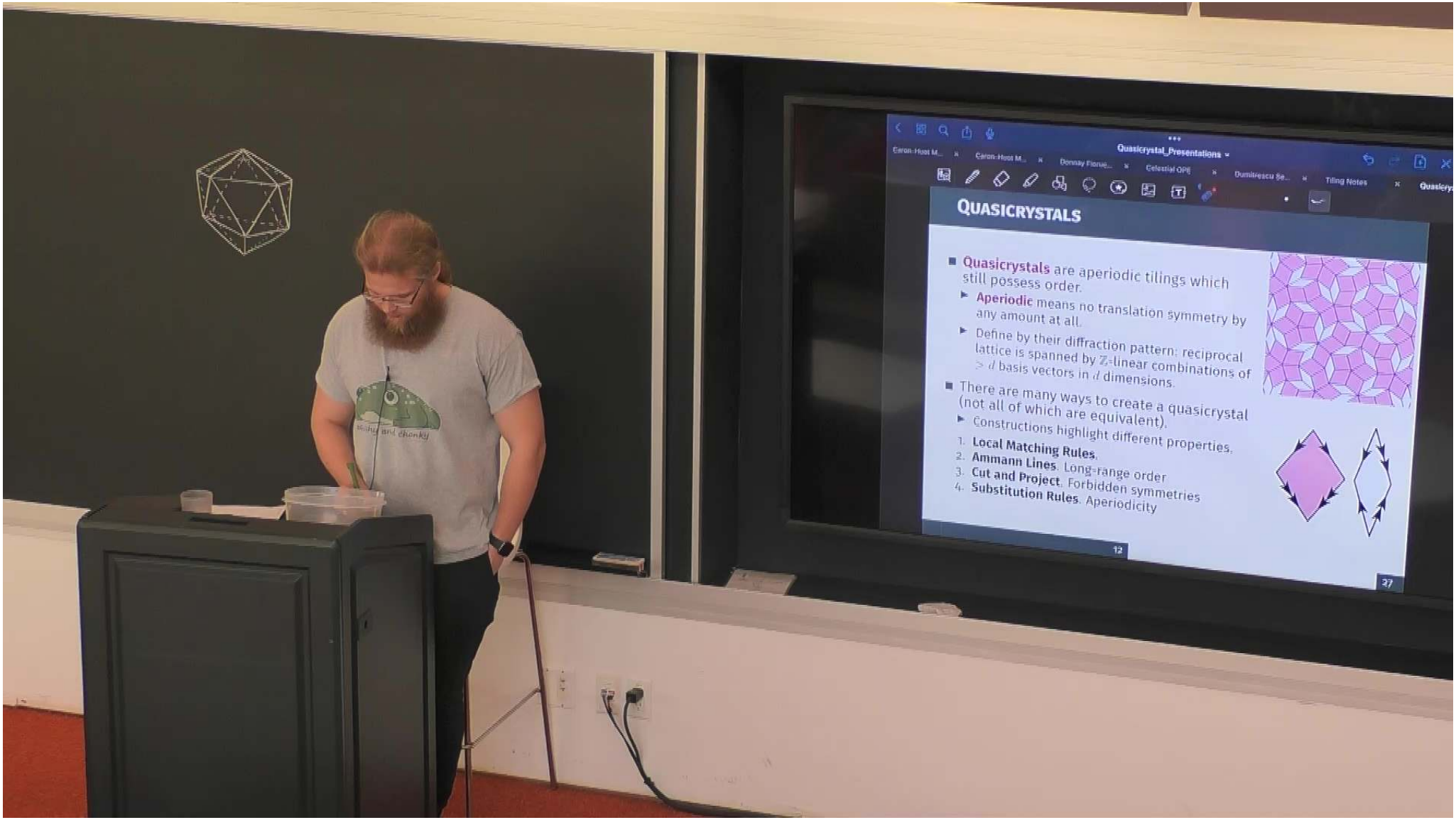

Quasicrystal_Presentations
QUASICRYSTALS
Quasicrystals are aperiodic tilings which still possess order.
Aperiodic means no translation symmetry by any amount at all.
Define by their diffraction pattern: reciprocal lattice is spanned by $\mathbb{Z}$-linear combinations of $> d$ basis vectors in d dimensions.
There are many ways to create a quasicrystal (not all of which are equivalent).
Constructions highlight different properties.
1. Local Matching Rules.
2. Ammann Lines. Long-range order
3. Cut and Project. Forbidden symmetries
4. Substitution Rules. Aperiodicity
12
37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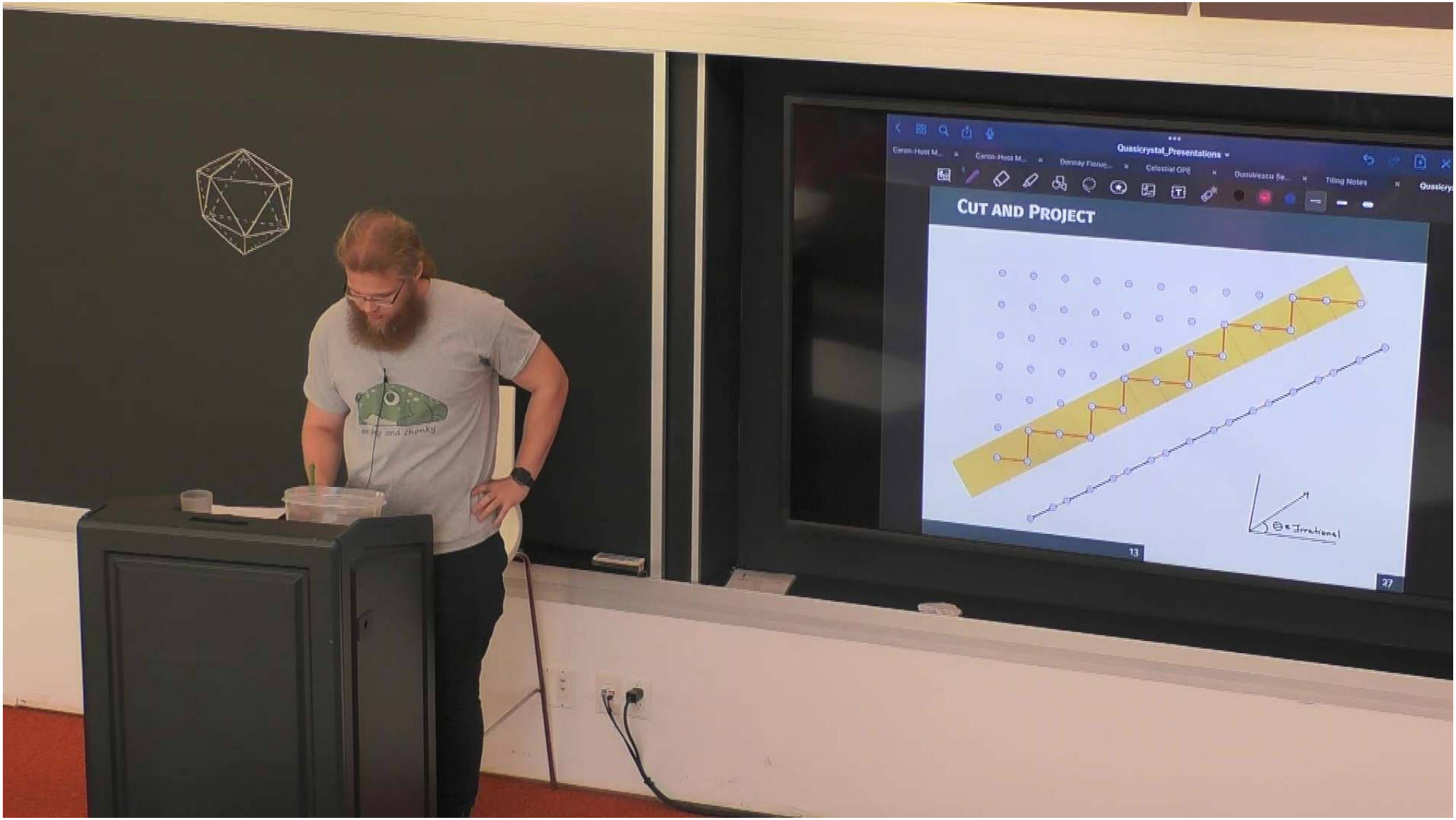
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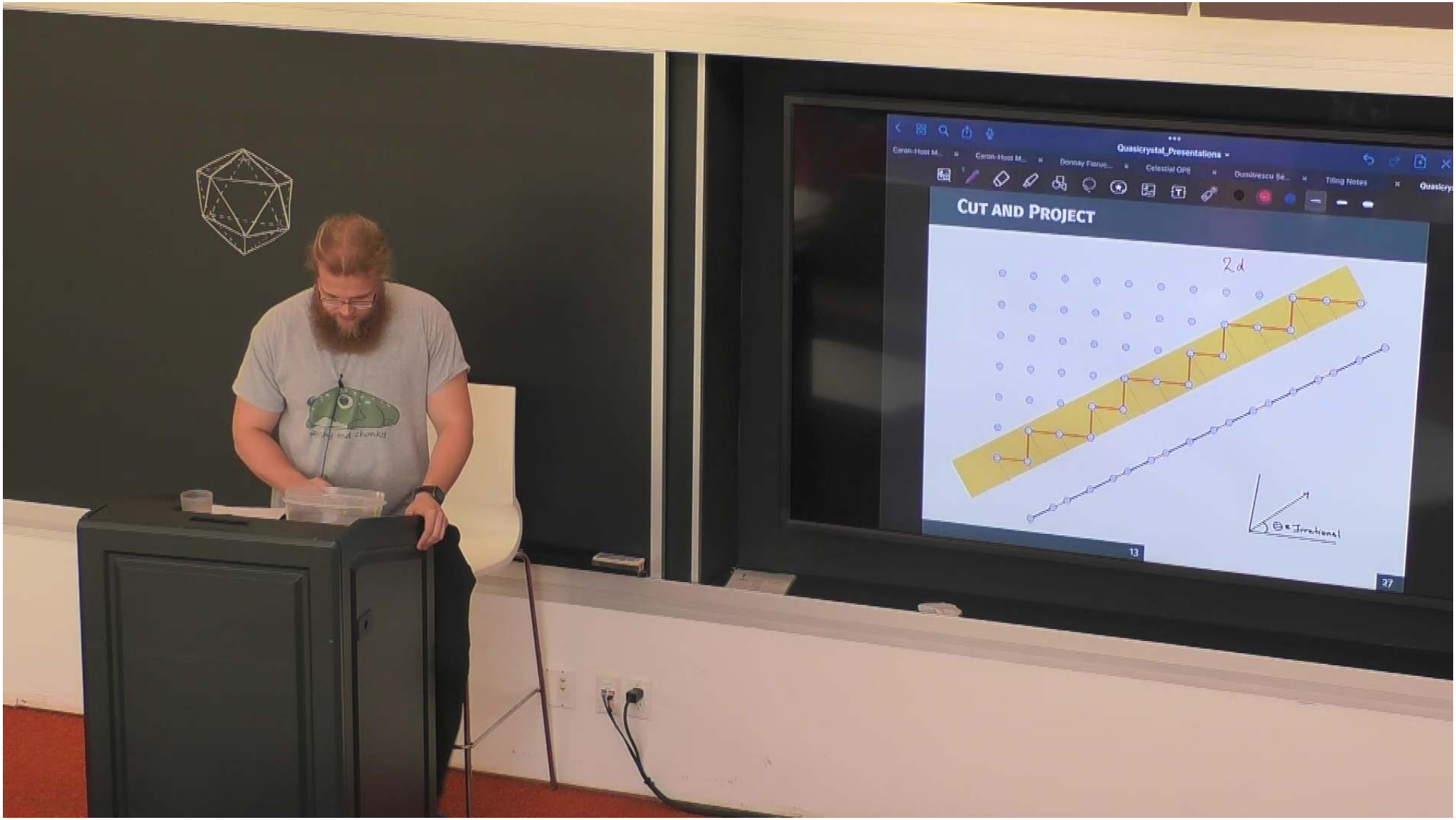
Quasicrystal_Presentations
CUT AND PROJECT
2d
θ = Irrational
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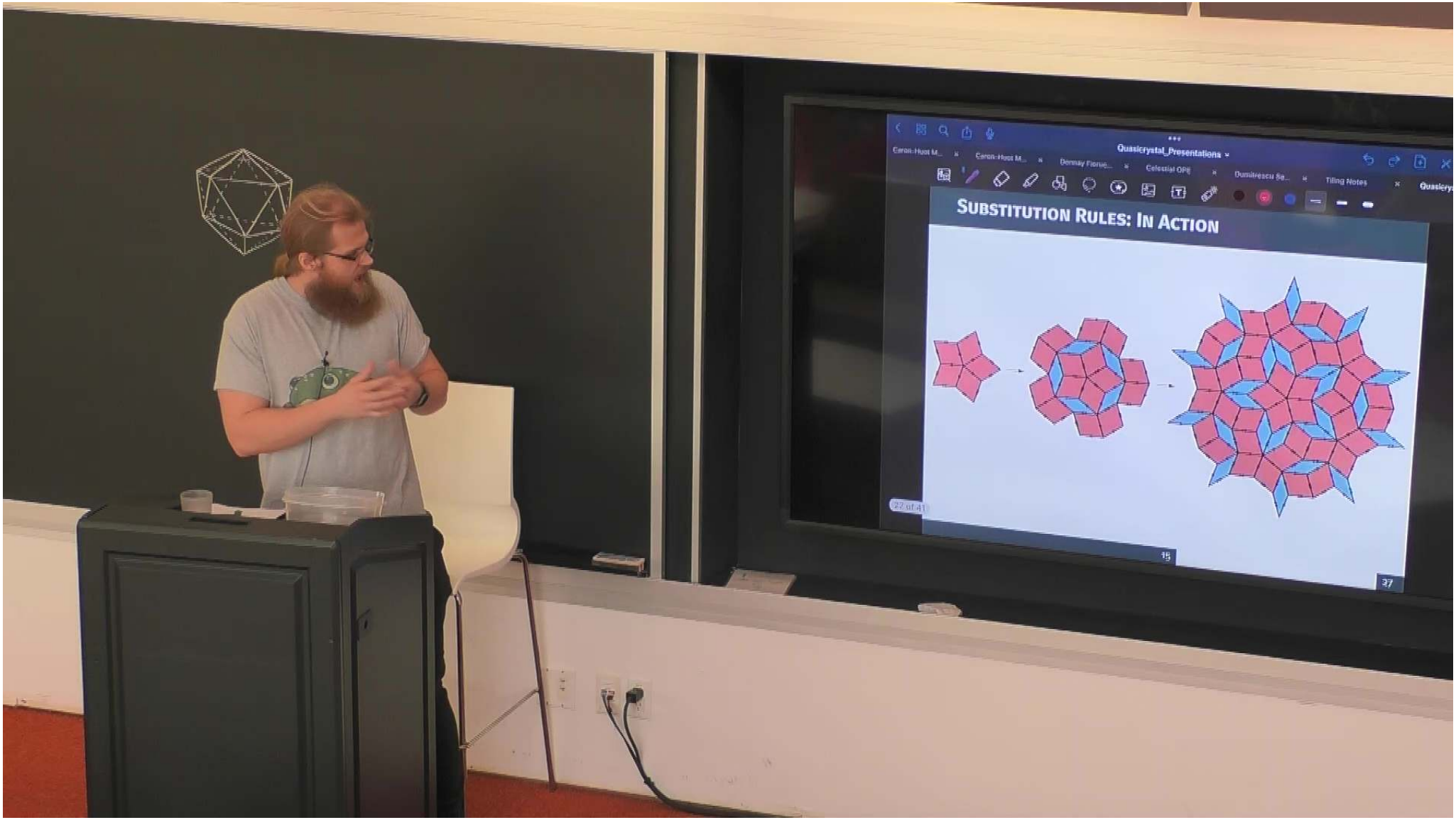
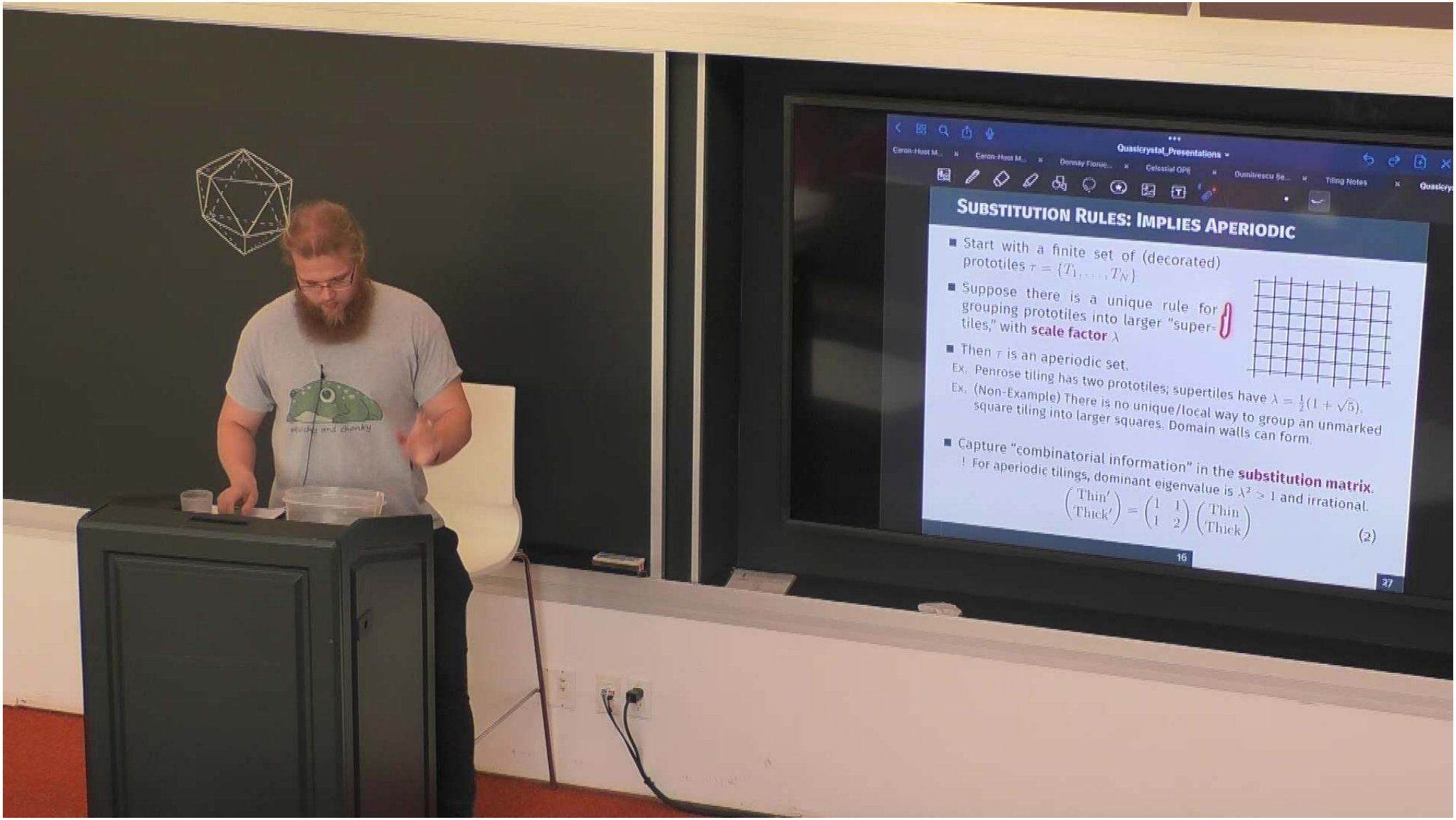

Quasicrystal_Presentations
Tiling Notes
SUBSTITUTION RULES: IN ACTION
22 of 41
37


Quasicrystal_Presentations
Tiling Notes
SUBSTITUTION RULES: IMPLIES APERIODIC
Start with a finite set of (decorated) prototiles $\tau = \{T_1, \ldots, T_N\}$
Suppose there is a unique rule for grouping prototiles into larger "super-tiles," with scale factor λ
Then τ is an aperiodic set.
Ex. Penrose tiling has two prototiles; supertiles have $\lambda = \frac{1}{2}(1+\sqrt{5})$.
Ex. (Non-Example) There is no unique/local way to group an unmarked square tiling into larger squares. Domain walls can form.
Capture "combinatorial information" in the substitution matrix.
! For aperiodic tilings, dominant eigenvalue is $\lambda^2 > 1$ and irrational.
$$\begin{pmatrix} \text{Thin}' \\ \text{Thick}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \text{Thin} \\ \text{Thick} \end{pmatrix} \quad (2)$$
16
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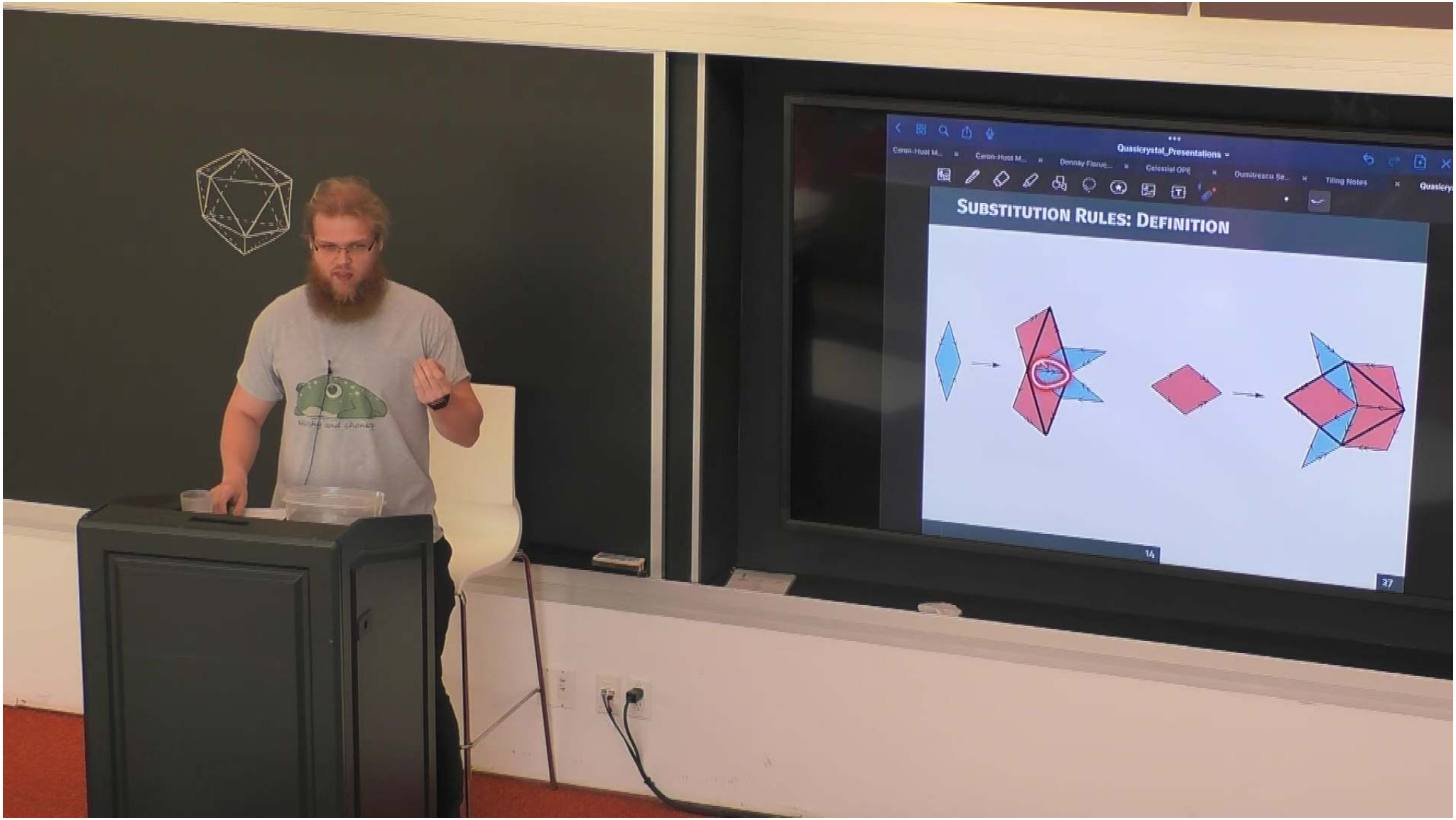
Quasicrystal_Presentations
Tiling Notes
SUBSTITUTION RULES: DEFINITION
14
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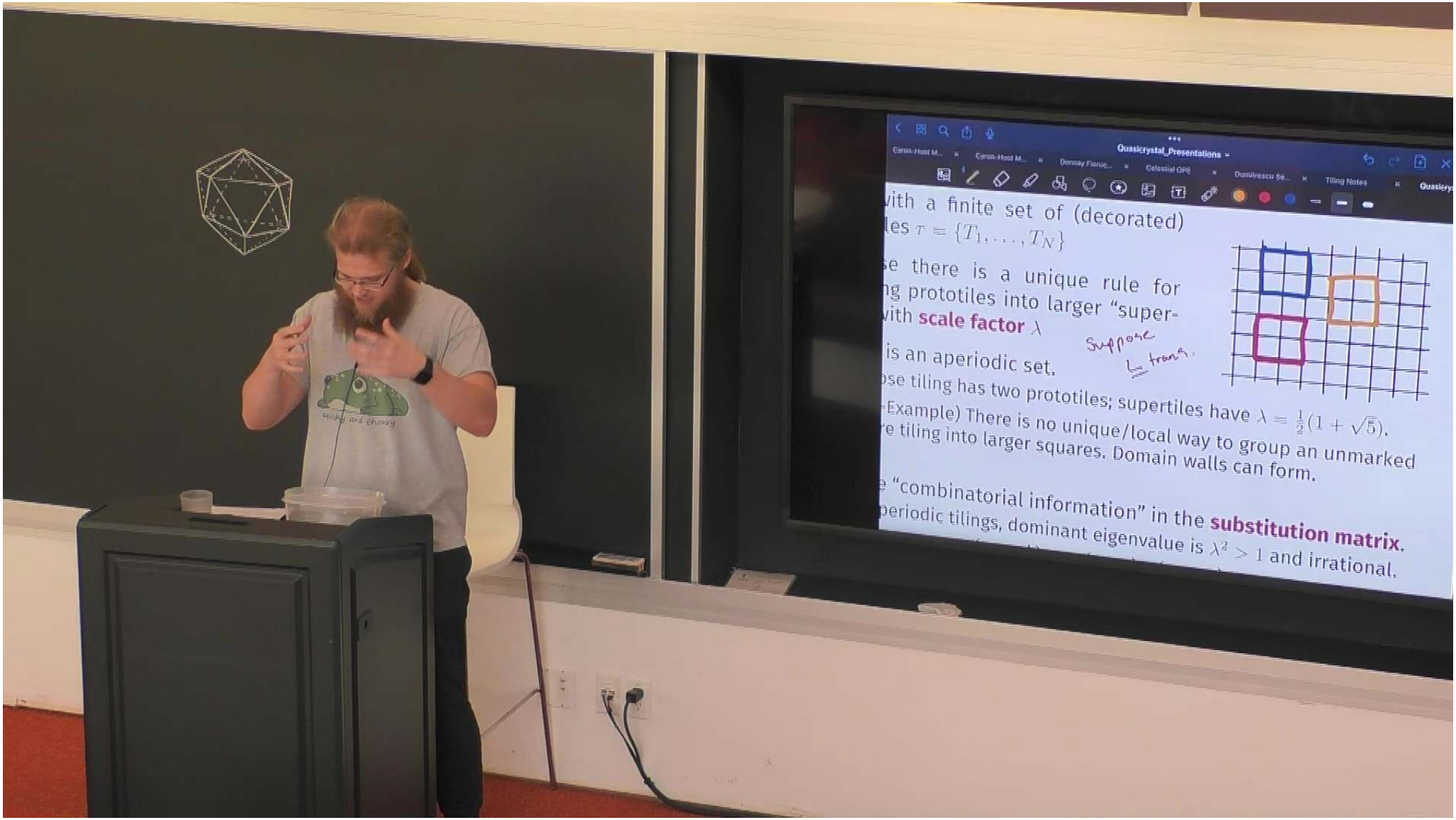
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Example) There is no unique/local way to group an unmarked
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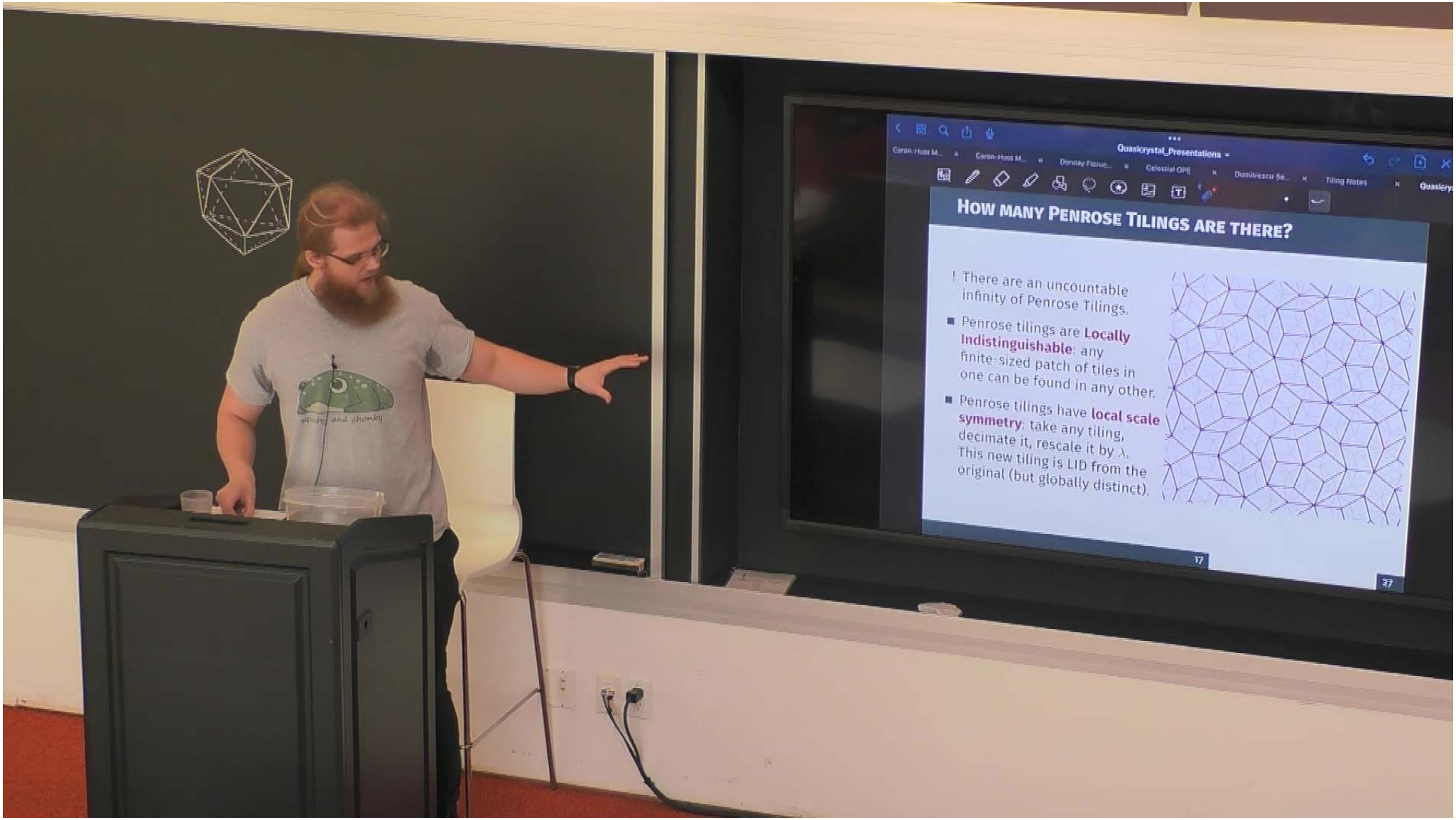
Quasicrystal_Presentations
How many Penrose Tilings are there?
There are an uncountable infinity of Penrose Tilings.
Penrose tilings are Locally Indistinguishable: any finite-sized patch of tiles in one can be found in any other.
Penrose tilings have local scale symmetry: take any tiling, decimate it, rescale it by λ. This new tiling is LID from the original (but globally distinct).
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Quasicrystal_Presentations
Tiling Notes
SUBSTITUTION RULES: IN ACTION
19
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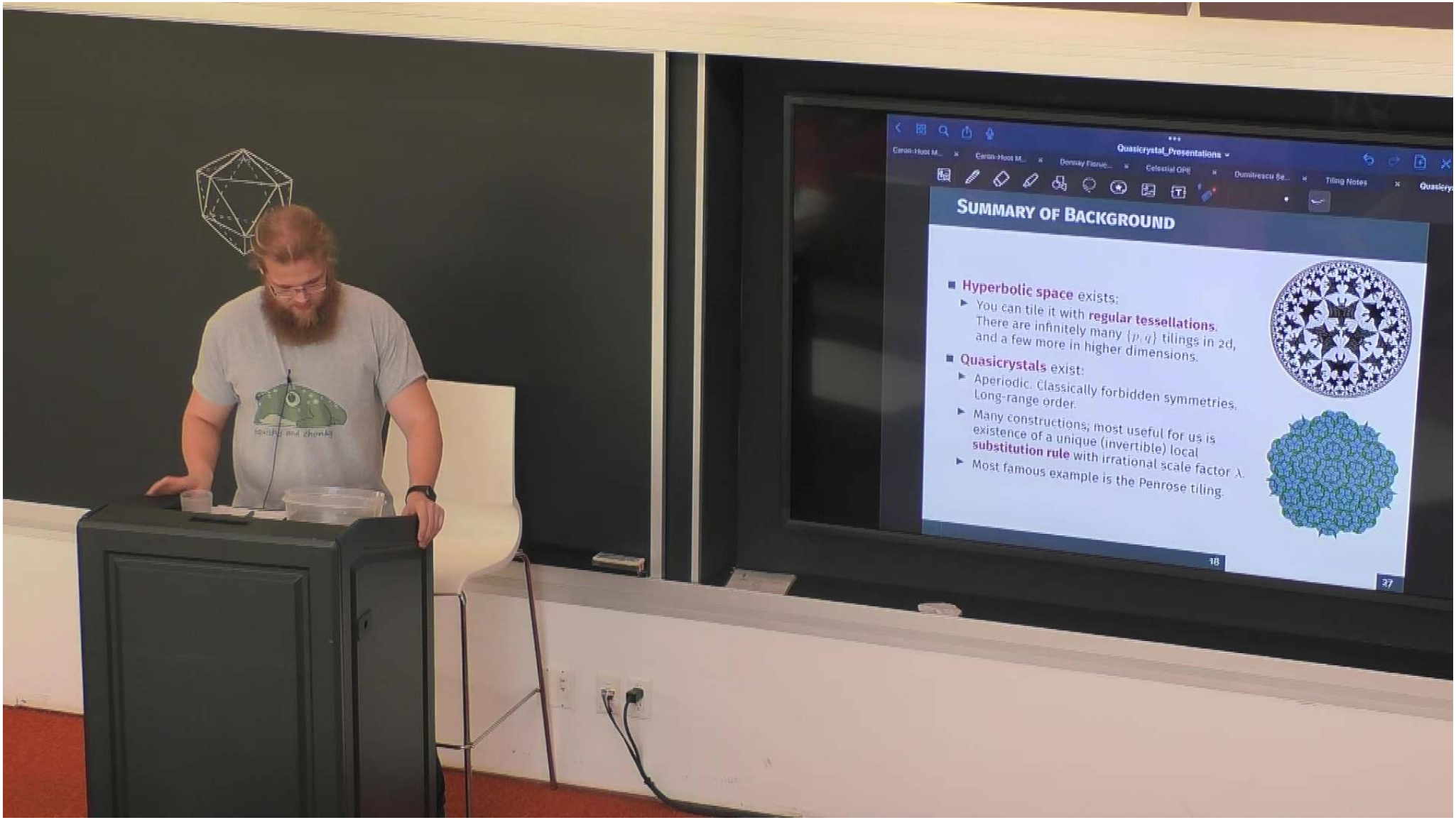
Quasicrystal_Presentations
SUMMARY OF BACKGROUND
Hyperbolic space exists:
You can tile it with regular tessellations.
There are infinitely many $\{p, q\}$ tilings in 2d, and a few more in higher dimensions.
Quasicrystals exist:
Aperiodic. Classically forbidden symmetries. Long-range order.
Many constructions, most useful for us is existence of a unique (invertible) local substitution rule with irrational scale factor λ.
Most famous example is the Penrose tiling.
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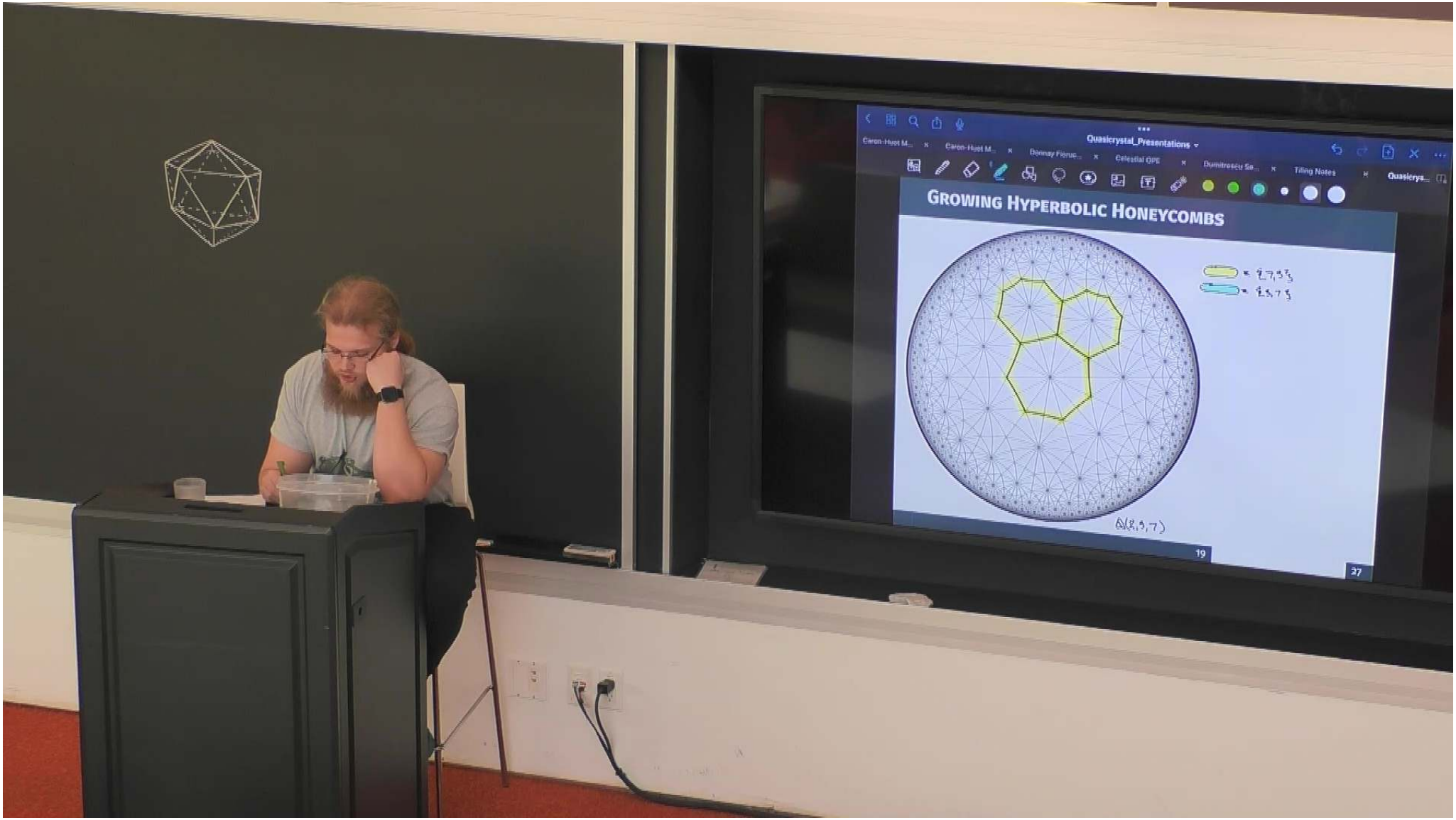
Quasicrystal_Presentations
GROWING HYPERBOLIC HONEYCOMBS
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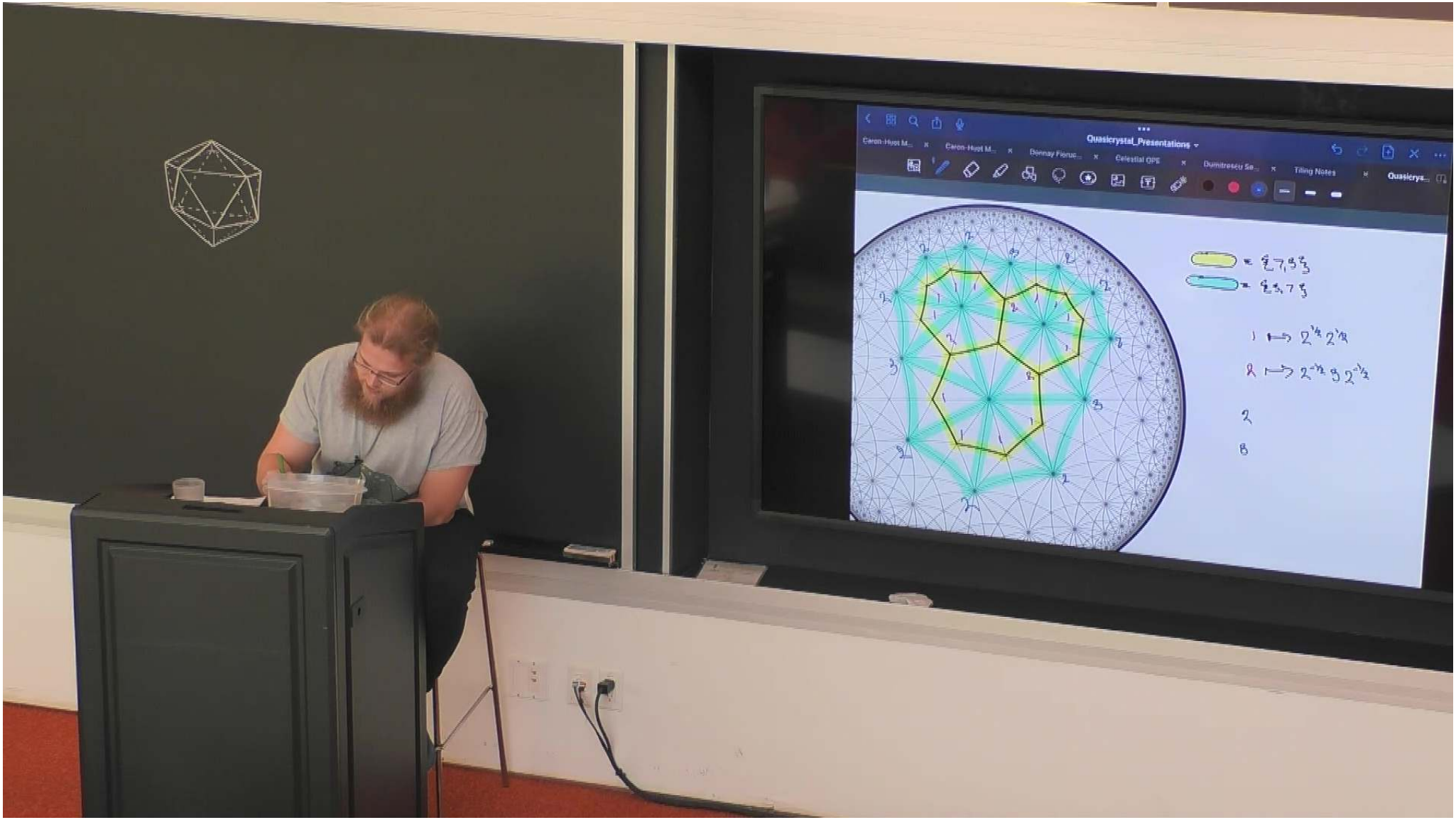
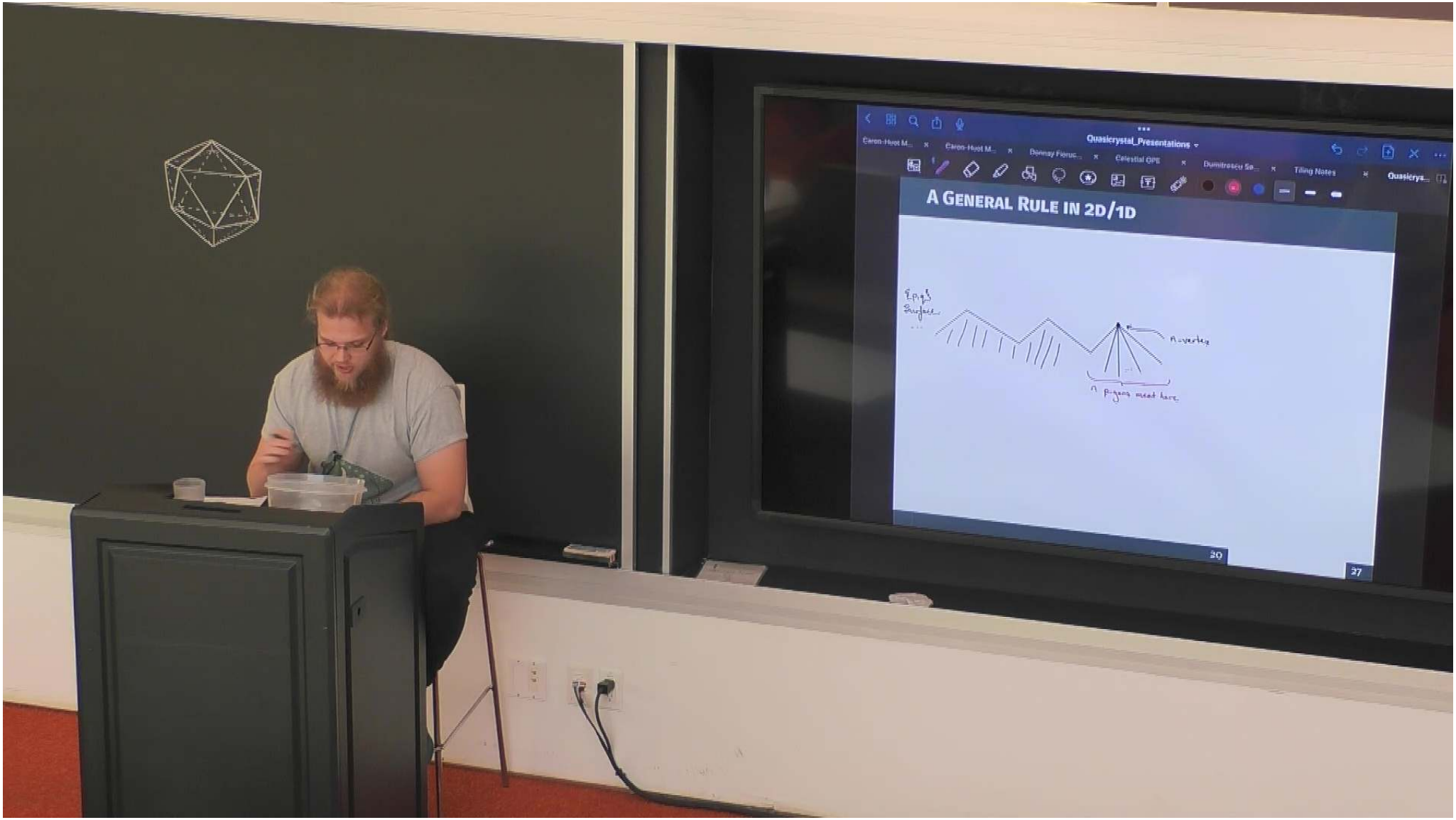

Quasicrystal_Presentations
Caron-Huot M...
Caron-Huot M...
Donnay Fleruc...
Celestial OPE
Dumitrescu Se...
Tiling Notes
Quasicrys...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A GENERAL RULE IN 2D/1D
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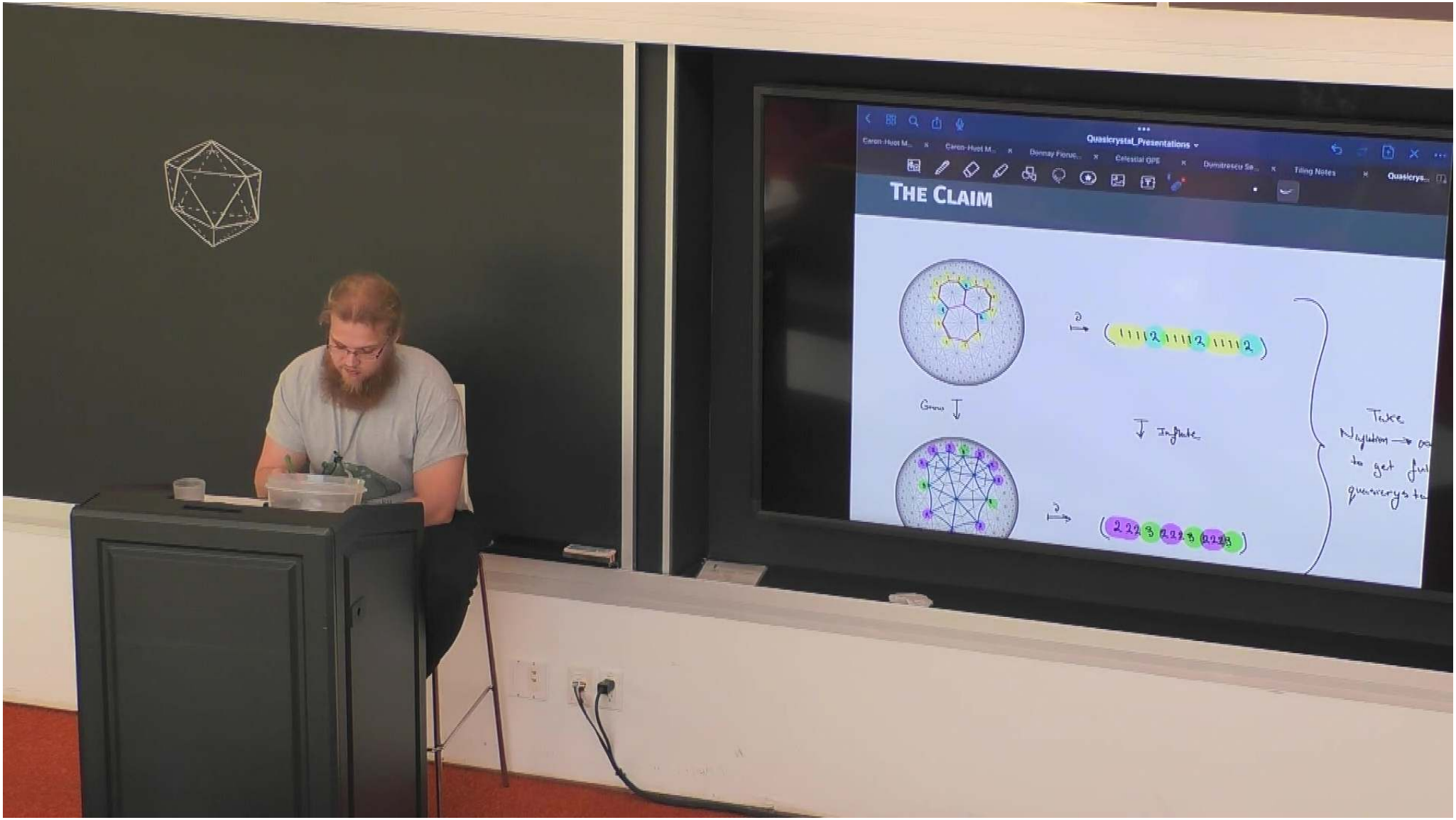
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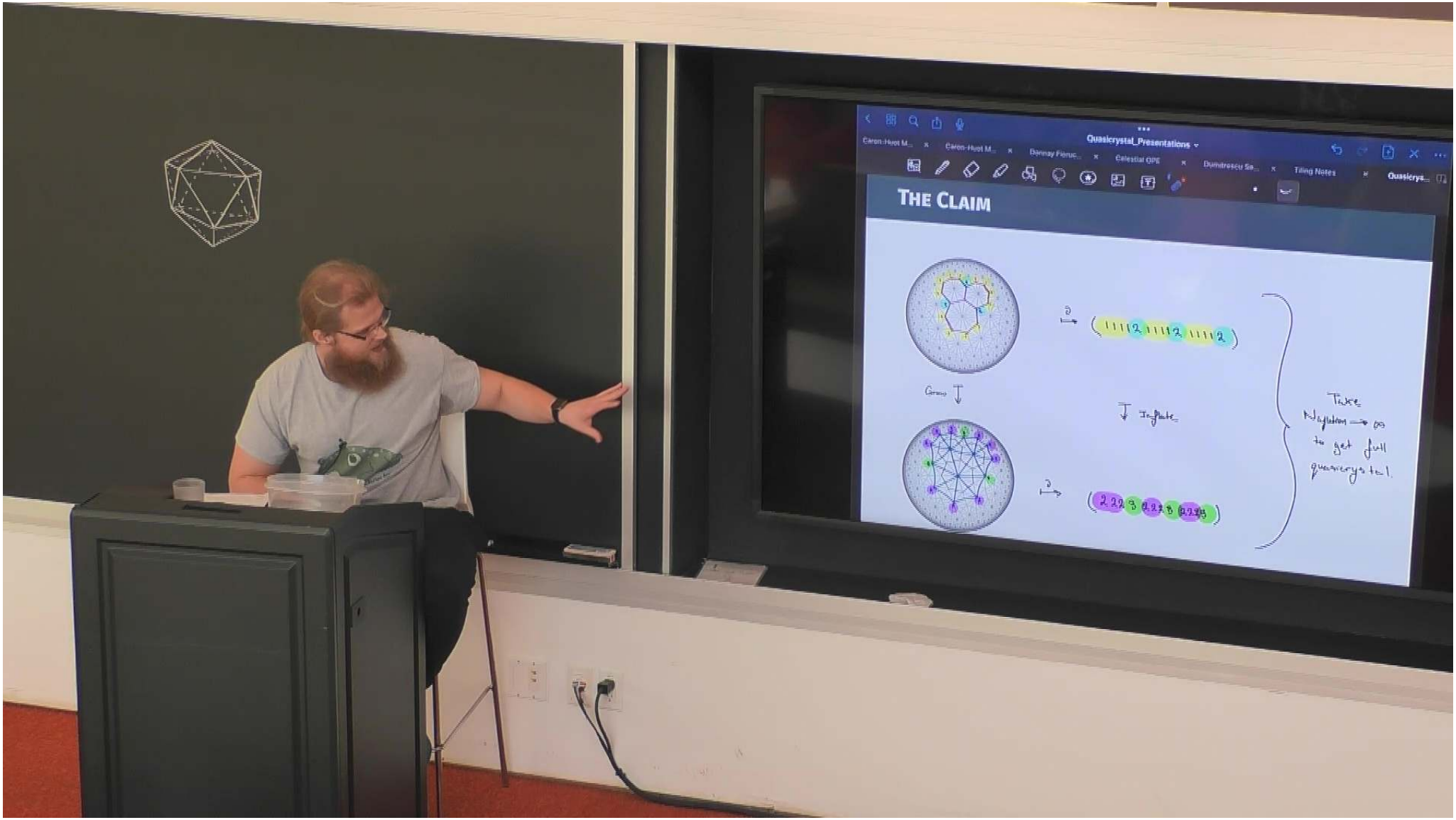
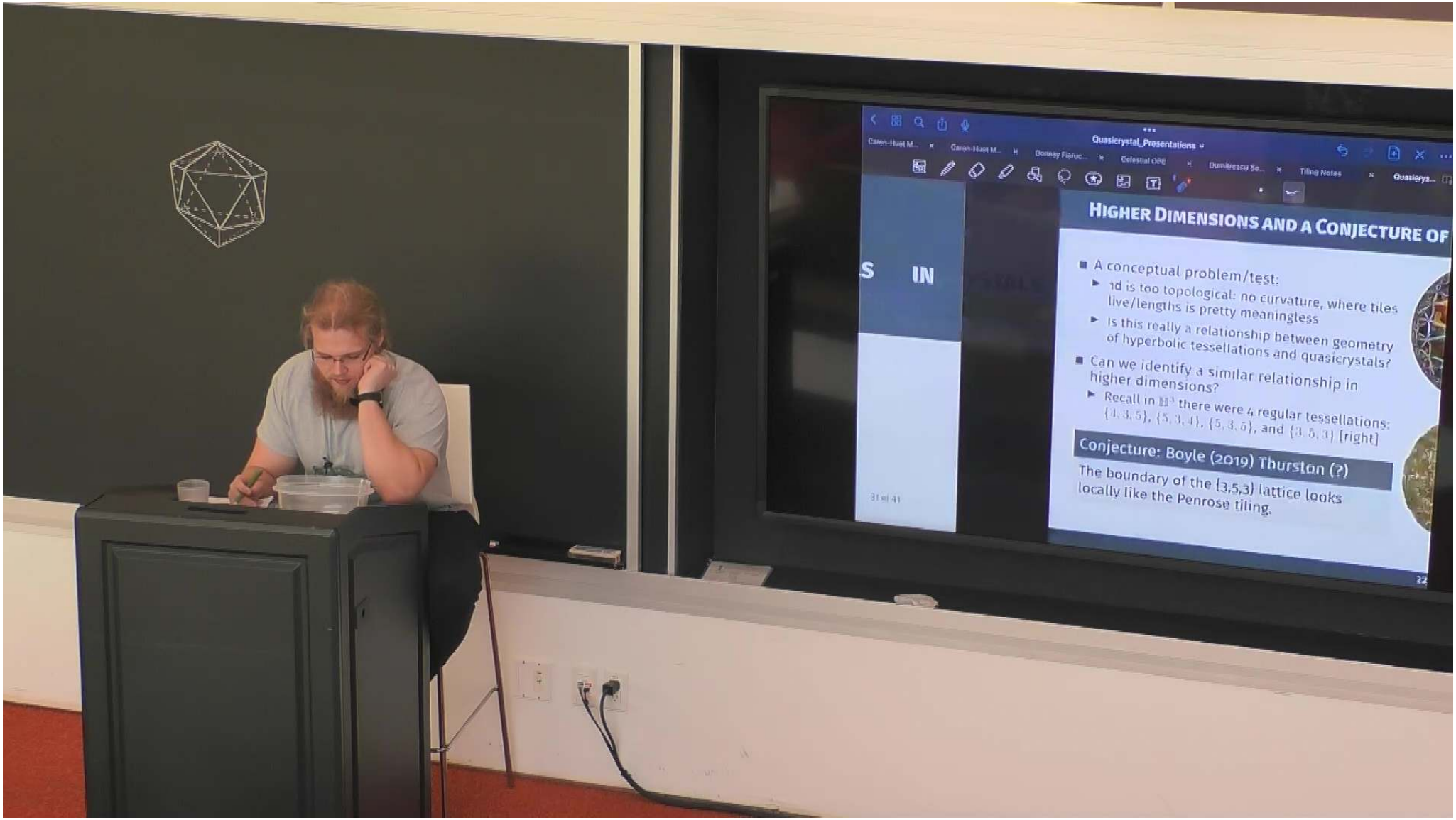
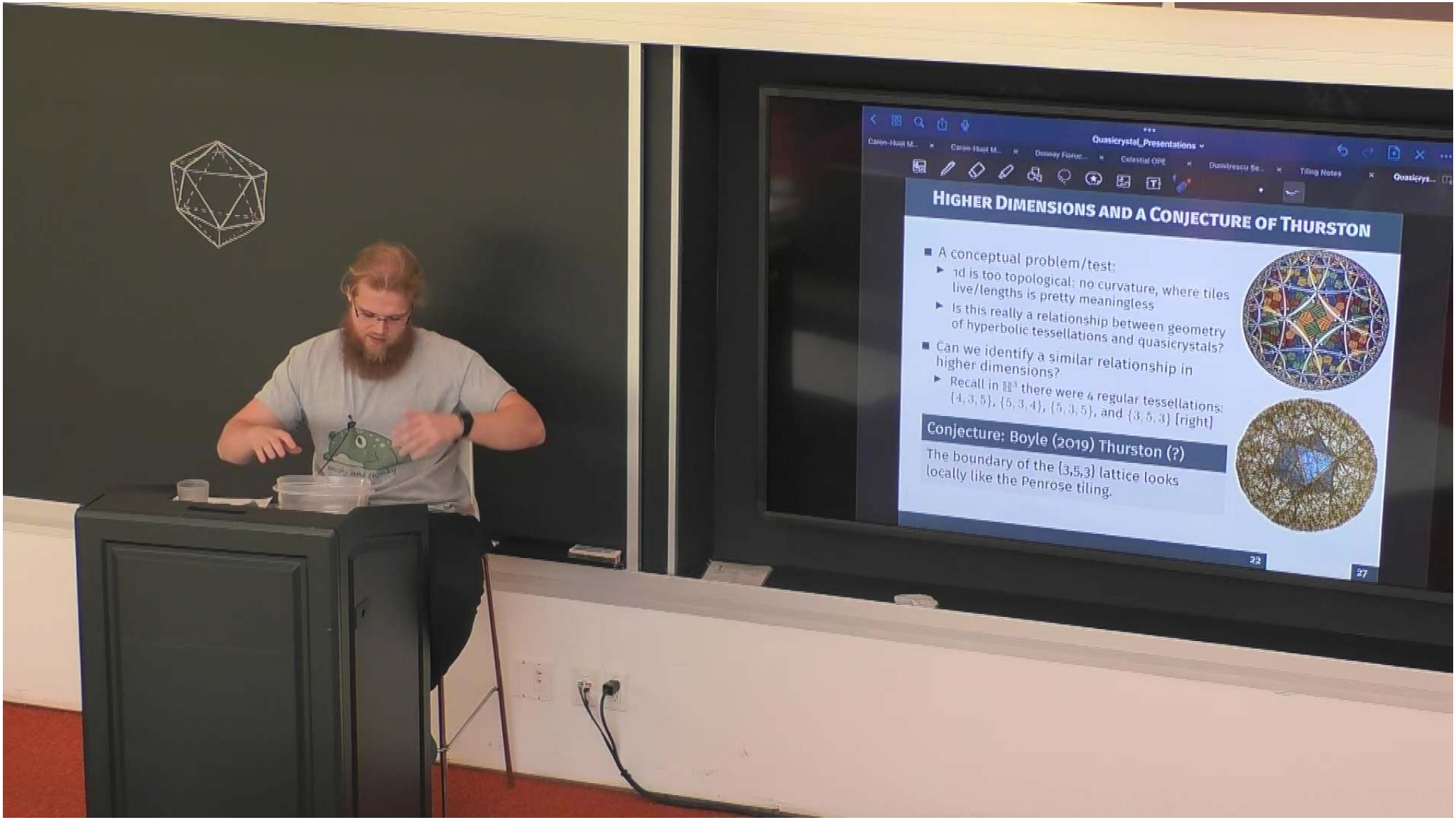
THE CLAIM
Tiling Notes
Inflate


Quasicrystal_Presentations
THE CLAIM


Quasicrystal_Presentations
HIGHER DIMENSIONS AND A CONJECTURE OF
A conceptual problem/test:
Is this really a relationship between geometry of hyperbolic tessellations and quasicrystals?
Can we identify a similar relationship in higher dimensions?
Conjecture: Boyle (2019) Thurston (?)
The boundary of the {3,5,3} lattice looks locally like the Penrose tiling.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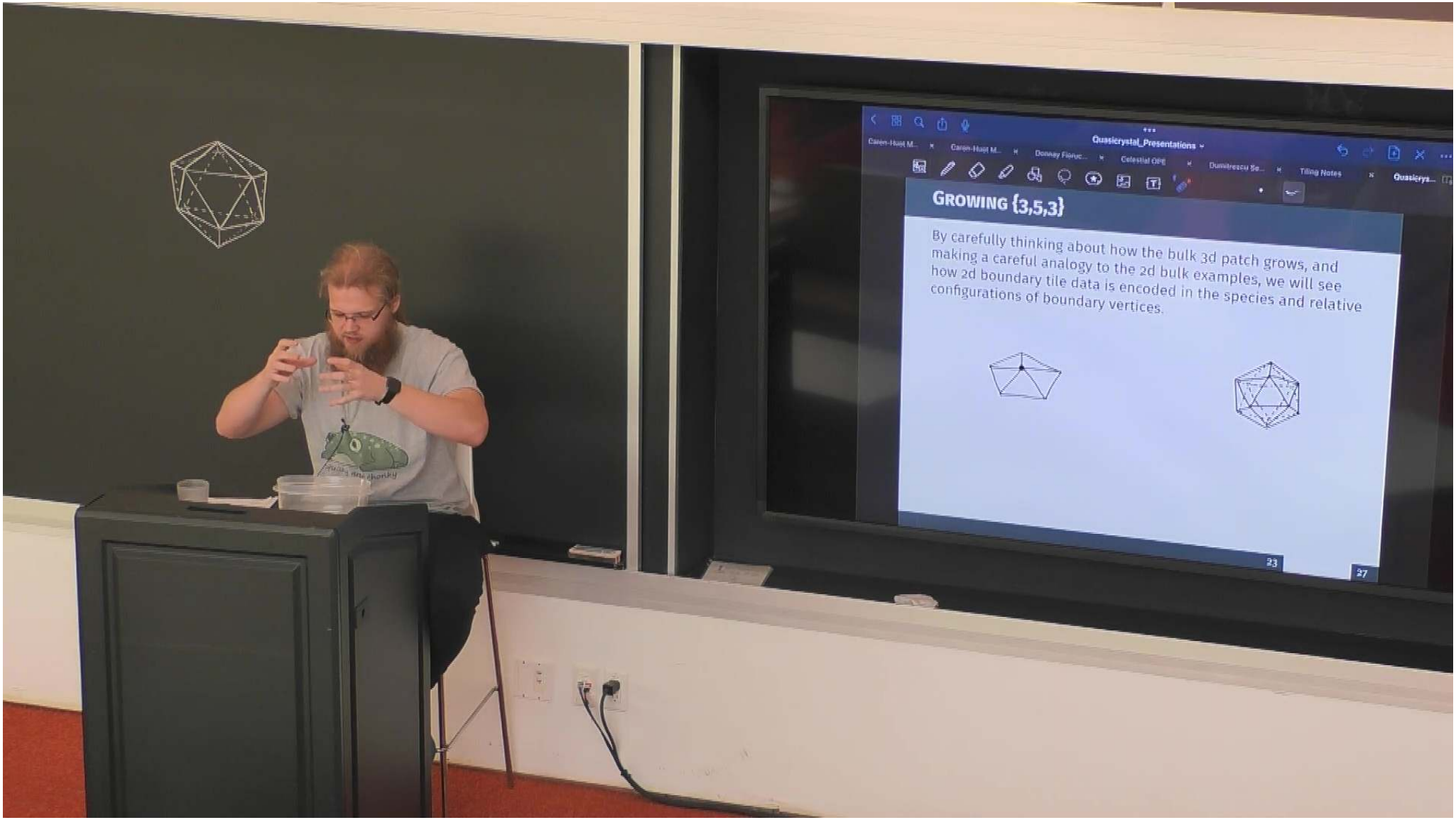
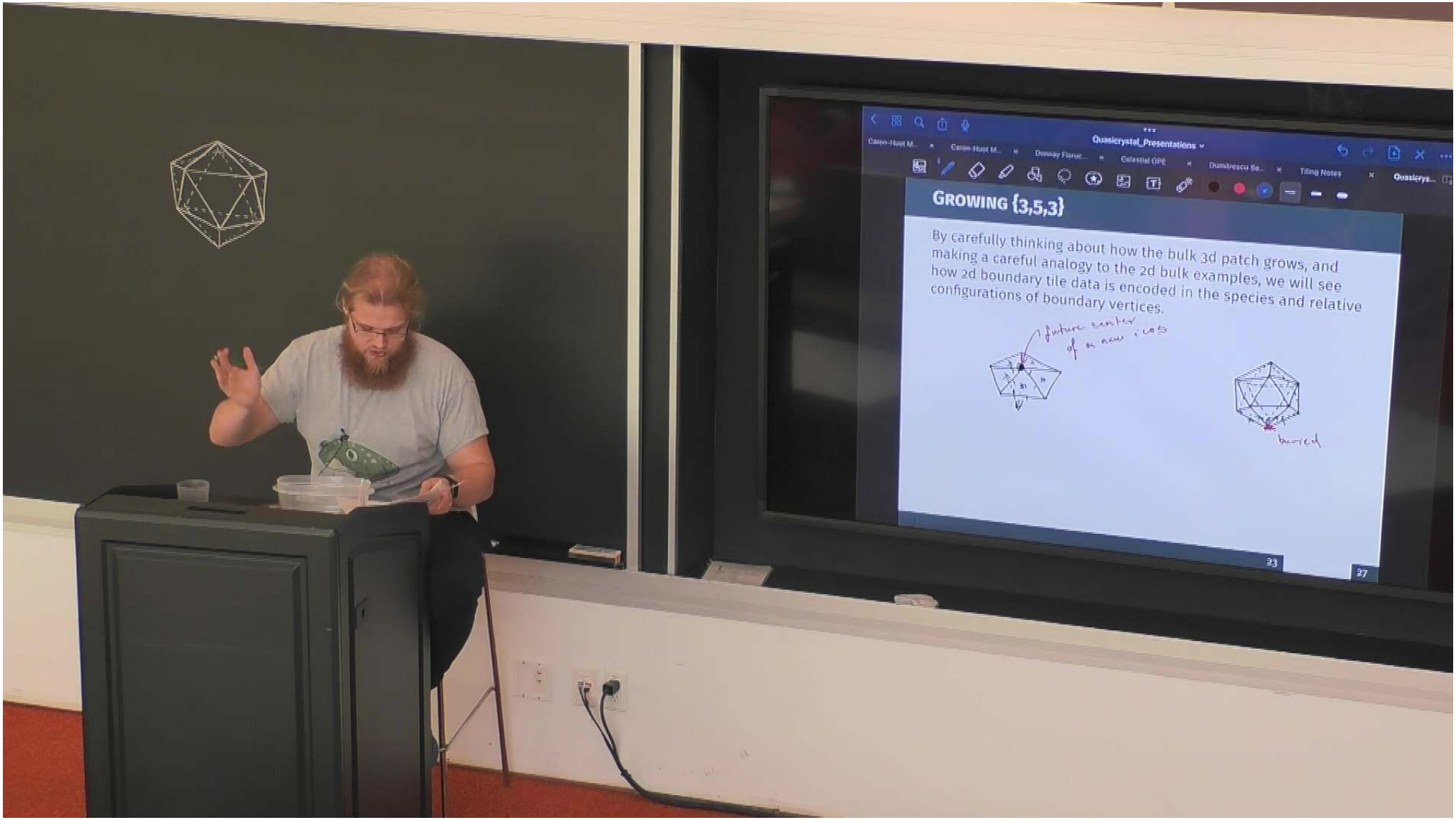
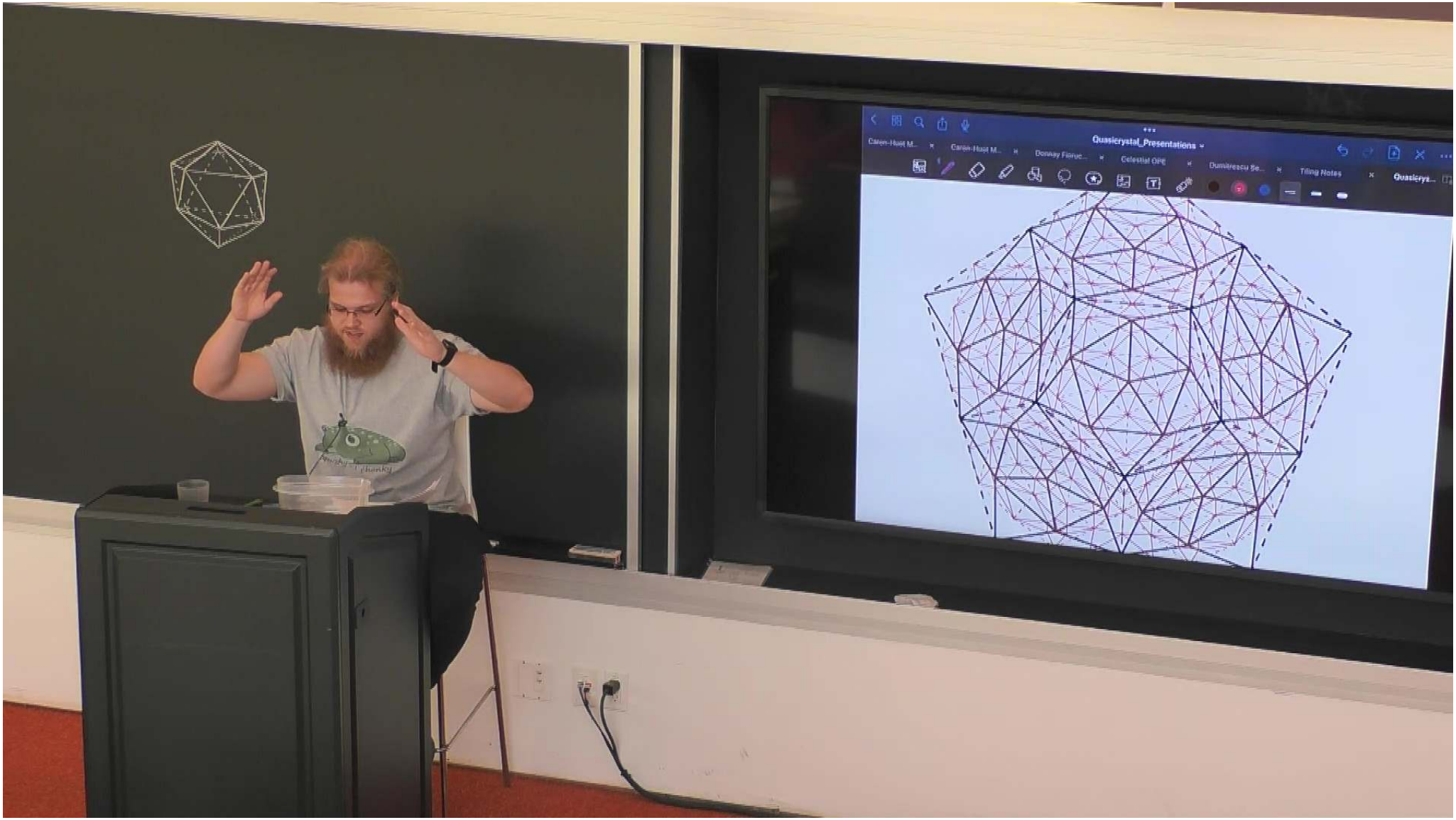
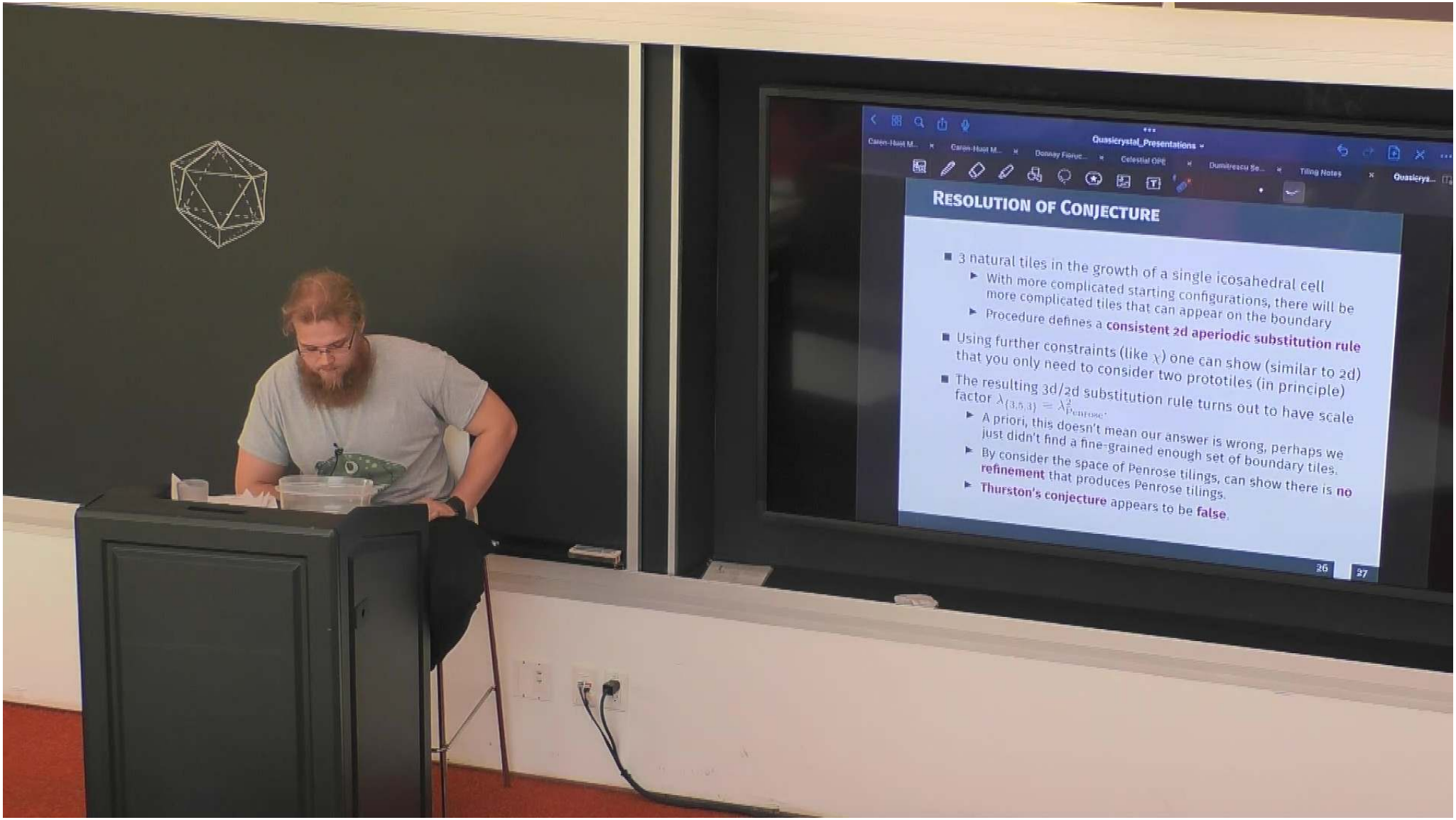
HIGHER DIMENSIONS AND A CONJECTURE OF THURSTON
A conceptual problem/test:
1d is too topological: no curvature, where tiles live/lengths is pretty meaningless
Is this really a relationship between geometry of hyperbolic tessellations and quasicrystals?
Can we identify a similar relationship in higher dimensions?
Recall in $\mathbb{H}^3$ there were 4 regular tessellations: $\{4,3,5\}$, $\{5,3,4\}$, $\{5,3,5\}$, and $\{3,5,3\}$ [right]
Conjecture: Boyle (2019) Thurston (?)
The boundary of the {3,5,3} lattice looks locally like the Penrose tiling.
33
27


Quasicrystal_Presentations
Caren-Hoist M...
Caren-Hoist M...
Donvay Fioruc...
Celestial OPE
Dumitrescu Se...
Tiling Notes
Quasicrys...
GROWING {3,5,3}
By carefully thinking about how the bulk 3d patch grows, and making a careful analogy to the 2d bulk examples, we will see how 2d boundary tile data is encoded in the species and relative configurations of boundary vertices.
23
27


Quasicrystal_Presentations
GROWING {3,5,3}
By carefully thinking about how the bulk 3d patch grows, and making a careful analogy to the 2d bulk examples, we will see how 2d boundary tile data is encoded in the species and relative configurations of boundary vertices.
future center of a new tile
boring
23
27


Quasicrystal_Presentations
Tiling Notes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Tiling Notes
RESOLUTION OF CONJECTURE
3 natural tiles in the growth of a single icosahedral cell
With more complicated starting configurations, there will be more complicated tiles that can appear on the boundary
Procedure defines a consistent 2d aperiodic substitution rule
Using further constraints (like χ) one can show (similar to 2d) that you only need to consider two prototiles (in principle)
The resulting 3d/2d substitution rule turns out to have scale factor $\lambda_{\{3,5,3\}} = \lambda^2_{\text{Penrose}}$.
A priori, this doesn't mean our answer is wrong, perhaps we just didn't find a fine-grained enough set of boundary tiles.
By consider the space of Penrose tilings, can show there is no refinement that produces Penrose tilings.
Thurston's conjecture appears to be false.
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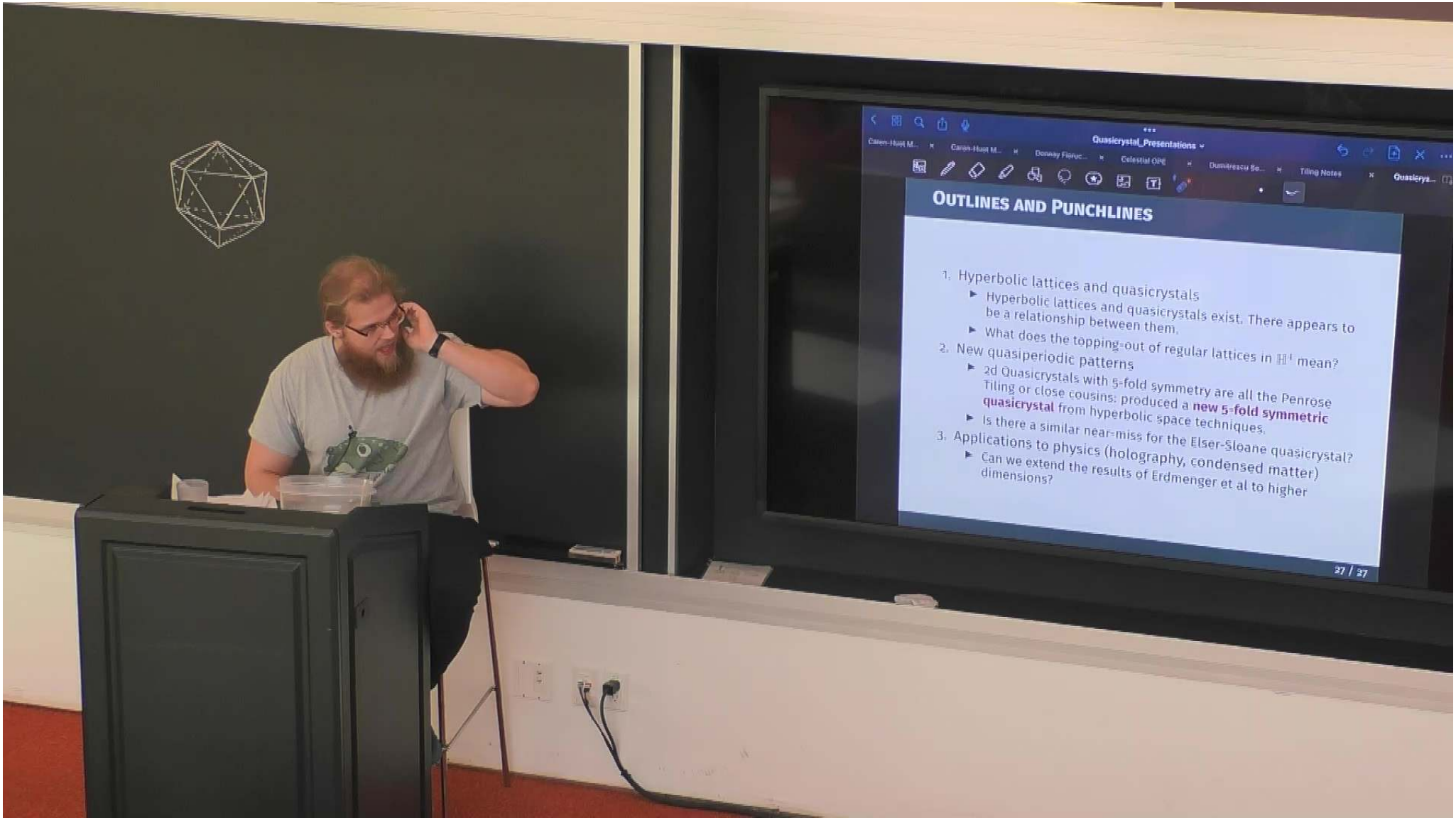
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OUTLINES AND PUNCHLINES
1. Hyperbolic lattices and quasicrystals
Hyperbolic lattices and quasicrystals exist. There appears to be a relationship between them.
What does the topping-out of regular lattices in $\mathbb{H}^4$ mean?
2. New quasiperiodic patterns
2d Quasicrystals with 5-fold symmetry are all the Penrose Tiling or close cousins: produced a new 5-fold symmetric quasicrystal from hyperbolic space techniques.
Is there a similar near-miss for the Elser-Sloane quasicrystal?
3. Applications to physics (holography, condensed matter)
Can we extend the results of Erdmenger et al to higher dimensions?
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