Title: Quantum Complexity of Kronecker Coefficients

Speakers: VojtÄ›ch HavlÃÄ•ek

Series: Perimeter Institute Quantum Discussions

Date: January 25, 2023 - 11:00 AM

URL: https://pirsa.org/23010108

Abstract: Kronecker coefficients appear in representation of the symmetric group in the decomposition of tensor products of irreducible representations. They are notoriously difficult to compute and it is a long standing problem to find a combinatorial expression for them.

We study the problem of computing Kronecker coefficients from quantum computational perspective. First, we show that the coefficients can be expressed as a dimension of a subspace given by intersection of two commuting, efficiently implementable projectors and relate their computation to the recently introduced quantum approximate counting class (QAPC). Using similar construction, we show that deciding positivity of Kronecker coefficients is contained in QMA. We give similar results for a related problem of approximating row sums in a character table of the symmetric group and show that its decision variant is in QMA. We then discuss two quantum algorithms - one that samples a distribution over squared characters and another one that approximates normalized Kronecker coefficients to inverse-polynomial additive error. We show that under a conjecture about average-case hardness of computing Kronecker coefficients, the resulting distribution is hard to sample from classically.

Our work explores new structures for quantum algorithms and improved characterization of the quantum approximate counting.

Joint work with David Gossett, Sergey Bravyi, Anirban Chowdhury and Guanyu Zhu

Zoom link: https://pitp.zoom.us/j/95976938016?pwd=eDV3TXZReHo5UHdvZ0hIbkhXOFcxQT09

Quantum Complexity of Kronecker Coefficients

Vojtech Havlicek [1] **David Gosset** [2] **Sergey Bravyi** [1]
Anirban Chowdhury [2] **Guanyu Zhu** [1]

[1] IBM Quantum, T. J. Watson Research Laboratory

[2] IQC University of Waterloo and Perimeter Institute

January 25, 2023

SUMMARY

- Kronecker coefficients appear in the representation theory of the symmetric group.
- They are notoriously difficult to compute

SUMMARY

- Kronecker coefficients appear in the representation theory of the symmetric group.
- They are notoriously difficult to compute
- It is a long standing problem to find a combinatorial formula for them.

SUMMARY

- We show that the coefficients can be expressed by a polynomially sized quantum circuit.
- This places the problem of computing them to relative precision into the recently introduced quantum approximate counting class (QAPC)
- We then show that deciding positivity of Kronecker coefficients is in QMA, which is to our knowledge the best known upper bound for the problem.
- We give similar results for a related problem of approximating row sums in a character table of the symmetric group and show that its decision variant is in QMA.

SUMMARY

- We give two quantum algorithms; one that samples from a distribution over squared characters and another one that approximates Kronecker coefficients normalized by irrep dimension to inverse-polynomial additive error.
- We are not aware of any efficient classical algorithms for these two tasks. We show that under a conjecture about average-case hardness of computing Kronecker coefficients, the resulting distribution is hard to sample from classically.
- Our work characterizes quantum approximate counting and explores new avenues towards quantum algorithms.

OUTLINE

- I will talk about representation theory of S_n.

Symmetric Group and its Representations

- A group is a set S with associative multiplication $S \times S \to S$ and an identity element $e \in S$. For all $s \in S$ exists unique $r \in S$ so that $s \times r = e$.

Symmetric Group and its Representations

- A *unitary* representation of a group G is a homomorphism $\rho : G \to U_d$, where U_d is the group of all (complex) unitary $d \times d$ matrices.

SYMMETRIC GROUP AND ITS REPRESENTATIONS

- A *unitary* representation of a group G is a homomorphism $\rho : G \to U_d$, where U_d is the group of all (complex) unitary $d \times d$ matrices.
- The vector space acted on by U_d is also called a representation (module).
- The identity element is *always* represented by the identity matrix.

(12)	(23)
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Permutation representation of S_3

SYMMETRIC GROUP AND ITS REPRESENTATIONS

- A *unitary* representation of a group G is a homomorphism $\rho : G \to U_d$, where U_d is the group of all (complex) unitary $d \times d$ matrices.
- The vector space acted on by U_d is also called a representation (module).
- The identity element is *always* represented by the identity matrix.

(12)	(23)
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Permutation representation of S_3

(12)	(23)
-1	-1

Sign representation of S_3

(12)	(23)
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

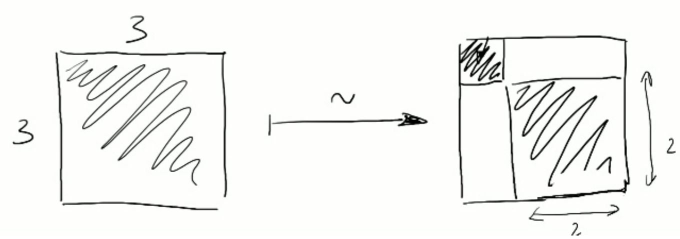
(Left) regular representation of S_3 in basis where $e = 1, (12) = 2, (23) = 3, (13) = 4, (123) = 5, (132) = 6$. This acts on itself by left group multiplication.

SYMMETRIC GROUP AND ITS REPRESENTATIONS

- Some representations act trivially on a non-trivial subspace of their representation modules.
- Example: Permutation representation of S_3 acts trivially on $(1, 1, 1)$ and nontrivially on $v \in \mathbb{C}_3$, such that $v_1 + v_2 + v_3 = 0$.

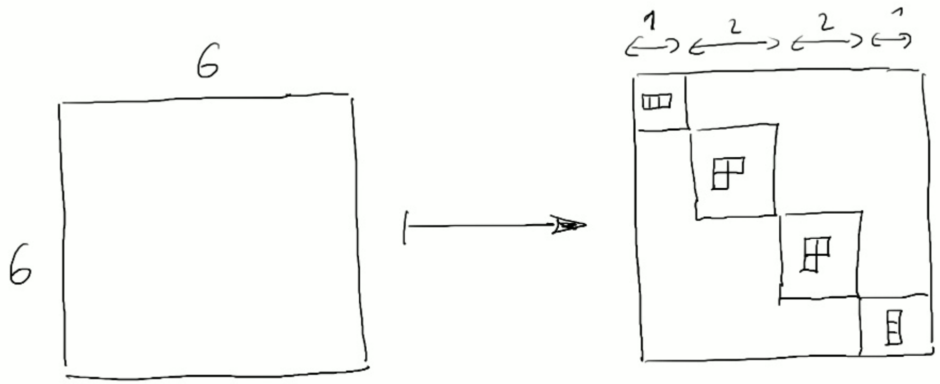
(12)	(23)
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

- There exists a unitary transformation that block-diagonalizes this representation. So I can decompose a 3 dimensional permutation representation of S_3 into $1 \oplus 2$ dimensional representations.
- These do not contain any such nontrivial subspace and are called *irreducible*.



Symmetric Group and its Representations

- Some representations contain multiple copies of the same irreducible representation.
- Example: The *regular* representation of G contains d_λ copies of every irreducible representation λ. It also contains all inequivalent irreducible representations of G
- For S_n, there is a nice way of indexing the irreducible representations: Young diagrams. The 6-dimensional regular representation of S_3 decomposes as:



Kronecker Coefficients

- We can build representations by taking tensor products.
- A tensor product of irreducible representations may no longer be irreducible! In general, for irreps λ, ν of G:

$$\rho_\lambda \otimes \rho_\nu = \bigoplus_{\mu \in \hat{G}} g_{\lambda\nu\mu} \rho_\mu, \tag{1}$$

where $g_{\lambda\nu\mu}$ is the multiplicity of μ in the tensor of ν and λ.

Kronecker Coefficients

$$\rho_\lambda \otimes \rho_\nu = \bigoplus_{\mu \in \hat{G}} g_{\lambda\nu\mu} \rho_\mu, \tag{2}$$

- Issue: the coefficients are very hard to compute in general!
- Major open problem in combinatorics: do the Kronecker coefficients *count* anything? Can they be expressed as a sum of some non-negative quantity?

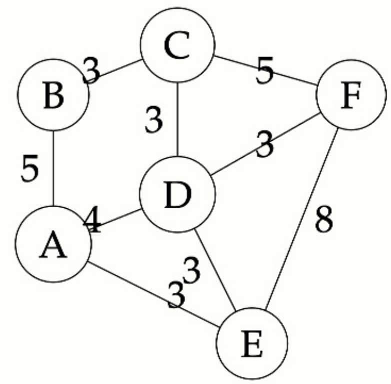
Counting Complexity Primer

- **Decision problem:** input is a bitstring of size n, the output is yes or no.
- **Counting problem:** input is a bistring of size n, the output is an integer.
- NP is the class of decision problem that can be efficiently checked by a classical algorithm.
- #P counts the number of accepting solutions to an NP problem.

Counting Complexity Primer

- **Decision problem:** input is a bitstring of size n, the output is yes or no.
- **Counting problem:** input is a bistring of size n, the output is an integer.
- NP is the class of decision problem that can be efficiently checked by a classical algorithm.
- #P counts the number of accepting solutions to an NP problem.

Example: Traveling Salesman:



- NP problem: Input a Graph G and an integer L. Is there a a tour of length at most L?
- #P problem: How many tours of lengths of at most L are there?

Counting Complexity Primer

- #P counts the number of accepting solutions to an NP problem.
- GapP counts the difference between the number of accepting and rejecting solutions to an NP problem.
- Approximate counting: find a *relative* approximation to counting problem. Relative approximation $\tilde{Q}$ of Q to error ϵ:

$$(1+\epsilon)^{-1}Q \leq \tilde{Q} \leq (1+\epsilon)Q. \tag{3}$$

- The problem of finding a $1/poly(n)$-relative approximation to any #P problem cannot be #P-hard (Stockmeyer's theorem).
- Does not hold for GapP!

Counting Complexity Primer

- #P counts the number of accepting solutions to an NP problem.
- GapP counts the difference between the number of accepting and rejecting solutions to an NP problem.
- Approximate counting: find a *relative* approximation to counting problem. Relative approximation $\tilde{Q}$ of Q to error ϵ:

$$(1+\epsilon)^{-1}Q \leq \tilde{Q} \leq (1+\epsilon)Q. \tag{3}$$

- The problem of finding a $1/poly(n)$-relative approximation to any #P problem cannot be #P-hard (Stockmeyer's theorem).
- Does not hold for GapP!

Counting Complexity Primer

What is known:

Problem	Complexity
Approximating Kronecker coefficients	GapP [Bürgisser and Ikenmeyer 2008]
Deciding if $g_{\lambda\mu\nu} > 0$	NP-hard [Pak and Panova 2014]

- NP-hard (loosely) means that if you solve that problem, you can solve anything in NP.

Quantum Complexity of Kronecker Coefficients

- QMA is a quantum version of the class NP.
- Let $A = (A_{\text{yes}}, A_{\text{no}})$ be a promise problem. A is in QMA, if there exists a polynomial-time generated family of quantum circuits such that [Watrous 2008]:
  - For all $x \in A_{\text{yes}}$, there exists a quantum state $|\psi\rangle$ on polynomially many qubits, such that the probability that the verifier accepts $|x, \psi\rangle$ is greater than $2/3$.
  - If $x \in A_{\text{no}}$, for all $|\psi\rangle$, the probability that the verifier accepts $|x, \psi\rangle$ is less than $1/3$.

Quantum Complexity of Kronecker Coefficients

- QMA is a quantum version of the class NP.
- Let $A = (A_{\text{yes}}, A_{\text{no}})$ be a promise problem. A is in QMA, if there exists a polynomial-time generated family of quantum circuits such that [Watrous 2008]:
  - For all $x \in A_{\text{yes}}$, there exists a quantum state $|\psi\rangle$ on polynomially many qubits, such that the probability that the verifier accepts $|x, \psi\rangle$ is greater than $2/3$.
  - If $x \in A_{\text{no}}$, for all $|\psi\rangle$, the probability that the verifier accepts $|x, \psi\rangle$ is less than $1/3$.
- QAPC is an approximate counting complexity class associated to QMA.
- Loosely: what is, approximately, the dimension of the subspace of valid witness states?

QUANTUM COMPLEXITY OF KRONECKER COEFFICIENTS

Why is this interesting?

- To our knowledge best known upper bound on the decision problem.
- Suggests that QAPC lie somewhere between approximate #P and GapP.

	Function contained in...	Approximation problem contained in...
Classical counting	#P	$\mathrm{FBPP}^{\mathrm{NP}}$
Quantum counting	#BQP	QAPC
Gap counting	GapP	GapP

- We learned new interesting things from the proof technique! Path to new quantum algorithms?

Positivity of Kronecker Coefficients is in QMA

- $g_{\lambda\mu\nu}$ is also multiplicity of the trivial irreducible representation in a tensor of irreducible representations $\rho_\lambda \otimes \rho_\mu \otimes \rho_\nu$.
- Proof strategy: Start with threefold copy of the regular representation $V \otimes V \otimes V$.
- We describe a measurement $\{\Pi_\lambda \otimes \Pi_\mu \otimes \Pi_\nu\}_{\lambda,\mu,\nu}$, where Π_λ projects from the regular representation onto the irreducible representation λ.
- We describe a quantum measurement that projects $V \otimes V \otimes V$ onto irreducible representations in this representation.
- (Details about the measurement later)

POSITIVITY OF KRONECKER COEFFICIENTS IS IN QMA

- Let the input be μ, ν, λ - labels of irreducible representations of S_n.
- The goal is to verify that $g_{\mu\nu\lambda} > 0$

A QMA verifier gets a witness state $|\psi\rangle \in V \otimes V \otimes V$ from the prover.

- Project each tensor factor into an irreducible representation. If you get outcomes μ, ν, λ, keep the post-measurement state and continue. Otherwise reject.
- Apply the projector onto irreps from the threefold tensor representation to the resulting state. If you measure the trivial irreducible representation, accept. Otherwise reject.

Positivity of Kronecker Coefficients is in QMA

- Let the input be μ, ν, λ - labels of irreducible representations of S_n.
- The goal is to verify that $g_{\mu\nu\lambda} > 0$

A QMA verifier gets a witness state $|\psi\rangle \in V \otimes V \otimes V$ from the prover.

- Project each tensor factor into an irreducible representation. If you get outcomes μ, ν, λ, keep the post-measurement state and continue. Otherwise reject.
- Apply the projector onto irreps from the threefold tensor representation to the resulting state. If you measure the trivial irreducible representation, accept. Otherwise reject.

The above test has the following properties:

- If the test passes, you know for sure that $g_{\mu,\nu,\lambda} > 0$.
- If $g_{\mu,\nu,\lambda} = 0$, any $|\psi\rangle$ is rejected with certainty.

Irrep Projectors

- Regular representation acts on a Hilbert space $\mathbb{C}[S_n]$ as $\sigma\,|\pi\rangle \mapsto |\sigma\pi\rangle$
- A projector from the regular representation onto an irreducible representation λ is given:

$$\Pi_\lambda = \frac{d_\lambda}{n!} \sum_{\alpha \in S_n} \chi_\lambda(\alpha)\alpha, \tag{4}$$

  where $\chi_\lambda(\alpha)$ is the character (trace) of an irreducible representation λ at α.
- Can be implemented as using the S_n QFT. For any function $f : S_n \to \mathbb{C}$ we have:

$$\mathrm{QFT}_n \sum_{\sigma \in S_n} f(\sigma)|\sigma\rangle = \sum_{\lambda \vdash n} \sum_{i,j} (\hat{f}(\lambda))_{ij} |\lambda, i, j\rangle, \tag{5}$$

- The projector is: $\Pi_\lambda = \mathrm{QFT}_n P_\lambda \mathrm{QFT}_n^\dagger$, where P_λ measures the label λ.
- The witness state will be a tensor of three states $|\psi\rangle \in V$. Given an input μ, ν, λ, an irrep measurement on a valid witness state will give an outcome μ, ν, λ. Additionally, the post-measurement state will have symmetries of the corresponding irreps.

Irrep Projectors

- We also need to implement a projector from the tensor triple of regular representations onto its irreducibles.
- This is given by:

$$Q_\lambda = \frac{d_\lambda}{n!} \sum_{\alpha \in S_n} \chi_\lambda(\alpha)(\alpha \otimes \alpha \otimes \alpha), \tag{6}$$

- We care about checking if the witness state has enough weight at the trivial irreducible rep:

$$Q = \frac{1}{n!} \sum_{\alpha \in S_n} (\alpha \otimes \alpha \otimes \alpha), \tag{7}$$

- This projector is harder to implement - we use Generalized Phase Estimation to do that [Harrow05]. See the paper for details.

QMA PROTOCOL REVISITED

Consider a QMA verifier which starts from a witness state $|\psi\rangle \in V \otimes V \otimes V$.

- Project each tensor factor into an irreducible representation by applying $\prod \otimes \prod \otimes \prod$. If you get outcomes μ, ν, λ, keep the post-measurement state and continue. Otherwise reject.
- Apply the projector Q onto irreps from the threefold tensor representation to the resulting state. If you measure the trivial irreducible representation, accept. Otherwise reject.

The above test has the following properties:

- If the state is accepted, the multiplicity of the trivial irrep in $g_{\mu,\nu,\lambda}$ is nonzero.
- If the multiplicity of the trivial irrep in $g_{\mu,\nu,\lambda}$ is zero, $|\psi\rangle$ is rejected with certainty.

Approximating Kronecker Coefficients and QAPC

- The dimension of the subspace that $Q[\prod_\lambda \otimes \prod_\mu \otimes \prod_\nu]$ projects onto is given by:

$$\mathrm{Tr}(Q\ \Pi^{\otimes 3}) = d_\lambda d_\mu d_\nu g_{\mu\nu\lambda}, \tag{8}$$

  where $g_{\mu\nu\lambda}$ is the Kronecker coefficient.
- The dimension factors d_λ, d_μ, d_ν are easily computable by the Frame-Robinson-Thrall (hook-length) formula.
- A mutliplicative approximation to the dimension of this subspace gives a multiplicative approximation to $g_{\mu\nu\lambda}$. This places the problem to QAPC.

Row Sums

We can use similar analysis for another open problem in algebraic combinatorics:

- The table of character values for all possible conjugacy classes of a group is called the character table.

The following problem was proposed by Stanley in [Stanley 1999]: Given a group G and a irreducible representation λ, compute $\sum_c \chi_c(\lambda)$, where c runs over conjugacy classes of G.

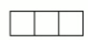
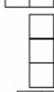
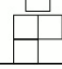
S_3	1	(12)	(123)	row sum
	1	1	1	3
	1	-1	1	1
	2	0	-1	1

Table. S_3 character table. Row sums of S_3. Irreps are labelled by Young diagrams, conjugacy classes by their representatives in cycle notation.

Row Sums

Define a conjugation representation:

$$\rho : S_n \to S_n \qquad \rho(\pi)\sigma \mapsto \pi\sigma\pi^{-1} \tag{9}$$

ROW SUMS

Define a conjugation representation:

$$\rho : S_n \to S_n \qquad \rho(\pi)\sigma \mapsto \pi\sigma\pi^{-1} \tag{9}$$

- Key fact: Row sums appear as multiplicities of an irreducible representation λ in a conjugation representation ρ.
- We place the problem into QMA and QAPC by similar strategy as for the Kronecker coefficients.
- We have to be a little more creative with the irrep projectors :)

Quantum Algorithm for Normalized Kroneckers

- Observation: Given an irrep λ, we can prepare a quantum state:

$$|\psi_\lambda\rangle = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} \chi_\lambda(\sigma) |\sigma\rangle \in \mathbb{C}[S_n]. \tag{10}$$

- This can be used to get a quantum algorithm that gives $1/poly(n)$ additive approximation to:

$$g'_{\mu,\nu,\lambda} = \frac{g_{\mu,\nu,\lambda}}{\min(d_\mu, d_\nu, d_\lambda)} = \langle\psi_\lambda|D_\mu|\psi_\nu\rangle \tag{11}$$

- Here D_μ acts as $D_\mu |\sigma\rangle = \frac{\chi_\mu(\sigma)}{d_\mu} |\sigma\rangle$ and can be again implemented by S_n QFT. (Details in the paper)
- Does this mean we have a new *interesting* quantum algorithm?
- Good: it does something nontrivial!
- Not so good: This range of approximation is not well studied. There is also a classical algorithm for approximating characters to similar precision [Jordan 2008].

CHARACTER DISTRIBUTIONS

- Using the same trick, we found an efficient quantum algorithm for sampling from:

$$P_\lambda(g) = \frac{\chi_\lambda^2(g)}{n!} \tag{12}$$

  for a fixed irrep λ.
- If approximating $\chi_\lambda^2(g)$ is in $\#\mathsf{P}$, then the polynomial hierarchy collapses to the second level [Ikenmeyer, Pak, and Panova 2022].
- This is a worst case result. If we conjecture that the approximation is also hard on most inputs, then the distribution becomes hard to sample from classically.

Conclusion and Open Problems

We improved characterization of QAPC by understanding the quantum complexity of Kronecker coefficients and the Row Sums problem and sketched two quantum algorithms.

Conclusion and Open Problems

We improved characterization of QAPC by understanding the quantum complexity of Kronecker coefficients and the Row Sums problem and sketched two quantum algorithms.

- Normalized Kroneckers have close relationship with quantum marginal problem [Christandl and Mitchison 2005]. Is there another natural interepretation of *unnormalized* Kroneckers in this context?
- Can we obtain a stronger hardness of sampling result over the character distributions?
- Can we give an evidence that the estimation quantum algorithm is classically hard/easy?

Thank you!