Title: Melting crystals and cluster integrable systems

Speakers: Mykola Semenyakin

Series: Quantum Fields and Strings

Date: January 17, 2023 - 2:00 PM

URL: https://pirsa.org/23010102

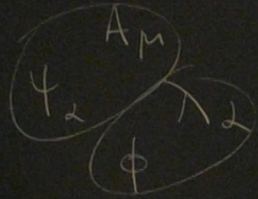
Abstract: The language of integrable systems is widely applicable to string theory. One context where it is useful is the Seiberg-Witten theory, describing low-energy dynamics of confined 4d N=2 supersymmetric gauge theories: the families of complex curves with differentials, playing a central role in this description, appeared to be spectral curves, solving the integrable systems of interacting particles. Moreover, the spectrum of stable BPS particles appears from the consideration of hyperkahler structures on the phase spaces of integrable systems. And the full partition functions of instantons, regularized by Omega-background, solve deuatonomized systems of particles.

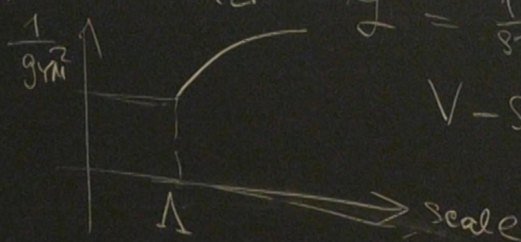
In my talk, I will explain correspondence unifying to some extent two latter ones. It relates discrete dynamics of so-called cluster integrable systems and partition functions of 5d N=1 supersymmetric gauge theories, or more generally of topological stings on corresponding local Calabi-Yau manifolds. Based on the simplest non-trivial example, I will show how both "equations" and "solutions" sides of correspondence naturally appear in the simple statistical models of dimers and "melting crystals" made out of them.

Zoom link: https://pitp.zoom.us/j/91449709915?pwd=eUVDeDdHVGI2ZVRwQ0hneXFpQk1wQT09

Relationships between Seiberg-Witten theory for $\mathcal{N}=2$ SYM and integrable systems

SU(N) $\mathcal{N}=2$ SYM in 4d





$$\mathcal{L} = \frac{1}{8\pi} \operatorname{Im}\left(\tau \operatorname{Tr}\left[\int d^2\theta\, W^\alpha W_\alpha + 2\int d^2\theta\, d^2\bar{\theta}\, \Phi^\dagger e^{-2V} \Phi\right]\right) + \text{c.c.}$$

$V - SU(N)$

$(U(1))^{N-1}$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

$$V(\phi) = \frac{1}{2g^2} \operatorname{Tr}\left([\phi, \phi^\dagger]^2\right)$$

$$[\phi, \phi^\dagger] = 0$$

$$\phi = \operatorname{diag}(a_1, a_2, \ldots, a_N)$$

$$\sum a_i = 0$$

$$\mathcal{L}_{\text{low energy}} = \frac{1}{4\pi}\left[\int d^4\theta\, \frac{\partial \mathcal{F}}{\partial A_i} \bar{A}^i + \int d^2\theta\, \frac{1}{2} \frac{\partial^2 \mathcal{F}}{\partial A_i \partial A_j} W^{\alpha i} W_{\alpha j}\right]$$

$\mathcal{F}(A_i)$ $\quad A_i - U(1)$ chiral $\mathcal{N}=1$ multiplets.

$+ \, c.c.$

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}$$

$$V(\phi) = \frac{1}{2g^2}\mathrm{Tr}\left([\phi,\phi^\dagger]^2\right)$$

$$[\phi,\phi^\dagger] = 0$$

$$\phi = \mathrm{diag}(a_1, a_2, \ldots, a_N)$$

$$\sum a_i = 0$$

Coulomb branch parameters.

$(U(1))^{N-1}$

$$\frac{\partial F}{\partial A_i}\bar{A}^i + \int d^2\theta \, \frac{1}{2} \frac{\partial^2 F}{\partial A_i \partial A_j} W^{\alpha i} W_{\alpha j}$$

A_i – $U(1)$ chiral $\mathcal{N}=1$ multiplets.

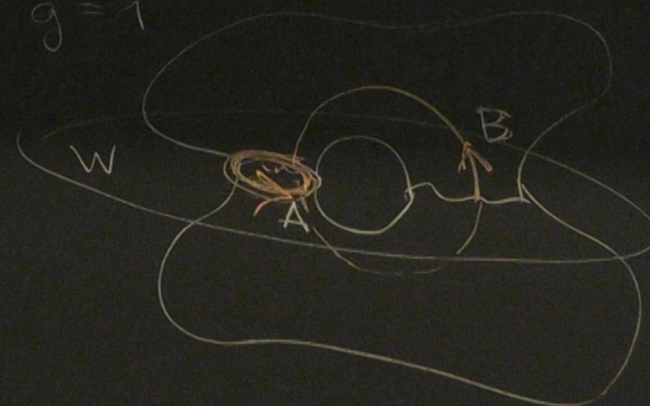
SU(2) $\quad \phi = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix}$

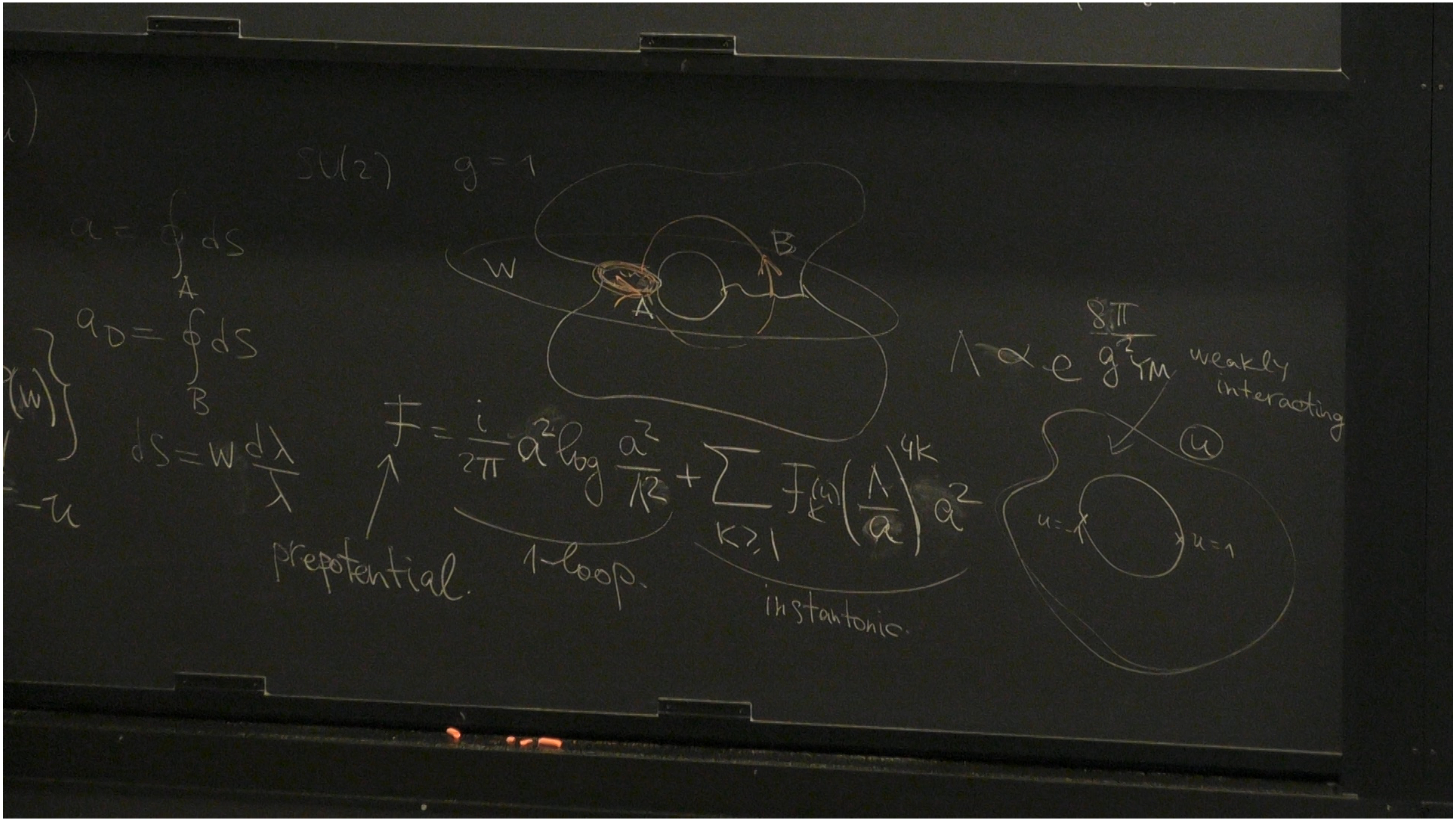
$$\frac{\partial \mathcal{F}(a)}{\partial a} = a_D \qquad a = \oint_A dS$$

$$C = \left\{ \lambda + \frac{\Lambda}{\lambda} = \underbrace{P(w)}_{w^2 - u} \right\} \qquad a_D = \oint_B dS$$

$$\frac{\partial \mathcal{F}}{\partial u} = a_D \frac{\partial a}{\partial u} \qquad dS = w \frac{d\lambda}{\lambda}$$

SU(2) $\quad g = 1$





SU(2) g=1
$a = \oint_A dS$
$a_D = \oint_B dS$
$dS = W \frac{d\lambda}{\lambda}$
W
A
B
$\mathcal{F} = \frac{i}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} + \sum_{k \geq 1} \mathcal{F}_k^{(u)} \left(\frac{\Lambda}{a}\right)^{4k} a^2$
prepotential.
1-loop.
instantonic.
$\Lambda \propto e^{-\frac{8\pi}{g^2_{YM}}}$
weakly interacting
u
u=-1
u=1

Integrable systems

$\mathbb{R}^{2d}$ $\quad \{q_i ; p_j\} = \delta_{ij}$

$H_\alpha = H_\alpha(p, q)$

d Hamiltonians , $\{H_\alpha ; H_\beta\} = 0$

$\Downarrow$

Liouville integrability.

Toda chain $L_i(z) = \begin{pmatrix} p_i + z & e^{q_i} \\ e^{-q_i} & 0 \end{pmatrix}$

$N = 2$ $q_2 = -q_1 = q$
$p_2 = -p_1 = p$
$\mathrm{tr}\left[L_1(z)L_2(z)\begin{pmatrix}1&0\\0&\Lambda\end{pmatrix}\right] = z^2 - H$

$\{L(z_1) \overset{\otimes}{,} L(z_2)\} = [r(z_1 - z_2); L(z_1) \otimes L(z_2)]$ - classical "RLL" equation.

- $T(z) = L_1(z)L_2(z)\ldots L_N(z)$ - also same property holds.
- $\{\mathrm{tr}T(z_1); \mathrm{tr}T(z_2)\} = 0$ $\quad \mathrm{tr}T(z) = \sum_a z^a t_a$

$\quad \parallel$

$\sum_{a,b} z_1^a z_2^b \{t_a, t_b\}$

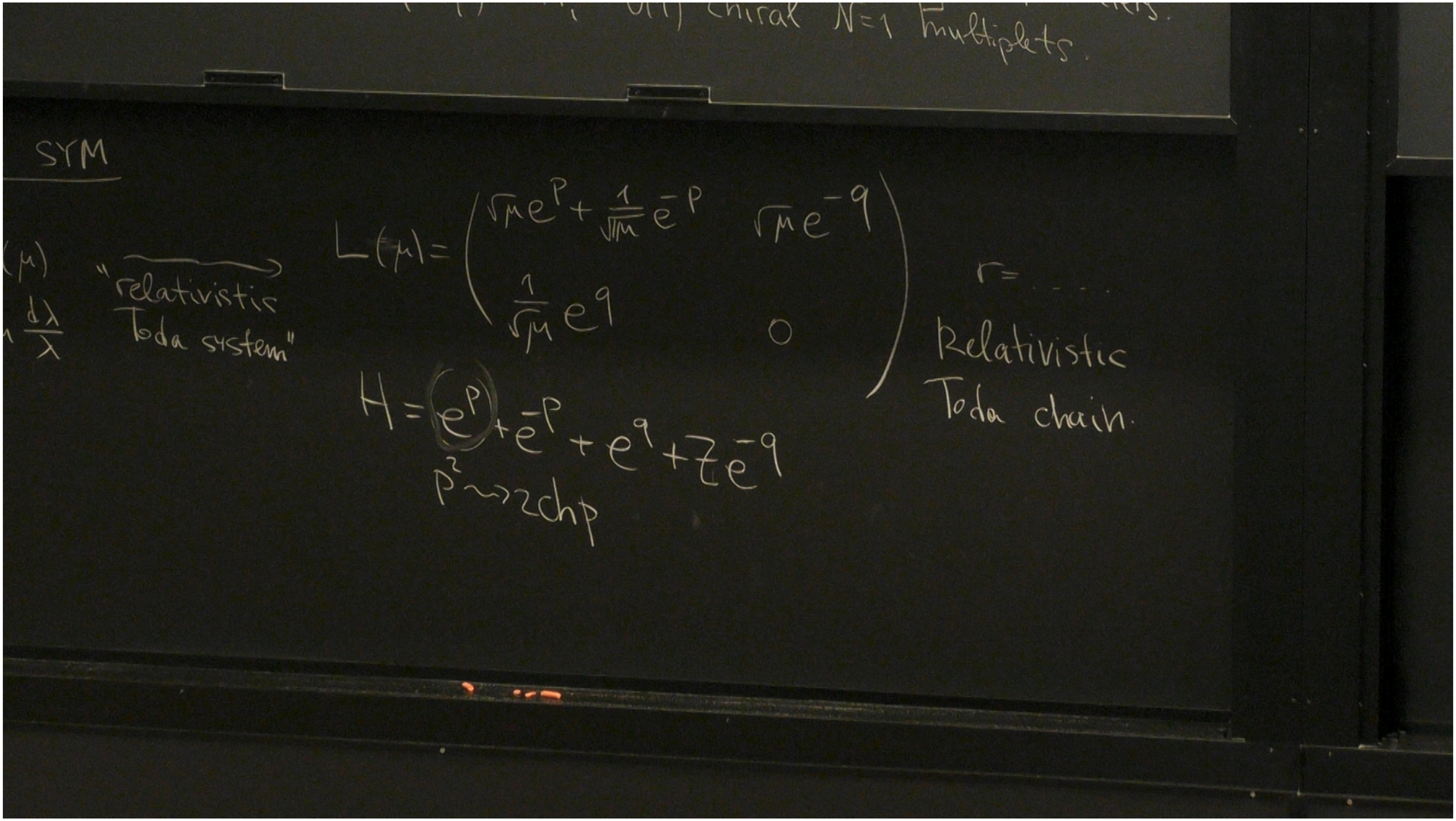
$H = p^2 + e^q + \Lambda e^{-q}$ Toda system.

$$\det\left(T(z)-\lambda\right)=0$$

$$\lambda^2-\lambda\,\underbrace{\mathrm{Tr}\left(T(z)\right)}_{P(z)}+\underbrace{\det T(z)}_{\Lambda}=0$$

$$\lambda+\frac{\Lambda}{\lambda}=P(z)$$

$$a_i=\oint_{A_i} w\,\frac{d\lambda}{\lambda}$$ – action variables of integrable system.

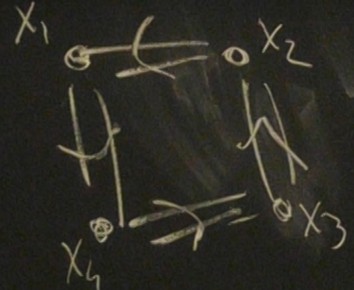

U(1) chiral N=1 multiplets.
SYM
"relativistic Toda system"
$L(\mu) = \begin{pmatrix} \sqrt{\mu}e^{p} + \frac{1}{\sqrt{\mu}}e^{-p} & \sqrt{\mu}e^{-q} \\ \frac{1}{\sqrt{\mu}}e^{q} & 0 \end{pmatrix}$
r= . . . .
Relativistic Toda chain.
$H = e^{p} + e^{-p} + e^{q} + Z e^{-q}$
$p^2 \rightsquigarrow 2\cosh p$

Cluster integrable systems

$\mathcal{X}$-cluster variety $\quad (\mathbb{C}^\times)^d$-charts

$$\{x_i, x_j\} = \varepsilon_{ij}\, x_i x_j \qquad \mathbb{C}^\times = \mathbb{C}\setminus\{0\}$$

$\varepsilon_{ij} \in \mathbb{Z}$



$$\{x_1, x_2\} = \underline{-2}\, x_1 x_2$$

integrable systems

variety $(\mathbb{C}^\times)^d$ - charts

$\{x_i, x_j\} = \varepsilon_{ij} x_i x_j \qquad \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$

$\mathbb{Z}$

$\{x_1, x_2\} = \underline{-2}\, x_1 x_2$

$(\mathbb{C}^\times)^d \xrightarrow{\mu_j} (\mathbb{C}^\times)^d \qquad \{x_i', x_j'\} = \varepsilon_{ij}' x_i' x_j'$

$$\begin{cases} x_j \to \dfrac{1}{x_j} = x_j' \\ x_i \to x_i \left(1 + x_j^{\operatorname{sgn} \varepsilon_{ij}}\right)^{\varepsilon_{ij}} = x_i' \quad i \neq j \end{cases}$$

$$\varepsilon_{ik}' = \varepsilon_{ik} + \frac{\varepsilon_{ij} |\varepsilon_{jk}| + \varepsilon_{jk} |\varepsilon_{ij}|}{2}$$

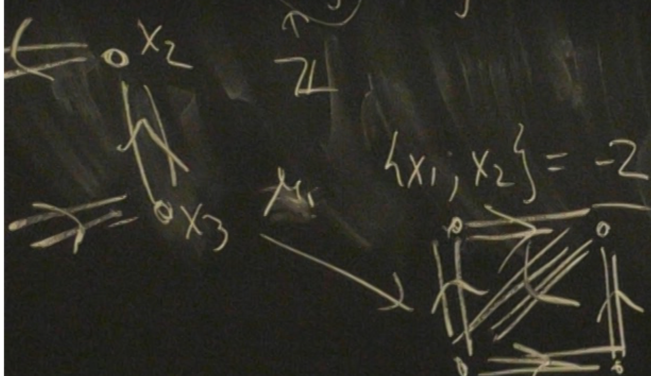
...ster integrable systems

...uster variety $(\mathbb{C}^\times)^d$ - charts.

$\{x_i, x_j\} = \varepsilon_{ij} x_i x_j$ $\qquad \mathbb{C}^\times = \mathbb{C}\setminus\{0\}$

$\uparrow$

$\mathbb{Z}$



$\{x_1, x_2\} = -2\, x_1 x_2$

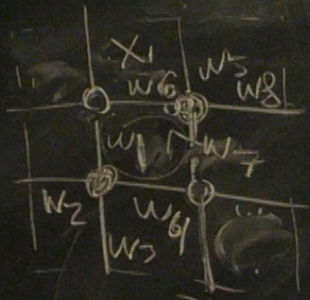
$(\mathbb{C}^\times)^d \xrightarrow{\mu_j} (\mathbb{C}^\times)^d \qquad \{x_i', x_j'\} = \varepsilon_{ij}' x_i' x_j'$

$$\begin{cases} x_j \to \dfrac{1}{x_j} = x_j' \\ x_i \to x_i\left(1 + x_j^{\operatorname{sgn}\varepsilon_{ij}}\right)^{\varepsilon_{ij}} = x_i' \quad i \neq j \end{cases}$$

$$\varepsilon_{ik}' = \varepsilon_{ik} + \frac{\varepsilon_{ij}|\varepsilon_{jk}| + \varepsilon_{ik}|\varepsilon_{ij}|}{2}$$

Cluster integrable systems

$H_\alpha = H_\alpha(x)$



$\mathbb{T}^2$

$$K = \begin{pmatrix} w_1 + w_3 \mu^{-1} & -w_6 - w_8 \lambda \\ w_4 + w_2 \lambda^{-1} & w_7 + w_5 \mu \end{pmatrix}$$

$$X_1 = \frac{w_2 w_8}{w_3 w_5}$$

$\det K ($

$P(\lambda, \mu)$

discrete $U(1)$ connection

discrete Dirac-Kasteleyn operator

$$-w_6 - w_8\lambda$$

$$w_7 + w_5\mu$$

$$\frac{w_8}{3w_5}$$

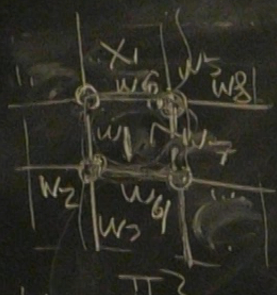
$\det K(\lambda,\mu) = 0$ – spectral curve of integrable sys.

$$P(\lambda,\mu) = \sum \lambda^i \mu^j H_{ij}(x)$$

$$\{H_{ij}(x), H_{k\ell}(x)\} = 0$$

$$H = \sqrt{x_1x_4} + \frac{1}{\sqrt{x_1x_4}} + \sqrt{x_1x_4}\,x_3 + \sqrt{\frac{x_1}{x_4}}$$

$$e^p \qquad e^{-p} \qquad e^q \qquad z e^{-q}$$



$$K = \begin{pmatrix} w_1 + w_3\mu^{-1} & -w_6 - w_8\lambda \\ w_4 + w_2\lambda^{-1} & w_7 + w_5\mu \end{pmatrix}$$

$\det K(\lambda, \mu) =$

$P(\lambda, \mu) = \sum \lambda^i$

discrete $U(1)$ connection

discrete Dirac-Kasteleyn operator

$$x_1 = \frac{w_2 w_8}{w_3 w_5}$$

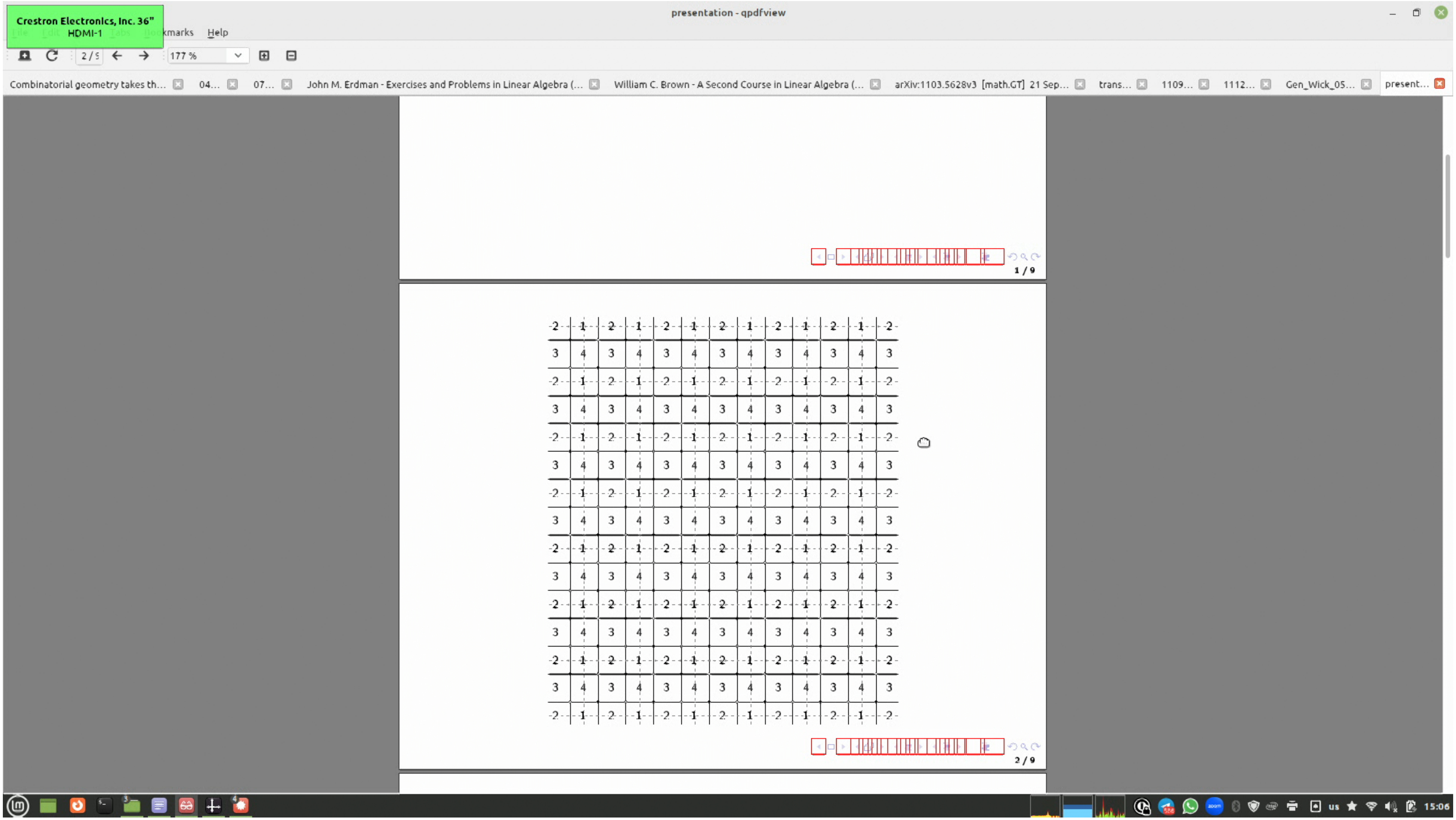
$$H = \sqrt{x_1 x_4} + \frac{1}{\sqrt{x_1 x_4}} + \sqrt{x_1 x_4}\, x$$

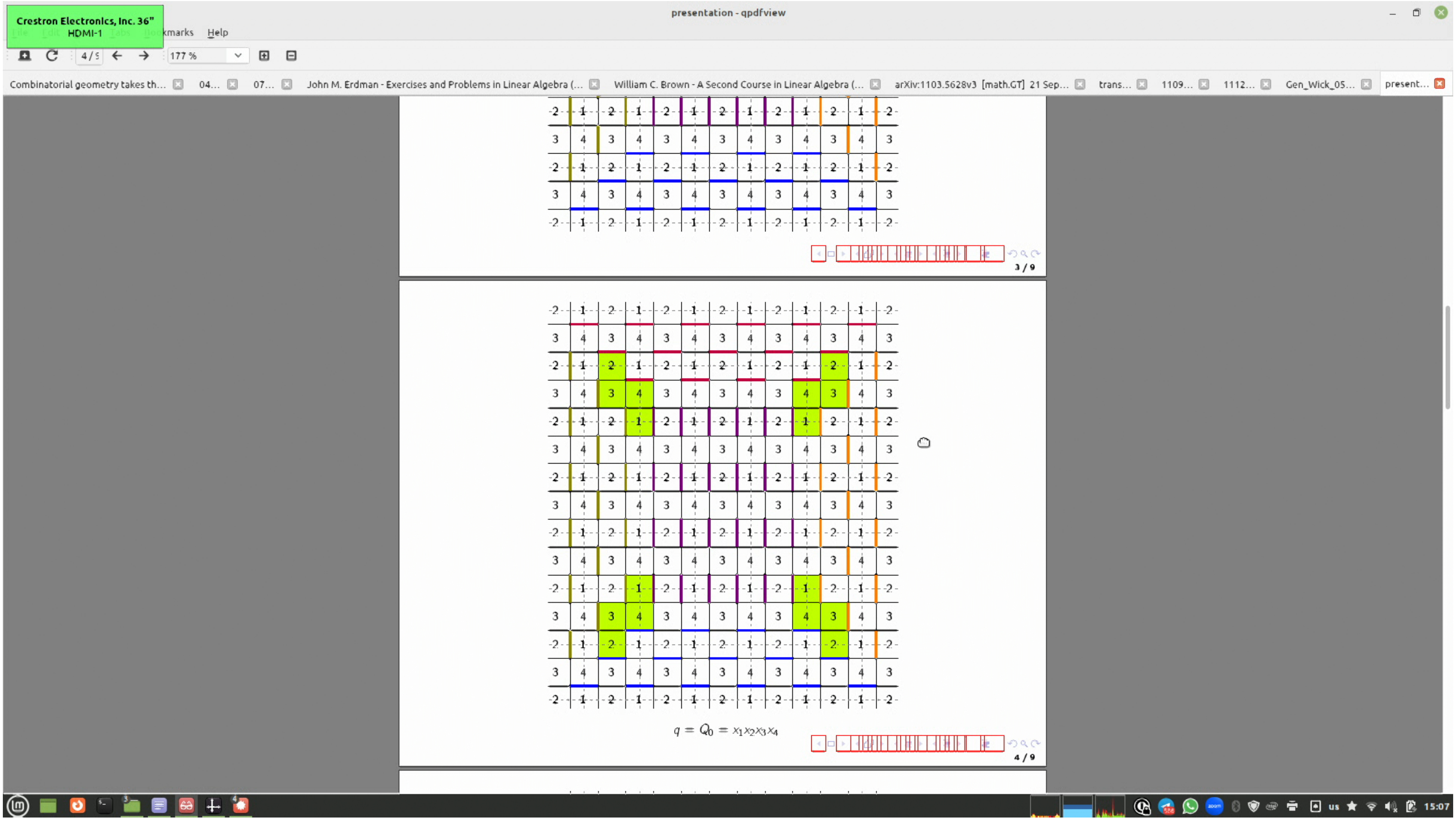
$\{ H_i(x),$

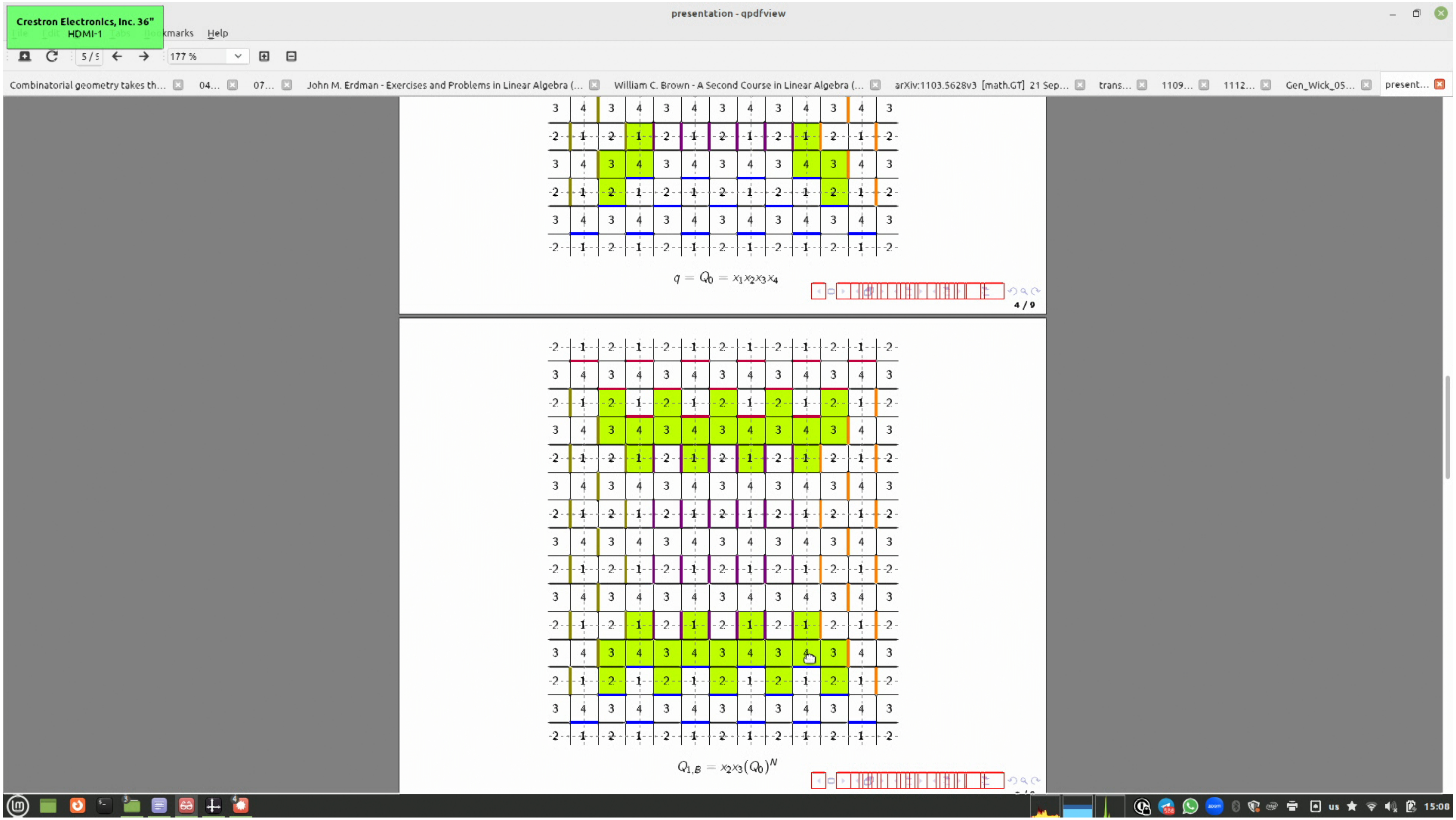
$$x_1 x_2 x_3 x_4 \neq 1 = q$$

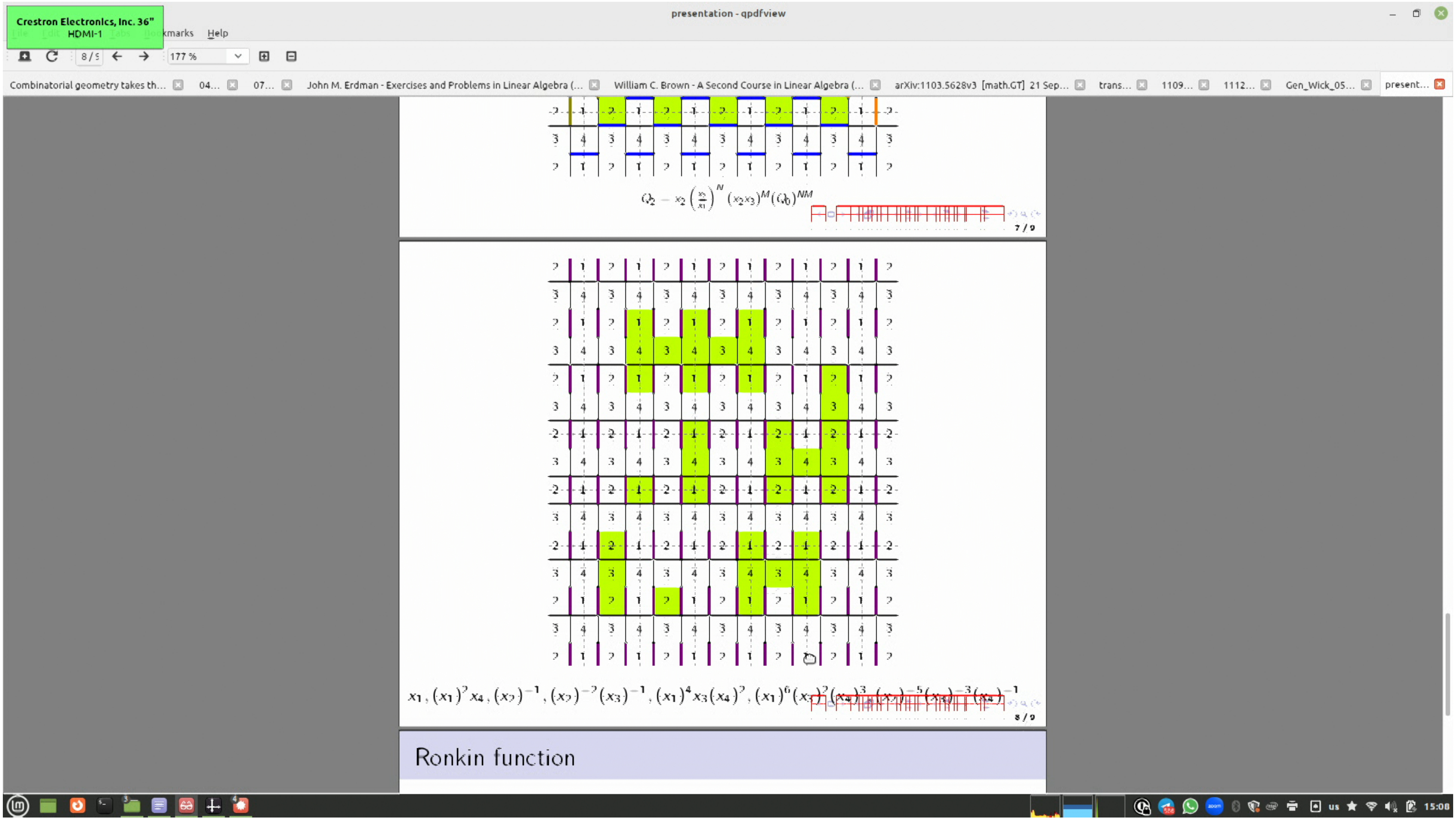
$e^p \quad e^{-p} \quad e^q$

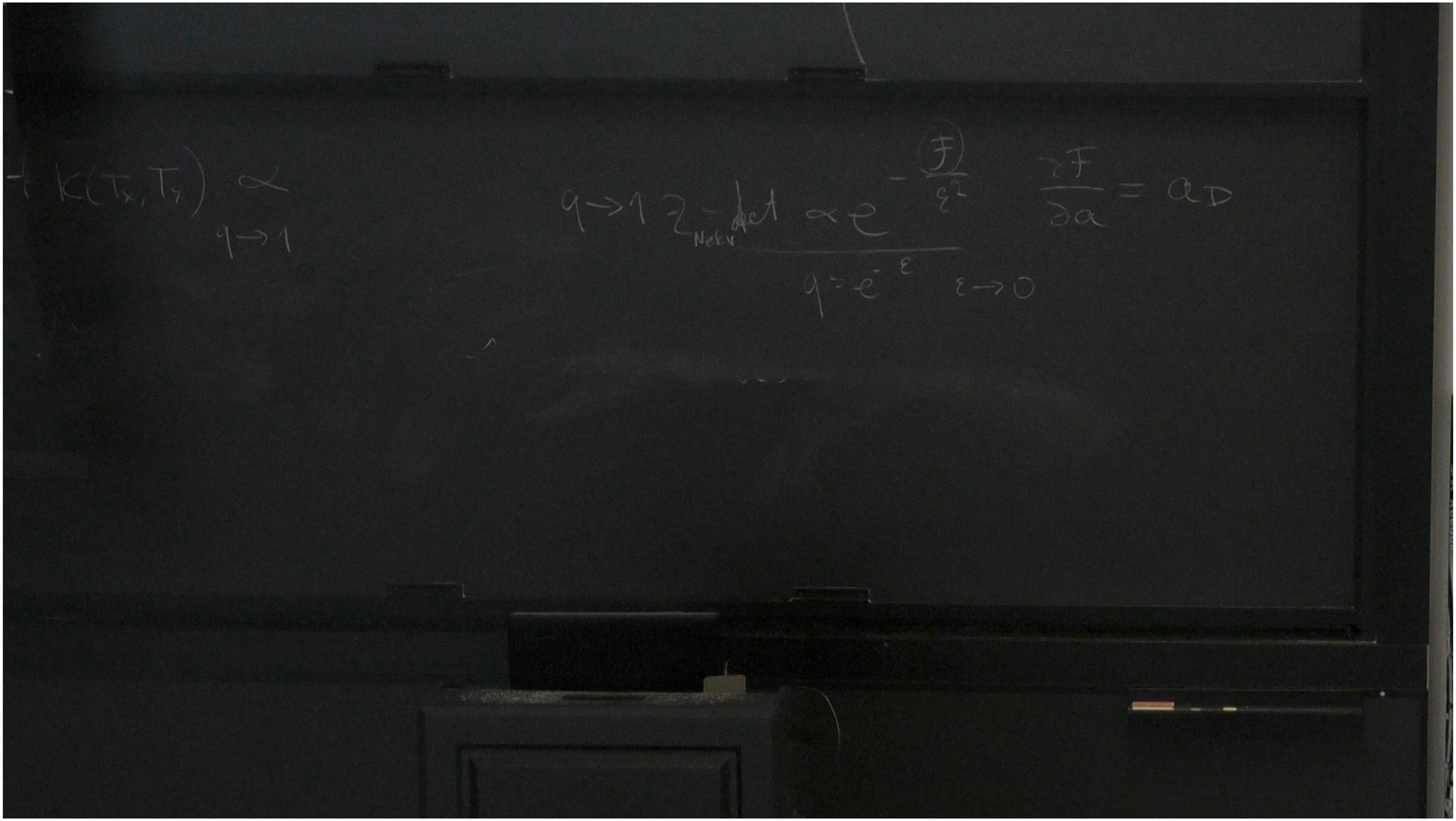
$$\lambda, \mu \longrightarrow T_x, T_y \qquad T_x T_y = q T_y T_x$$


Crestron Electronics, Inc. 36"
HDMI-1
presentation - qpdfview
Bookmarks Help
177 %
Combinatorial geometry takes th...
04...
07...
John M. Erdman - Exercises and Problems in Linear Algebra (...
William C. Brown - A Second Course in Linear Algebra (...
arXiv:1103.5628v3 [math.GT] 21 Sep...
trans...
1109...
1112...
Gen_Wick_05...
present...
1 / 9
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
3 4 3 4 3 4 3 4 3 4 3 4 3
2 1 2 1 2 1 2 1 2 1 2 1 2
2 / 9
15:06


Crestron Electronics, Inc. 36"
HDMI-1
presentation - qpdfview
Help
177 %
Combinatorial geometry takes th...
John M. Erdman - Exercises and Problems in Linear Algebra (...
William C. Brown - A Second Course in Linear Algebra (...
arXiv:1103.5628v3 [math.GT] 21 Sep...
trans...
Gen_Wick_05...
present...
3 / 9
$q = Q_0 = x_1x_2x_3x_4$
4 / 9
15:07


Crestron Electronics, Inc. 36"
HDMI-1
presentation - qpdfview
Combinatorial geometry takes th...
John M. Erdman - Exercises and Problems in Linear Algebra (...
William C. Brown - A Second Course in Linear Algebra (...
arXiv:1103.5628v3 [math.GT] 21 Sep...
$q = Q_0 = x_1 x_2 x_3 x_4$
4 / 9
$Q_{1,B} = x_2 x_3 (Q_0)^N$
15:08


Crestron Electronics, Inc. 36"
HDMI-1
presentation - qpdfview
Help
177 %
Combinatorial geometry takes th...
John M. Erdman - Exercises and Problems in Linear Algebra (...
William C. Brown - A Second Course in Linear Algebra (...
arXiv:1103.5628v3 [math.GT] 21 Sep...
Gen_Wick_05...
present...
7 / 9
8 / 9
Ronkin function
15:08


K(Tx, Ty) ∝
q → 1
q → 1 Z_Nekr = det ∝ e^(−F/ε²)
∂F/∂a = a_D
q = e^(−ε)   ε → 0