Title: Physical interpretation of non-normalizable quantum states and a new notion of equilibrium in pilot-wave theory

Speakers: Indrajit Sen

Series: Quantum Foundations

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Abstract: Non-normalizable quantum states are usually discarded as mathematical artefacts in quantum mechanics. However, such states naturally occur in quantum gravity as solutions to physical constraints. This suggests reconsidering the interpretation of such states. Some of the existing approaches to this question seek to redefine the inner product, but this arguably leads to further challenges.

In this talk, I will propose an alternative interpretation of non-normalizable states using pilot-wave theory. First, I will argue that the basic conceptual structure of the theory contains a straightforward interpretation of these states. Second, to better understand such states, I will discuss non-normalizable states of the quantum harmonic oscillator from a pilot-wave perspective. I will show that, contrary to intuitions from orthodox quantum mechanics, the non-normalizable eigenstates and their superpositions are bound states in the sense that the pilot-wave velocity field vy->0 at large ±y. Third, I will introduce a new notion of equilibrium, named pilot-wave equilibrium, and use it to define physically-meaningful equilibrium densities for such states. I will show, via an H-theorem, that an arbitrary initial density with compact support relaxes to pilot-wave equilibrium at a coarse-grained level, under assumptions similar to those for relaxation to quantum equilibrium. I will conclude by discussing the implications for pilot-wave theory, quantum gravity and quantum foundations in general.

Based on:

I. Sen. "Physical interpretation of non-normalizable harmonic oscillator states and relaxation to pilot-wave equilibrium" arXiv:2208.08945 (2022)

Zoom link:  https://pitp.zoom.us/j/93736627504?pwd=VGtxZE5rTFdnT1dqZlFRWTFvWlFQUT09

# Physical interpretation of non-normalizable quantum states and a new notion of equilibrium in pilot-wave theory

Indrajit Sen

Institute for Quantum Studies, Chapman University

Jan 11, 2023

# External motivation

Solutions to the Wheeler-deWitt equation are generically non-normalizable.

Example: Kodama State[1]

[1] H. Kodama, *Phys. Rev. D* **1990**, *42*, 2548, E. Witten, *arXiv gr-qc/0306083* **2003**.

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Traditional approach: Redefine the inner product[2]

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[2] S. Alexander et al., *arXiv:2212.07446* **2022**.

# External motivation

Solutions to the Wheeler-deWitt equation are generically non-normalizable.

Example: Kodama State

Traditional approach: Redefine the inner product

Challenges:
1. Closed-form expression difficult to obtain.
2. Interpretation of $\psi$ not clear.

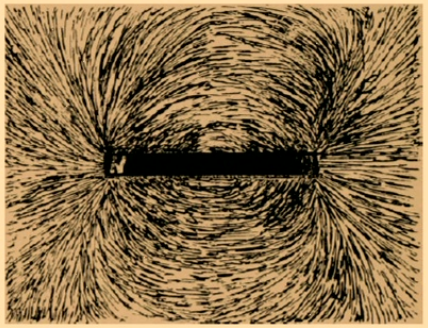
Can foundational thinking inform the discussion?

$\psi$-epistemic v/s $\psi$-ontic theories[3].

[3] L. Hardy, *Stud. Hist. Phil. Sci. B* **2004**, *35*, N. Harrigan, R. W. Spekkens, *Found. Phys.* **2010**, *40*, M. S. Leifer, *arXiv:1409.1570* **2014**, M. F. Pusey et al., *Nat. Phys.* **2012**, *8*, 475–478.

# Internal motivation

$$\hat{H}|\psi\rangle = i\hbar|\psi\rangle$$
$$\vec{v} = \vec{\nabla}S/m$$



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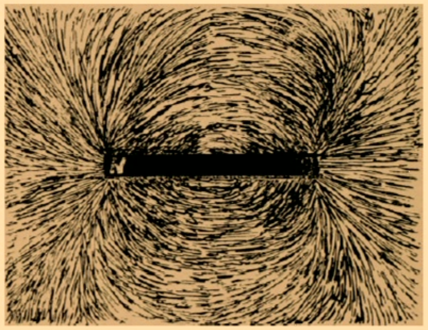

$$\left.\begin{array}{l} \hat{H}|\psi\rangle = i\hbar|\psi\rangle \\ \vec{v} = \vec{\nabla}S/m \end{array}\right\} \text{Laws of nature}$$

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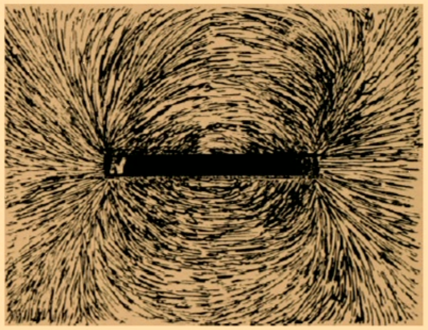

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$x(0) \longrightarrow$ initial condition

# Internal motivation

$$\left.\begin{aligned}\hat{H}|\psi\rangle &= i\hbar|\psi\rangle \\ m\vec{a} &= -\vec{\nabla}(V+Q)\end{aligned}\right\}\text{Laws of nature}$$

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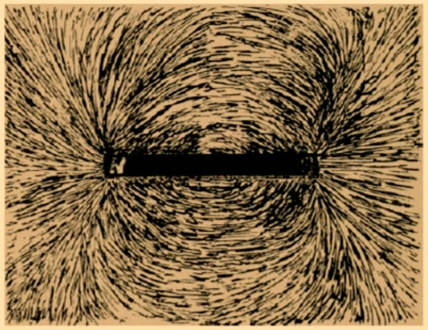

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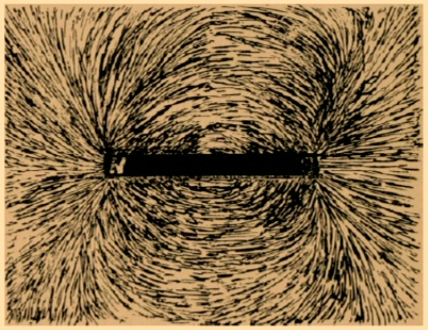

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**Pilot-wave theory**

$x(0)$ and $\psi(0)$ are logically independent.

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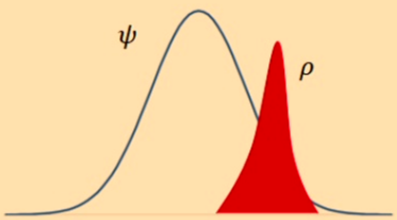
$\rho(x,0)$ and $\psi(0)$ are logically independent.




# Internal motivation

$$\left.\begin{array}{l}\hat{H}|\psi\rangle = i\hbar|\psi\rangle \\ \vec{v} = \vec{\nabla}S/m + \vec{v}_f\end{array}\right\} \text{Laws of nature}$$

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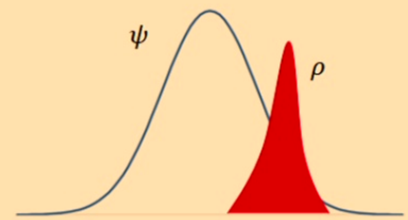
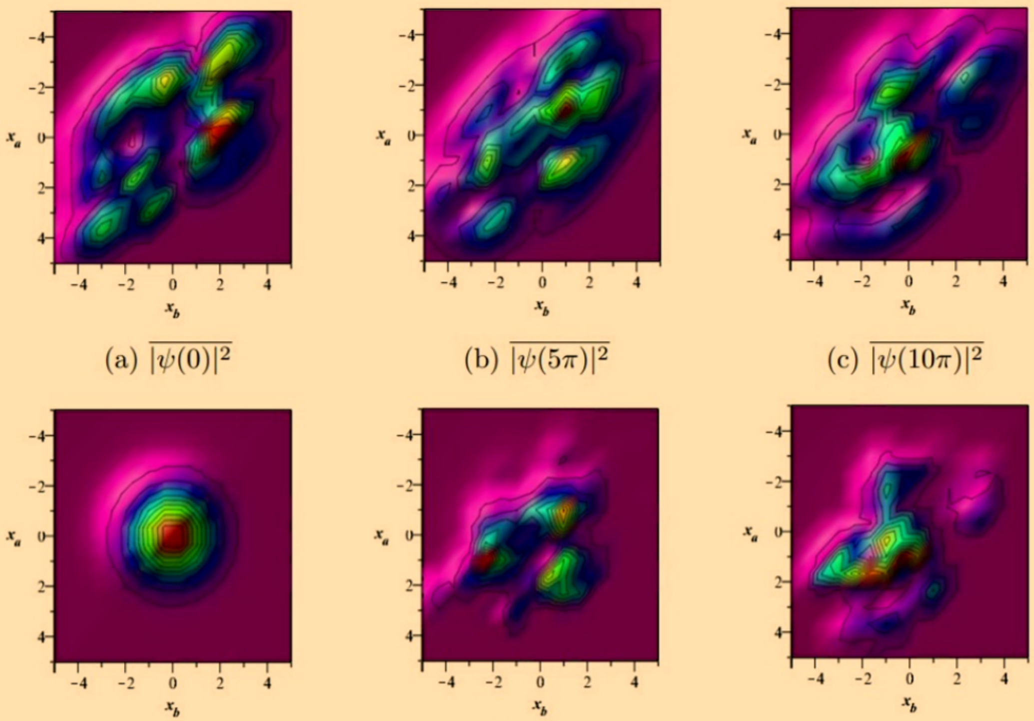
**Pilot-wave theory**

$x(0)$ and $\psi(0)$ are logically independent.

$\rho(x, 0)$ and $\psi(0)$ are logically independent[a].

[a] D. Bohm, *Phys. Rev.* **1953**, *89*, 458, D. Bohm, J.-P. Vigier, *Phys. Rev.* **1954**.



(a) $\overline{|\psi(0)|^2}$ (b) $\overline{|\psi(5\pi)|^2}$ (c) $\overline{|\psi(10\pi)|^2}$

(d) $\overline{\rho(0)}$ (e) $\overline{\rho(5\pi)}$ (f) $\overline{\rho(10\pi)}$

5

[5] F. B. Lustosa et al., *arXiv:2205.13701* **2022**.

**External motivation** ⟷ **Internal motivation** ✓[6]

Non-normalizable states that satisfy physical constraints but need interpretation

Basic conceptual structure allows non-normalizable states but unexplored

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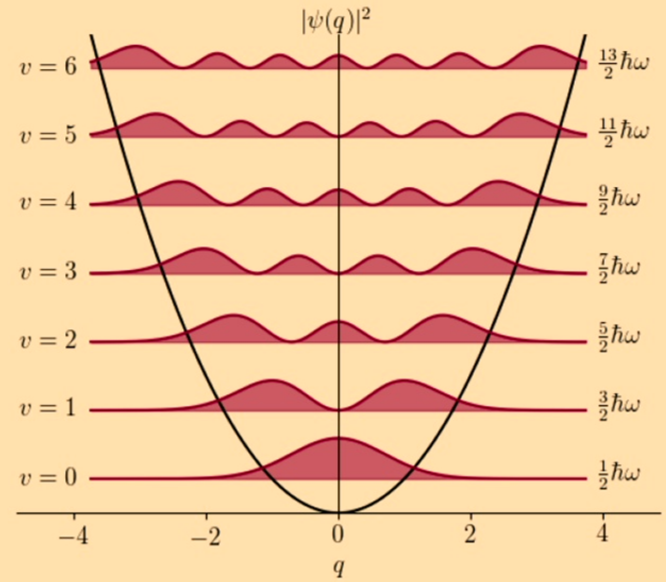
# Quantum Harmonic Oscillator

$$-\frac{d^2\psi}{dy^2} + y^2\psi = K\psi,\ y \equiv \sqrt{m\omega/\hbar}x \text{ and } K \equiv 2E/\hbar\omega.$$

Use ansatz $\psi(y) = e^{-y^2/2}h^K(y)$



# Quantum Harmonic Oscillator

$$\boxed{\frac{d^2h^K}{dy^2} - 2y\frac{dh^K}{dy} + (K-1)h^K = 0}$$

$$\psi^K(y) = a_0\varphi_0^K(y) + a_1\varphi_1^K(y)$$

where $\varphi_0^K \equiv e^{-y^2/2}h_0^K(y)$ and $\varphi_1^K \equiv e^{-y^2/2}h_1^K(y)$.

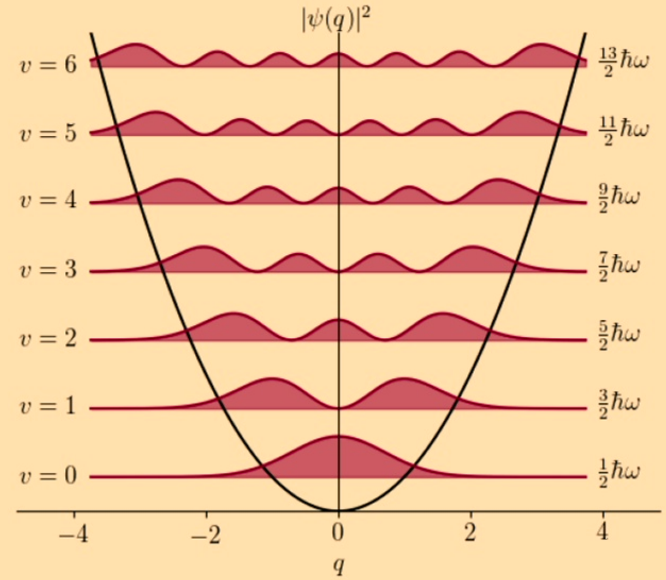
$$h_0^K(y) = M(\frac{1}{4}(1-K), \frac{1}{2}, y^2)$$

$$h_1^K(y) = yM(\frac{1}{4}(3-K), \frac{3}{2}, y^2)$$

$M(c,d,y) \equiv \sum_{j=0}^{\infty} \frac{(c)_j}{(d)_j}\frac{y^j}{j!}$, $\quad (t)_j \equiv \Gamma(t+j)/\Gamma(t)$

Pochhammer symbol

Confluent hypergeometric function of the first kind



# Velocity field: eigenstates

$$\psi^K_{\theta,\phi}(y) = \cos\theta\varphi^K_0(y) + \sin\theta e^{i\phi}\varphi^K_1(y)$$

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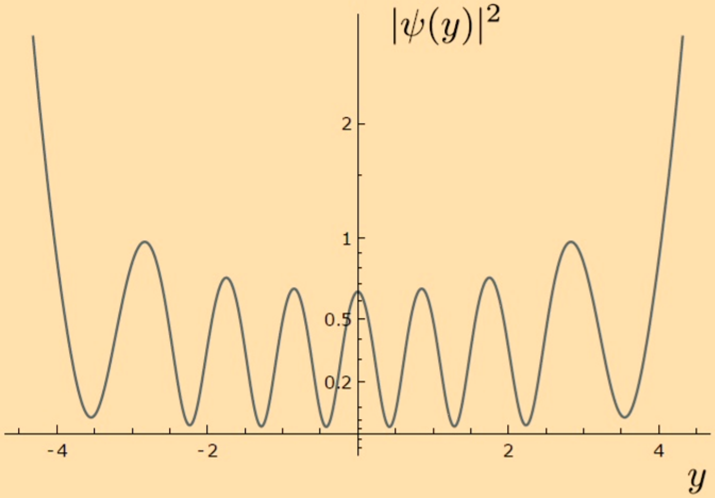


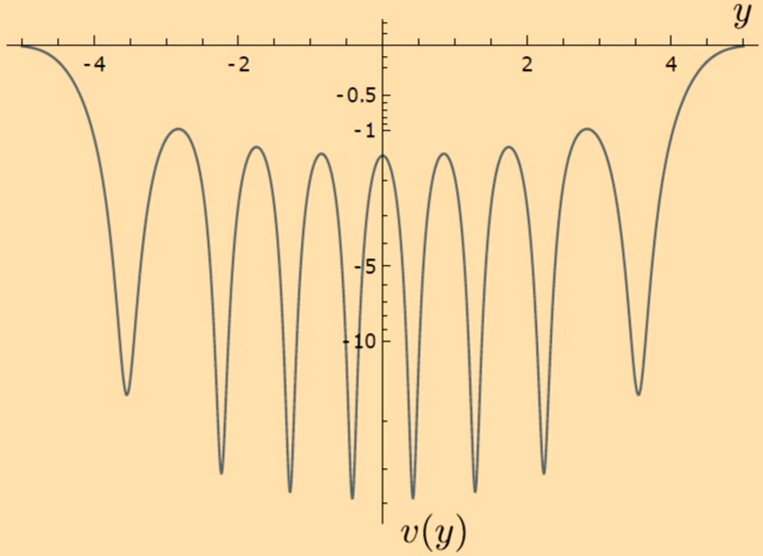


$K = 14$, $\theta = 3.2\pi$ and $\phi = 1.5\pi$

*a)* *b)*

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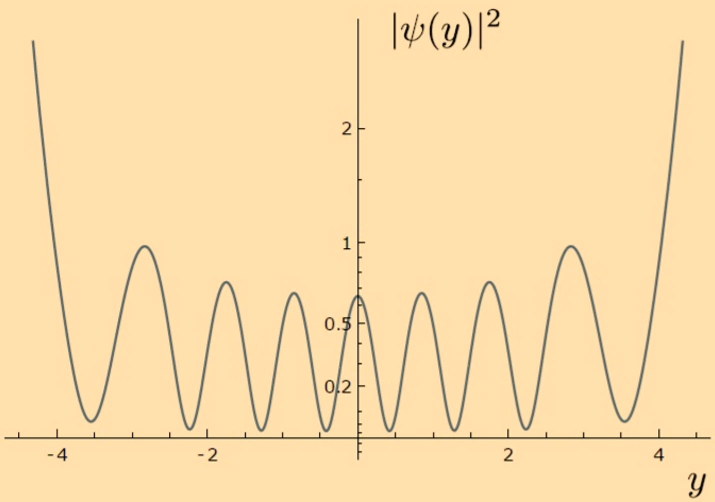


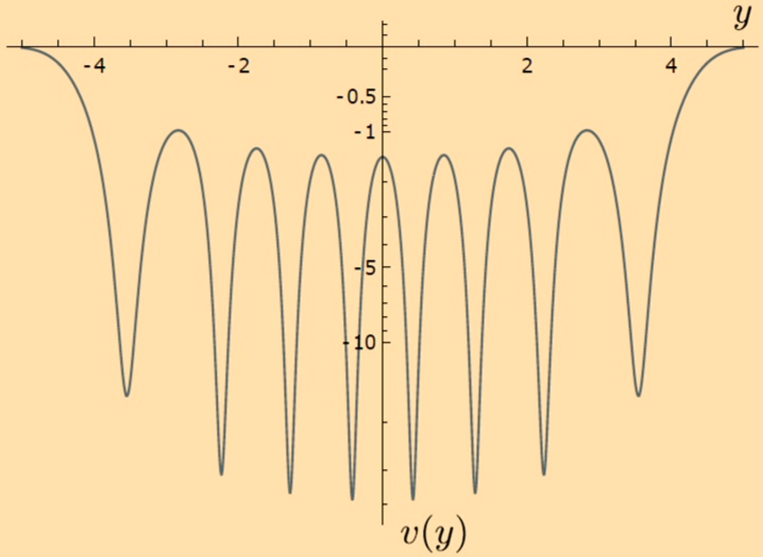


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*a)* *b)*

# Velocity field: superpositions

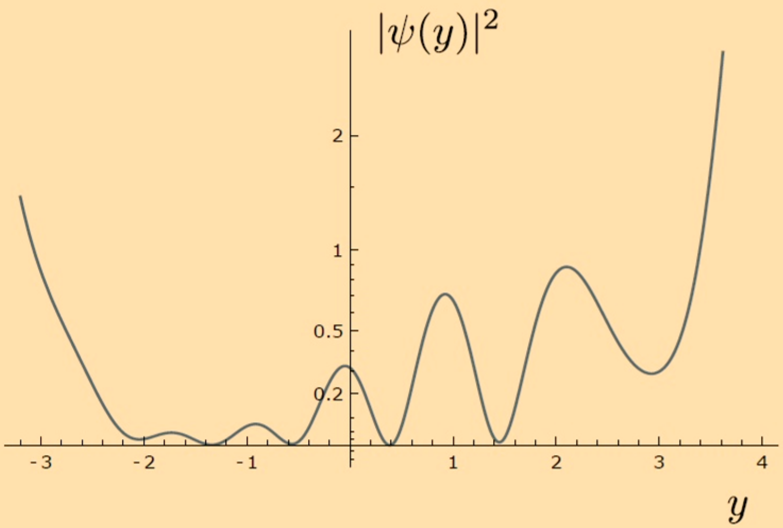
Asymptotic approximation ($|y| \to \infty$)

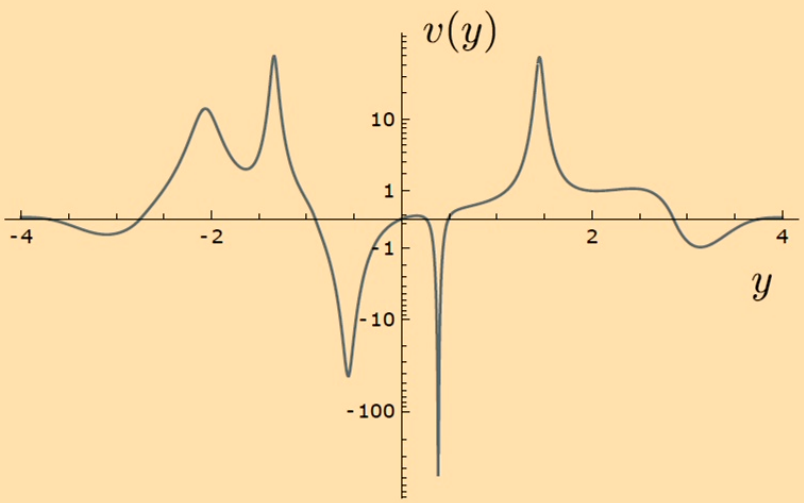
$$\psi^{K,m}(y) = \cos\theta_m^K \varphi_0(y) + \sin\theta_m^K e^{i\phi_m^K}\varphi_1(y)$$

where $\varphi_0(y) = y^2 e^{y^2/2}$, $\varphi_1(y) = y e^{y^2/2}$

Asymptotic approximation

$$\lim_{|y|\to\infty} v(y,t) = \frac{j(y,t)}{|\psi(y,t)|^2} \sim \frac{y^2 e^{y^2}}{y^4 e^{y^2}} = \frac{1}{y^2}$$




$$\psi(y) = 1/\sqrt{6}\psi^{15.2}_{\pi/3,\pi/4}(y) + \sqrt{2/3}e^{i\pi/5}\psi^{5.8}_{\pi/2,\pi}(y) + 1/\sqrt{6}e^{i\pi/8}\psi^{10.2}_{\pi/7,\pi/5}(y)$$

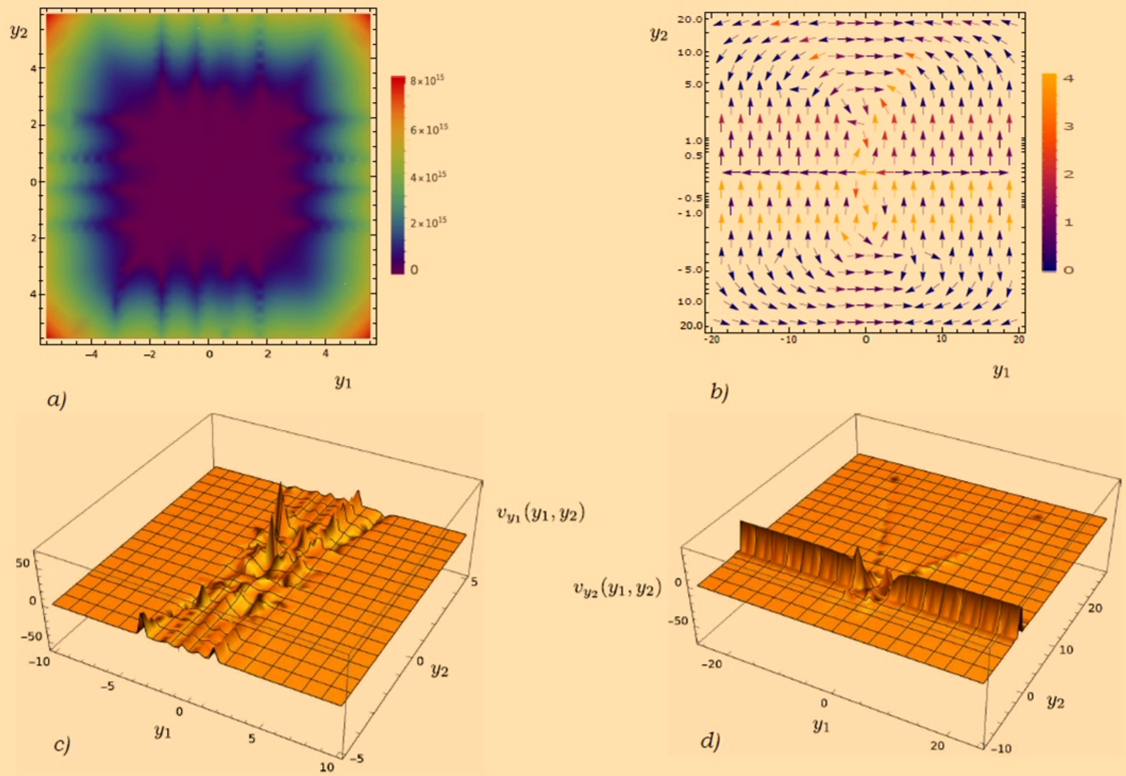
a) b)

# Velocity field: multiple dimensions

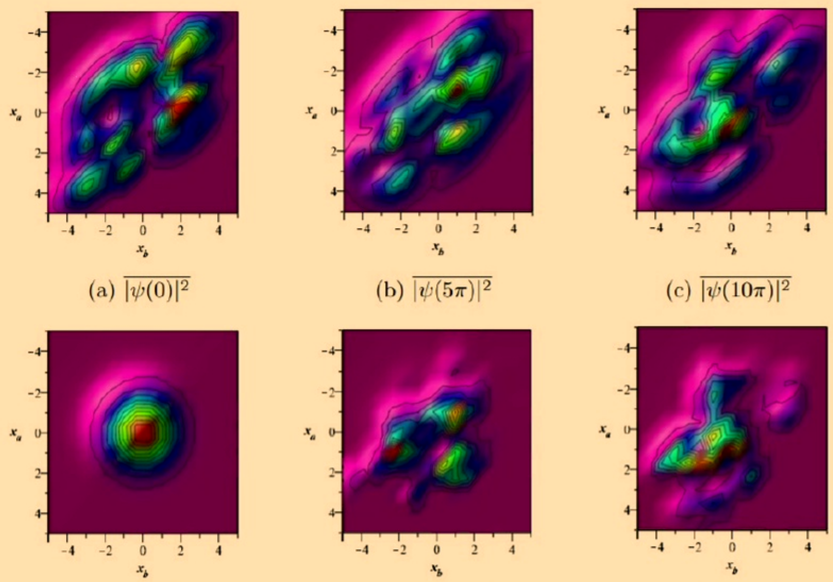
$$\psi(\vec{y},t) = \sum_{j=1}^{n} c_j(t) \prod_{g=1}^{N} \psi^{K_j^g}(y_g)$$

$$v_r(\vec{y},t) = \frac{j_r(\vec{y},t)}{|\psi(\vec{y},t)|^2} \sim \frac{1}{y_r^2} \text{ at large } |y_r|$$

$$\psi(y_1,y_2) =$$
$$\sqrt{2}/3\psi^{1.4}_{3\pi/4,4\pi/3}(y_1)\psi^{8}_{2.2\pi,4.1\pi}(y_2) +$$
$$\sqrt{2}/3e^{i\pi/5}\psi^{5}_{8\pi/5,5.8\pi}(y_1)\psi^{15.6}_{2\pi/5,9\pi/16}(y_2) +$$
$$1/3e^{i\pi/8}\psi^{9}_{\pi/5,\pi/7}(y_1)\psi^{0.75}_{\pi/6,\pi/9}(y_2) +$$
$$2/3e^{i\pi/9}\psi^{11.4}_{5\pi/3,6\pi/7}(y_1)\psi^{12.6}_{2\pi/5,7\pi/16}(y_2)$$



# Equilibrium density?

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Normalizable case: $\overline{\rho} \longrightarrow \overline{|\psi|^2}$

# Pilot-wave equilibrium

$$H_q \equiv \int_{\mathcal{C}} \rho(\vec{y}) \ln \frac{\rho(\vec{y})}{|\psi(\vec{y})|^2} d\vec{y}$$

not a well-defined relative entropy

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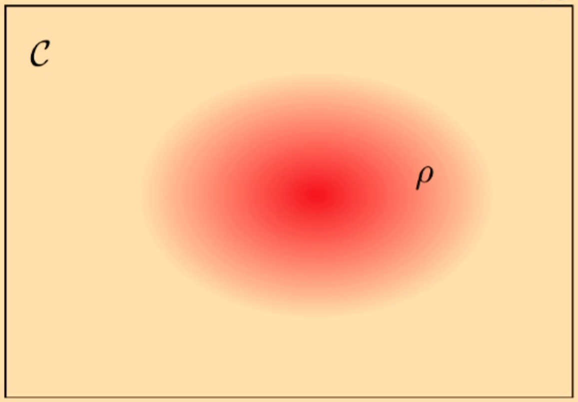


$$\boxed{H_{pw} \equiv \int_{\mathcal{C}} \rho(\vec{y}) \ln \frac{\rho(\vec{y})}{\rho_{pw}(\vec{y})} d\vec{y}}$$

where,

$$\rho_{pw}(\vec{y}) \equiv \begin{cases} |\psi(\vec{y})|^2/\mathcal{N} & , \text{ for } \vec{y} \in \Omega \\ 0 & , \text{ for } \vec{y} \in \mathcal{C} \setminus \Omega \end{cases}$$

and $\mathcal{N} \equiv \int_\Omega |\psi(\vec{y})|^2 d\vec{y}$.

compact support of $\rho$ on $\mathcal{C}$

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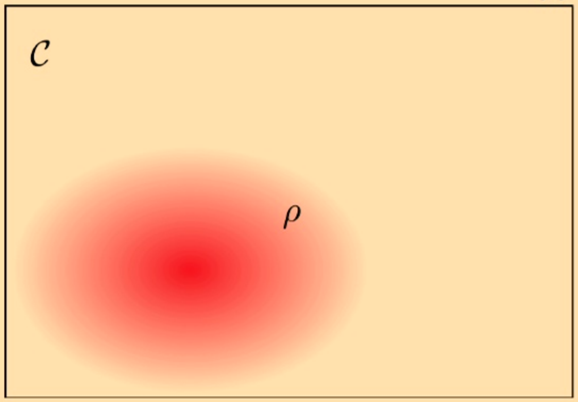


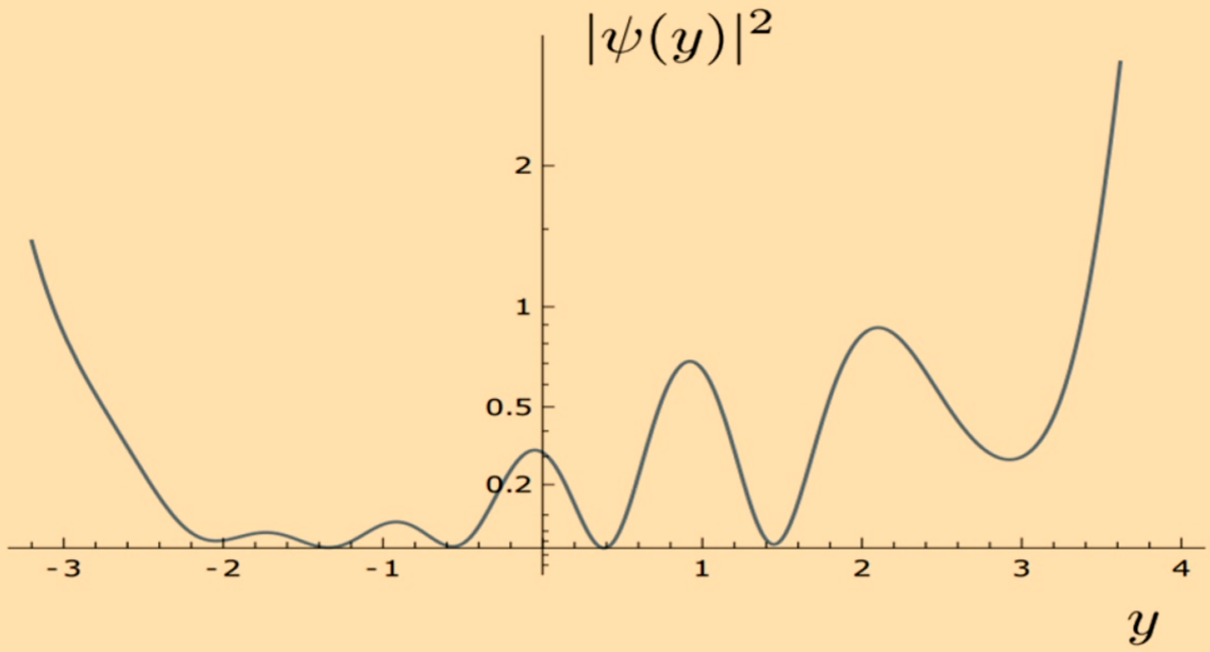

$$\boxed{H_{pw}(t) \equiv \int_{\mathcal{C}} \rho(\vec{y}, t) \ln \frac{\rho(\vec{y}, t)}{\rho_{pw}(\vec{y}, t)} d\vec{y}}$$

where,

$$\rho_{pw}(\vec{y}, t) = \begin{cases} |\psi(\vec{y}, t)|^2 / \mathcal{N}(t) & , \text{for } \vec{y} \in \Omega_t \\ 0 & , \text{for } \vec{y} \in \mathcal{C} \setminus \Omega_t \end{cases}$$

and $\mathcal{N}(t) = \int_{\Omega_t} |\psi(\vec{y}, t)|^2 d\vec{y}$.

compact time-dependent support of $\rho$ on $\mathcal{C}$

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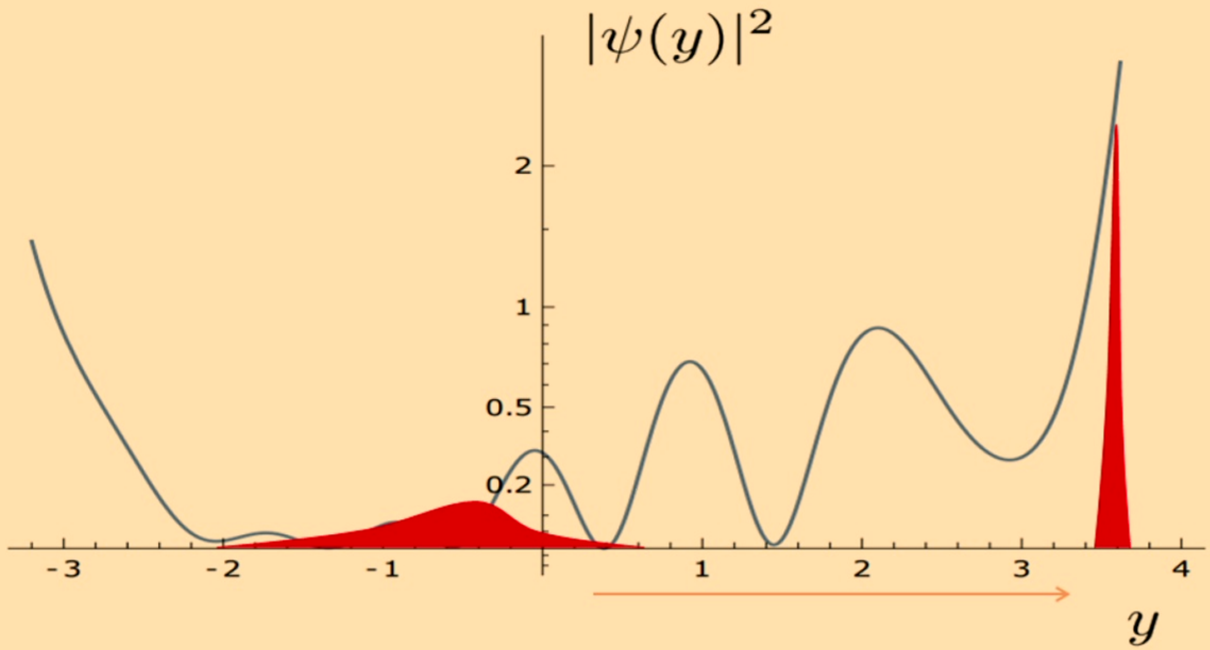


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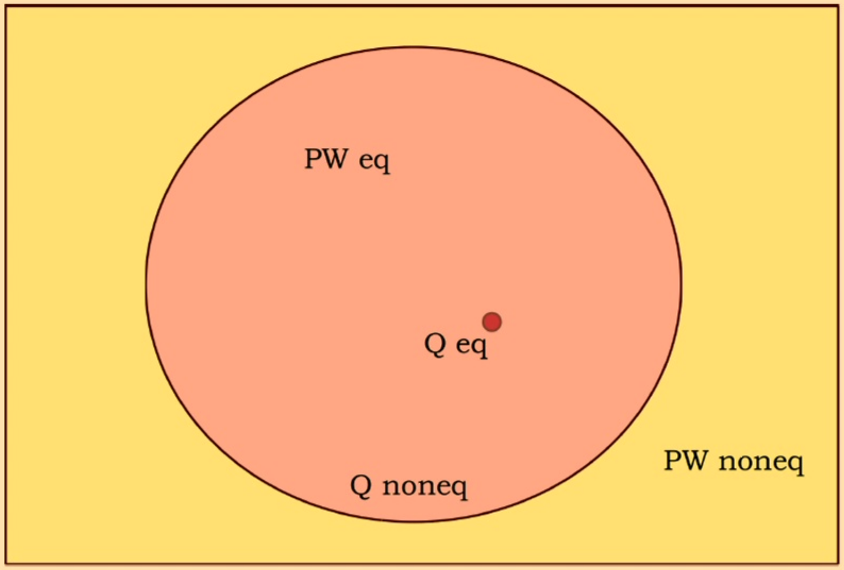


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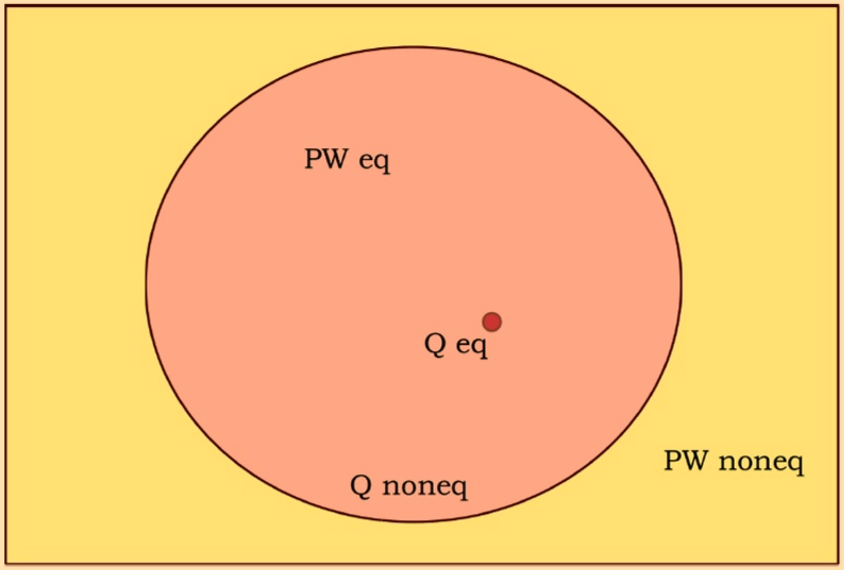


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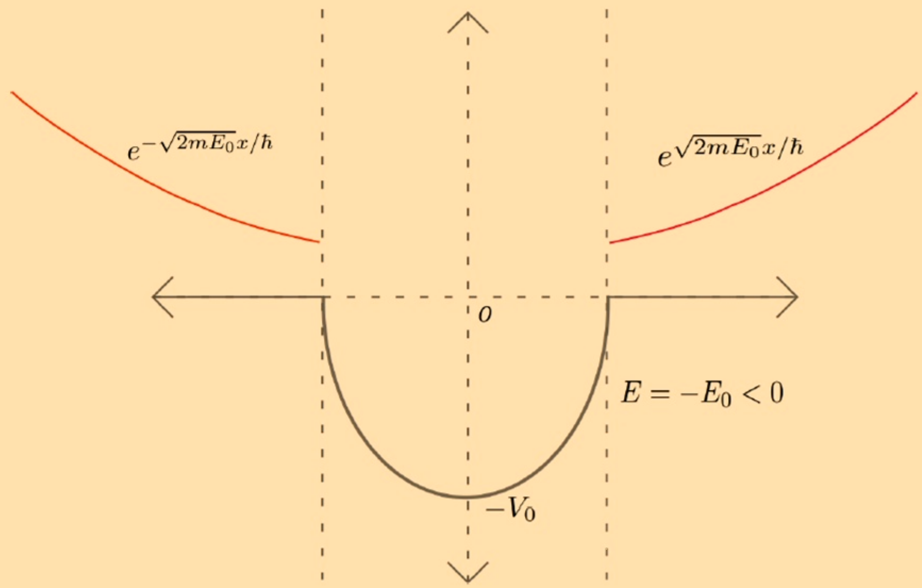

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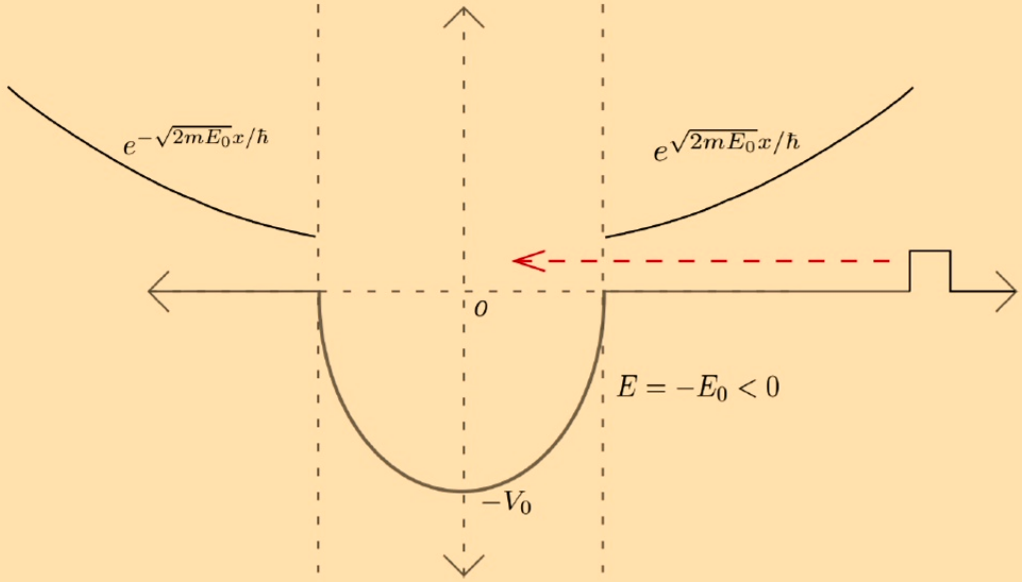
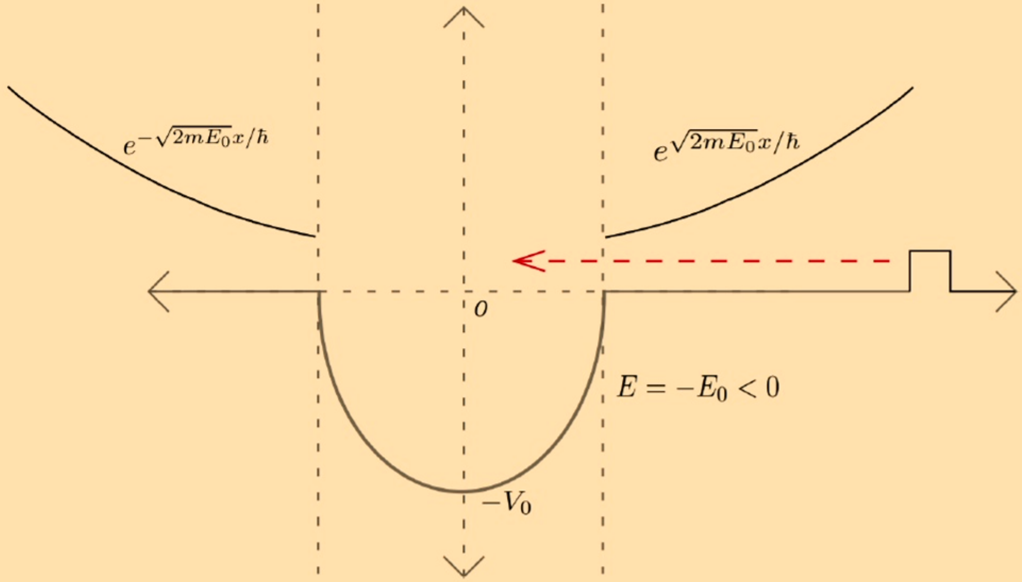



$$\psi(x,t) = \int_{-\infty}^{\infty} \psi(x',0)\big(K(x,t;x',0) + \Delta K(x,t;x',0)\big)\,dx'$$

# Implications: Non-relativistic quantum mechanics

Why don't we observe these states in the lab?
Answer: Pilot-wave theory predicts such states will be unstable.

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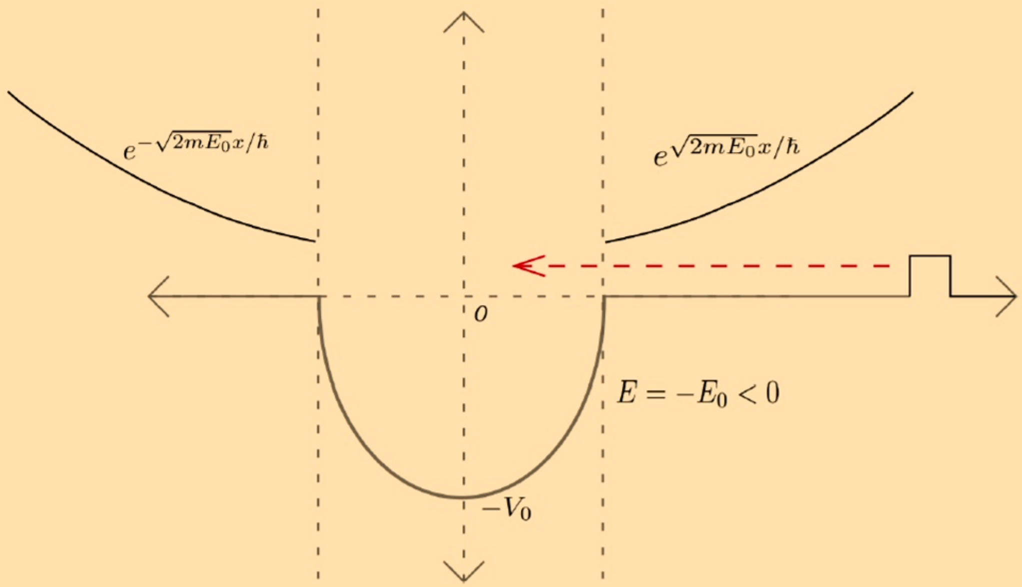


Emergence of the appearance of quantization.
Experimental prediction? $\psi(x) \sim 1/\sqrt{x}$

# Implications: Quantum field theory

Non-normalizable states in Fourier space.

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Non-normalizable states in Fourier space.

Example: Scalar field on a flat expanding space-time

$$\sum_{\vec{k},r} \left(\frac{1}{2a^3}\pi^2_{\vec{k},r} + \frac{ak^2}{2} q^2_{\vec{k},r}\right)\psi = i\frac{\partial\psi}{\partial t}$$

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need not have a Fourier transform

has a Fourier transform

$$\phi(\vec{k},t)\equiv\frac{1}{(2\pi)^{3/2}}\int\phi(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}d\vec{x}=\frac{\sqrt{V}}{(2\pi)^{3/2}}\left(q_{\vec{k},1}(t)+iq_{\vec{k},2}(t)\right)$$

# Implications: Quantum field theory

Non-normalizable states in Fourier space.

Example: Scalar field on a flat expanding space-time$\longleftarrow$ non-unitarity

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need not have a Fourier transform

has a Fourier transform

$$\phi(\vec{k},t)\equiv\frac{1}{(2\pi)^{3/2}}\int\phi(\vec{x},t)e^{-i\vec{k}\cdot\vec{x}}d\vec{x}=\frac{\sqrt{V}}{(2\pi)^{3/2}}\left(q_{\vec{k},1}(t)+iq_{\vec{k},2}(t)\right)$$

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# Implications: Quantum field theory and relativistic quantum mechanics

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Example: Electromagnetic field $\longleftrightarrow$ atom (Frank-Hertz experiment)
Possible experimental prediction.

Particle interpretation of Klein-Gordon equation.

# Implications: Quantum Gravity

May give physical interpretation to:

1. generic solutions to Wheeler-deWitt equation.
2. the Kodama state.
3. states in shape-dynamics formulation[9] of pilot-wave theory.

[9] D. Dürr et al., *J. Stat. Phys.* **2019**, 1–43.

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Rethink the normalizability condition in general $\psi$-ontic models.

*THANK YOU!*