

Title: Noether's theorems and gauge symmetries

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Collection: Symmetries Graduate School 2023

Date: January 26, 2023 - 9:00 AM

URL: <https://pirsa.org/23010083>

### \* Maxwell

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$\delta L = -\frac{1}{2} F^{\mu\nu} \delta F_{\mu\nu}$$

$$= -F^{\mu\nu} \partial_\mu \delta A_\nu$$

$$= \partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\mu (F^{\mu\nu} \delta A_\nu)$$

$$= E \delta \phi + \partial_\mu \sigma^\mu$$

$$\delta_\alpha A_\mu = \partial_\mu \alpha$$

$$\begin{aligned} \text{Here } E \delta \phi &= \partial_\mu F^{\mu\nu} \delta_\alpha A_\nu \\ &= \partial_\mu F^{\mu\nu} \partial_\nu \alpha \end{aligned}$$



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$$= \partial_\mu F^{\mu\nu} \delta A_\nu - \partial_\mu (F^{\mu\nu} \delta A_\nu)$$

$$= E \delta \phi + \partial_\mu \sigma^\mu$$

$$\delta_\alpha A_\mu = \partial_\mu \alpha$$

$$\text{Hence } E \delta \phi = \partial_\mu F^{\mu\nu} \delta_\alpha A_\nu$$

$$= \partial_\mu F^{\mu\nu} \partial_\nu \alpha$$

$$= -\partial_\nu \partial_\mu F^{\mu\nu} \alpha + \partial_\nu (\partial_\mu F^{\mu\nu} \alpha)$$

$$= 0 - \underbrace{\partial_\mu (\partial_\nu F^{\mu\nu} \alpha)}_{\text{CM}}$$



$$\delta_\alpha A_\mu = \partial_\mu \alpha$$

$$\text{Here } E\delta p = \partial_\mu F^{\mu\nu} \delta_\alpha A_\nu$$

$$= \partial_\mu F^{\mu\nu} \partial_\nu \alpha$$

$$= -\partial_\nu \partial_\mu F^{\mu\nu} \alpha + \partial_\nu (\partial_\mu F^{\mu\nu} \alpha)$$

$$= 0 - \underbrace{\partial_\mu (\partial_\nu F^{\mu\nu} \alpha)}_{C^\mu}$$

$$\delta_\alpha L = 0$$

Noether current:

$$J^\mu = \Theta^\mu[\delta_\alpha] - 0$$

$$= -F^{\mu\nu} \delta_\alpha A_\nu$$

$$= -F^{\mu\nu} \partial_\nu \alpha$$

$$= \partial_\nu F^{\mu\nu} \alpha - \partial_\nu (F^{\mu\nu} \alpha)$$

$$= C^\mu + \partial_\nu (F^{\mu\nu} \alpha)$$

Has the structure expected from  
Noether 2.



$$\delta_\alpha L = 0$$

Noether current:

$$\begin{aligned} J^\mu &= \frac{\partial L}{\partial (\partial_\mu A_\nu)} \delta A_\nu - \delta L \\ &= -F^{\mu\nu} \delta A_\nu \\ &= -F^{\mu\nu} \partial_\nu \alpha \\ &= \partial_\nu F^{\mu\nu} \alpha - \partial_\nu (F^{\mu\nu} \alpha) \\ &= C^\mu + \partial_\nu (F^{\mu\nu} \alpha) \end{aligned}$$

Has the structure expected from  
Noether 2.

Electric charge:

$$\begin{aligned} Q_E &= \int_S d^3x (-F^{0r}) \\ &= \int_S d^3x E^r \end{aligned}$$

$$\frac{dQ_E}{dt} = \int_S d^3x (-\partial_t F^{0r})$$



$$\delta_\alpha L = 0$$

Noether current:

$$\begin{aligned} J^\mu &= \Theta^\mu[\delta_\alpha] - 0 \\ &= -F^{\mu\nu} \delta_\alpha A_\nu \\ &= -F^{\mu\nu} \partial_\nu \alpha \\ &= \partial_\nu F^{\mu\nu} \alpha - \partial_\nu (F^{\mu\nu} \alpha) \\ &= C^\mu + \partial_\nu (F^{\mu\nu} \alpha) \end{aligned}$$

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Electric charge:

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$$\frac{dQ_E}{dt} = \oint_S (d^3x) (-\partial_t F^{0r})$$

$$\hat{=} \oint_S d^3x \partial_a F^{ar}$$

$$= 0$$



$$\delta_\alpha L = 0$$

Noether current:

$$\begin{aligned} J^\mu &= \Theta^\mu[\delta_\alpha] - 0 \\ &= -F^{\mu\nu} \delta_\alpha A_\nu \\ &= -F^{\mu\nu} \partial_\nu \alpha \\ &= \partial_\nu F^{\mu\nu} \alpha - \partial_\nu (F^{\mu\nu} \alpha) \\ &= C^\mu + \partial_\nu (F^{\mu\nu} \alpha) \end{aligned}$$

Has the structure expected from  
Noether 2.

Electric charge:

$$\begin{aligned} Q_E &= \oint_S d^3x (-F^{0r}) \Big|_{\alpha=1} \\ &= \oint_S d^3x E^r \\ \frac{dQ_E}{dt} &= \oint_S (d^3x) (-\partial_t F^{0r}) \\ &\hat{=} \oint_S d^3x \partial_a F^{ar} \\ &= 0 \end{aligned}$$



\* Gravity

$$L = \sqrt{g} (R - 2\Lambda)$$

$$\begin{aligned} \delta L &= -\frac{1}{2} \sqrt{g} g_{\mu\nu} \delta g^{\mu\nu} (R - 2\Lambda) \\ &\quad + \sqrt{g} (\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu}) \\ &= \sqrt{g} \delta g^{\mu\nu} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu} \\ &= E_{\mu\nu} \delta g^{\mu\nu} + 2 \delta \sigma_{\mu\nu} \end{aligned}$$



# \* Gravity

$$L = \sqrt{g} (R - 2\Lambda)$$

$$\delta L = -\frac{1}{2} \sqrt{g} g_{\mu\nu} \delta g^{\mu\nu} (R - 2\Lambda) + \sqrt{g} (\delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu})$$

$$= \sqrt{g} \delta g^{\mu\nu} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) + \sqrt{g} g^{\mu\nu} \delta R_{\mu\nu}$$

$$= E_{\mu\nu} \delta g^{\mu\nu} + \partial_\mu \theta^\mu$$

$$\theta^\mu = \sqrt{g} (g^{\alpha\beta} \delta \Gamma^\mu_{\alpha\beta} - g^{\alpha\mu} \delta \Gamma^\beta_{\beta\alpha})$$

=



$$\begin{aligned}\sigma^\mu &= \sqrt{g} (g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\mu - g^{\alpha\mu} \delta \Gamma_{\beta\alpha}^\beta) \\ &= \sqrt{g} \nabla_\nu (g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta} - \delta g^{\mu\nu})\end{aligned}$$

$$\delta \Gamma_{\mu\nu}^\alpha = \frac{1}{2} g^{\alpha\tau} (\nabla_\mu \delta g_{\tau\nu} + \nabla_\nu \delta g_{\tau\mu} - \nabla_\tau \delta g_{\mu\nu})$$

Noether identity:  $\delta_\xi = \mathcal{L}_\xi$

$$E \delta_\xi \phi = \sqrt{g} \mathcal{L}_\xi g^{\mu\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu})$$

$$= -2\sqrt{g} \nabla^\mu \xi^\nu (G_{\mu\nu} + \Lambda g_{\mu\nu})$$

$$= +2\sqrt{g} \xi^\nu \nabla^\mu (G_{\mu\nu} + \Lambda g_{\mu\nu})$$

$$= 2\partial^\mu (\sqrt{g} \xi^\nu (G_{\mu\nu} + \Lambda g_{\mu\nu}))$$

$$= 0 - \partial_\mu c^\mu$$



$$\delta T_{\mu\nu}^{\alpha} = \frac{1}{2} g^{\alpha\sigma} (\nabla_{\mu} \delta g_{\sigma\nu} + \nabla_{\nu} \delta g_{\sigma\mu} - \nabla_{\sigma} \delta g_{\mu\nu})$$

Noether identity:  $\delta_{\xi} = \mathcal{L}_{\xi}$

$$\begin{aligned} E \delta_{\xi} \phi &= \sqrt{g} \mathcal{L}_{\xi} g^{\mu\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\ &= -2\sqrt{g} \nabla^{\mu} \xi^{\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\ &= +2\sqrt{g} \xi^{\nu} \nabla^{\mu} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\ &\quad - 2\partial^{\mu} (\sqrt{g} \xi^{\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu})) \\ &= 0 - \partial_{\mu} C^{\mu} \end{aligned}$$

$$\mathcal{L}_{\xi} L = \partial_{\mu} (\xi^{\mu} L)$$

$$T^{\mu} = \partial^{\mu} (\mathcal{L}_{\xi}) - \xi^{\mu} L$$



$$\begin{aligned}\Theta^\mu &= \sqrt{g} (g^{\alpha\beta} \delta T^\mu_{\alpha\beta} - g^{\alpha\mu} \delta T^\beta_{\beta\alpha}) \\ &= \sqrt{g} \nabla_\nu (g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta} - \delta g^{\mu\nu})\end{aligned}$$

$$\begin{aligned}\Theta^\mu[\mathcal{L}_\xi] &= \sqrt{g} \nabla_\nu (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu - 2g^{\mu\nu} \nabla_\alpha \xi^\alpha) \\ &= \sqrt{g} (\nabla_\nu \nabla^\mu \xi^\nu + \nabla_\nu \nabla^\nu \xi^\mu - 2\nabla_\nu \nabla_\alpha \xi^\alpha) \\ R_{\mu\nu} &= \sqrt{g} (2R^{\mu\nu} \xi_\nu + \nabla_\nu (\nabla^\nu \xi^\mu - \nabla^\mu \xi^\nu))\end{aligned}$$

$$\delta T^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (\nabla_\mu \delta g_{\sigma\nu} + \nabla_\nu \delta g_{\sigma\mu} - \nabla_\sigma \delta g_{\mu\nu})$$

Noether identity:  $\delta_\xi = \mathcal{L}_\xi$

$$\begin{aligned}E \delta_\xi \phi &= \sqrt{g} \mathcal{L}_\xi g^{\mu\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\ &= -2\sqrt{g} \nabla^\mu \xi^\nu (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\ &= +2\sqrt{g} \xi^\nu \nabla^\mu (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\ &\quad - 2\partial^\mu (\sqrt{g} \xi^\nu (G_{\mu\nu} + \Lambda g_{\mu\nu})) \\ &= 0 - \partial_\mu C^\mu\end{aligned}$$



$$\delta(\sqrt{g}F) = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}F + \sqrt{g}\delta F$$

\* Gravity

$$L = \sqrt{g}(R - 2\Lambda)$$

$$\delta L = -\frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}(R - 2\Lambda)$$

$$+ \sqrt{g}(\delta R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})$$

$$= \sqrt{g}(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})$$

$$= \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}$$

$$\delta L = 2\Lambda$$

$$\delta^\mu = \sqrt{g}(g^{\alpha\beta}\delta\Gamma_{\alpha\beta}^\mu - g^{\alpha\mu}\delta\Gamma_{\beta\alpha}^\beta)$$

$$= \sqrt{g}\nabla_\nu(g^{\mu\nu}g_{\alpha\beta}\delta g^{\alpha\beta} - \delta g^\mu_\nu g^{\alpha\beta})$$

$$\delta^\mu[\mathcal{L}_\xi] = \sqrt{g}\nabla_\nu(\nabla^\mu\xi^\nu + \nabla^\nu\xi^\mu - 2g^{\mu\nu}\nabla_\alpha\xi^\alpha)$$

$$= \sqrt{g}(\nabla_\nu\nabla^\mu\xi^\nu + \nabla_\nu\nabla^\nu\xi^\mu - 2\nabla_\nu\nabla^\mu\xi^\nu)$$

$$= \sqrt{g}(2R^{\mu\nu}\xi_\nu + \nabla_\nu(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu))$$



$$\delta(\sqrt{g}F) = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}F + \sqrt{g}\delta F$$

\* Gravity

$$L = \sqrt{g}(R - 2\Lambda)$$

$$\begin{aligned}\delta L &= -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}(R - 2\Lambda) \\ &\quad + \sqrt{g}(\delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}) \\ &= \sqrt{g}\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right) + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} \\ &= E_{\mu\nu}\delta g^{\mu\nu} + 2\delta_\mu\theta^\mu\end{aligned}$$

$$\begin{aligned}\delta^\mu &= \sqrt{g}(g^{\mu\nu}\delta g_{\nu\lambda} + \delta g^{\mu\nu}g_{\nu\lambda}) \\ &= \sqrt{g}\delta^\mu_\lambda\end{aligned}$$

$$\begin{aligned}\partial^\mu[L_\xi] &= \sqrt{g}\partial^\mu(\xi^\nu) \\ &= \sqrt{g}(\nabla_\nu\nabla^\mu\xi^\nu) \\ &= \sqrt{g}(2R^{\mu\nu}\xi_\nu)\end{aligned}$$



$$\delta \Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} g^{\alpha\sigma} (\nabla_{\mu} \delta g_{\sigma\nu} + \nabla_{\nu} \delta g_{\sigma\mu} - \nabla_{\sigma} \delta g_{\mu\nu})$$

Noether identity:  $\delta_{\xi} = \mathcal{L}_{\xi}$

$$E \delta_{\xi} \phi = \sqrt{g} \mathcal{L}_{\xi} g^{\mu\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu})$$

$$= -2\sqrt{g} \nabla^{\mu} \xi^{\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu})$$

$$= +2\sqrt{g} \xi^{\nu} \nabla^{\mu} (G_{\mu\nu} + \Lambda g_{\mu\nu})$$

$$- 2\partial^{\mu} (\sqrt{g} \xi^{\nu} (G_{\mu\nu} + \Lambda g_{\mu\nu}))$$

$$= 0 - \partial_{\mu} C^{\mu}$$

$$\mathcal{L}_{\xi} L = \partial_{\mu} (\xi^{\mu} L)$$

$$T^{\mu} = \partial^{\mu} (\mathcal{L}_{\xi}) - \xi^{\mu} L$$

$$= 2\sqrt{g} \xi_{\nu} (G^{\mu\nu} + \Lambda g^{\mu\nu}) + \partial_{\nu} Q^{(\mu\nu)}$$

$$\stackrel{\wedge}{=} \partial_{\nu} Q^{(\mu\nu)}$$

$$Q^{\mu\nu} = \sqrt{g} (\nabla^{\nu} \xi^{\mu} - \nabla^{\mu} \xi^{\nu})$$

Komar



#### ④ Covariant phase space formalism

\* (Variational bi-complex)

- Spacetime:  $dx^\mu$   $d^2 = 0$
- Space of field:  $\delta\phi$   $\delta^2 = 0$

$$\begin{aligned} \delta t &= \delta x \\ \sigma^\mu &= \sqrt{g} (g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^\mu - g^{\alpha\mu} \delta \Gamma_{\beta\alpha}^\beta) \\ &= \sqrt{g} \nabla_\nu (g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta} - \delta g^{\mu\nu}) \\ &= \sqrt{g} \nabla_\nu (\nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu - 2g^{\mu\nu} \nabla_\alpha \xi^\alpha) \\ &= \sqrt{g} (\nabla_\nu \nabla^\mu \xi^\nu + \nabla_\nu \nabla^\nu \xi^\mu - 2\nabla_\nu \nabla_\alpha \xi^\alpha) \\ &= \sqrt{g} (2R^{\mu\nu} \xi_\nu + \nabla_\nu (\nabla^\nu \xi^\mu - \nabla^\mu \xi^\nu)) \end{aligned}$$



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• Spacetime:  $dx^\mu$   $d^2 = 0$

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For example,  $L$  is a  $(4, 0)$ -form.

$\Theta[\delta]$  is a  $(3, 1)$ -form



#### ④ Covariant phase space formalism

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• Spacetime:  $dx^\mu$ ,  $d^2 = 0$ ,  $i, \lrcorner$

• Space of field  $\delta\phi$ ,  $\delta^2 = 0$ ,  $\mathbb{I}, \lrcorner$

For example,  $L$  is a  $(4, 0)$ -form.

$\Theta[\delta]$  is a  $(3, 1)$ -form

Between  $\xi^\mu$  vector field  
and  $P = P_\mu dx^\mu$

$$\left. \begin{array}{l} \xi^\mu \text{ vector field} \\ P = P_\mu dx^\mu \end{array} \right\} \begin{array}{l} (\xi \lrcorner P) \\ = (i_\xi P) = \xi^\mu P_\mu \end{array}$$

Between



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Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta L}{\delta \phi(x)}$

$$\mathbb{I}_{X_\epsilon} \Theta[\delta] = \Theta[\delta_\epsilon]$$



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For example,  $L$  is a  $(4, 0)$ -form.

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Between  $\xi^\mu$  vector field and  $P = P_\mu dx^\mu$   $\left\{ \begin{array}{l} (\xi, P) \\ (i_\xi P) = \xi^\mu P_\mu \end{array} \right.$  No

Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta L}{\delta \phi(x)}$

$$\mathbb{I}_{X_\epsilon} \Theta[\delta] = \Theta[\delta_\epsilon] = \mathbb{I}_\epsilon \Theta = \epsilon \lrcorner \Theta$$



$$\xi^\mu \text{ vector field } \left\{ \begin{array}{l} (\xi, p) \\ p = p_\mu dx^\mu \end{array} \right\} = (i_\xi p) = \xi^\mu p_\mu$$

$$X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta \mathcal{L}}{\delta \phi(x)}$$

$$\begin{aligned} \int_{X_\epsilon} \theta[\delta] &= \theta[\delta_\epsilon] = \int_\epsilon \theta \\ &= \epsilon \int \theta \end{aligned}$$

\* Potential and symplectic structure.

$$\delta L = E \delta \phi + \delta \theta$$

potential  $\theta$  is a  $(3,1)$ -form.

symplectic current :  $\omega = \delta \theta$

$(3,2)$ -form.

$$\omega[s, \delta]$$

$$\omega = \delta_1 \theta[\delta_2] - \delta_2 \theta[\delta_1]$$

$$\mathcal{L}_\xi L = \partial_\mu (\xi^\mu L)$$

$$\begin{aligned} T^\mu &= \partial^\mu (\mathcal{L}_\xi) \\ &= 2\sqrt{g} \xi_\nu (G^{\mu\nu}) \end{aligned}$$

$$\hat{=} \dots \partial_\nu Q^{(\mu\nu)}$$

$$Q^{\mu\nu} = \sqrt{g} (\nabla^\nu \xi^\mu)$$



#### ④ Covariant phase space formalism

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• Spacetime:  $dx^\mu$ ,  $d^2 = 0$ ,  $i, \lrcorner, \wedge$

• Space of field  $\delta\phi$ ,  $\delta^2 = 0$ ,  $I, \lrcorner, \wedge$

For example  $L$  is a  $(4, 0)$ -form

$\Theta[\delta]$  is a  $(3, 1)$ -form

Between  $\xi^\mu$  vector field and  $P = P_\mu dx^\mu$

$$\left\{ \begin{array}{l} (\xi \lrcorner P) \\ (i_\xi P) = \xi^\mu P_\mu \end{array} \right.$$

Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta L}{\delta \phi(x)}$

$$I_{X_\epsilon} \Theta[\delta] = \Theta[\delta_\epsilon] = I_\epsilon \Theta = \epsilon_\lrcorner \Theta$$

$$\left\{ \begin{array}{l} p \wedge q = (-1)^{p_1} q \wedge p \end{array} \right.$$



#### ④ Covariant phase space formalism

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• Spacetime:  $dx^\mu$ ,  $d^2 = 0$ ,  $i, \lrcorner, \lrcorner$

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For example,  $L$  is a  $(4, 0)$ -form

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Between  $\xi^\mu$  vector field and  $P = P_\mu dx^\mu$

$$\left. \begin{array}{l} \xi^\mu \text{ vector field} \\ \text{and } P = P_\mu dx^\mu \end{array} \right\} \begin{array}{l} (\xi \lrcorner P) \\ = (i_\xi P) = \xi^\mu P_\mu \end{array}$$

Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta}{\delta \phi(x)}$

$$I_{X_\epsilon} \Theta[\delta] = \Theta[\delta_\epsilon] = I_\epsilon \Theta = \epsilon \lrcorner \Theta$$

$$\begin{cases} p \wedge q = (-1)^{p_1} q \wedge p \\ \delta p \wedge \delta q = (-1)(-1)^{p_1} \delta q \wedge \delta p \end{cases}$$



\* Potential and symplectic structure.

$$p_\mu = \frac{\delta L}{\delta \dot{\phi}^\mu}$$

$$SL = E \delta \phi + \delta \Theta$$

• potential  $\Theta$  is a  $(3,1)$ -form.

$$= \int \Theta$$

• symplectic current :  $\omega = \delta \Theta$

$$= \int \omega$$

$(3,2)$ -form.

$$\omega[s_1, s_2]$$

$$\omega = \delta_1 \Theta[s_2] - \delta_2 \Theta[s_1]$$

• Symplectic structure

$$\Omega = \int \omega$$

Generator of a gauge transformation is:



\* Potential and symplectic structure.

$$p_\mu = \frac{\delta L}{\delta \dot{\phi}^\mu}$$

$$SL = E \delta \phi + \delta \Theta$$

• potential  $\Theta$  is a  $(3,1)$ -form.  
"psq"

$$= \int \Theta$$

• symplectic current

$$\omega = \delta \Theta = \delta p \delta q$$

$(3,2)$ -form.

$$= \int \omega$$

• Symplectic structure  
 $\omega[s_1, s_2] = \delta_1 \Theta[s_2] - \delta_2 \Theta[s_1]$

$$\Omega = \int \omega$$

• Generator of a gauge transformation is:

$$\delta G(\epsilon) = - \epsilon \lrcorner \Omega$$

Why?

$$\Omega = \int \delta \Theta = \int \delta p \delta q$$

$$- \epsilon \lrcorner \Omega = \int$$



malism

Between  $\xi^\mu$  vector field and  $P = p_\mu dx^\mu$

$$\left. \begin{array}{l} \xi^\mu \\ P = p_\mu dx^\mu \end{array} \right\} \begin{array}{l} (\xi \lrcorner P) \\ = (i_\xi P) = \xi^\mu p_\mu \end{array}$$

Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta \mathcal{L}}{\delta \phi(x)}$

$$\begin{aligned} \mathbb{I}_{X_\epsilon} \Theta[\delta] &= \Theta[\delta_\epsilon] = \mathbb{I}_\epsilon \Theta \\ &= \epsilon \lrcorner \Theta \end{aligned}$$

$$\begin{cases} p \wedge q = (-1)^{p_1} q \wedge p \\ \delta p \wedge \delta q = (-1)(-1)^{p_1} \delta q \wedge \delta p \end{cases}$$

$$\epsilon \lrcorner (\delta p \delta q) = \delta_\epsilon p \delta q - \delta p \delta_\epsilon q$$

\* Potential and symplectic structure

$$\delta L = E \delta \phi + \delta \Theta$$

potential  $\Theta$  is a  $(3,1)$ -form.

symplectic current  $\omega = \delta \Theta = \delta p \wedge \delta q$   $(3,2)$ -form.

symplectic structure  $\omega[\delta_1, \delta_2] = \delta_1 \Theta[\delta_2] - \delta_2 \Theta[\delta_1]$

$$\Omega = \int \omega$$



# \* Potential and symplectic structure.

$$\delta L = E \delta \phi + \delta \Theta$$

• potential  $\Theta$  is a  $(3,1)$ -form.  
"psq"

• symplectic current:  $\omega = \delta \Theta = \delta p \delta q$   
 $(3,2)$ -form.

$$\omega[\delta_1, \delta_2]$$

$$\omega = \delta_1 \Theta[\delta_2] - \delta_2 \Theta[\delta_1]$$

• Symplectic structure

$$\Omega = \int_{\Sigma} \omega$$

• Generator of a gauge transformation is:

$$\delta G(\epsilon) = -\epsilon \int_{\Sigma} \Omega$$

Why?  $\Omega = \int_{\Sigma} \delta \Theta = \int_{\Sigma} \delta p \delta q$

$$-\epsilon \int_{\Sigma} \Omega = \int_{\Sigma} -\epsilon \int_{\Sigma} (\delta p \delta q)$$

$$= \int_{\Sigma} \delta_{\epsilon} q \delta p - \delta_{\epsilon} p \delta q$$

$$= \delta G(\epsilon)$$

$$\{q, G(\epsilon)\} = \delta_{\epsilon} q$$



$$L = \frac{1}{2} * F \wedge F$$

$$\delta L = *F \wedge \delta F$$

$$= *F \wedge d\delta A$$

$$= -d(*F) \wedge \delta A + \underbrace{d(*F \wedge \delta A)}_0$$

$$\Theta = *F \wedge \delta A$$

$$W = \delta \Theta = \delta(*F) \wedge \delta A$$

$$\mathcal{L} = \int_{\Sigma} \delta(*F) \wedge \delta A$$

Between  $\xi^\mu$  vector field and  $P = P_\mu dx^\mu$

$$\left. \begin{array}{l} \xi^\mu \text{ vector field} \\ \text{and } P = P_\mu dx^\mu \end{array} \right\} \begin{array}{l} (\xi \lrcorner P) \\ = (i_\xi P) = \xi^\mu P_\mu \end{array}$$

Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta \mathcal{L}}{\delta \phi(x)}$

$$\mathbb{I}_{X_\epsilon} \Theta[\delta] = \Theta[\delta_\epsilon] = \mathbb{I}_\epsilon \Theta = \epsilon \lrcorner \Theta$$

$$\begin{cases} p \wedge q = (-1)^{pq} q \wedge p \\ \delta p \wedge \delta q = (-1)(-1)^{pq} \delta q \wedge \delta p \end{cases}$$

$$\epsilon \lrcorner (\delta p \delta q) = \delta_\epsilon p \delta q - \delta p \delta_\epsilon q$$



$$L = \frac{1}{2} * F \wedge F$$

$$\delta L = *F \wedge \delta F$$

$$= *F \wedge d\delta A$$

$$= -d(*F) \wedge \delta A + \underbrace{d(*F \wedge \delta A)}_{\theta}$$

$$\theta = *F \wedge \delta A$$

$$\omega = \delta \theta = \delta(*F) \wedge \delta A$$

$$\mathcal{L} = \int_{\Sigma} \delta(*F) \wedge \delta A$$

$$-d \lrcorner \mathcal{L} = \int_{\Sigma} \delta(*F) \wedge \delta A - \delta(*F) \wedge \delta A$$

$$= \int_{\Sigma} \delta(*F) \wedge d\alpha$$

Between  $\xi^\mu$  vector field and  $P = P_\mu dx^\mu$

$$\left. \begin{array}{l} \xi^\mu \\ P = P_\mu dx^\mu \end{array} \right\} \begin{array}{l} (\xi \lrcorner P) \\ = (i_\xi P) = \xi^\mu P_\mu \end{array}$$

Between  $X_\epsilon = \int d^4x \delta_\epsilon \phi \frac{\delta \mathcal{L}}{\delta \phi(x)}$

$$\mathbb{I}_{X_\epsilon} \theta[\delta] = \theta[\delta_\epsilon] = \mathbb{I}_\epsilon \theta = \epsilon \lrcorner \theta$$

$$\begin{cases} p \wedge q = (-1)^{pq} q \wedge p \\ \delta p \wedge \delta q = (-1)(-1)^{pq} \delta q \wedge \delta p \end{cases}$$

$$\epsilon \lrcorner (\delta p \delta q) = \delta_\epsilon p \delta q - \delta p \delta_\epsilon q$$

\* Potential

SL

potential

symplectic

ω Symplectic



$$\begin{aligned}
 L &= \frac{1}{2} *F \wedge F & \int \alpha \partial_a E^a \\
 & & \Downarrow \\
 \delta L &= *F \wedge \delta F & \int \alpha d *F \\
 &= *F \wedge d \delta A \\
 &= -d(*F) \wedge \delta A + \underbrace{d(*F \wedge \delta A)}_{\Theta} \\
 \Theta &= *F \wedge \delta A \\
 \omega &= \delta \Theta = \delta(*F) \wedge \delta A \\
 \Omega &= \int_{\Sigma} \delta(*F) \wedge \delta A \\
 -d \lrcorner \Omega &= \int_{\Sigma} \delta(*F) \wedge \delta A - \delta(*F) \wedge \delta A \\
 &= \int_{\Sigma} \delta(*F) \wedge d\alpha \\
 &= \delta \left( \int_{\Sigma} *F \wedge d\alpha \right) \\
 &= \delta G(\alpha)
 \end{aligned}$$

$$G(\alpha) = \int_{\Sigma} *F \wedge d\alpha$$

\* Potential a

$$\delta L = E$$

potential.  $\Theta$  is  
"psq"  
symplectic current

$\omega[s, \delta]$   
electric skur

$$\Omega =$$



$$L = \frac{1}{2} *F \wedge F$$

$$\delta L = *F \wedge \delta F$$

$$= *F \wedge d\delta A$$

$$= -d(*F) \wedge \delta A + \underbrace{d(*F \wedge \delta A)}_{\Theta}$$

$$\Theta = *F \wedge \delta A$$

$$\omega = \delta \Theta = \delta(*F) \wedge \delta A$$

$$= \int_{\Sigma} \delta(*F) \wedge \delta A$$

$$-\alpha \mapsto -\Omega = \int_{\Sigma} \delta(*F) \wedge \delta A - \delta(*F) \wedge \delta A$$

$$= \int_{\Sigma} \delta(*F) \cdot d\alpha$$

$$= \delta \left( \int_{\Sigma} *F \cdot d\alpha \right)$$

$$G(\alpha) = \int_{\Sigma} *F \wedge d\alpha$$

Is differentiable

$$\text{If we write } G(\alpha) = - \int_{\Sigma} d*F \alpha + \int_{\Sigma} \alpha *F$$

$$\hat{=} \int_{\Sigma} \alpha *F$$

\* Potential

$$\delta L =$$

• potential  $\Theta$

• symplectic  $\omega$

$$\omega[s, s]$$

• Symplectic



$$E_K = \frac{-c^4}{8\pi G} \int (d^4x) \sqrt{-g} \mathcal{L}_K$$

→ M

$$\mathcal{L}_\xi = \xi^\mu (d_\mu \cdot) + d(\xi^\mu \cdot)$$

\* Grawky

$$\delta_\xi L = \mathcal{L}_\xi L = d(\xi^\mu L)$$

Noether current:  $J = C + dQ \triangleq dQ$

$$\delta J = \delta \theta [\delta_\xi] - \xi^\mu \delta L$$

$$= \delta \theta [\delta_\xi] - \xi^\mu (E \delta q + d\theta)$$

$$\triangleq \delta \theta [\delta_\xi] - \xi^\mu d\theta$$

$$= \delta \theta [\delta_\xi] - \mathcal{L}_\xi \theta + d(\xi^\mu \theta)$$

$$= -\xi^\mu \omega + d(\xi^\mu \theta)$$

$$\triangleq dQ$$