Title: Noetherâ€™s theorems and gauge symmetries

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$$\sqrt{g}\,\nabla_\mu T^{[\mu\omega]} = \partial_\mu\left(\sqrt{g}\,T^{[\mu\omega]}\right)$$

- $\forall\ \xi^\mu$ we have $\nabla_\mu\Big(\underbrace{\nabla_\nu \nabla^{[\mu}\xi^{\omega]}}_{J^\mu_{\text{Koma}}}\Big) = 0$

- If ξ^μ is Killing then $J^\mu_{\text{Koma}} = R^{\mu\omega}\xi_\omega$

$$\oint_S (dx)^2_{\mu\omega}\,\nabla^\mu\xi^\omega \;\hat{=}\; 0$$

$$\sqrt{g}\,\nabla_\mu T^{[\mu\omega]} = \partial_\mu\left(\sqrt{g}\,T^{[\mu\omega]}\right)$$

- $\forall\ \xi^\mu$ we have $\nabla_\mu\underbrace{\left(\nabla_\nu \nabla^{[\mu}\xi^{\omega]}\right)}_{J^\mu_{Komar}} = 0$

- If ξ^μ is Killing then $J^\mu_{Komar} = R^\mu{}_\nu \xi^\nu$

$$E_{Komar} = -\frac{c^4}{8\pi G}\oint_S (dx)^2_{\mu\nu}\,\nabla^\mu \xi^\nu \quad \hat{=} 0$$

* Weak fields:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad h \ll \eta$$

$$\eta = \mathrm{diag}(-1,+1,+1,+1)$$

$$\partial_0 h_{\mu\nu} = 0$$

$$G_{00} = \frac{1}{2}\left(\partial_a\partial_b h^{ab} - \Delta h^a_a\right) + \mathcal{O}(h^2)$$

$$\hat{=} -\Delta h_{00} = -\partial_a\partial^a h_{00}$$

$h \ll \eta$

$+1, +', +1)$

$h^a_a) + \mathcal{O}(h^2)$

$_{00}$

Define

$$E = \int_\Sigma d^3x\, T_{00} = \frac{c^4}{8\pi G} \int_\Sigma d^3x\, G_{00}$$

$$= \frac{c^4}{16\pi G} \oint_S (dx)_a \left(\partial_b h^{ab} - \partial^a h^b_b \right)$$

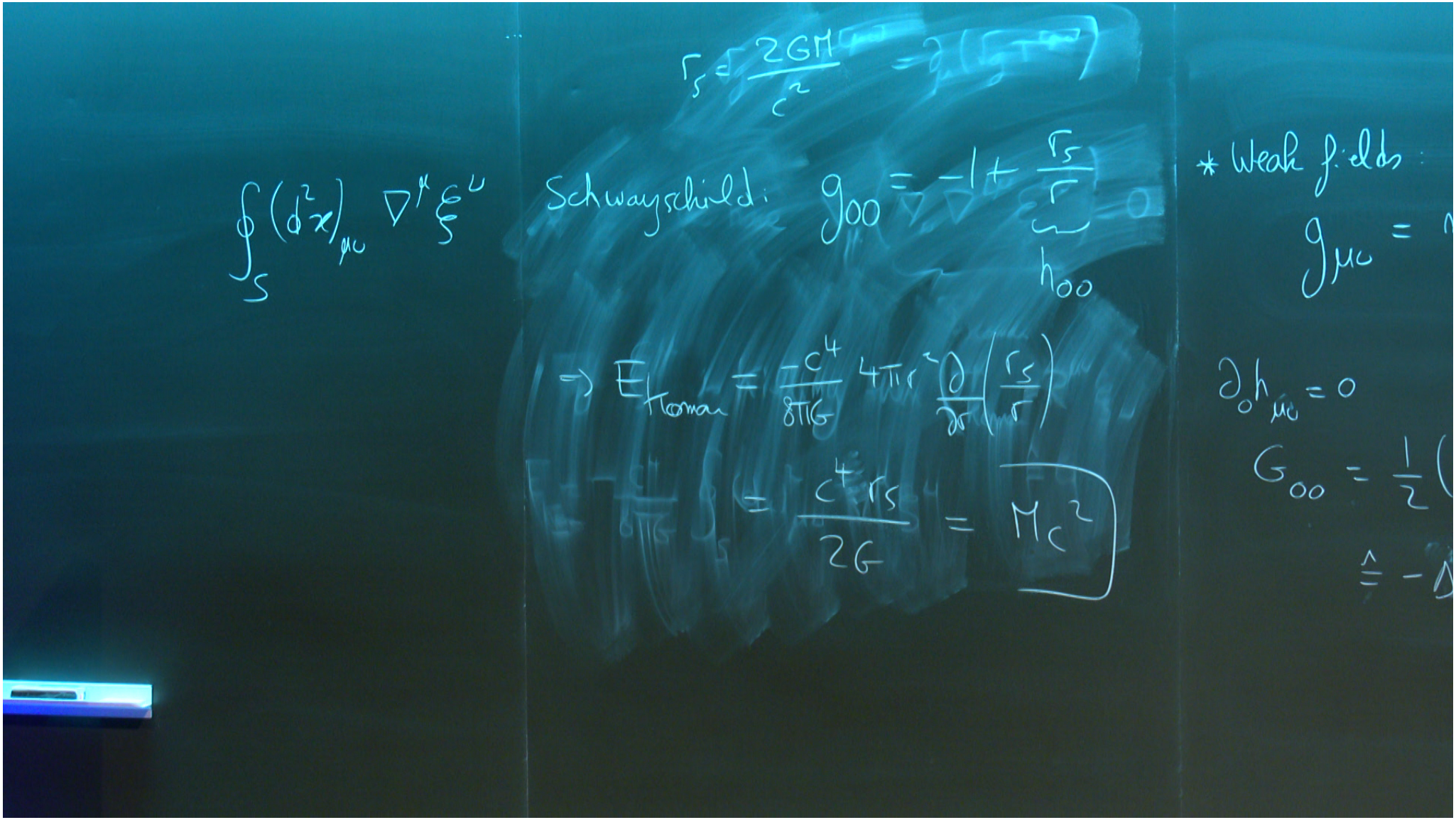
$$\hat{=} -\frac{c^4}{8\pi G} \oint_S (d^2x)_a\, \partial^a h_{00}$$

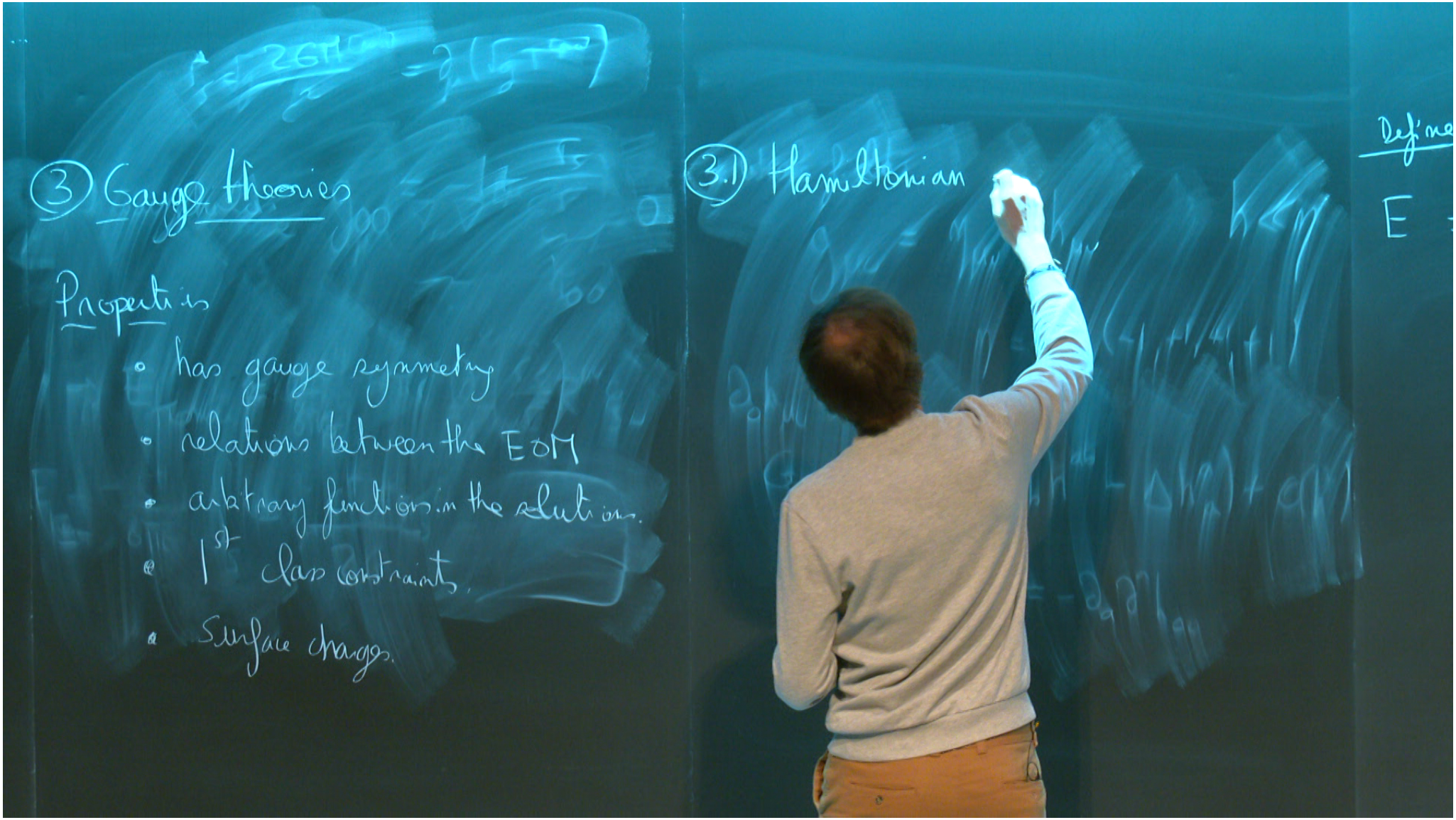
For $\xi^\mu = \partial_t$

we can show that $\nabla^a \xi^0 = -\frac{1}{2} \partial_a h_{00}$

$$\nabla^0 \xi^a = \frac{1}{2} \partial_a h_{00}$$

$$\Rightarrow E_{\text{Komar}} = -\frac{c^4}{8\pi G} \oint_S (d^2x)_{0a}\, \partial^a h_{00}$$


$r_s = \frac{2GM}{c^2}$
$\oint_S (d^2x)_{\mu\nu} \nabla^\mu \xi^\nu$
Schwarzschild: $g_{00} = -1 + \frac{r_s}{r}$
h_{00}
* Weak fields:
$g_{\mu\nu} =$
$\Rightarrow E_{Komar} = \frac{-c^4}{8\pi G} 4\pi r^2 \frac{\partial}{\partial r}\left(\frac{r_s}{r}\right)$
$= \frac{c^4 r_s}{2G} = Mc^2$
$\partial_0 h_{\mu\nu} = 0$
$G_{00} = \frac{1}{2}($
$\hat{=} -\Delta$


3 Gauge theories
Properties
has gauge symmetry
relations between the EOM
arbitrary functions in the solutions.
1st class constraints.
Surface charges.
3.1 Hamiltonian
Define

③ Gauge theories

Properties

- has gauge symmetry
- relations between the EOM
- arbitrary functions in the solutions.
- 1st class constraints.
- Surface charges.

③.1 Hamiltonian formulation.

$$L = p_a \dot{q}^a - H_0 - \lambda^i C_i$$

$$= p_a \dot{q}^a - H$$

1st class constraint means:

$$\{C_i, C_j\} = f_{ij}^{\ k} C_k$$

$$\{C_i, H_0\} = f_i^{\ j} C_j$$

Define

E

(3.1) Hamiltonian formulation

$$L = p_a \dot{q}^a - H_0 - \lambda^i C_i$$

$$p_a \dot{q}^a - H$$

1st class constraint means:

$$\{C_i, C_j\} = f_{ij}^{k} C_k$$

$$\{C_i, H_0\} = f_i^j C_j$$

$$\delta_\epsilon q = \epsilon^i \{q, C_i\} = \epsilon^i \frac{\partial C_i}{\partial p}$$

$$\delta_\epsilon p = \epsilon^i \{p, C_i\} = -\epsilon^i \frac{\partial C_i}{\partial q}$$

$$\delta_\epsilon \lambda^i = \dot{\epsilon}^i + f^i_{jk} \epsilon^j - f^i_{jk} \lambda^j \epsilon^k$$

$$\Rightarrow \delta_\epsilon L = \frac{d}{dt}\left(p_a \delta_\epsilon q^a - \epsilon^i C_i\right)$$

For

we

$\Rightarrow E_{ko}$

$$\delta_\epsilon q = \epsilon^i \{q, C_i\} = \epsilon^i \frac{\partial C_i}{\partial p}$$

$$\delta_\epsilon p = \epsilon^i \{p, C_i\} = -\epsilon^i \frac{\partial C_i}{\partial q}$$

$$\delta_\epsilon \lambda^i = \dot{\epsilon}^i + \rho^i{}_j \epsilon^j - \rho^i_{jk} \lambda^j \epsilon^k$$

$$\Rightarrow \delta_\epsilon L = \frac{d}{dt}\left(p_a \delta_\epsilon q^a - \epsilon^i C_i\right) = \frac{db_\epsilon}{dt}$$

Noether $\hat{=} 0$

* Time reparametrization:

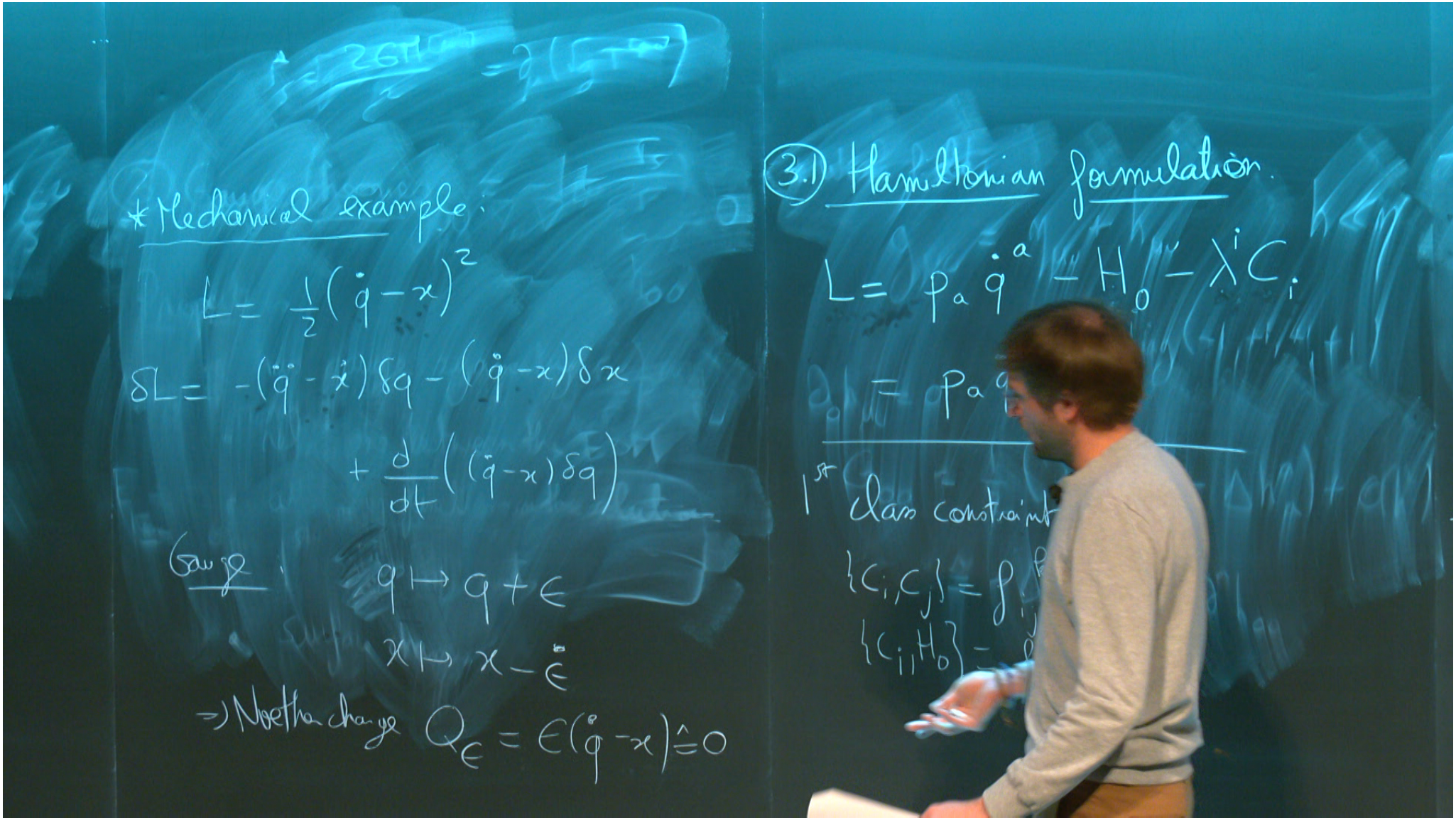
$$\delta_\epsilon q = \epsilon \dot{q}$$

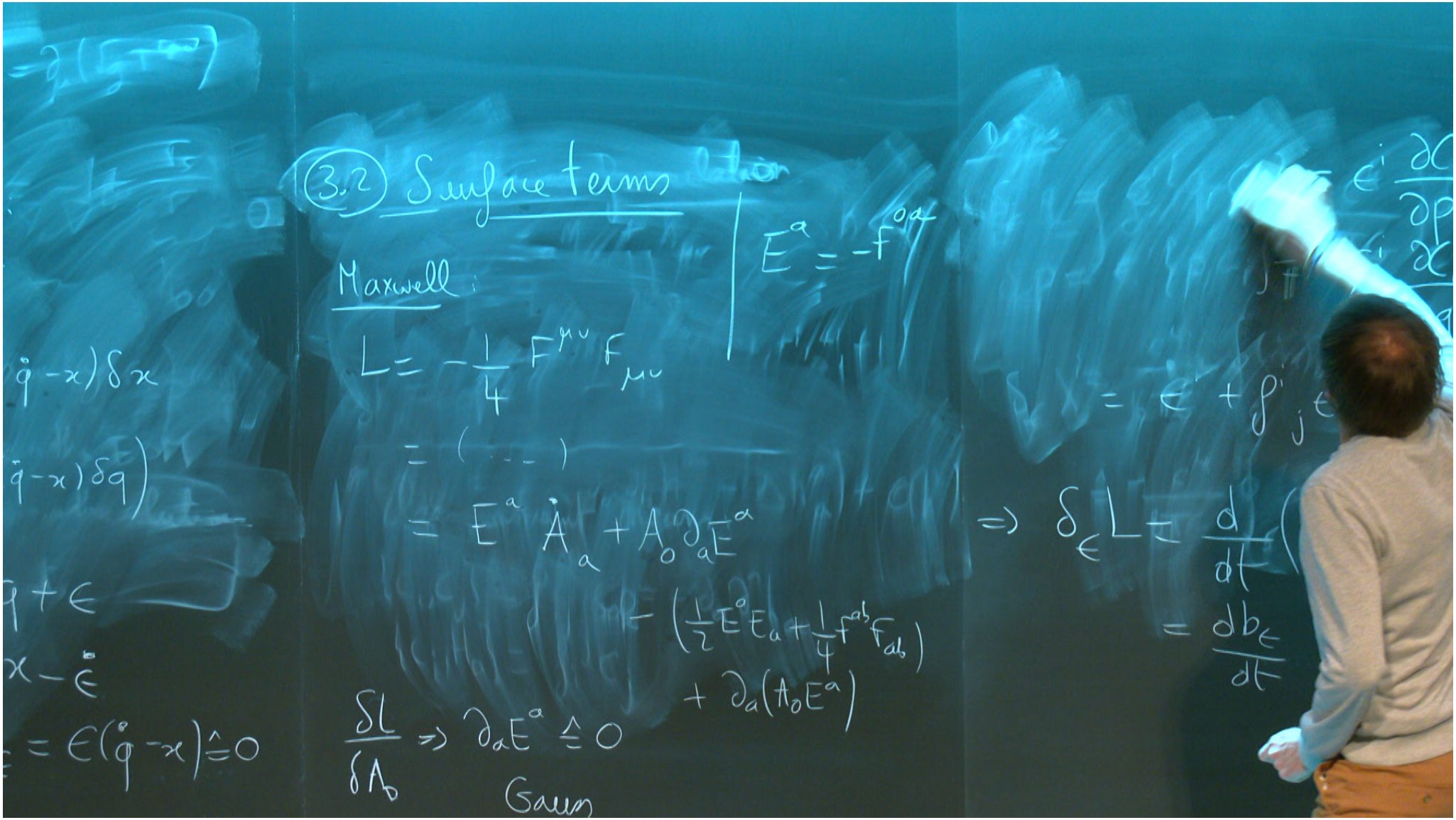
$$\delta_\epsilon p = \epsilon \dot{p}$$

$$\delta_\epsilon \lambda^i = \dot{\epsilon} \lambda^i + \epsilon \dot{\lambda}^i$$

$$\Rightarrow \delta_\epsilon C_i = \epsilon \dot{C}_i$$

If $H_0 = 0$, then $\delta_\epsilon L = \frac{d}{dt}\left(p \delta_\epsilon q - \epsilon \lambda^i C_i\right)$


* Mechanical example:
$L = \frac{1}{2}(\dot{q} - x)^2$
$\delta L = -(\ddot{q} - \dot{x})\delta q - (\dot{q} - x)\delta x + \frac{d}{dt}\left((\dot{q} - x)\delta q\right)$
Gauge: $q \mapsto q + \epsilon$
$x \mapsto x - \dot{\epsilon}$
⇒ Noether charge $Q_\epsilon = \epsilon(\dot{q} - x) \hat{=} 0$
(3.1) Hamiltonian formulation.
$L = p_a \dot{q}^a - H_0 - \lambda^i C_i$
1st class constraint
$\{C_i, C_j\} =$
$\{C_i, H_0\} =$


3.2 Surface terms
Maxwell:
$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$
$= (\dots)$
$= E^a \dot{A}_a + A_0 \partial_a E^a$
$- \left(\frac{1}{2} E^a E_a + \frac{1}{4} F^{ab} F_{ab}\right)$
$+ \partial_a (A_0 E^a)$
$\frac{\delta L}{\delta A_0} \Rightarrow \partial_a E^a \hat{=} 0$
Gaus
$E^a = -F^{0a}$
$\Rightarrow \delta_\epsilon L = \frac{d}{dt}$
$= \frac{db_\epsilon}{dt}$

$$\delta_\alpha A_\mu = \partial_\mu \alpha$$

(3.2) Surface terms

Maxwell:

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \qquad E^a = -F^{0a}$$

$$= (\ldots)$$

$$= E^a \dot{A}_a + A_0 \partial_a E^a - \left(\frac{1}{2} E^a E_a + \frac{1}{4} F^{ab} F_{ab}\right) + \partial_a (A_0 E^a)$$

$$\frac{\delta L}{\delta A_0} \Rightarrow \partial_a E^a \approx 0$$

Gaus

* Differentiability of the constraint:

Smeared Gauss constraint:

$$G(\alpha) = \int_\Sigma d^3x \; \alpha \, \partial_a E^a$$

With $\{A_a(x), E^b(y)\} = \delta^b_a \delta^3(x-y)$

We get $\{A_a, G(\alpha)\} = -\{A_a, \int_\Sigma d^3x \, \partial_b \alpha E^b\}$

$$= -\partial_a \alpha$$

* T

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If

* Differentiability of the constraint.

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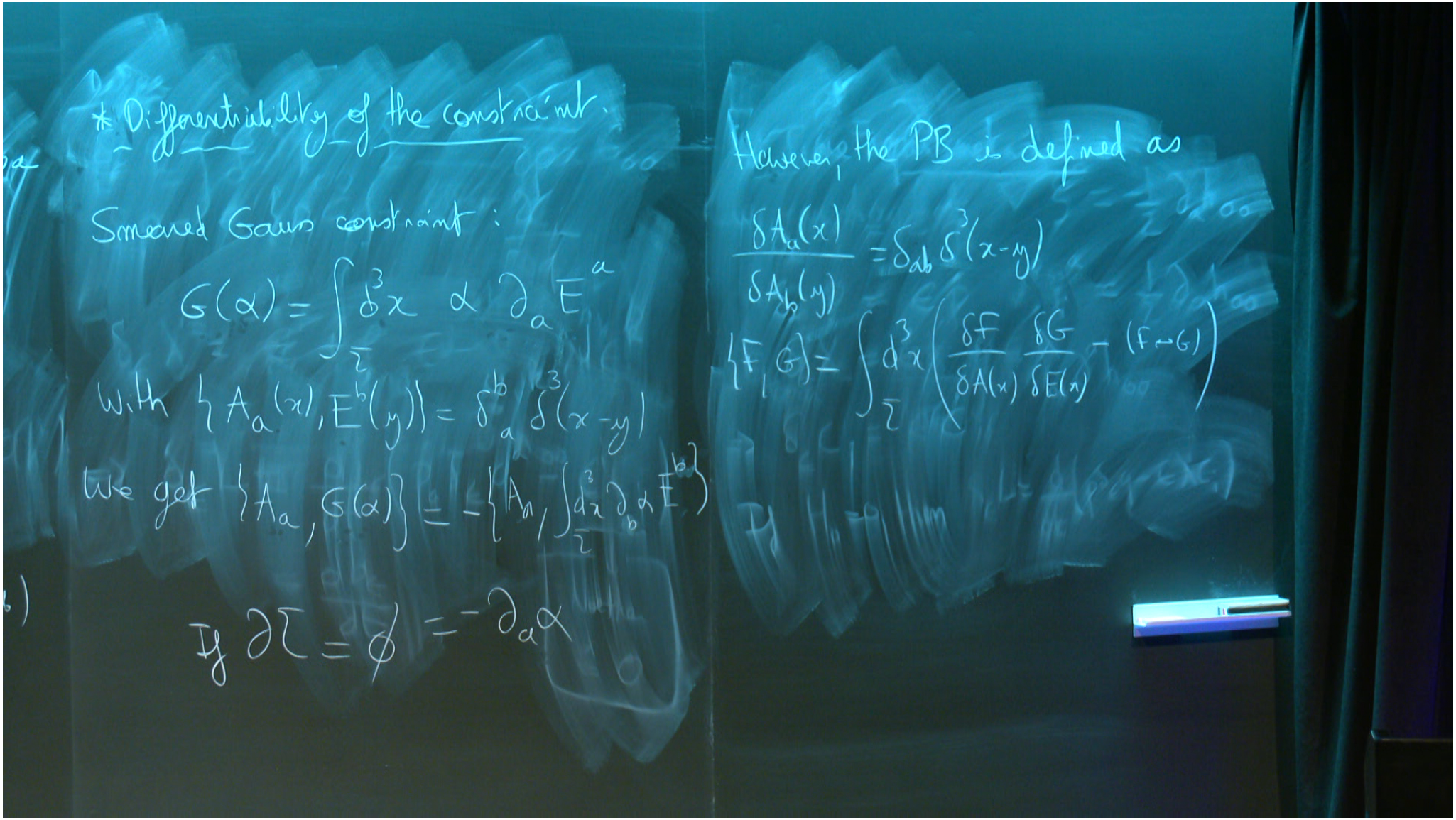
We get $\{A_a, G(\alpha)\} = -\{A_a, \int_\Sigma d^3x \, \partial_b \alpha \, E^b\}$

$$= -\partial_a \alpha$$

However, the PB is defined as

$$\frac{\delta A_a(x)}{\delta A_b(y)} = \delta_{ab} \delta^3(x-y)$$

$$\{F, G\} = \int_\Sigma d^3x \left(\frac{\delta F}{\delta A(x)} \frac{\delta G}{\delta E(x)} - (F \leftrightarrow G) \right)$$


* Differentiability of the constraint.
Smeared Gauss constraint:
$G(\alpha) = \int_\Sigma d^3x \; \alpha \, \partial_a E^a$
With $\{A_a(x), E^b(y)\} = \delta_a^b \delta^3(x-y)$
We get $\{A_a, G(\alpha)\} = -\{A_a, \int_\Sigma d^3x \, \partial_b \alpha \, E^b\}$
If $\partial\Sigma = \emptyset = -\partial_a \alpha$
However, the PB is defined as
$\frac{\delta A_a(x)}{\delta A_b(y)} = \delta_{ab} \delta^3(x-y)$
$\{F, G\} = \int_\Sigma d^3x \left(\frac{\delta F}{\delta A(x)} \frac{\delta G}{\delta E(x)} - (F \leftrightarrow G) \right)$

For the Gauss constraint:

$$\delta G(\alpha) = \int_\Sigma \alpha\, \partial_a \delta E^a$$

$$= -\int_\Sigma \partial_a \alpha\, \delta E^a + \oint_S n_a \alpha\, \delta E^a$$

If the surface term does not vanish, we define

$$\tilde{G}(\alpha) = G(\alpha) - \oint_S n_a \alpha E^a$$

→ Noether charge

$$\delta_\alpha A_\mu = \partial_\mu \alpha$$

(3.2) Surface terms

$$E^a = -F^{0a}$$

Maxwell:

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

$$= (\dots)$$

$$= E^a \dot{A}_a + A_0 \partial_a E^a$$

$$- \left(\frac{1}{2} E^a E_a + \frac{1}{4} F^{ab} F_{ab} \right)$$

$$+ \partial_a (A_0 E^a)$$

$$\frac{\delta L}{\delta A_0} \Rightarrow \partial_a E^a \doteq 0$$

Gauss constraint.

For the Gauss constraint:

$$\delta G(\alpha) = \int_\Sigma \alpha\, \partial_a \delta E^a$$

$$= -\int_\Sigma \partial_a \alpha\, \delta E^a + \oint_S n_a \alpha\, \delta E^a$$

If the surface term does not vanish, we define

$$\tilde{G}(\alpha) = G(\alpha) - \oint_S n_a \alpha E^a$$

↑ true gauge generator

on-shell: $\tilde{G}(\alpha) \stackrel{\wedge}{=} -\oint_S n_a \alpha E^a$

(3.2) Surface terms

Maxwell:

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu}$$

$$= (\ldots)$$

$$= E^a \dot{A}$$

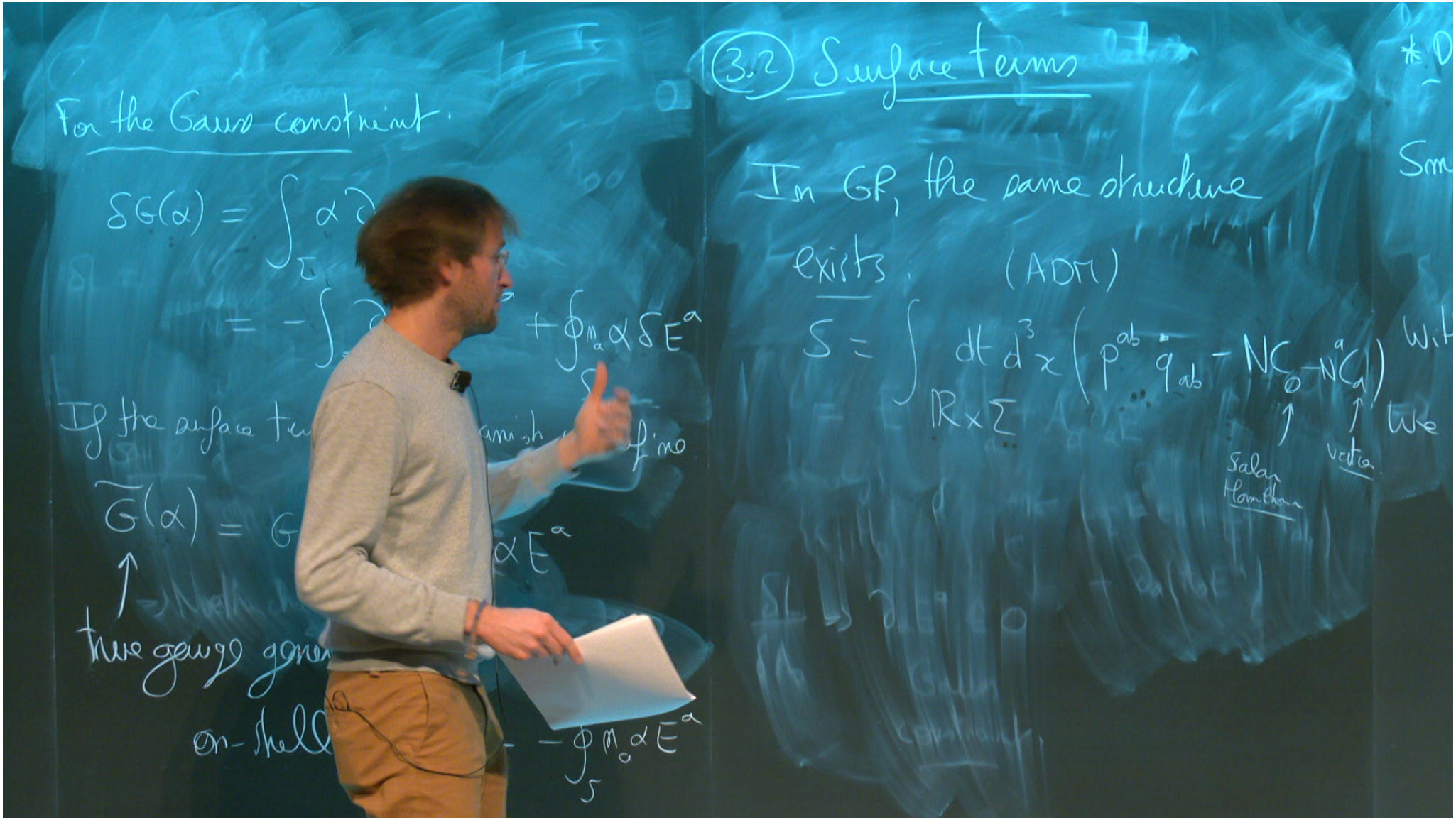
$$E^a = -f^{0a}$$

$$E_a + \frac{1}{4} f^{ab} F_{ab})$$

$$\frac{\delta L}{\delta A_0} \Rightarrow \partial_a E^a$$

Gau

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For the Gauss constraint:
$\delta G(\alpha) = \int_\Sigma \alpha \partial$
$= -\int \partial$ $+ \oint_{\partial\Sigma} n_a \alpha \delta E^a$
If the surface te
$\tilde{G}(\alpha) = G$ αE^a
true gauge gene
on-shell $- \oint_S n_a \alpha E^a$
(3.2) Surface terms
In GR, the same structure
exists. (ADM)
$S = \int_{\mathbb{R}\times\Sigma} dt\, d^3x \left(p^{ab} \dot{q}_{ab} - N C_0 - N^a C_a \right)$
Scalar
Hamiltonian
Vector
Wit
We

For the Gauss constraint:

$$\delta G(\alpha) = \int_\Sigma \alpha\, \partial_a \delta E^a$$

$$= -\int_\Sigma \partial_a \alpha\, \delta E^a + \oint_S n_a \alpha\, \delta E^a$$

If the surface term does not vanish, we define

$$\tilde{G}(\alpha) = G(\alpha) - \oint_S n_a \alpha E^a$$

↑ true gauge generator

on-shell: $\tilde{G}(\alpha) \hat{=} -\oint_S n_a \alpha E^a$

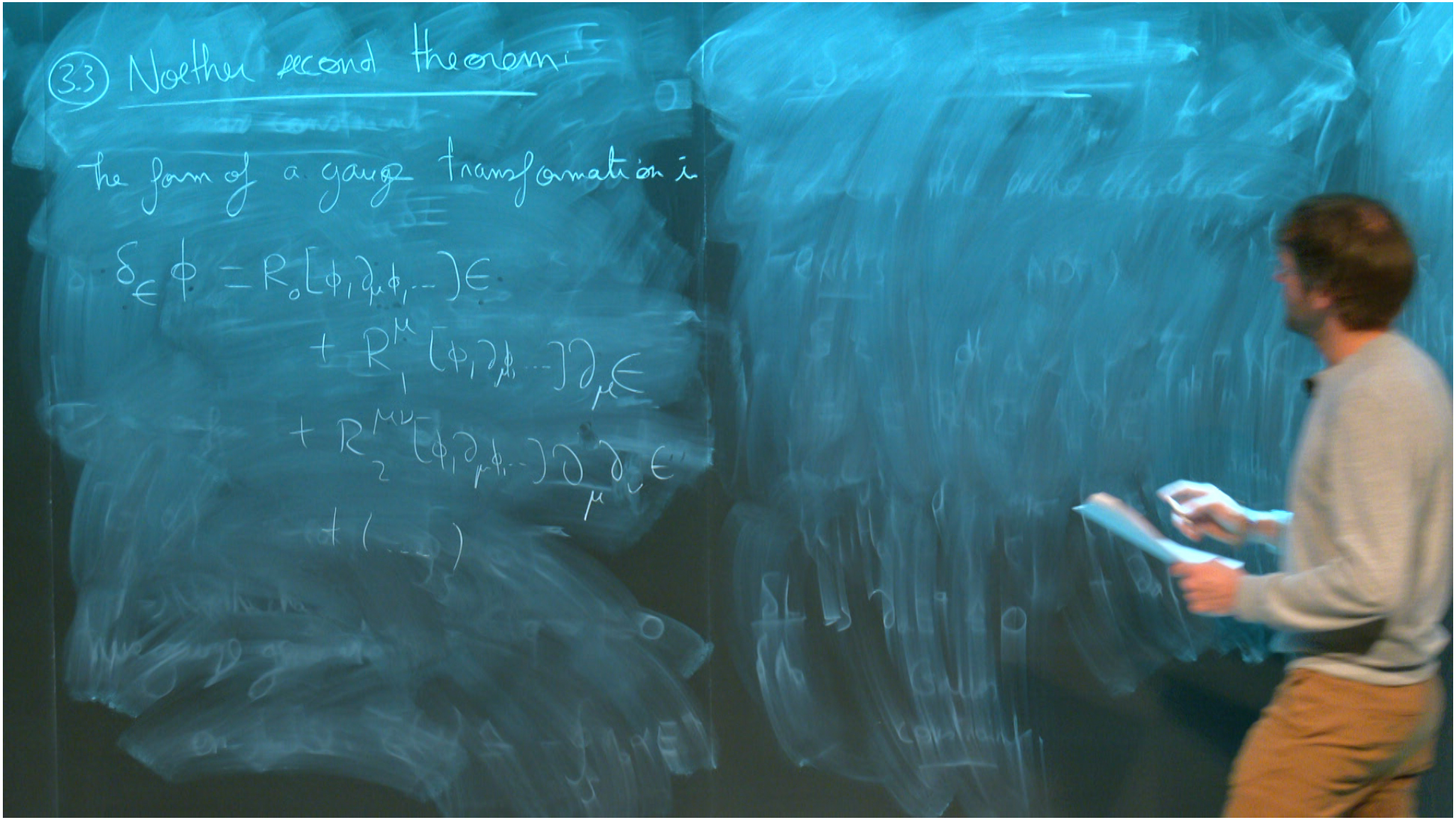
(3.2) Surface terms

In GR, the same structure exists. (ADM)

$$S = \int_{\mathbb{R}\times\Sigma} dt\, d^3x \left(p^{ab} \dot{q}_{ab} - N C_0 - N^a C_a \right)$$

scalar, vector

$$H = H_0 + \lambda^i C_i$$


(3.3) Noether second theorem:
the form of a gauge transformation is
$\delta_\epsilon \phi = R_0[\phi, \partial_\mu \phi, \ldots]\epsilon$
$+ R_1^\mu[\phi, \partial_\mu \phi, \ldots]\partial_\mu \epsilon$
$+ R_2^{\mu\nu}[\phi, \partial_\mu \phi, \ldots]\partial_\mu \partial_\nu \epsilon$
$+ (\ldots)$

(3.3) Noether second theorem

the form of a gauge transformation is

$$\delta_\epsilon \phi = R_0[\phi, \partial_\mu \phi, \dots]\epsilon + R_1^\mu[\phi, \partial_\mu \phi, \dots]\partial_\mu \epsilon + R_2^{\mu\nu}[\phi, \partial_\mu \phi, \dots]\partial_\mu \partial_\nu \epsilon + (\dots)$$

$$\delta L = E\delta\phi + \partial_\mu \theta^\mu$$

For a gauge transformation:

$$\delta_\epsilon L = E\delta_\epsilon \phi + \partial_\mu \theta^\mu(\delta_\epsilon) = \partial_\mu b^\mu_\epsilon$$

Integrating the bulk term by parts, we get

$$E\delta_\epsilon \phi = \left[ER_0 - \partial_\mu(ER_1^\mu) + (\dots)\right]\epsilon - \partial_\mu C^\mu_\epsilon$$

with $C^\mu_\epsilon \stackrel{\wedge}{=} 0$.

If δ_ϵ is a gauge symmetry, then we have the Noether identities:

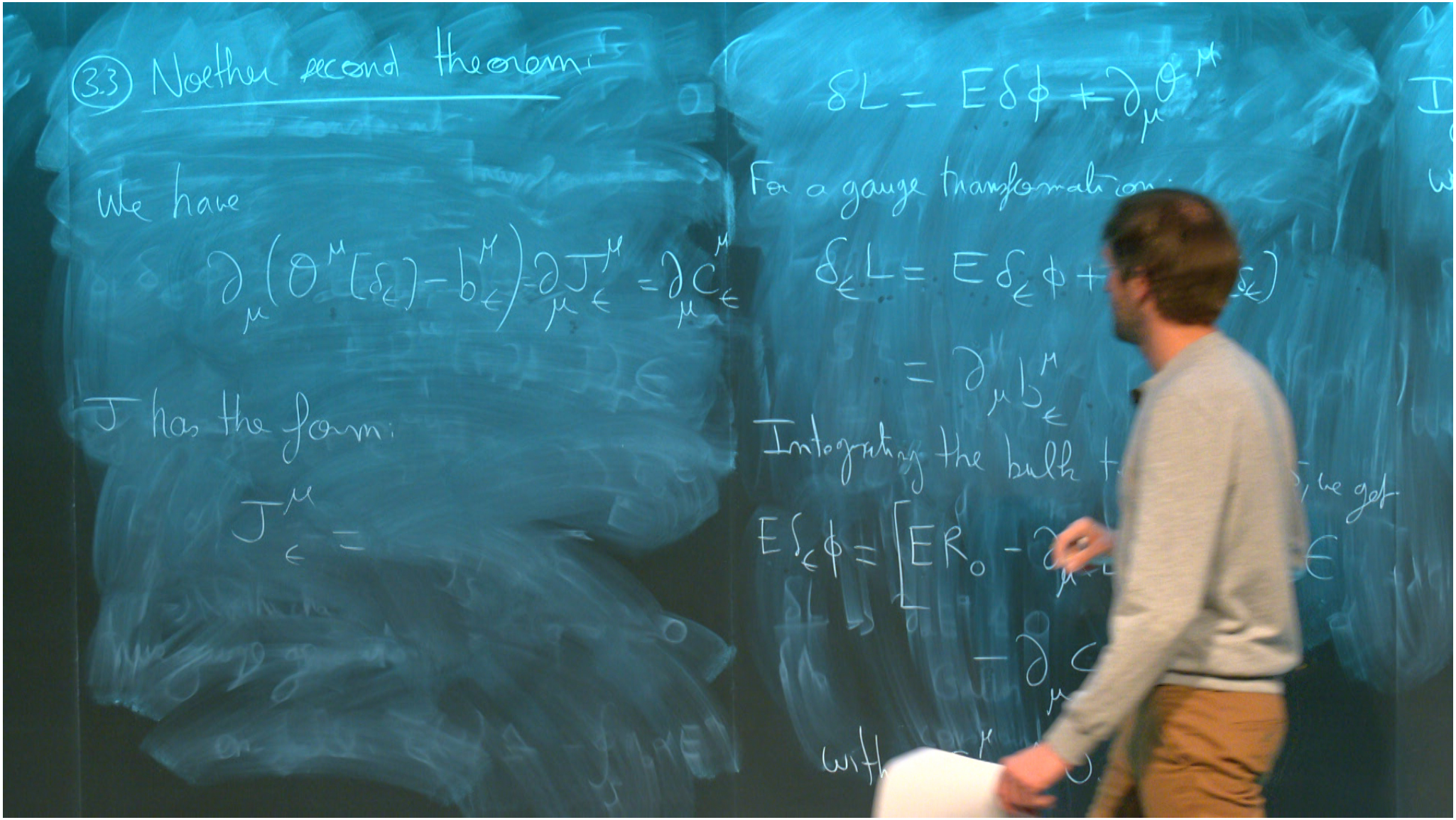
$$ER_0 - \partial_\mu(ER^\mu_1) + (\dots) = 0$$

off-shell

2nd theorem:

$$E\delta_\epsilon\phi = -\partial_\mu C^\mu_\epsilon$$

with $C^\mu_\epsilon \stackrel{\wedge}{=} 0$


(3.3) Noether second theorem:
We have
$\partial_\mu(\theta^\mu(\delta_\epsilon) - b^\mu_\epsilon) = \partial_\mu J^\mu_\epsilon = \partial_\mu C^\mu_\epsilon$
J has the form:
$J^\mu_\epsilon =$
$\delta L = E\delta\phi + \partial_\mu \theta^\mu$
For a gauge transformation:
$\delta_\epsilon L = E\delta_\epsilon\phi +$
$= \partial_\mu b^\mu_\epsilon$
Integrating the bulk t
we get
$E\delta_\epsilon\phi = [ER_0 -$
$-\partial_\mu$
with

(3.3) Noether second theorem:

We have

$$\partial_\mu\left(\theta^\mu[\delta_\epsilon\phi] - b^\mu_\epsilon\right) = \partial_\mu J^\mu_\epsilon = \partial_\mu C^\mu_\epsilon$$

J has the form:

$$\boxed{J^\mu_\epsilon = C^\mu_\epsilon + \partial_\nu Q^{[\mu\nu]}}$$

Noether-Wald "charge"

$$\delta L = E\delta\phi + \partial_\mu \theta^\mu$$

For a gauge transformation:

$$\delta_\epsilon L = E\,\delta_\epsilon\phi + \partial_\mu \theta^\mu[\delta_\epsilon\phi]$$

$$= \partial_\mu b^\mu_\epsilon$$

Integrating the bulk term by parts, we get

$$E\,\delta_\epsilon\phi = \left[ER_0 - \partial_\mu(ER^\mu_1) + (\dots)\right]\epsilon - \partial_\mu C^\mu_\epsilon$$

with $C^\mu_\epsilon \overset{\wedge}{=} 0$.