Title: Noetherâ€™s theorems and gauge symmetries

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Collection: Symmetries Graduate School 2023

Date: January 24, 2023 - 9:00 AM

URL: https://pirsa.org/23010078

(2) Field theory

(2.1) First theorem

$L[\phi(x), \partial_\mu \phi(x)]$

$$\delta L = \left[\frac{\partial L}{\partial \phi} - \partial_\mu\left(\frac{\partial L}{\partial \partial_\mu \phi}\right)\right]\delta\phi + \partial_\mu \Theta^\mu$$

$$= E\,\delta\phi + \partial_\mu \Theta^\mu$$

(pre-) symplectic potential

$$\frac{\partial L}{\partial \partial_\mu \phi}\,\delta\phi$$

Symmetry: $\delta_\epsilon L = \partial_\mu b^\mu_\epsilon$

$$\Rightarrow \partial_\mu J^\mu_\epsilon = -E\,\delta\phi \hat{=} 0$$

where $J^\mu_\epsilon := \Theta^\mu[\delta_\epsilon] - b^\mu_\epsilon$

↑ conserved Noether current.

Ambiguity: $J^\mu \mapsto J^\mu + \partial_\nu a^{[\mu\nu]} + k^\mu$

$k^\mu \hat{=} 0$

Conserved charge:

$$Q_\epsilon = \int_\Sigma d^3x\, J^0_\epsilon$$

$$\frac{dQ_\epsilon}{dt} = \int_\Sigma d^3x\, \partial_t J^0_\epsilon = -\int_\Sigma d^3x\, \partial_a J^a_\epsilon$$

$$= -\oint_{\partial\Sigma} d^2x\, n_a J^a_\epsilon$$

$= 0$ for appropriate boundary conditions.

(2.2)

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$\frac{\partial L}{\partial \partial_\mu \phi}\delta\phi$

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* with a variational principle.

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(2.2) Energy-momentum tensor.

From invariance under $x^\mu \longmapsto x^\mu - \epsilon^\mu$

we can build $J^\mu_\epsilon = -T^\mu{}_\nu \epsilon^\nu$

$\partial_a J^a_\epsilon$

$n_a J^a_\epsilon$

ary

In Maxwell theory

For a global symmetry, we have

$$\delta_\epsilon A_\mu = \epsilon^\alpha \partial_\alpha A_\mu$$

Improved transformation:

$$\delta_\epsilon A_\mu = \epsilon^\alpha \partial_\alpha A_\mu - \partial_\mu (\epsilon^\alpha A_\alpha)$$

$$= -F_{\mu\nu}\epsilon^\nu$$

$$\Rightarrow \delta_\epsilon L = \delta_\epsilon \left(-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}\right) = \partial_\mu (\epsilon^\nu L)$$

$$\Rightarrow J^\mu_\epsilon = \left(4 F^{\mu\rho} F_{\nu\rho} - \delta^\mu_\nu F^2\right)\epsilon_\nu$$

$\Rightarrow$ For $\epsilon^\mu = (1,0,0,0)$

we can build $P^\mu = \int_\Sigma d^3x\, T^\mu{}_0$

$$\Rightarrow P^0 = \frac{1}{2}\int_\Sigma d^3x \left(E^2 + B^2\right)$$

$$= \text{Hamiltonian.}$$



General covariance

$$\delta(\sqrt{g}F) = \frac{1}{2}\sqrt{g}\,g^{\mu\nu}\delta g_{\mu\nu}F + \sqrt{g}\,\delta F$$

$$\Rightarrow \mathcal{L}_\xi(\sqrt{g}F) = (\dots) = \partial_\mu(\xi^\mu\sqrt{g}F)$$

$$\mathcal{L}_\xi L_m = \frac{\delta L_m}{\delta\phi}\mathcal{L}_\xi\phi + \frac{\delta L_m}{\delta g_{\mu\nu}}\mathcal{L}_\xi g_{\mu\nu} + \partial_\mu O^\mu[\mathcal{L}_\xi]$$

$$= (\quad) + 2\frac{\delta L_m}{\delta g_{\mu\nu}}\nabla_\mu\xi_\nu + \partial_\mu O^\mu[\mathcal{L}_\xi]$$

$$= (\quad) + \sqrt{g}\,T^{\mu\nu}\nabla_\mu\xi_\nu + \partial_\mu O^\mu$$

$$= (\quad) - \sqrt{g}\,\nabla_\mu T^{\mu\nu}\xi_\nu + \partial_\mu(O^\mu + \sqrt{g}\,T^{\mu\nu}\xi_\nu)$$

$\Rightarrow$ general covariance implies $\boxed{\nabla_\mu T^{\mu\nu} \hat{=} 0}$

For gravity, we have a "problem":

$$T^{EH+m}_{\mu\nu} = \frac{-2}{\sqrt{g}}\frac{\delta L}{\delta g^{\mu\nu}} = T_{\mu\nu} - \frac{c^4}{8\pi G}G_{\mu\nu}$$

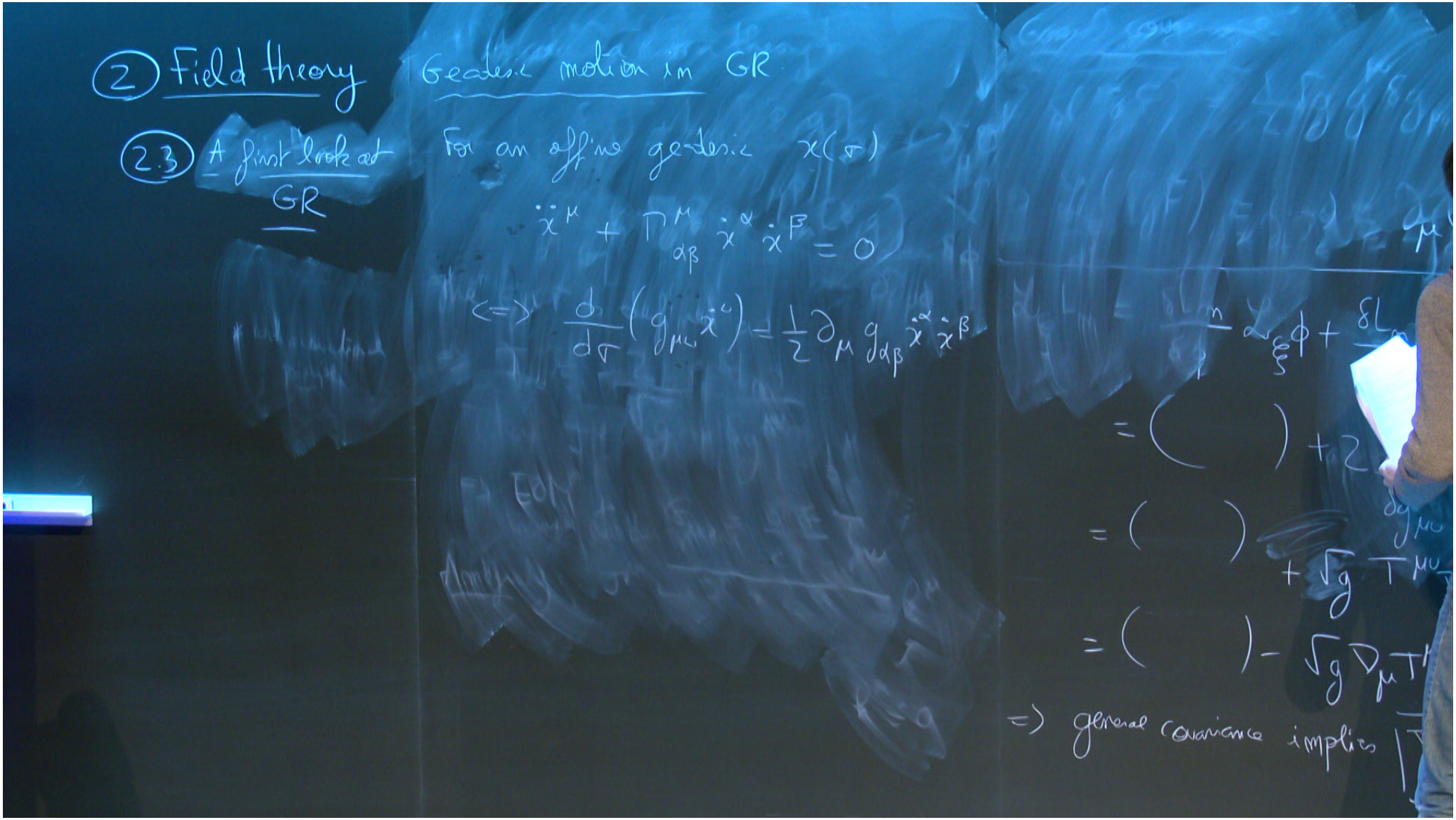
$$\hat{=} 0$$

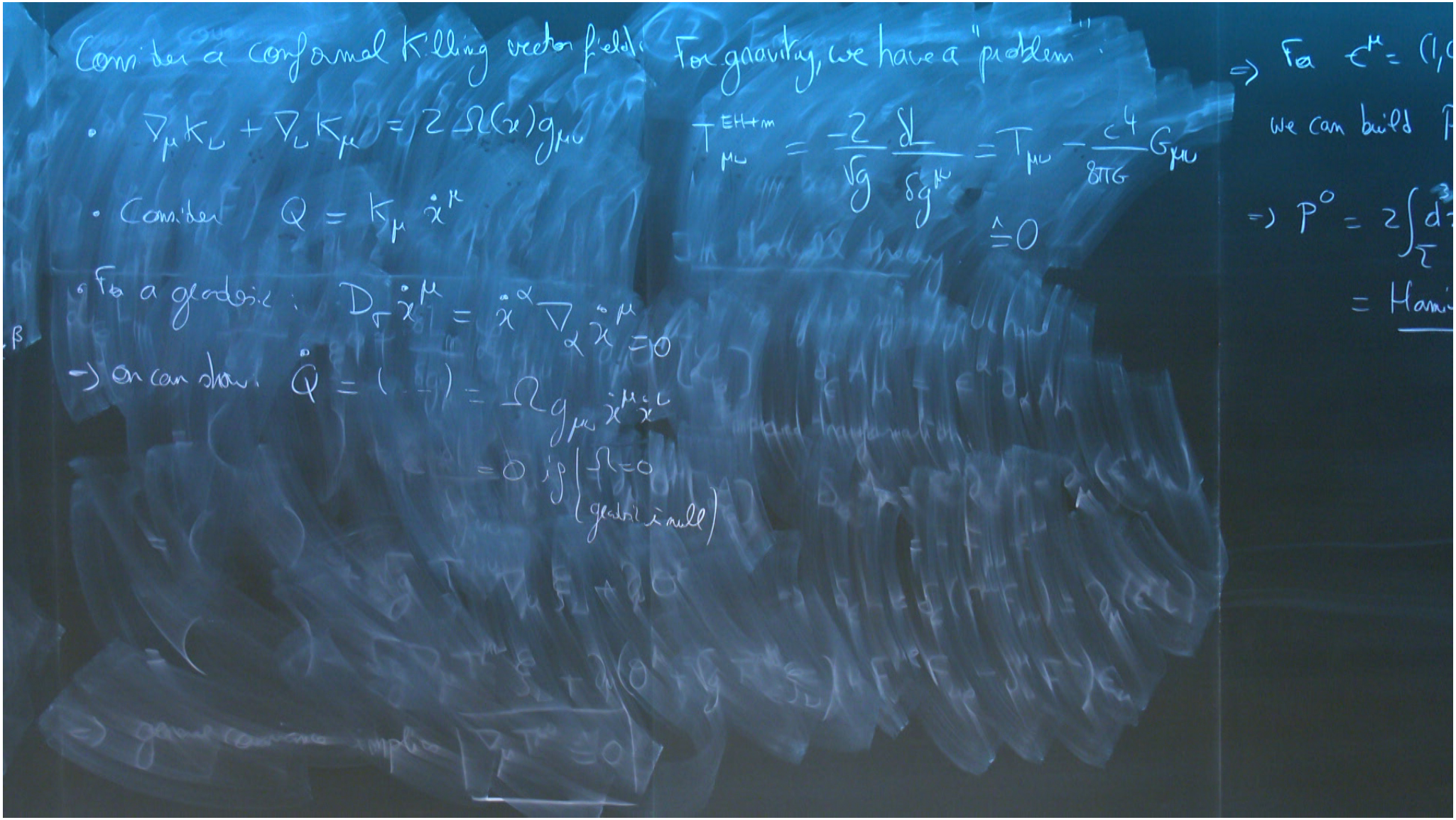
$\Rightarrow$ Fa ϵ^μ =

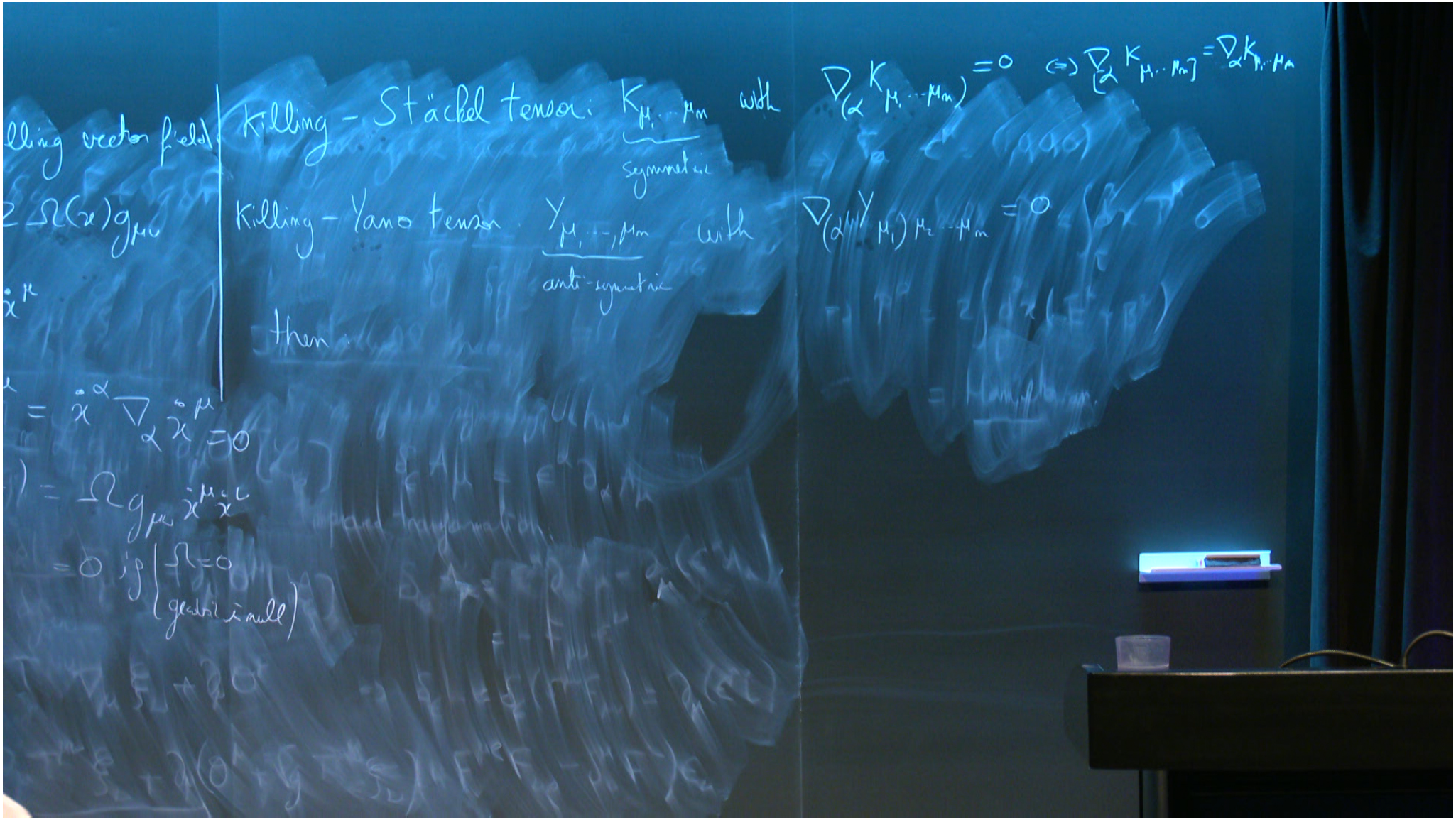
we can build

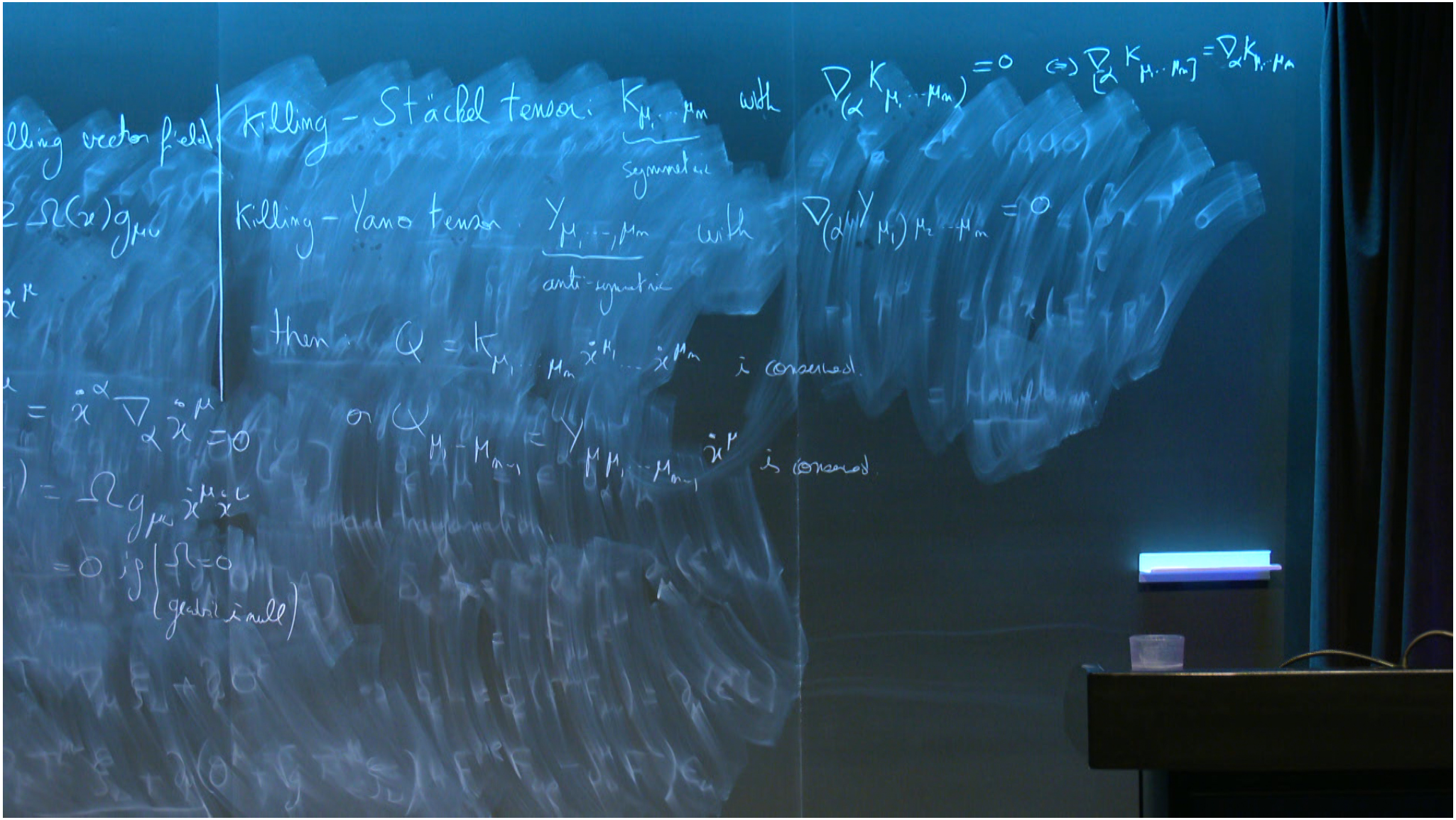
$\Rightarrow P^0 = 2$

$= H$


2 Field theory
2.3 A first look at GR
Geodesic motion in GR
For an affine geodesic $x(\tau)$
$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$
$\Leftrightarrow \frac{d}{d\tau}(g_{\mu\nu}\dot{x}^\nu) = \frac{1}{2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$
$\Rightarrow$ general covariance implies


Consider a conformal Killing vector field:
$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 2\Omega(x) g_{\mu\nu}$
Consider $Q = K_\mu \dot{x}^\mu$
For a geodesic: $D_\tau \dot{x}^\mu = \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0$
$\dot{Q} = (\ ,\ ) = \Omega g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$
$= 0$ if $\Omega = 0$
(geodesic is null)
For gravity, we have a "problem":
$T^{EH+m}_{\mu\nu} = \frac{-2}{\sqrt{g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = T_{\mu\nu} - \frac{c^4}{8\pi G} G_{\mu\nu}$
$\hat{=} 0$
$\Rightarrow$ For $\epsilon^\mu = (1,$
we can build
$\Rightarrow P^0 = 2\int_\Sigma d^3$
$=$ Hamil


Killing vector field
Killing – Stäckel tensor: $K_{\mu_1 \dots \mu_n}$ with $\nabla_{(\alpha} K_{\mu_1 \dots \mu_n)} = 0$ $\Leftrightarrow$ $\nabla_{[\alpha} K_{\mu_1 \dots \mu_n]} = \nabla_\alpha K_{\mu_1 \dots \mu_n}$
symmetric
Killing – Yano tensor $Y_{\mu_1 \dots \mu_n}$ with $\nabla_{(\alpha} Y_{\mu_1) \mu_2 \dots \mu_n} = 0$
anti-symmetric
then
$\dot{x}^\mu$
$= \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0$
$= \Omega g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$
$= 0$ if $\Omega = 0$
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Killing – Yano tensor $Y_{\mu_1 \dots \mu_n}$ with $\nabla_{(\alpha} Y_{\mu_1) \mu_2 \dots \mu_n} = 0$
anti-symmetric
then $Q = K_{\mu_1 \dots \mu_n} \dot{x}^{\mu_1} \dots \dot{x}^{\mu_n}$ is conserved
or $Q_{\mu_1 \dots \mu_{n-1}} = Y_{\mu \mu_1 \dots \mu_{n-1}} \dot{x}^\mu$ is conserved
$= \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0$
$= \Omega g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$
$= 0$ if $\Omega = 0$
(gradient is null)

For Kerr, there are two null vectors

l and k $\quad (l^2 = 0 = k^2)$

$l \cdot k = -1$ $\quad \rho^2 = r^2 + a^2 \cos^2\theta$

and $K_{\mu\nu} = \rho^2 (l_\mu k_\nu + l_\nu k_\mu) + r^2 g_{\mu\nu}$

$\Rightarrow Q_C = K_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$ is the Carter constant of Kerr spacetime

Consider a conformal Killing vector field

- $\nabla_\mu K_\nu + \nabla_\nu K_\mu = 2\Omega(x) g_{\mu\nu}$
- Consider $Q = K_\mu \dot{x}^\mu$
- For a geodesic: $D_\tau \dot{x}^\mu = \dot{x}^\alpha \nabla_\alpha \dot{x}^\mu = 0$

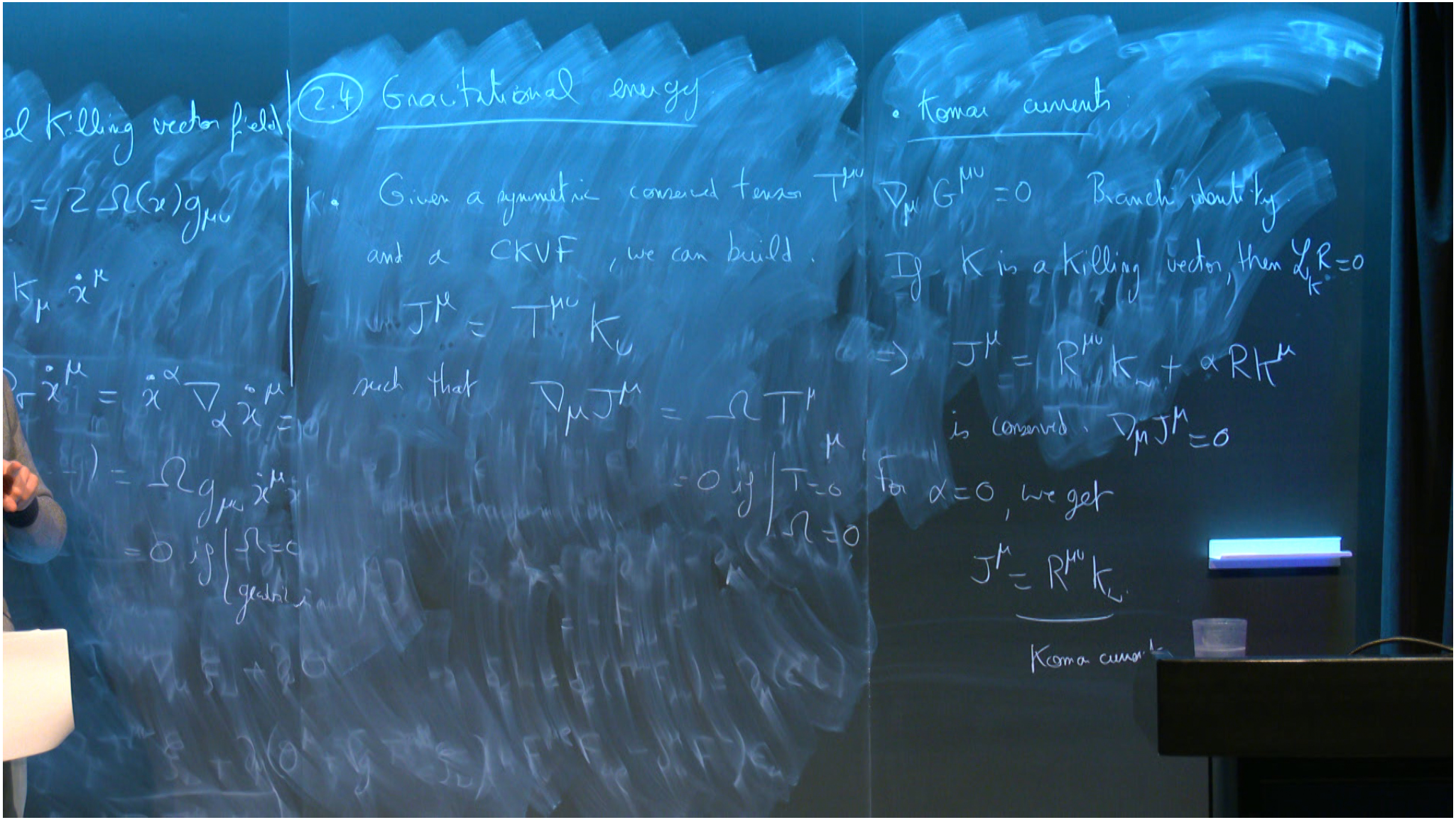
$\Rightarrow$ one can show: $\dot{Q} = (\ldots) = \Omega g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$

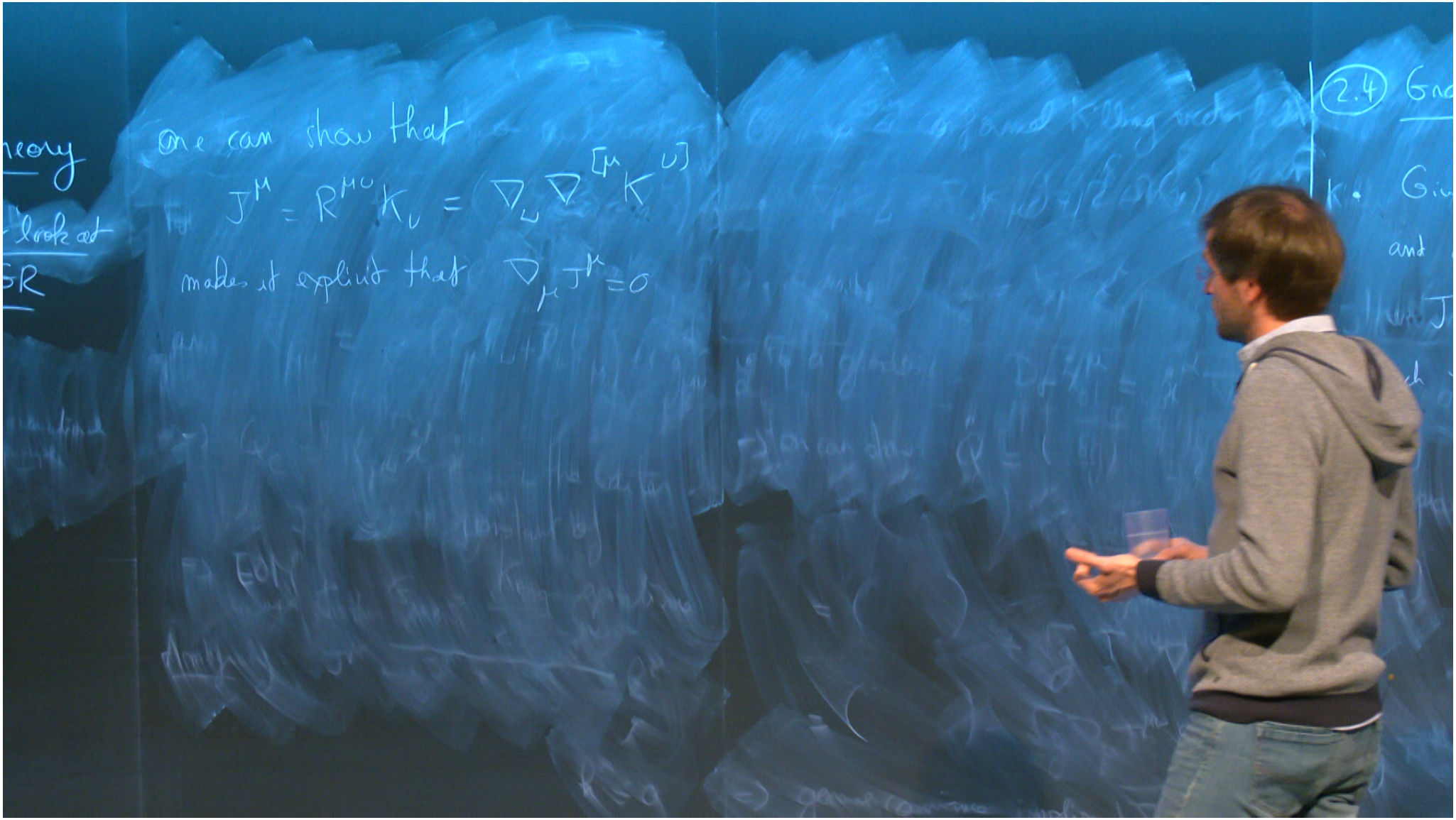
$= 0$ if $\begin{cases} \Omega = 0 \\ \text{(geodesic is null)} \end{cases}$

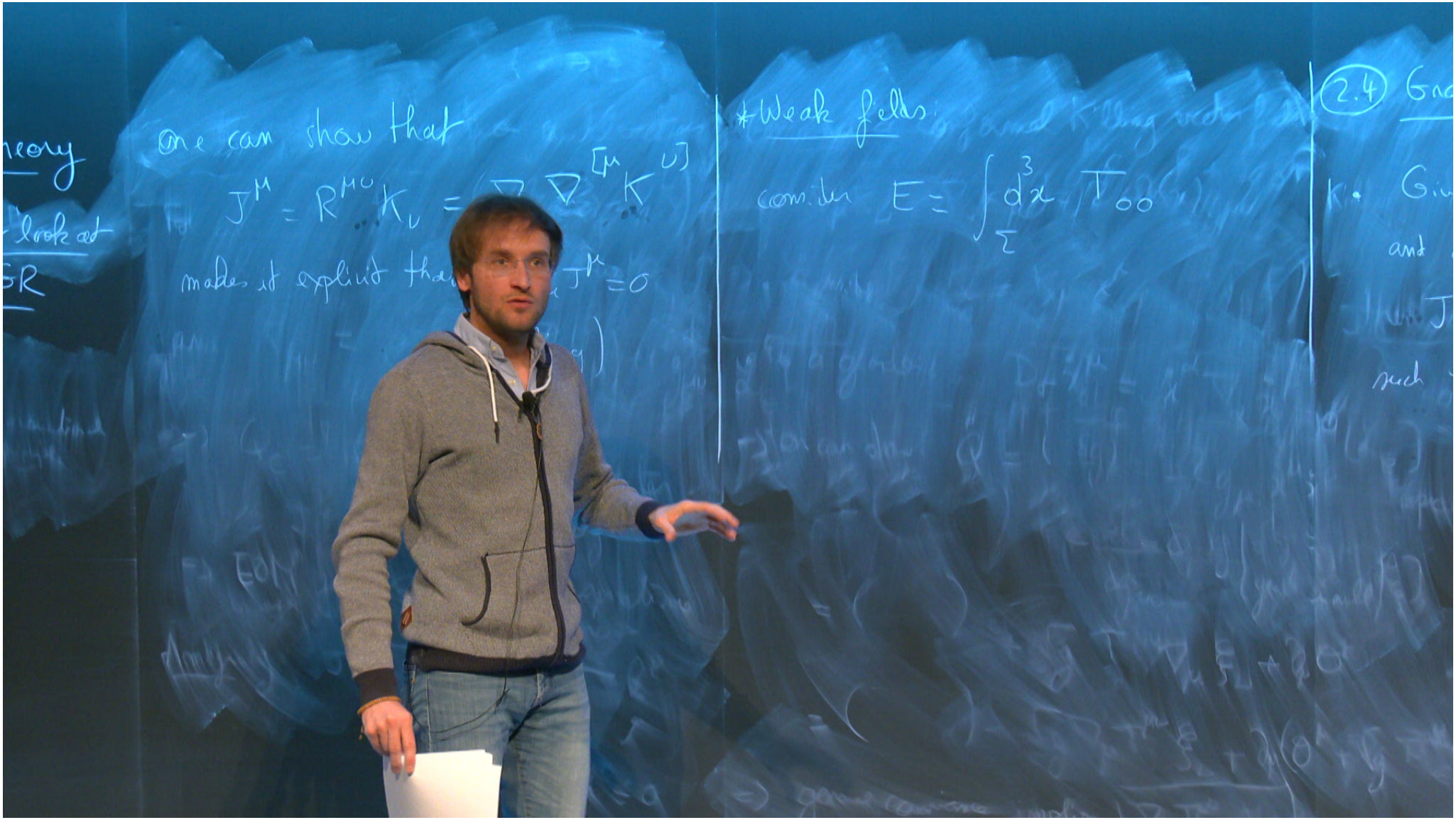
Killing – S

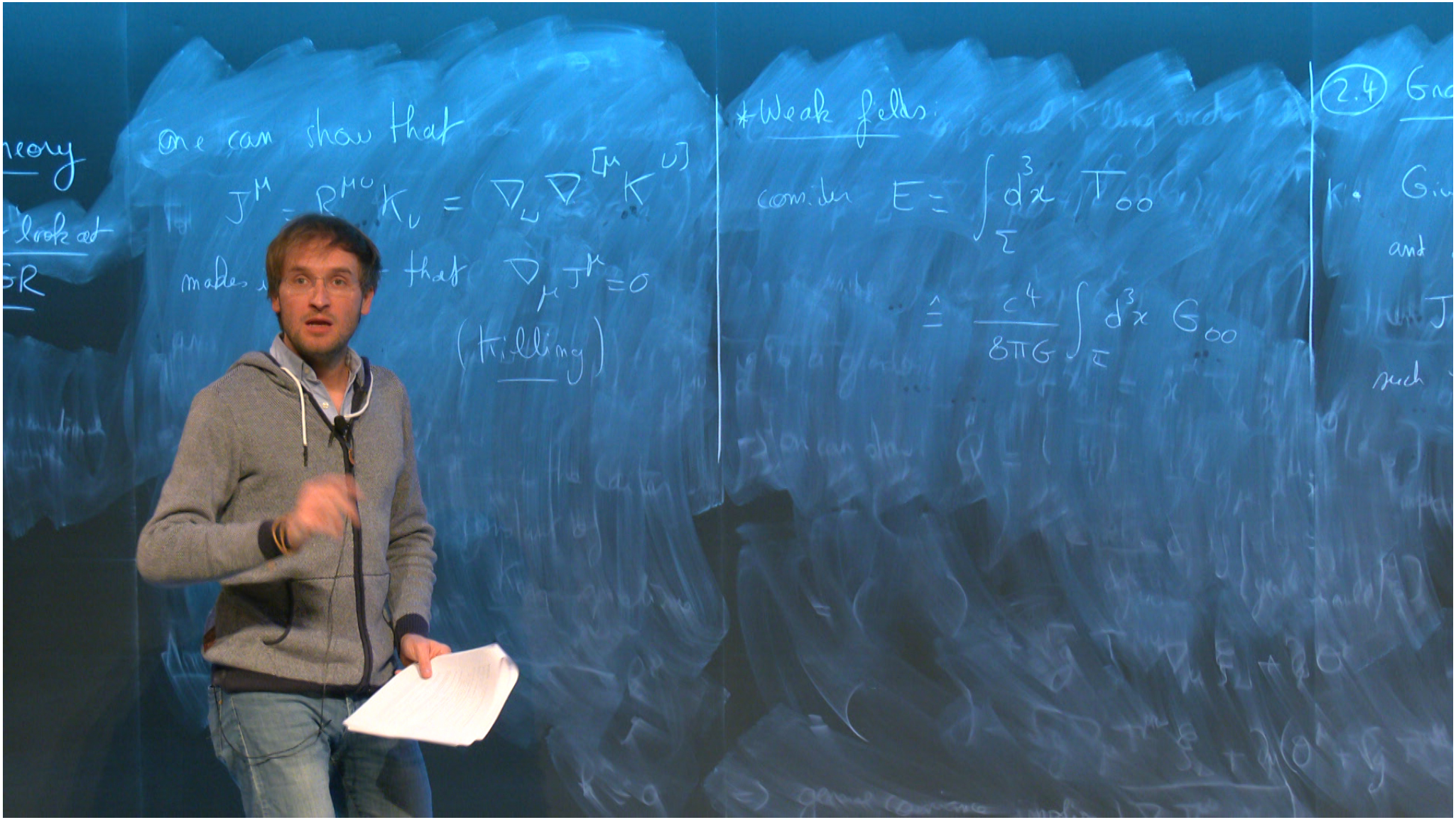
Killing – Yan

then


Killing vector field
2.4 Gravitational energy
Given a symmetric conserved tensor $T^{\mu\nu}$ and a CKVF, we can build
$J^\mu = T^{\mu\nu} K_\nu$
such that $\nabla_\mu J^\mu = \Omega T^\mu_\mu$
$=0$ if $T=0$ or $\Omega=0$
Komar currents
$\nabla_\mu G^{\mu\nu} = 0$ Bianchi identity
If K is a Killing vector, then $\mathcal{L}_K R = 0$
$\Rightarrow J^\mu = R^{\mu\nu} K_\nu + \alpha R K^\mu$
is conserved. $\nabla_\mu J^\mu = 0$
for $\alpha = 0$, we get
$J^\mu = R^{\mu\nu} K_\nu$
Komar current


One can show that
$J^\mu = R^{\mu\nu} K_\nu = \nabla_\nu \nabla^{[\mu} K^{\nu]}$
makes it explicit that $\nabla_\mu J^\mu = 0$


One can show that
$J^\mu = R^{\mu\nu}{}_{\dots} K_\nu = \nabla_\nu \nabla^{[\mu} K^{\nu]}$
$\nabla_\mu J^\mu = 0$
*Weak fields:
consider $E = \int_\Sigma d^3x \, T_{00}$
2.4


One can show that
$J^\mu = R^{\mu\nu} K_\nu = \nabla_\nu \nabla^{[\mu} K^{\nu]}$
$\nabla_\mu J^\mu = 0$
(Killing)
*Weak fields:
consider $E = \int_\Sigma d^3x \, T_{00}$
$\hat{=} \frac{c^4}{8\pi G} \int_\Sigma d^3x \, G_{00}$
2.4