Title: Noetherâ€™s theorems and gauge symmetries

Speakers: Marc Geiller

Collection: Symmetries Graduate School 2023

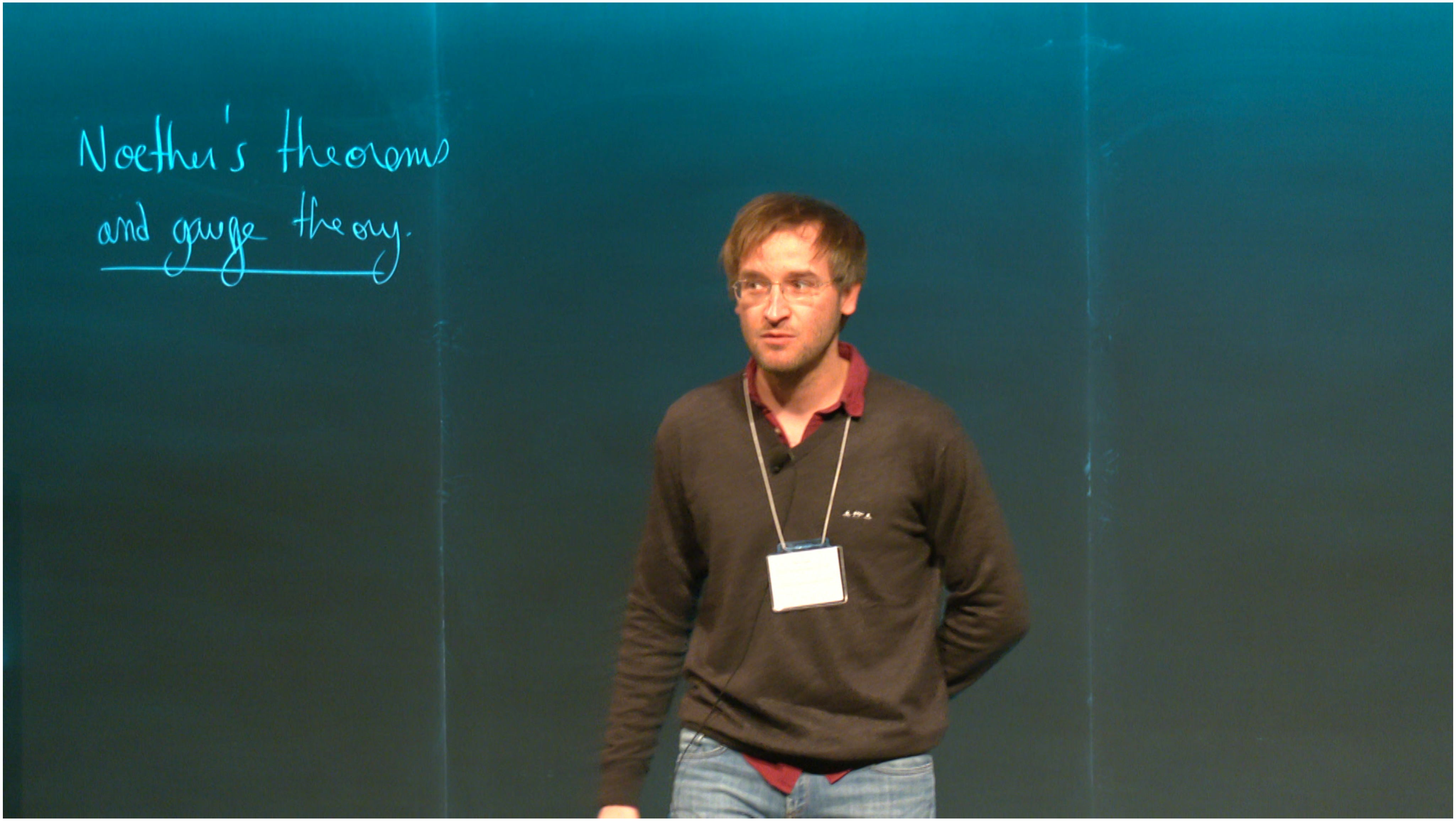
Date: January 23, 2023 - 9:10 AM

URL: https://pirsa.org/23010074

Abstract: This lecture is devoted to Noether's theorems and the study of the interplay between symmetries and conservation laws, from ordinary mechanics to general relativity. In order to start on a common ground and interest a broad audience, we will begin with a review of Noether's (first) theorem in ordinary non-relativistic mechanics. This will enable us to settle some subtleties, agree on conventions, and especially explore some curious and lesser-known symmetry features of familiar models (such as particles and celestial mechanics). We will then move on to field theory, and discuss the construction of conserved currents and energy-momentum tensors. This will include a discussion of conserved quantities in general relativity. Finally, we will turn to the core of the topic, which is Noether's (second) theorem for gauge symmetries. After recalling the basic properties of gauge theories in Lagrangian and Hamiltonian form, we will derive the consequences of gauge symmetry for the construction of conserved charges. For this, we will introduce the so-called covariant phase space formalism, which enables to construct symmetry charges and algebras, and derive (non) conservation laws. This will be illustrated in Maxwell's theory and in general relativity. In particular, we will focus in depth on the example of three-dimensional gravity as an exactly soluble model in which all aspects of symmetries can be understood. We will end with an outlook towards the notion of asymptotic symmetries and their use in classical and quantum gravity.

Ideally, the audience should be familiar with:

Hamiltonian mechanics
differential forms
basic features of general relativity


Noether's theorems
and gauge theory.

(1) Mechanics

(1.1) Noether's 1st theorem

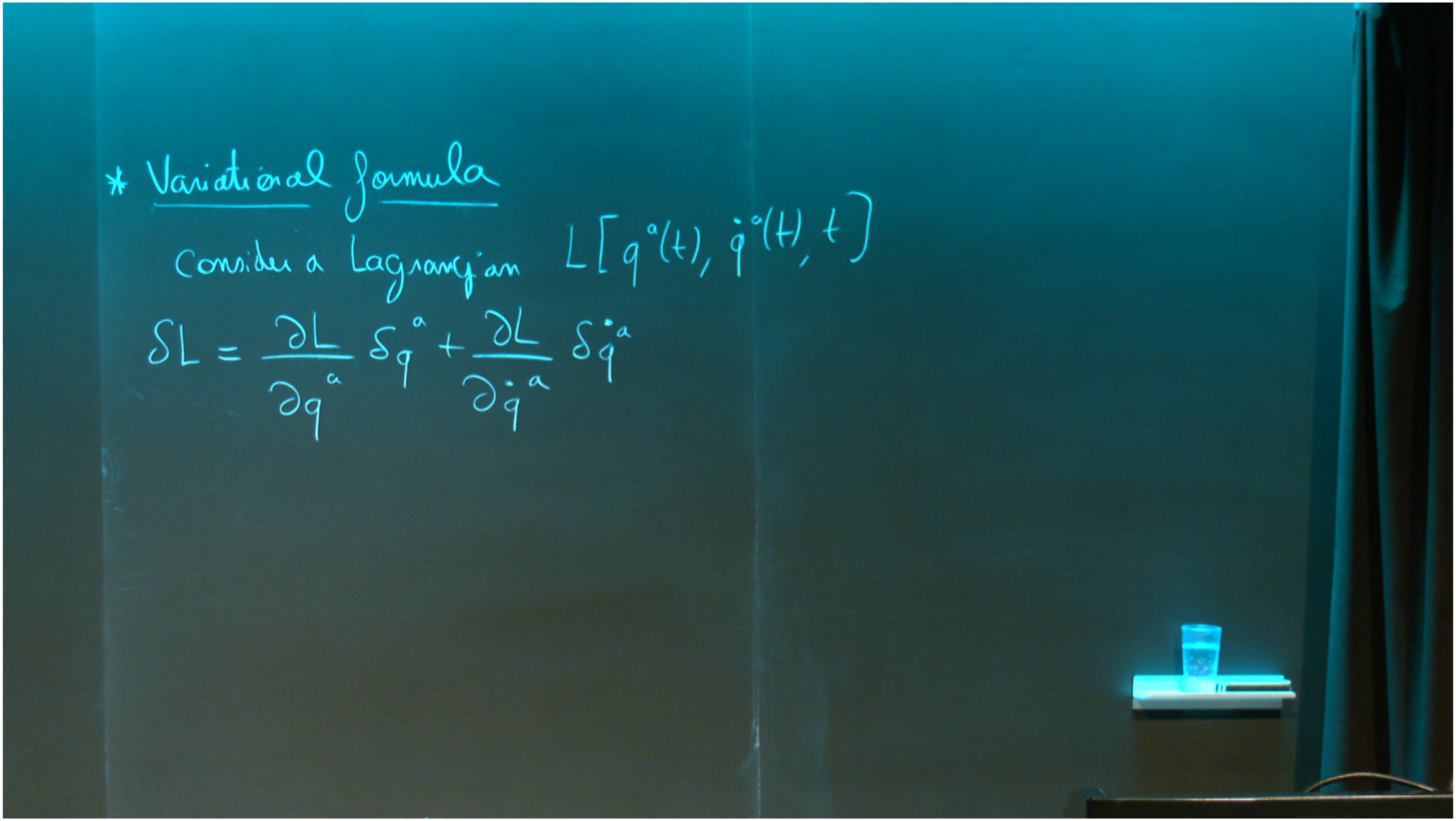
* Variations

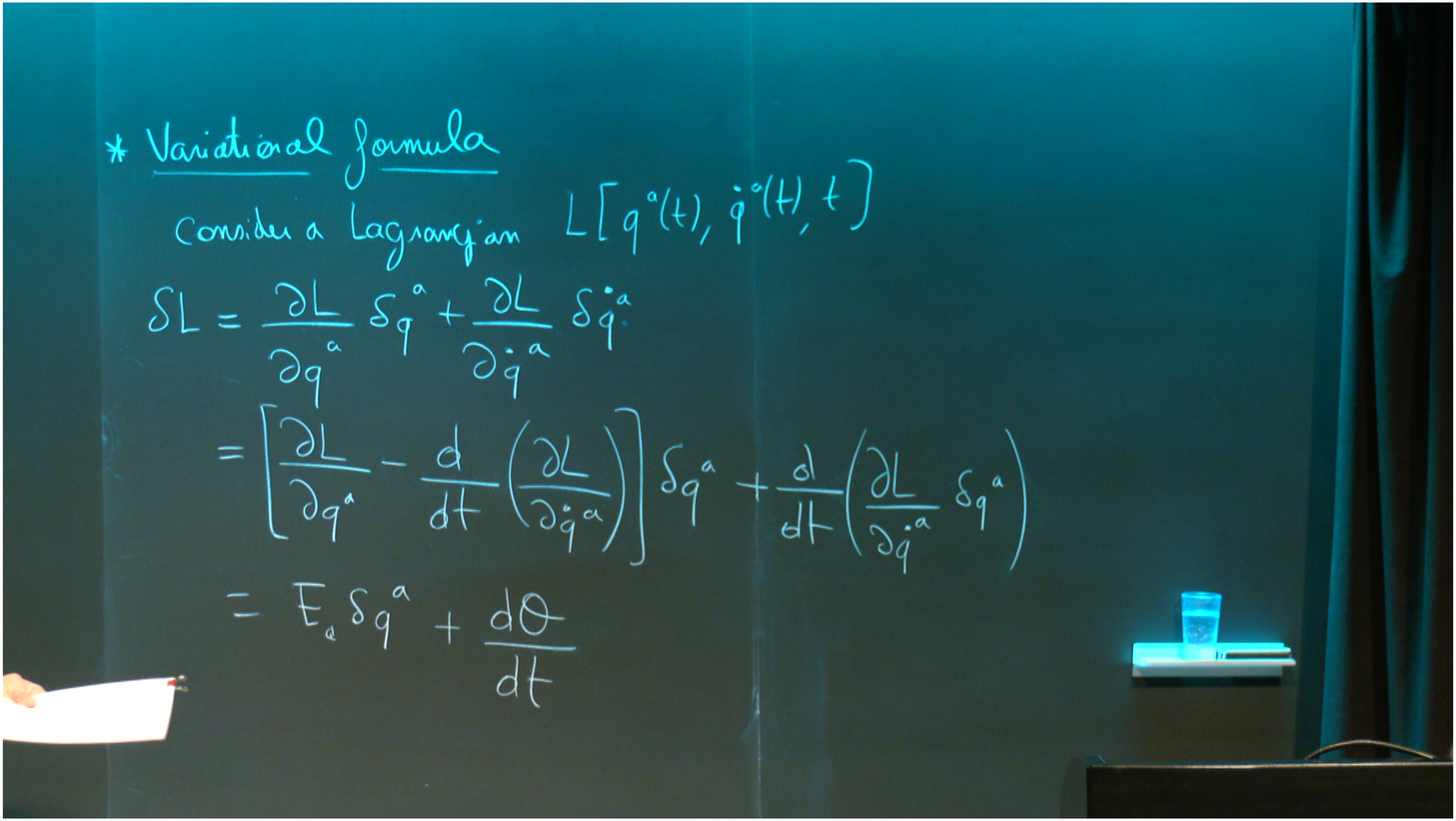
Coordinates q^a.

A variation is a displacement $q^a(t) \mapsto q^a(t) + \delta q^a(t)$

* Variational formula

Consider a Lagrangian $L[q^a(t), \dot{q}^a(t), t]$


* Variational formula
Consider a Lagrangian $L[q^a(t), \dot{q}^a(t), t]$
$\delta L = \frac{\partial L}{\partial q^a} \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a$


* Variational formula
consider a Lagrangian $L[q^a(t), \dot{q}^a(t), t]$
$\delta L = \frac{\partial L}{\partial q^a} \delta q^a + \frac{\partial L}{\partial \dot{q}^a} \delta \dot{q}^a$
$= \left[\frac{\partial L}{\partial q^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a}\right)\right] \delta q^a + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^a} \delta q^a\right)$
$= E_a \delta q^a + \frac{d\Theta}{dt}$

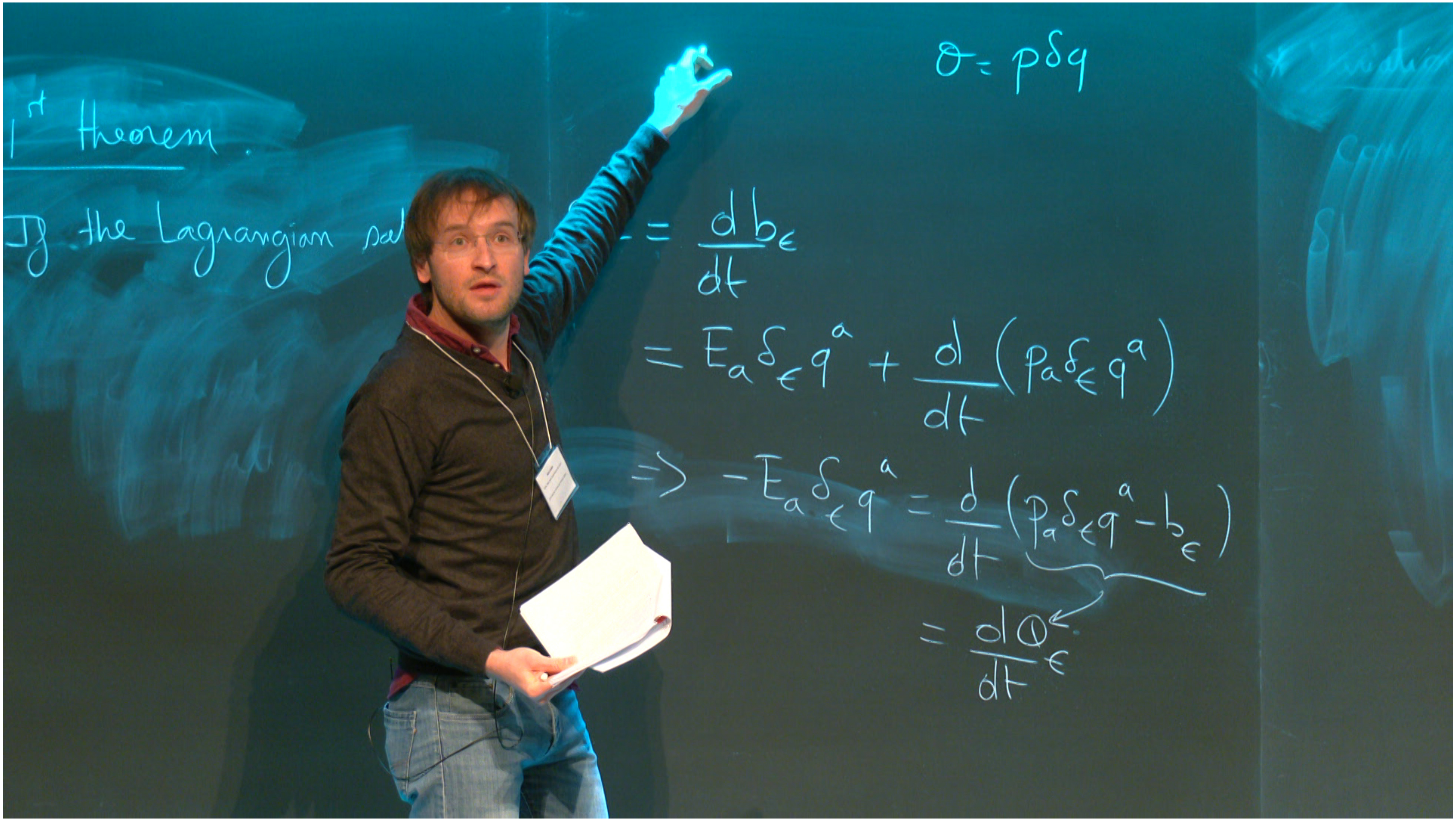
* 1st theorem.

If the Lagrangian satisfies $\delta_\epsilon L = \frac{d b_\epsilon}{dt}$

$$= E_a \delta_\epsilon q^a + \frac{d}{dt}\left(p_a \delta_\epsilon q^a\right)$$

$$\Rightarrow -E_a \delta_\epsilon q^a = \frac{d}{dt}\left(p_a \delta_\epsilon q^a - b_\epsilon\right)$$

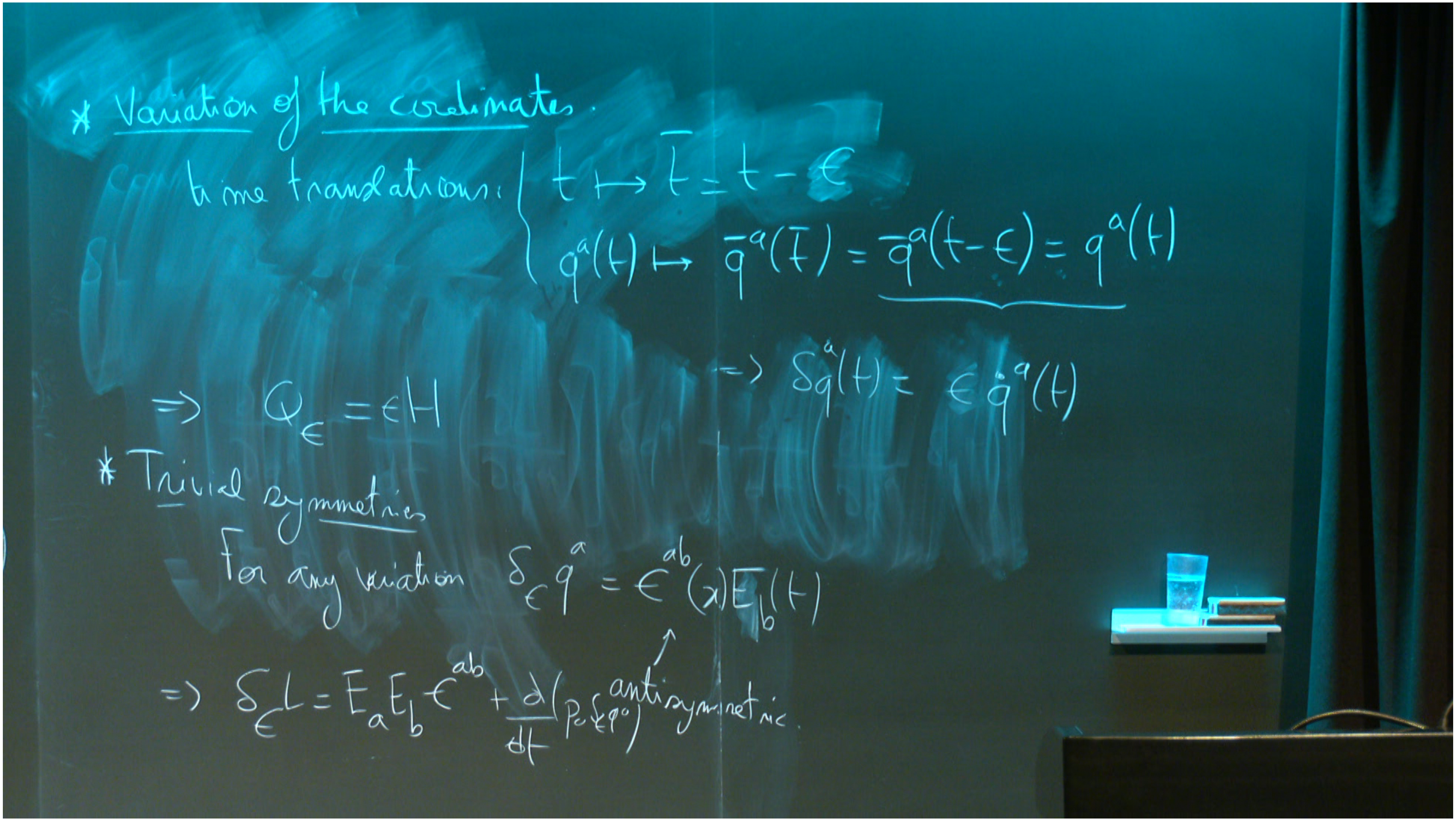
$$= \frac{dQ_\epsilon}{dt}$$


1st theorem.
If the Lagrangian sat
$\theta = p\delta q$
$= \frac{d b_\epsilon}{dt}$
$= E_a \delta_\epsilon q^a + \frac{d}{dt}(p_a \delta_\epsilon q^a)$
$\Rightarrow -E_a \delta_\epsilon q^a = \frac{d}{dt}(p_a \delta_\epsilon q^a - b_\epsilon)$
$= \frac{d Q_\epsilon}{dt}$

* Variation of the coordinates.

time translations:
$$\begin{cases} t \mapsto \bar{t} = t - \epsilon \\ q^a(t) \mapsto \bar{q}^a(\bar{t}) = \underbrace{\bar{q}^a(t-\epsilon) = q^a(t)} \end{cases}$$

$$\Rightarrow \delta q^a(t) = \epsilon \dot{q}^a(t)$$


* Variation of the coordinates.
time translations: $t \mapsto \bar{t} = t - \epsilon$
$q^a(t) \mapsto \bar{q}^a(\bar{t}) = \bar{q}^a(t-\epsilon) = q^a(t)$
$\Rightarrow \delta q^a(t) = \epsilon \dot{q}^a(t)$
$\Rightarrow Q_\epsilon = \epsilon H$
* Trivial symmetries
For any variation $\delta_\epsilon q^a = \epsilon^{ab}(x) E_b(t)$
antisymmetric
$\Rightarrow \delta_\epsilon L = E_a E_b \epsilon^{ab} + \frac{d}{dt}(p_a \delta_\epsilon q^a)$

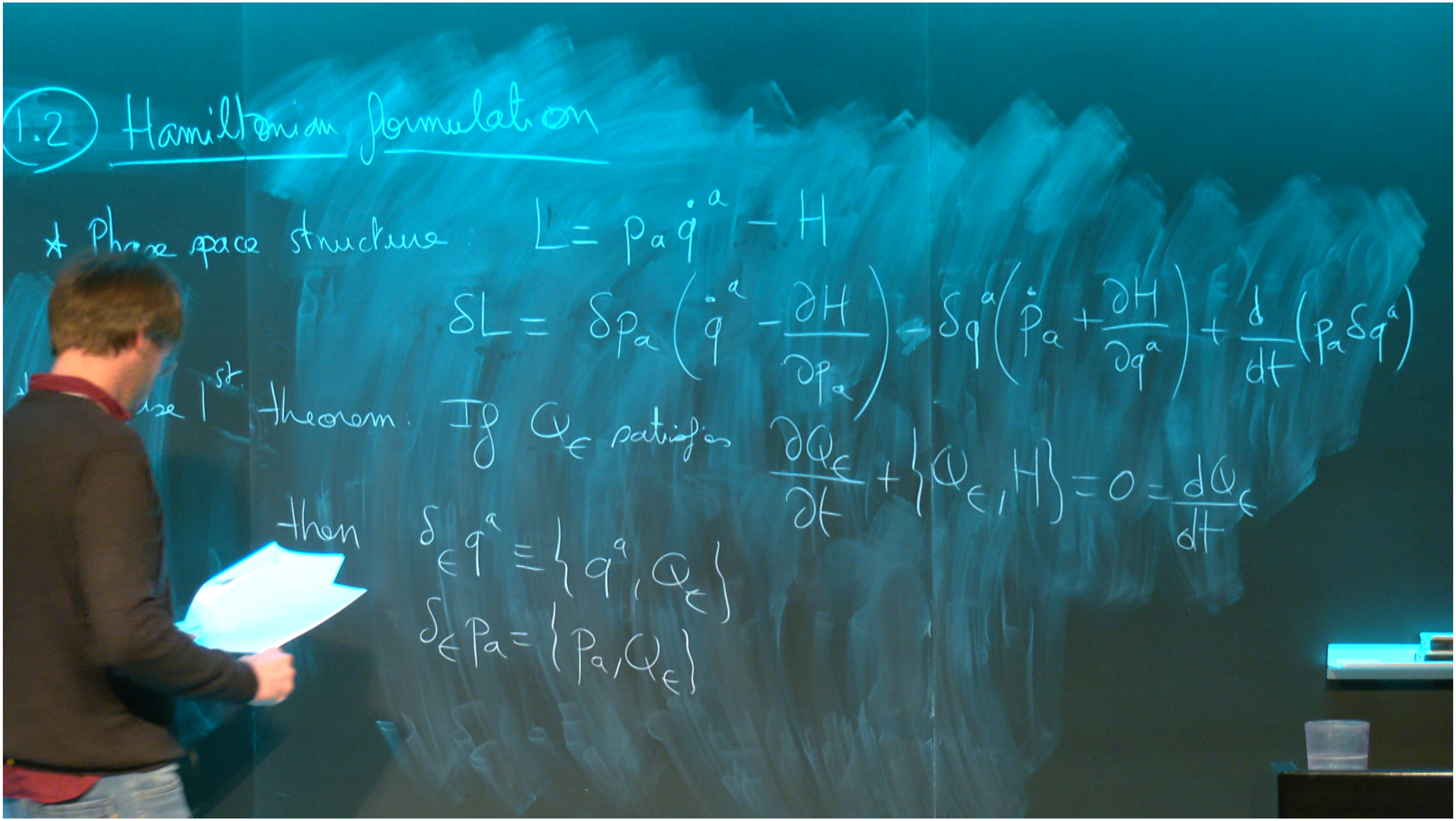
* Short cut

We can compute $-E_a \delta_\epsilon q^a = \frac{d Q_\epsilon}{dt}$

Noether charge (Q_ϵ)

* Lagrangian gauge transformations

If we define $L' = L + \frac{d l}{dt}$


1.2 Hamiltonian formulation
* Phase space structure: $L = p_a \dot{q}^a - H$
$\delta L = \delta p_a \left(\dot{q}^a - \frac{\partial H}{\partial p_a} \right) - \delta q^a \left(\dot{p}_a + \frac{\partial H}{\partial q^a} \right) + \frac{d}{dt}\left(p_a \delta q^a \right)$
1st theorem: If Q_ϵ satisfies $\frac{\partial Q_\epsilon}{\partial t} + \{ Q_\epsilon , H \} = 0 = \frac{dQ_\epsilon}{dt}$
then $\delta_\epsilon q^a \equiv \{ q^a , Q_\epsilon \}$
$\delta_\epsilon p_a = \{ p_a , Q_\epsilon \}$

1.2 Hamiltonian formulation

* Phase space structure: $L = p_a \dot{q}^a - H$

$$\delta L = \delta p_a \left(\dot{q}^a - \frac{\partial H}{\partial p_a} \right) - \delta q^a \left(\dot{p}_a + \frac{\partial H}{\partial q^a} \right) + \frac{d}{dt}\left(p_a \delta q^a \right)$$

* Inverse 1st theorem: If Q_ϵ satisfies $\frac{\partial Q_\epsilon}{\partial t} + \{ Q_\epsilon, H \} = 0 = \frac{d Q_\epsilon}{dt}$

then $\delta_\epsilon q^a \equiv \{ q^a, Q_\epsilon \}$

$\delta_\epsilon p_a = \{ p_a, Q_\epsilon \} \Rightarrow \delta_\epsilon L = \frac{d}{dt}\left(p_a \delta_\epsilon q^a - Q_\epsilon \right)$

1.2 Hamiltonian formulation

* Phase space structure: $L = p_a \dot{q}^a - H$

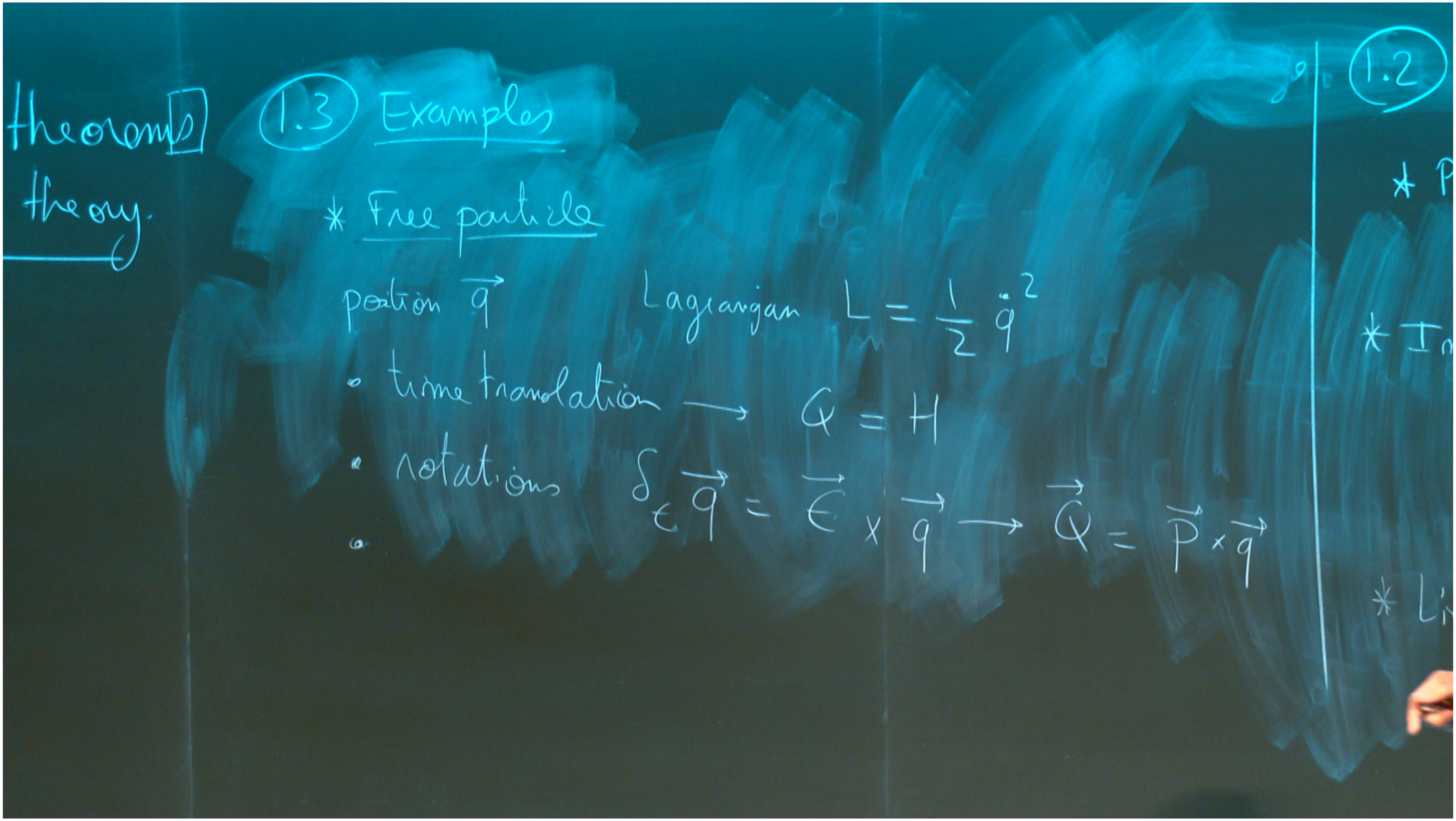
$$\delta L = \delta p_a \left(\dot{q}^a - \frac{\partial H}{\partial p_a} \right) - \delta q^a \left(\dot{p}_a + \frac{\partial H}{\partial q^a} \right) + \frac{d}{dt}\left(p_a \delta q^a \right)$$

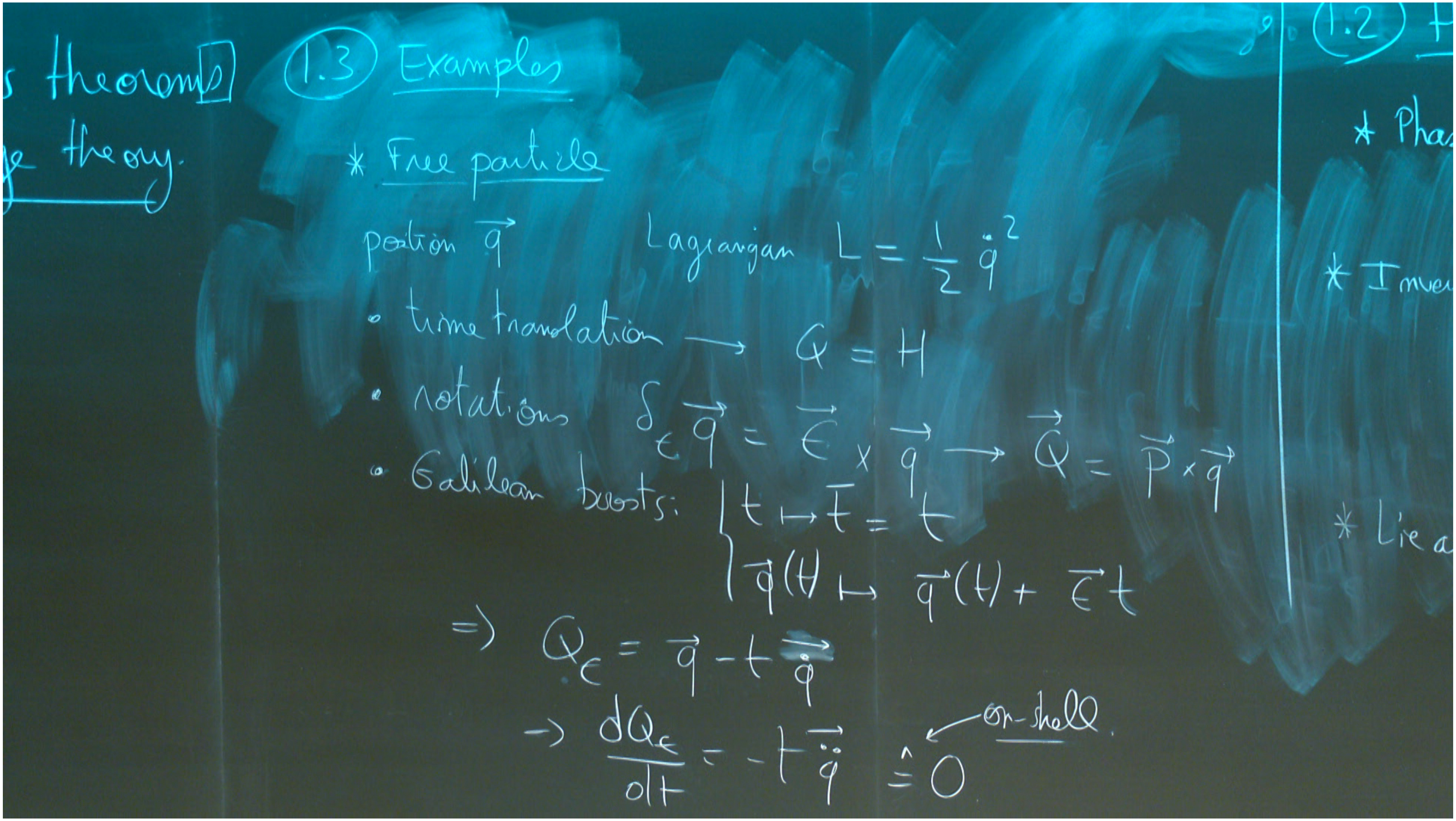
* Inverse 1st theorem: If Q_ϵ satisfies $\frac{\partial Q_\epsilon}{\partial t} + \{ Q_\epsilon, H \} = 0 = \frac{d Q_\epsilon}{dt}$

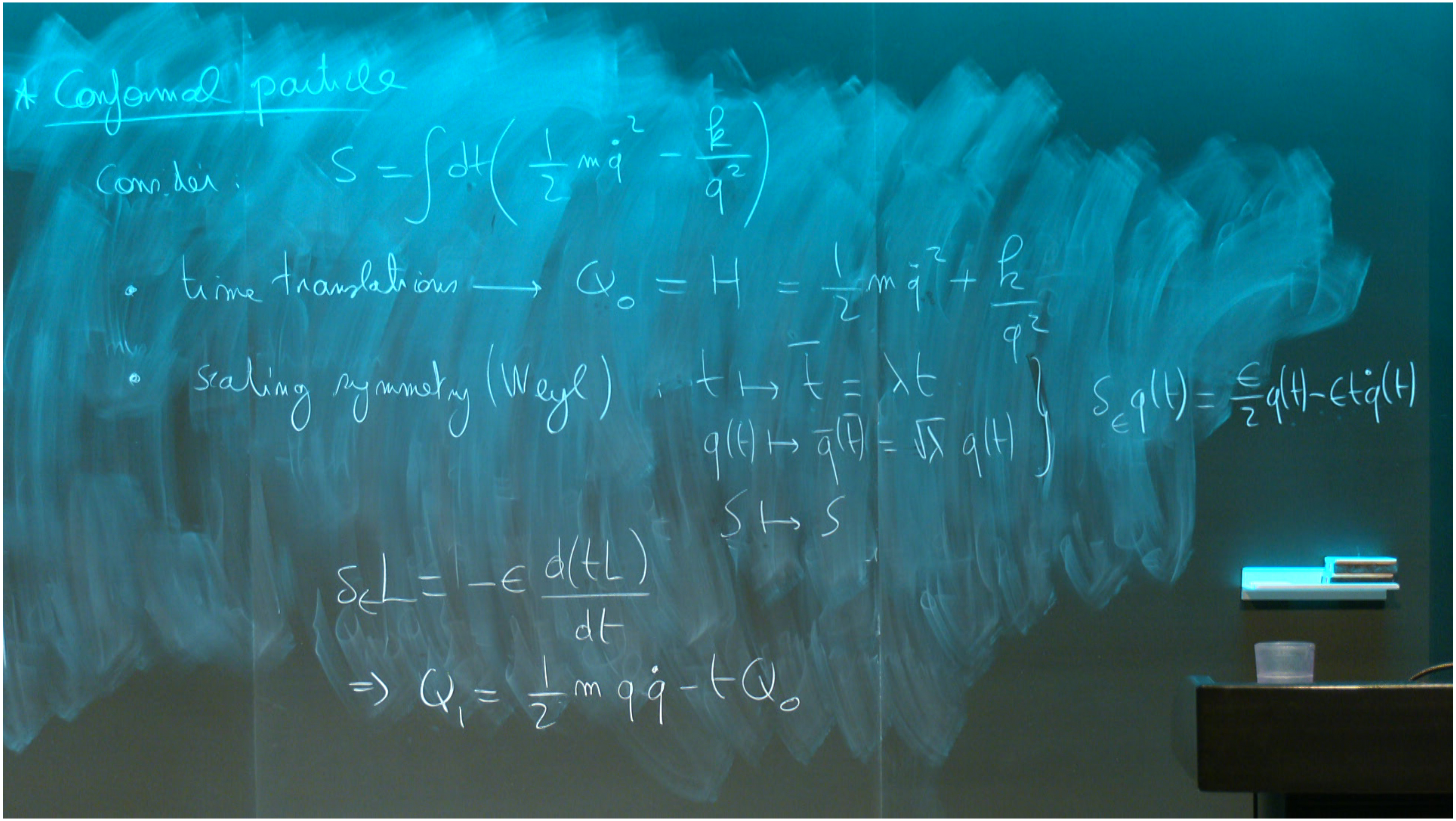
then $\delta_\epsilon q^a \equiv \{ q^a, Q_\epsilon \}$

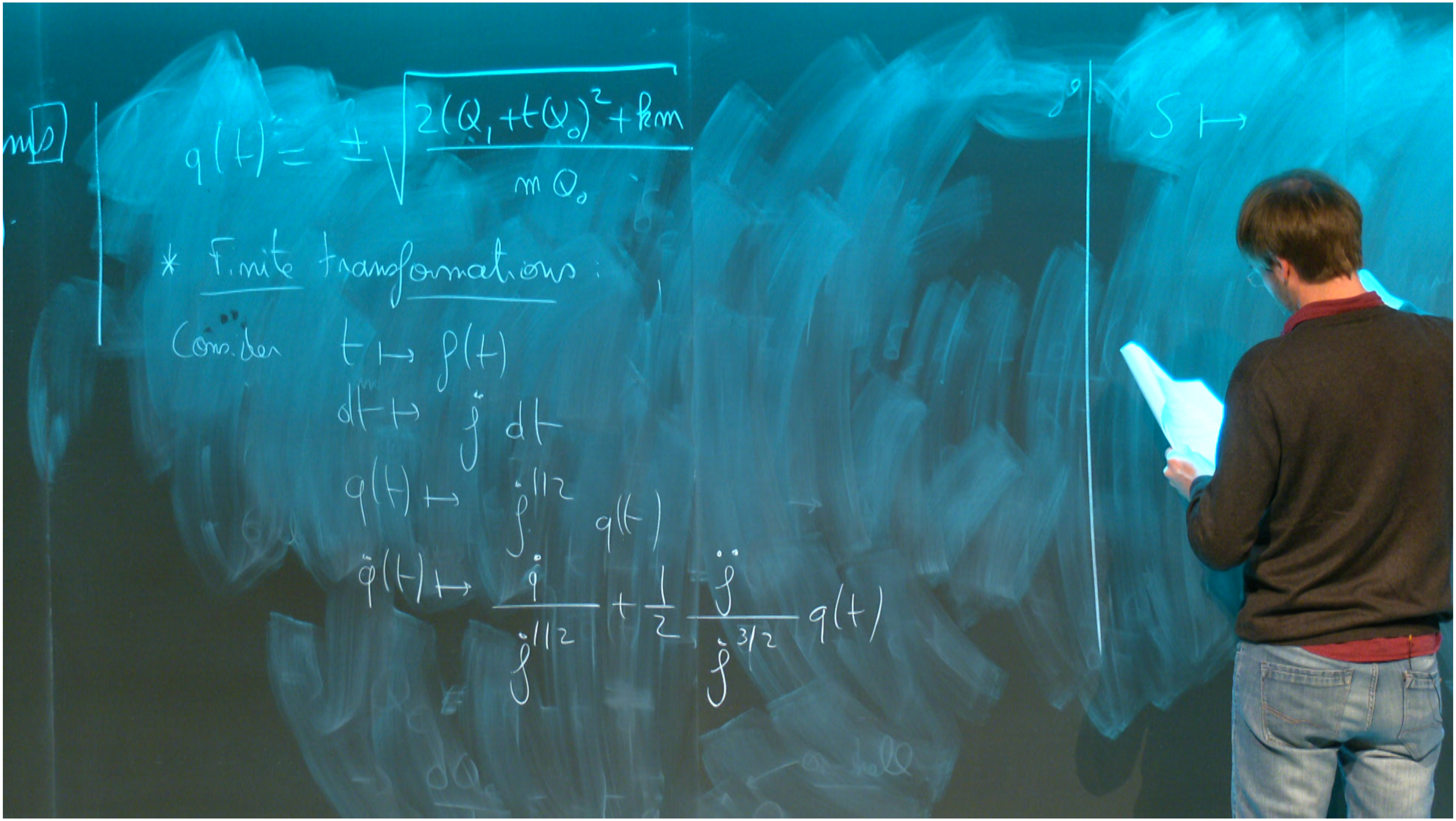
$\delta_\epsilon p_a = \{ p_a, Q_\epsilon \}$ $\Rightarrow$ $\delta_\epsilon L = \frac{d}{dt}\left(p_a \delta_\epsilon q^a - Q_\epsilon \right)$

* Lie algebra: $\{ Q_i, Q_j \} = f_{ij}{}^{k} Q_k$

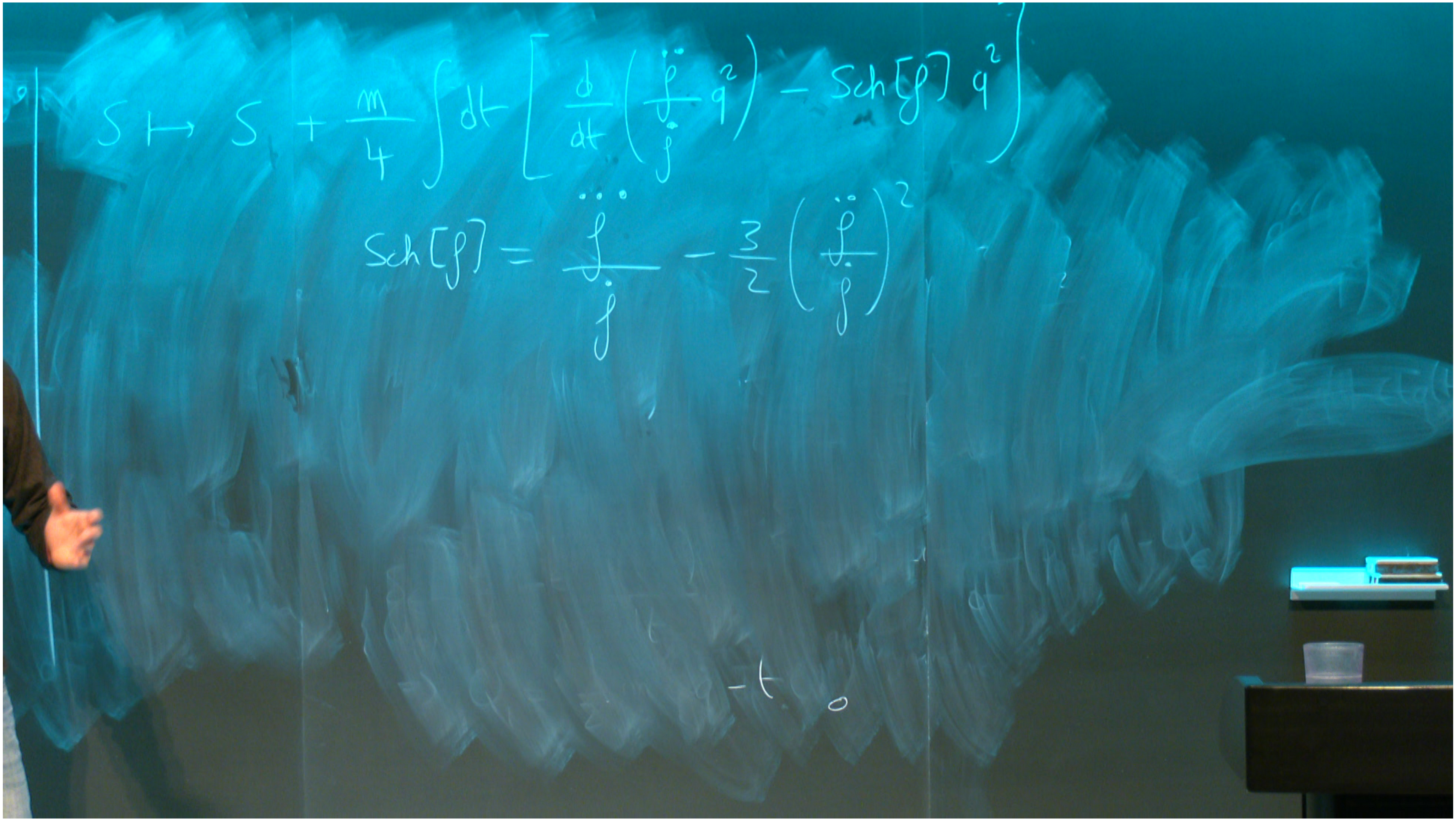

theorem[s]
theory.
1.3 Examples
* Free particle
position $\vec{q}$
Lagrangian $L = \frac{1}{2}\dot{q}^2$
• time translation $\longrightarrow$ $Q = H$
• rotations $\delta_\epsilon \vec{q} = \vec{\epsilon} \times \vec{q} \longrightarrow \vec{Q} = \vec{p} \times \vec{q}$
•
1.2
* P
* In
* Li

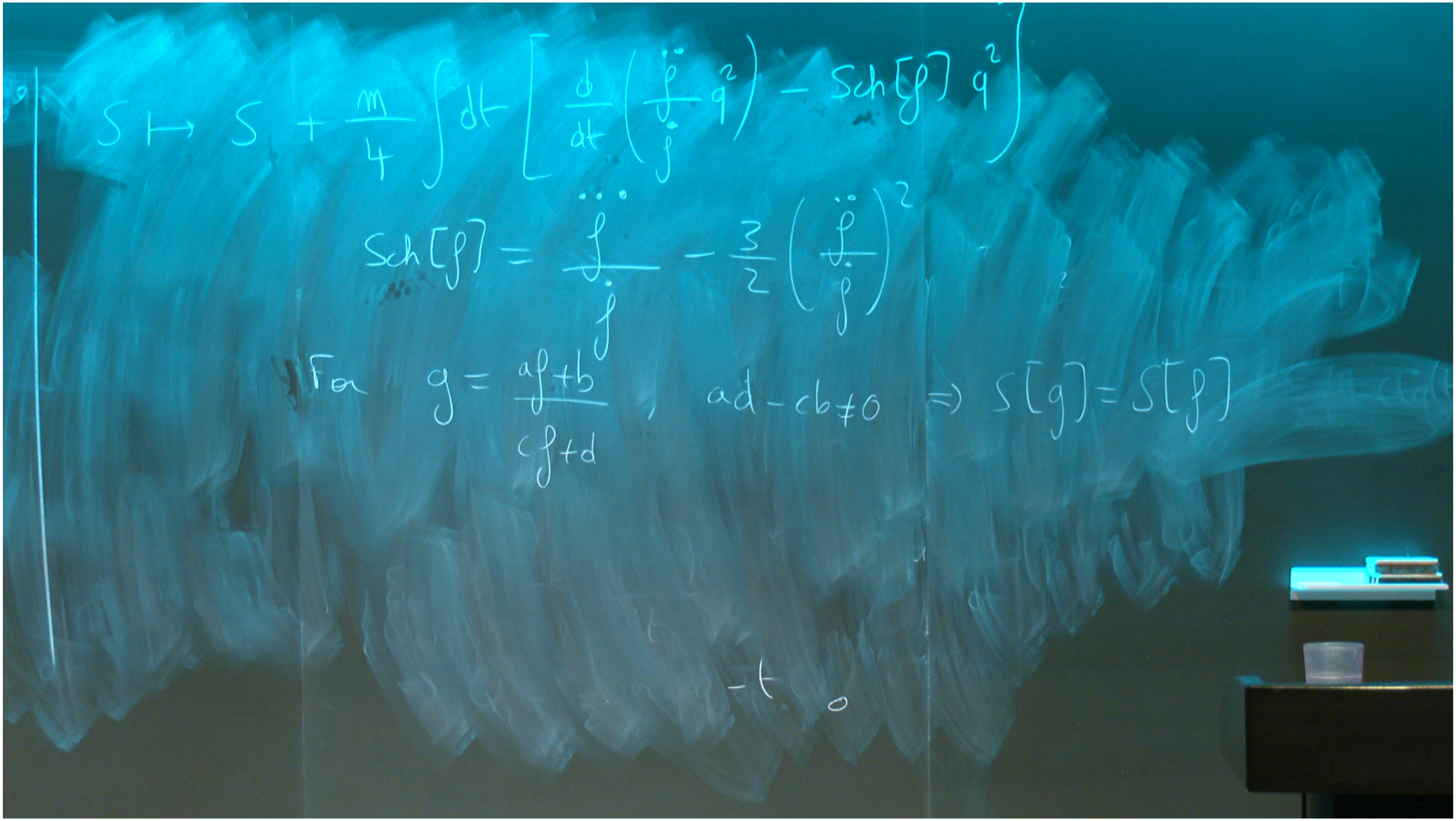

theorems
theory.
1.3 Examples
* Free particle
position $\vec{q}$ Lagrangian $L = \frac{1}{2}\dot{q}^2$
• time translation → $Q = H$
• rotations $\delta_\epsilon \vec{q} = \vec{\epsilon} \times \vec{q} \to \vec{Q} = \vec{p} \times \vec{q}$
• Galilean boosts: $t \mapsto \bar{t} = t$, $\vec{q}(t) \mapsto \vec{q}(t) + \vec{\epsilon} t$
$\Rightarrow Q_\epsilon = \vec{q} - t\dot{\vec{q}}$
$\to \frac{dQ_\epsilon}{dt} = -t\ddot{\vec{q}} \hat{=} 0$
on-shell
1.2
* Pha
* Inver
* Lie a


* Conformal particle
Consider: $S = \int dt \left(\frac{1}{2} m \dot{q}^2 - \frac{k}{q^2} \right)$
• time translations ⟶ $Q_0 = H = \frac{1}{2} m \dot{q}^2 + \frac{k}{q^2}$
• scaling symmetry (Weyl): $t \mapsto \bar{t} = \lambda t$, $q(t) \mapsto \bar{q}(\bar{t}) = \sqrt{\lambda}\, q(t)$
$\delta_\epsilon q(t) = \frac{\epsilon}{2} q(t) - \epsilon t \dot{q}(t)$
$S \mapsto S$
$\delta_\epsilon L = -\epsilon \frac{d(tL)}{dt}$
$\Rightarrow Q_1 = \frac{1}{2} m q \dot{q} - t Q_0$



$q(t) = \pm\sqrt{\frac{2(Q_1 + tQ_0)^2 + km}{m Q_0}}$
* Finite transformations:
Consider $t \mapsto f(t)$
$dt \mapsto \dot{f}\, dt$
$q(t) \mapsto \dot{f}^{1/2} q(t)$
$\dot{q}(t) \mapsto \frac{\dot{q}}{\dot{f}^{1/2}} + \frac{1}{2}\frac{\ddot{f}}{\dot{f}^{3/2}} q(t)$
$S \mapsto$


$S \mapsto S + \frac{m}{4}\int dt \left[\frac{d}{dt}\left(\frac{\ddot{f}}{\dot{f}} q^2\right) - \mathrm{Sch}[f]\, q^2 \right]$
$\mathrm{Sch}[f] = \frac{\dddot{f}}{\dot{f}} - \frac{3}{2}\left(\frac{\ddot{f}}{\dot{f}}\right)^2$



$S \mapsto S + \frac{m}{4}\int dt \left[\frac{d}{dt}\left(\frac{\ddot{f}}{\dot{f}} q^2\right) - Sch[f]\, q^2 \right]$
$Sch[f] = \frac{\dddot{f}}{\dot{f}} - \frac{3}{2}\left(\frac{\ddot{f}}{\dot{f}}\right)^2$
For $g = \frac{af+b}{cf+d}$, $ad - cb \neq 0 \Rightarrow S[g] = S[f]$

$$S \mapsto S + \frac{m}{4}\int dt \left[\frac{d}{dt}\left(\frac{\ddot f}{\dot f}\, q^2\right) - \mathrm{Sch}[f]\, q^2\right]$$

$$\mathrm{Sch}[f] = \frac{\dddot f}{\dot f} - \frac{3}{2}\left(\frac{\ddot f}{\dot f}\right)^2$$

For $g = \dfrac{af+b}{cf+d}$, $ad - cb \neq 0 \Rightarrow S[g] = S[f]$

If the symmetry is $\delta_\epsilon q(t) = \frac{1}{2}\dot\epsilon(t)\, q(t) - \epsilon(t)\, \dot q(t)$

with $\epsilon(t) = -\epsilon_0 + \epsilon_1 t - \epsilon_2 t^2$

$\Rightarrow$

$$S \mapsto S + \frac{m}{4}\int dt \left[\frac{d}{dt}\left(\frac{\ddot{f}}{\dot{f}}\, q^2\right) - \mathrm{Sch}[f]\, q^2\right]$$

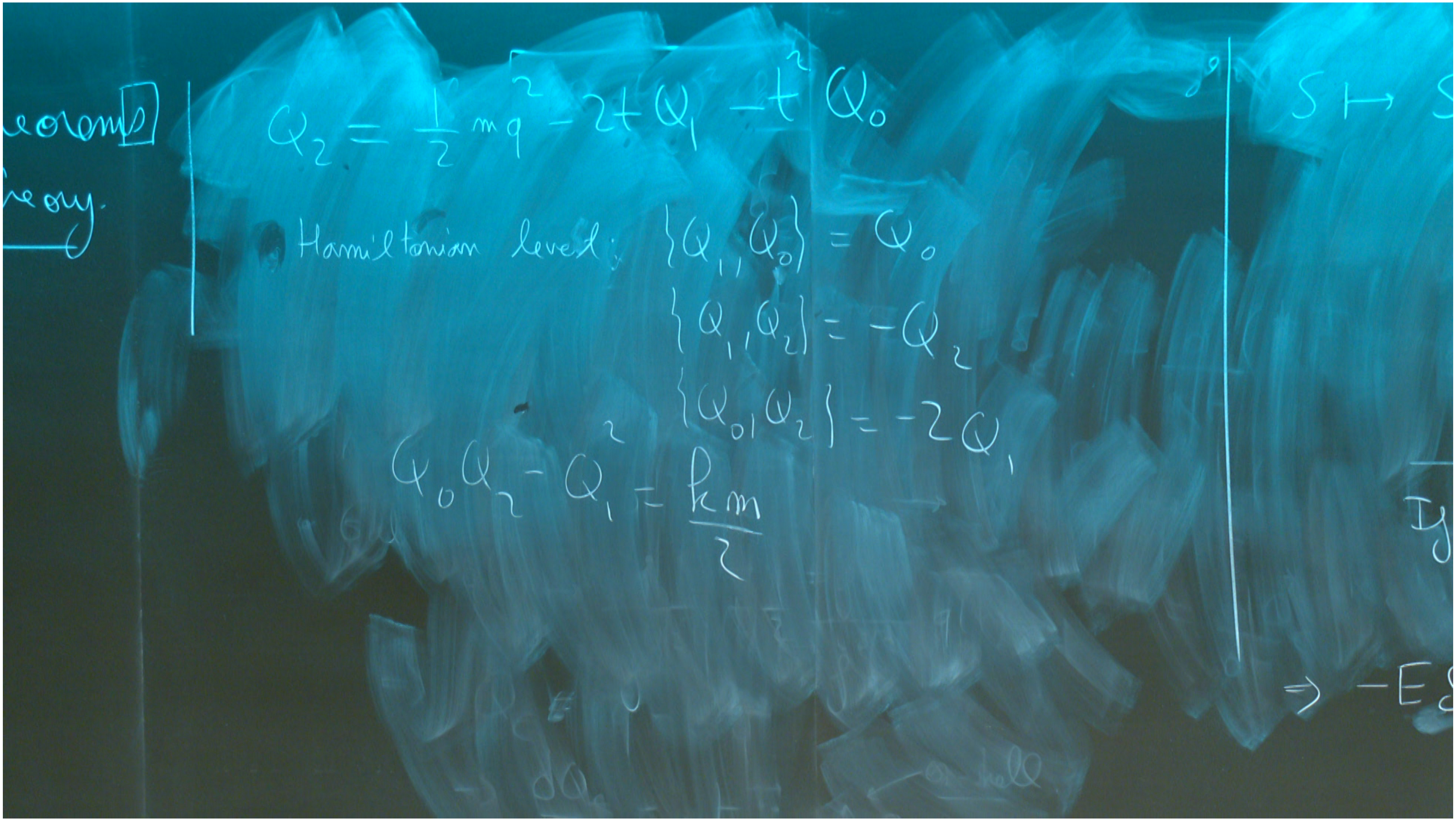
$$\mathrm{Sch}[f] = \frac{\dddot{f}}{\dot{f}} - \frac{3}{2}\left(\frac{\ddot{f}}{\dot{f}}\right)^2$$

For $g = \frac{af+b}{cf+d}$, $ad - cb \neq 0 \Rightarrow S[g] = S[f]$

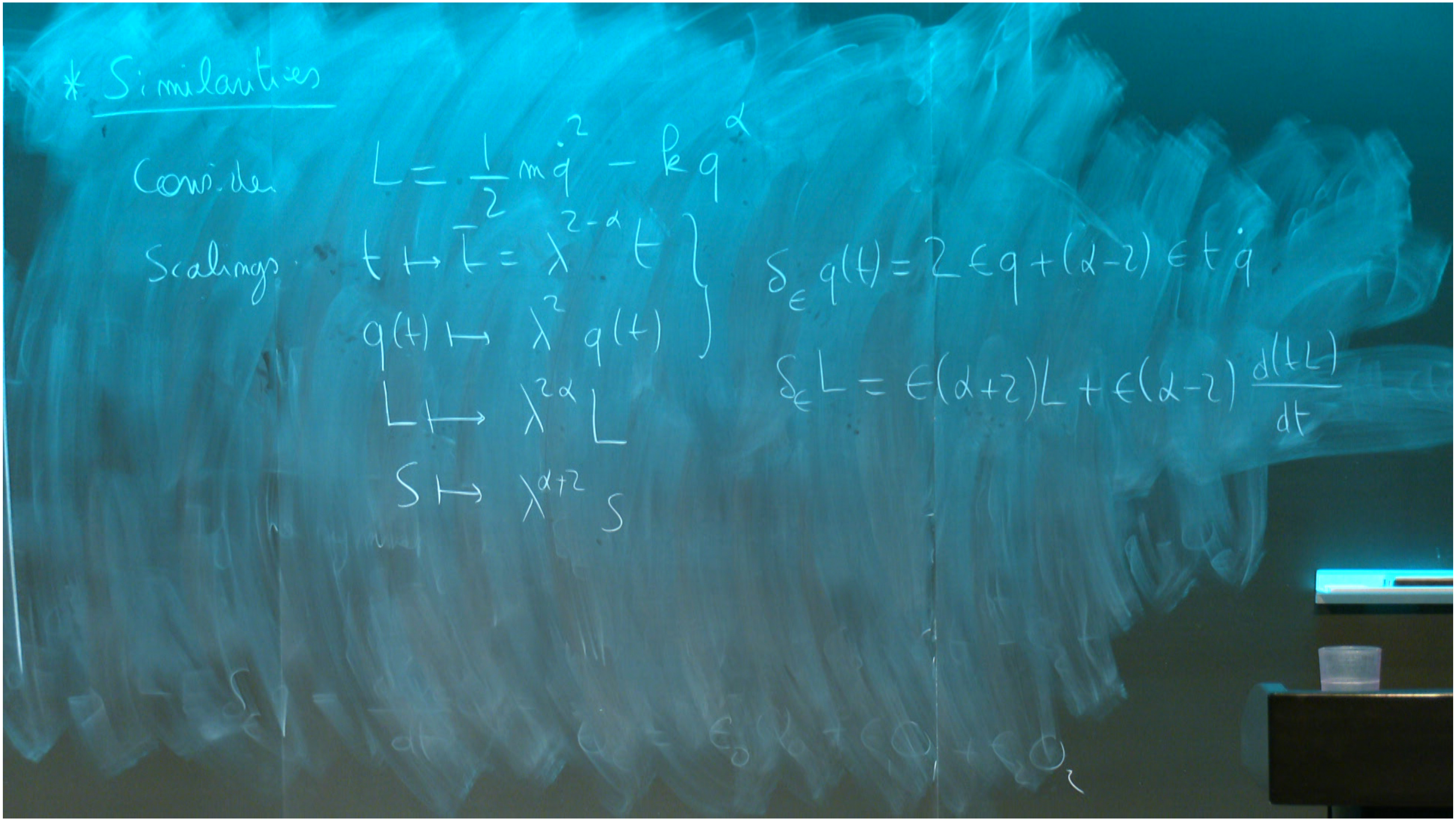
If the symmetry is $\delta_\epsilon q(t) = \frac{1}{2}\dot{\epsilon}(t)\, q(t) - \epsilon(t)\, \dot{q}(t)$

with $\epsilon(t) = -\epsilon_0 + \epsilon_1 t - \epsilon_2 t^2$

$$\Rightarrow -E\,\delta_\epsilon q = \frac{dQ_\epsilon}{dt} \qquad Q_\epsilon = \epsilon_0 Q_0 + \epsilon_1 Q_1 + \epsilon_2 Q_2$$



$Q_2 = \frac{1}{2} m q^2 - 2t Q_1 - t^2 Q_0$
Hamiltonian level: $\{Q_1, Q_0\} = Q_0$
$\{Q_1, Q_2\} = -Q_2$
$\{Q_0, Q_2\} = -2Q_1$
$Q_0 Q_2 - Q_1^2 = \frac{km}{2}$


* Similarities
Consider $L = \frac{1}{2} m \dot{q}^2 - k q^\alpha$
Scalings: $t \mapsto \bar{t} = \lambda^{2-\alpha} t$
$q(t) \mapsto \lambda^2 q(t)$
$L \mapsto \lambda^{2\alpha} L$
$S \mapsto \lambda^{\alpha+2} S$
$\delta_\epsilon q(t) = 2\epsilon q + (\alpha-2)\epsilon t \dot{q}$
$\delta_\epsilon L = \epsilon(\alpha+2)L + \epsilon(\alpha-2)\frac{d(tL)}{dt}$

Nöther theorem:

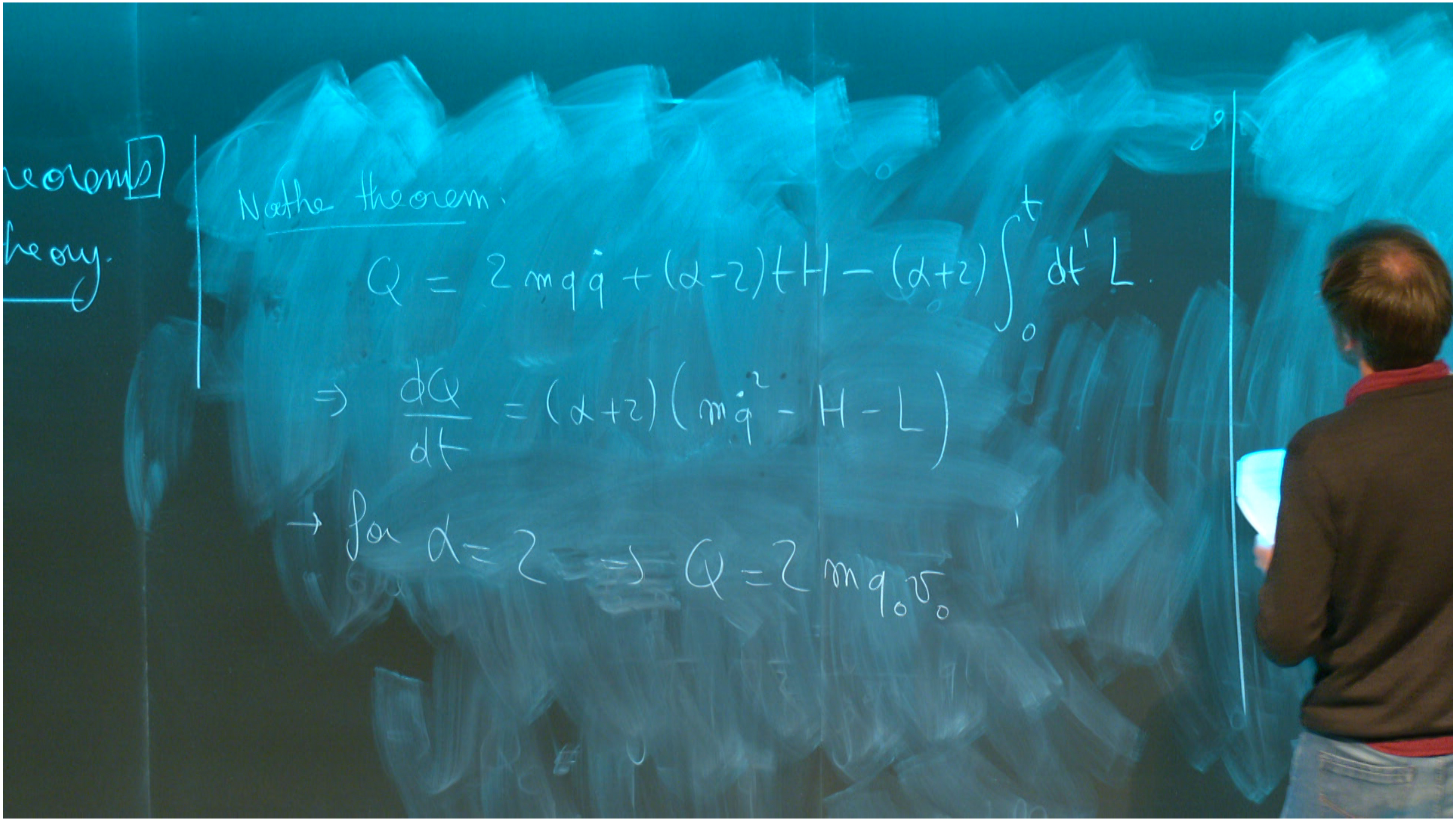
$$Q = 2mq\dot{q} + (\alpha-2)tH - (\alpha+2)\int_0^t dt'\, L .$$

$$\Rightarrow \frac{dQ}{dt} = (\alpha+2)\left(m\dot{q}^2 - H - L\right)$$

* Similariti

Conside

Scalings


Nœthe theorem:
$Q = 2mq\dot{q} + (\alpha-2)tH - (\alpha+2)\int_0^t dt' L.$
$\Rightarrow \frac{dQ}{dt} = (\alpha+2)(m\dot{q}^2 - H - L)$
$\rightarrow$ for $\alpha = 2 \Rightarrow Q = 2mq_0v_0$

* Newtonian potential $\quad L = \frac{1}{2} m \dot{q}^2 + \frac{k}{q}$

Using the angular momentum $\quad \vec{J} = \vec{q} \times \vec{p}$

We define the · Laplace-Runge-Lenz (LRL) vector: $\vec{A} := \vec{p} \times \vec{J} - \frac{km}{q} \vec{q}$

· Hamilton vector $\quad \vec{B} := \vec{p} - \frac{km}{qJ^2} \vec{J} \times \vec{q}$

$\vec{A} \cdot \vec{J} = \vec{B} \cdot \vec{J} = \vec{A} \cdot \vec{B} = 0$

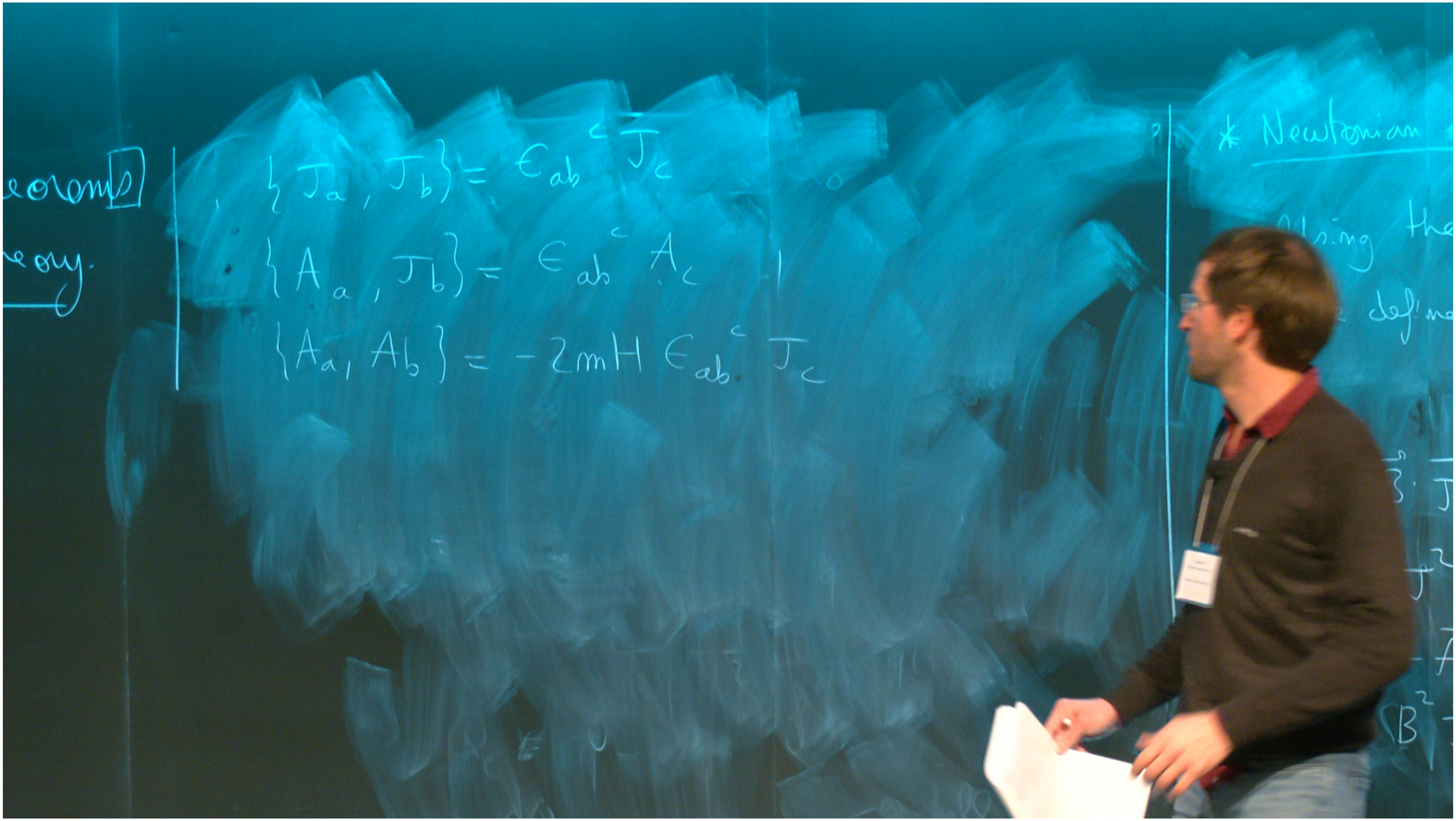
$\vec{J} \times \vec{A} = J^2 \vec{B}$

$\vec{J} \times \vec{B} = -\vec{A}$

$\vec{A} \times \vec{B} = B^2 \vec{J}$

$\vec{A} \cdot \vec{q} = A q \cos\theta = J^2 - kmq$

$\Rightarrow \frac{1}{q} = \frac{km}{J^2}\left(1 + \frac{A}{km} \cos\theta\right)$


$\{J_a, J_b\} = \epsilon_{ab}{}^{c} J_c$
$\{A_a, J_b\} = \epsilon_{ab}{}^{c} A_c$
$\{A_a, A_b\} = -2mH\, \epsilon_{ab}{}^{c} J_c$
* Newtonian

…eorems [s]

…heory.

$$\{J_a, J_b\} = \epsilon_{ab}{}^{c} J_c$$

$$\{A_a, J_b\} = \epsilon_{ab}{}^{c} A_c$$

$$\{A_a, A_b\} = -2mH\, \epsilon_{ab}{}^{c} J_c$$

$$H<0 \longrightarrow so(4)$$

$$H=0 \longrightarrow so(2,1)$$

$$H>0 \longrightarrow so(3,1)$$

* Newtonian …

Using th…

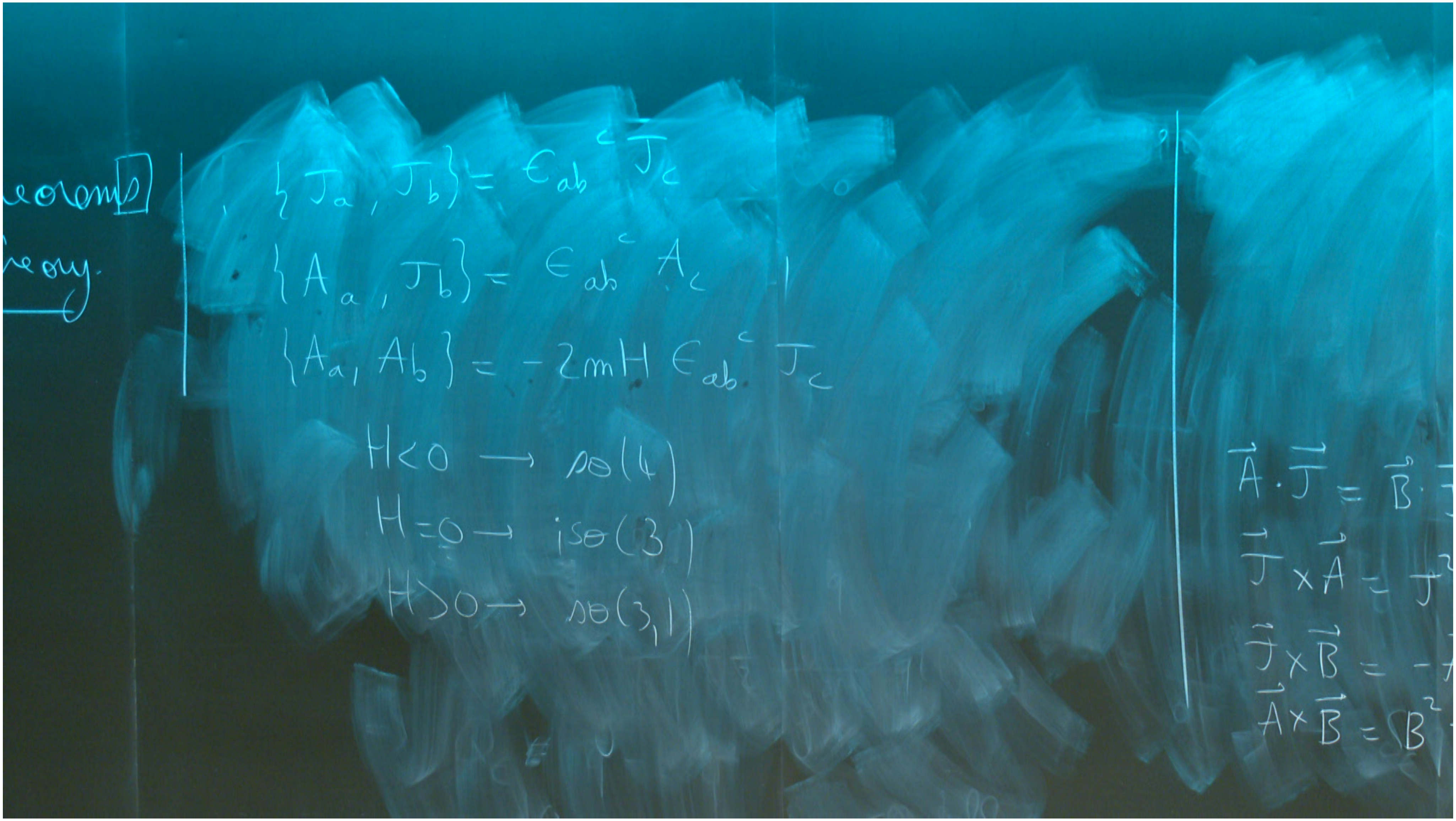
We defin…

$$\vec{A}\cdot\vec{J} = \vec{B}\cdot \ldots$$

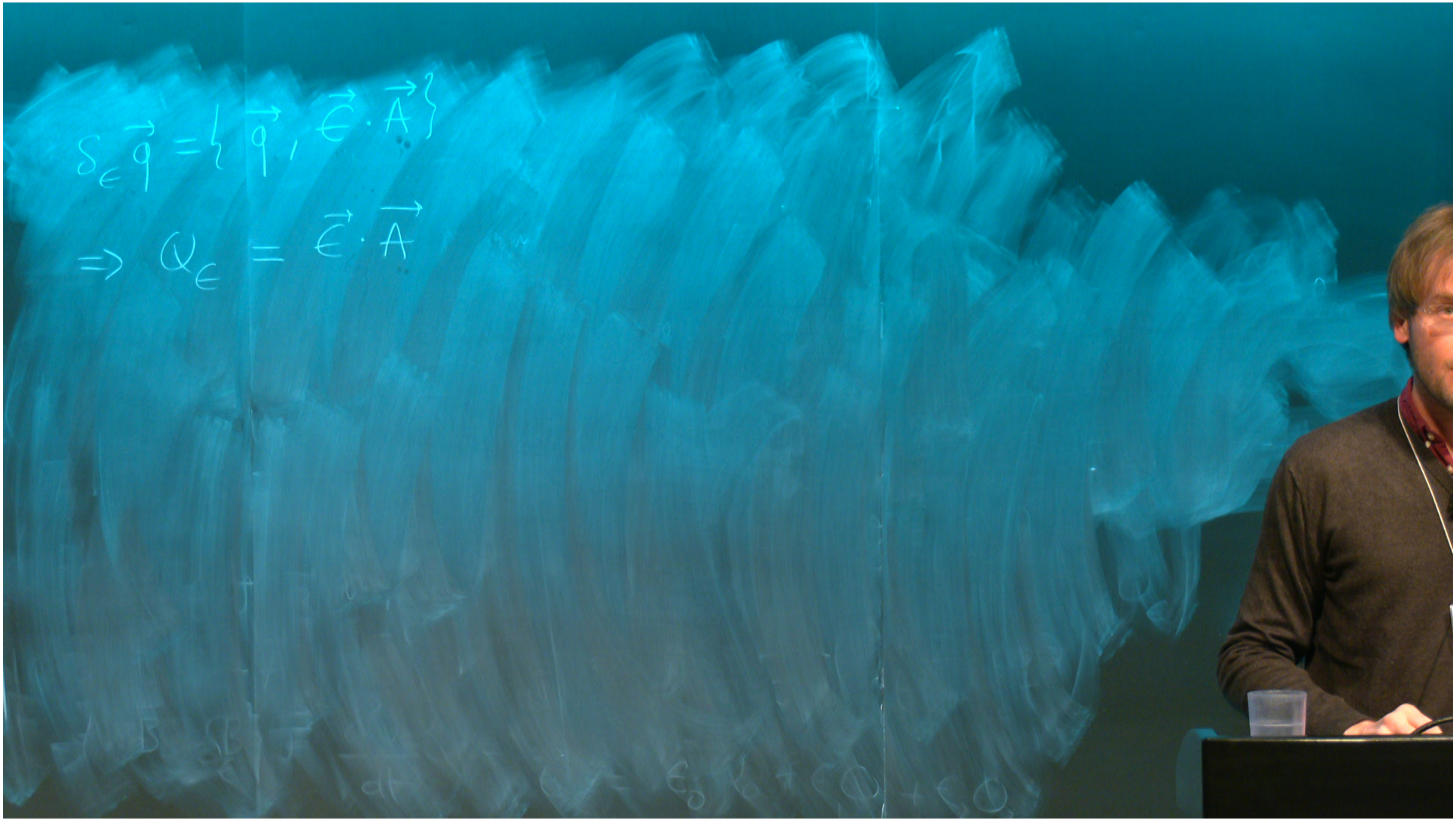
$$\vec{J}\times\vec{A} = J^2 \ldots$$

$$\vec{J}\times\vec{B} = - \ldots$$

$$\vec{A}\times\vec{B} = B^2 \ldots$$



$\{J_a, J_b\} = \epsilon_{ab}{}^{c} J_c$
$\{A_a, J_b\} = \epsilon_{ab}{}^{c} A_c$
$\{A_a, A_b\} = -2mH \epsilon_{ab}{}^{c} J_c$
$H<0 \longrightarrow so(4)$
$H=0 \longrightarrow iso(3)$
$H>0 \longrightarrow so(3,1)$
$\vec{A} \cdot \vec{J} = \vec{B} \cdot$
$\vec{J} \times \vec{A} =$
$\vec{J} \times \vec{B} = -$


$\delta_\epsilon \vec{q} = \{\vec{q}, \vec{\epsilon}\cdot\vec{A}\}$
$\Rightarrow Q_\epsilon = \vec{\epsilon}\cdot\vec{A}$