Title: Bipartite entanglement and the arrow of time

Speakers: Markus Frembs

Series: Quantum Foundations

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Abstract: Quantum correlations in general and quantum entanglement in particular embody both our continued struggle towards a foundational understanding of quantum theory as well as the latter's advantage over classical physics in various information processing tasks. Consequently, the problems of classifying (i) quantum states from more general (non-signalling) correlations, and (ii) entangled states within the set of all quantum states, are at the heart of the subject of quantum information theory.

In this talk I will present two recent results (from https://journals.aps.org/pra/abstract/10.1103/PhysRevA.106.062420 and https://arxiv.org/abs/2207.00024) that shed new light on these problems, by exploiting a surprising connection with time in quantum theory:

First, I will sketch a solution to problem (i) for the bipartite case, which identifies a key physical principle obeyed by quantum theory: quantum states preserve local time orientations--roughly, the unitary evolution in local subsystems.

Second, I will show that time orientations are intimately connected with quantum entanglement: a bipartite quantum state is separable if and only if it preserves arbitrary local time orientations. As a variant of Peres's well-known entanglement criterion, this provides a solution to problem (ii).

Zoom link: https://pitp.zoom.us/j/97607837999?pwd=cXBYUmFVaDRpeFJSZ0JzVmhSajdwQT09

Bipartite entanglement and the arrow of time

based on:
MF and A. Döring, Phys. Rev. A, 106, 062420 (2022)
MF, arXiv: 2207.00024

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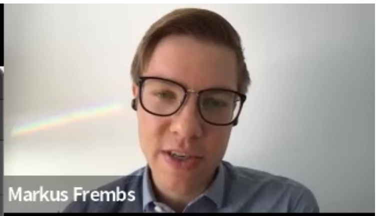
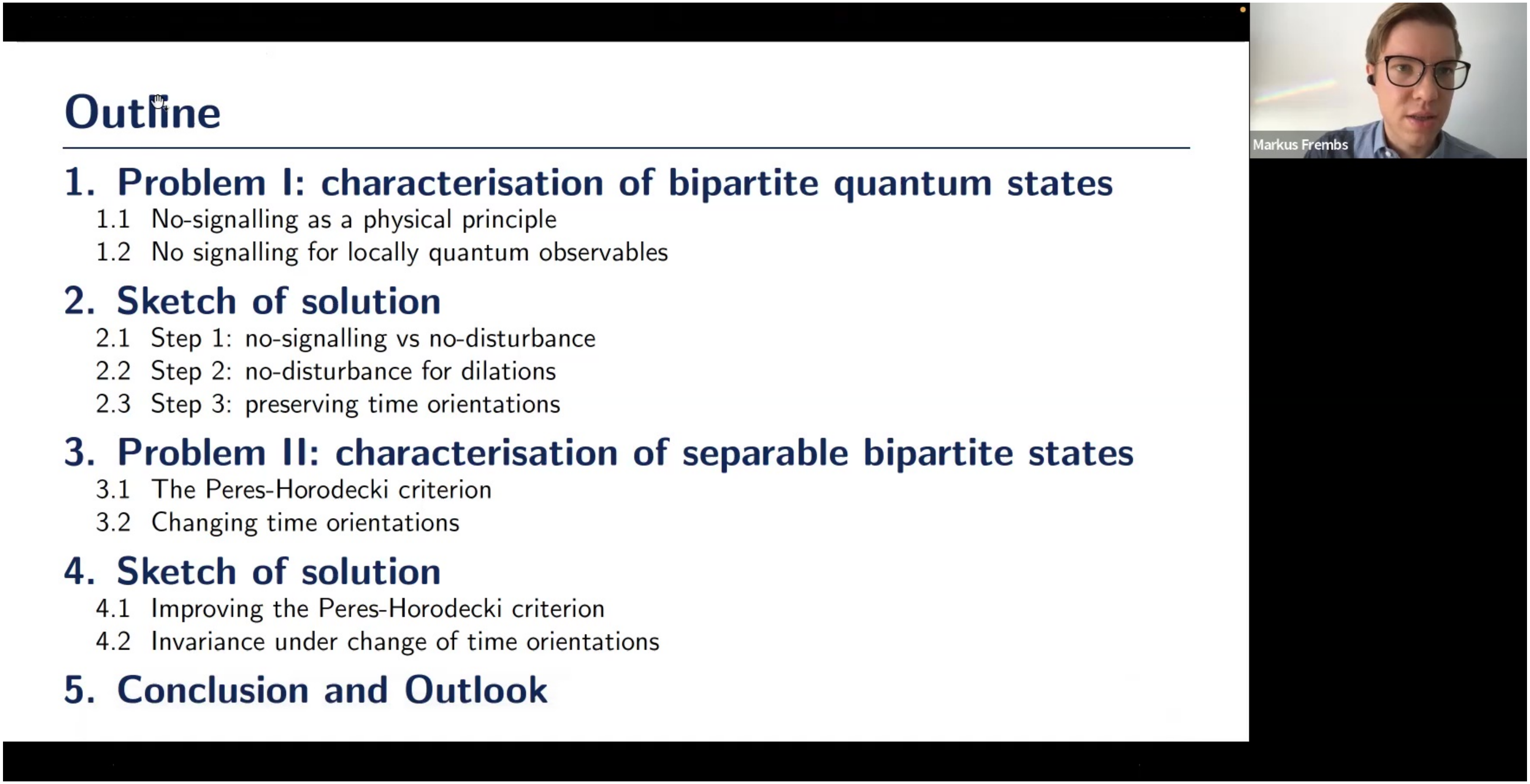
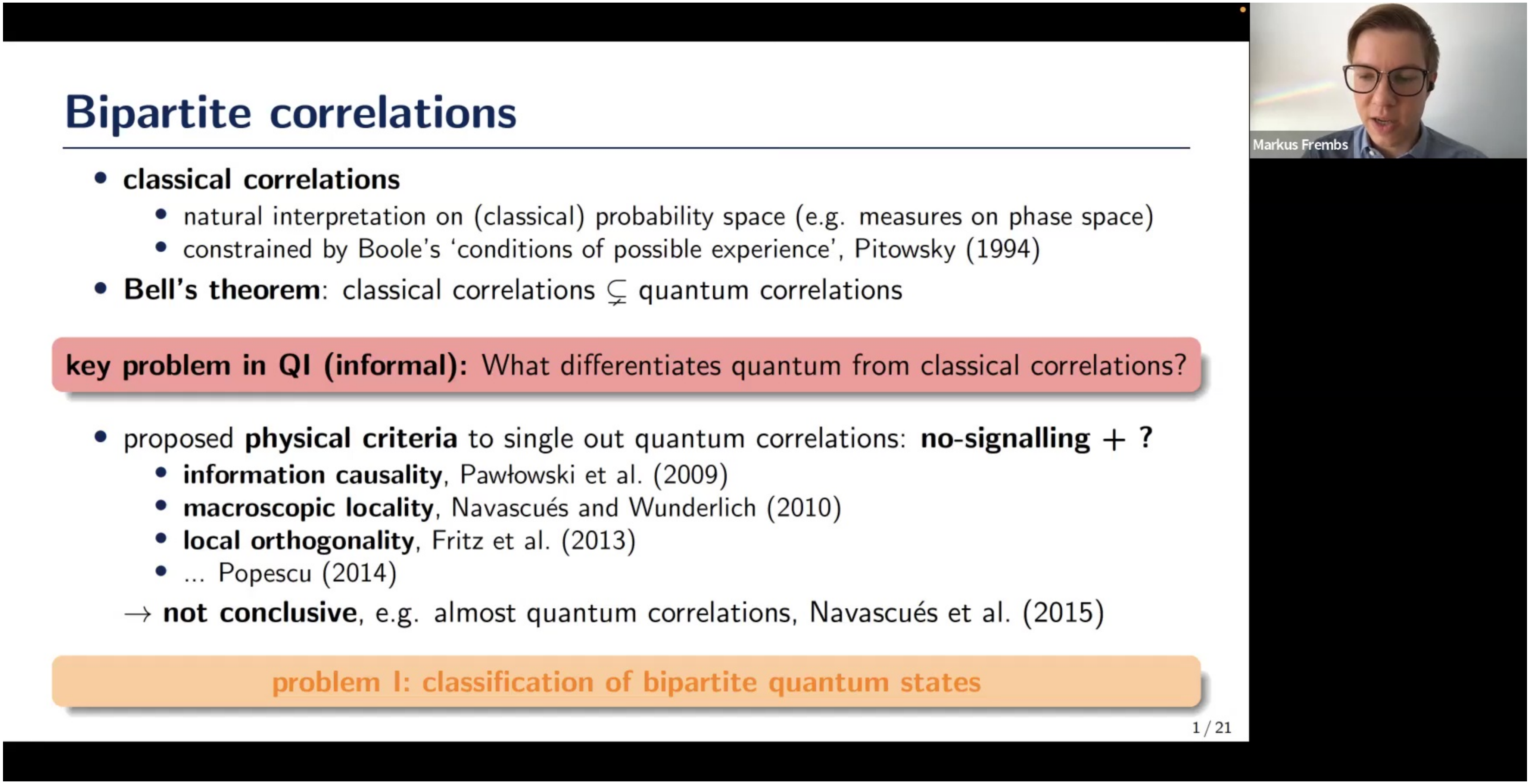
Quantum Foundations Seminar
Perimeter Institute
10 January 2023





Outline

1. **Problem I: characterisation of bipartite quantum states**
    - 1.1 No-signalling as a physical principle
    - 1.2 No signalling for locally quantum observables
2. **Sketch of solution**
    - 2.1 Step 1: no-signalling vs no-disturbance
    - 2.2 Step 2: no-disturbance for dilations
    - 2.3 Step 3: preserving time orientations
3. **Problem II: characterisation of separable bipartite states**
    - 3.1 The Peres-Horodecki criterion
    - 3.2 Changing time orientations
4. **Sketch of solution**
    - 4.1 Improving the Peres-Horodecki criterion
    - 4.2 Invariance under change of time orientations
5. **Conclusion and Outlook**



Bipartite correlations

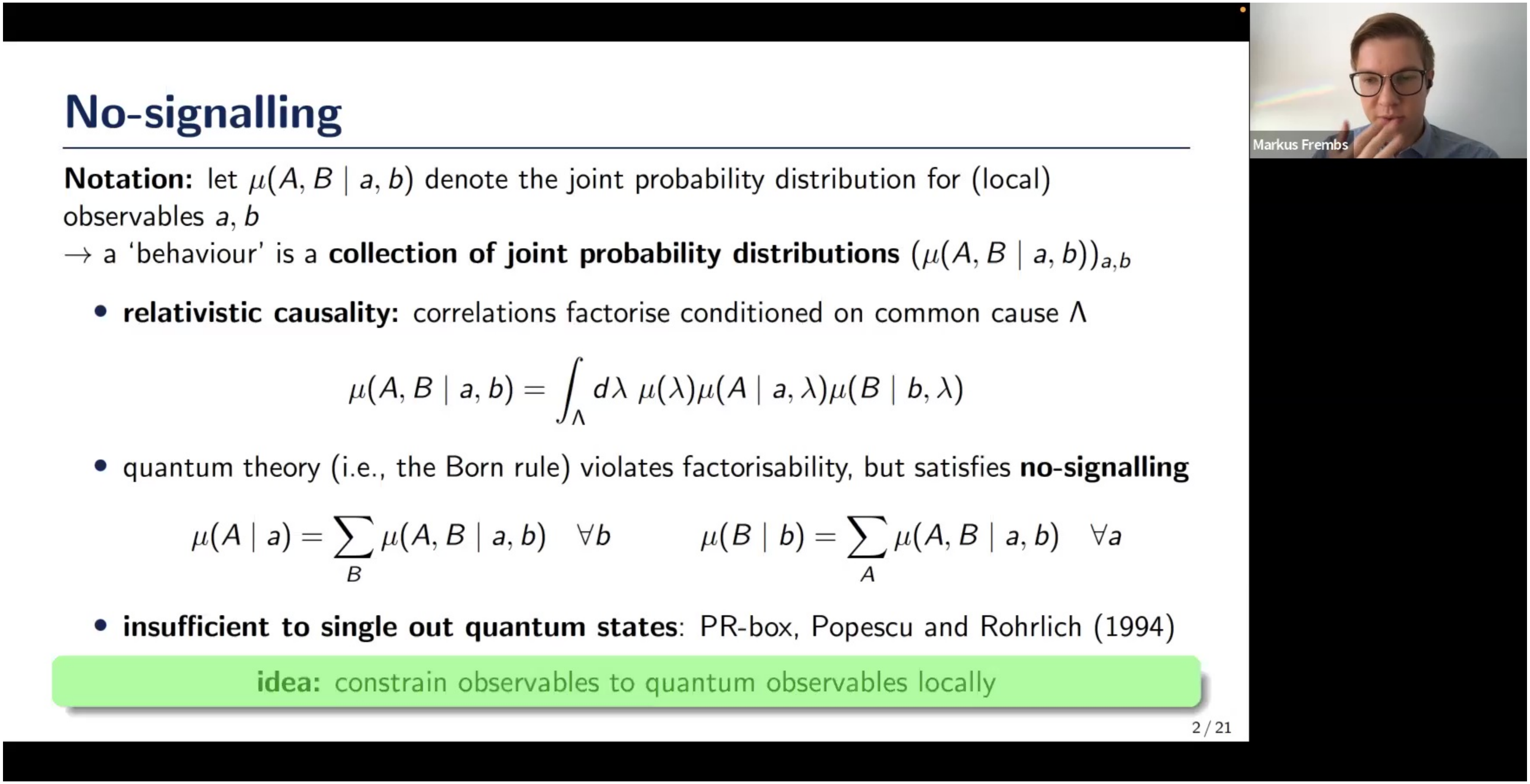
- **classical correlations**
  - natural interpretation on (classical) probability space (e.g. measures on phase space)
  - constrained by Boole's 'conditions of possible experience', Pitowsky (1994)
- **Bell's theorem**: classical correlations $\subsetneq$ quantum correlations

key problem in QI (informal): What differentiates quantum from classical correlations?

- proposed **physical criteria** to single out quantum correlations: **no-signalling + ?**
  - **information causality**, Pawłowski et al. (2009)
  - **macroscopic locality**, Navascués and Wunderlich (2010)
  - **local orthogonality**, Fritz et al. (2013)
  - ... Popescu (2014)

→ **not conclusive**, e.g. almost quantum correlations, Navascués et al. (2015)

problem I: classification of bipartite quantum states



No-signalling

Notation: let $\mu(A, B \mid a, b)$ denote the joint probability distribution for (local) observables a, b
→ a 'behaviour' is a **collection of joint probability distributions** $(\mu(A, B \mid a, b))_{a,b}$

- **relativistic causality:** correlations factorise conditioned on common cause Λ

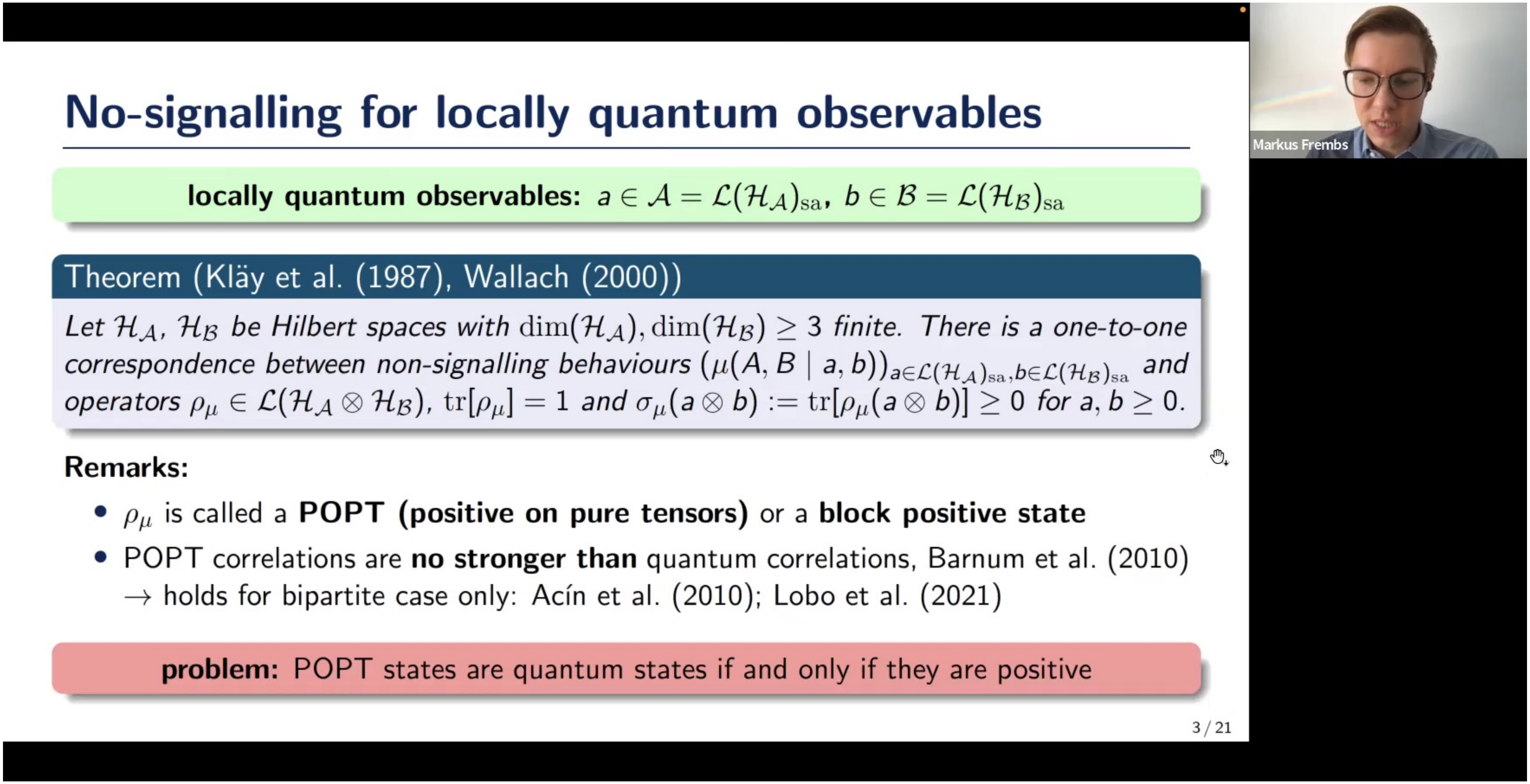
$$\mu(A, B \mid a, b) = \int_\Lambda d\lambda \; \mu(\lambda)\mu(A \mid a, \lambda)\mu(B \mid b, \lambda)$$

- quantum theory (i.e., the Born rule) violates factorisability, but satisfies **no-signalling**

$$\mu(A \mid a) = \sum_B \mu(A, B \mid a, b) \quad \forall b \qquad\qquad \mu(B \mid b) = \sum_A \mu(A, B \mid a, b) \quad \forall a$$

- **insufficient to single out quantum states**: PR-box, Popescu and Rohrlich (1994)

idea: constrain observables to quantum observables locally



No-signalling for locally quantum observables

locally quantum observables: $a \in \mathcal{A} = \mathcal{L}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}}$, $b \in \mathcal{B} = \mathcal{L}(\mathcal{H}_\mathcal{B})_{\mathrm{sa}}$

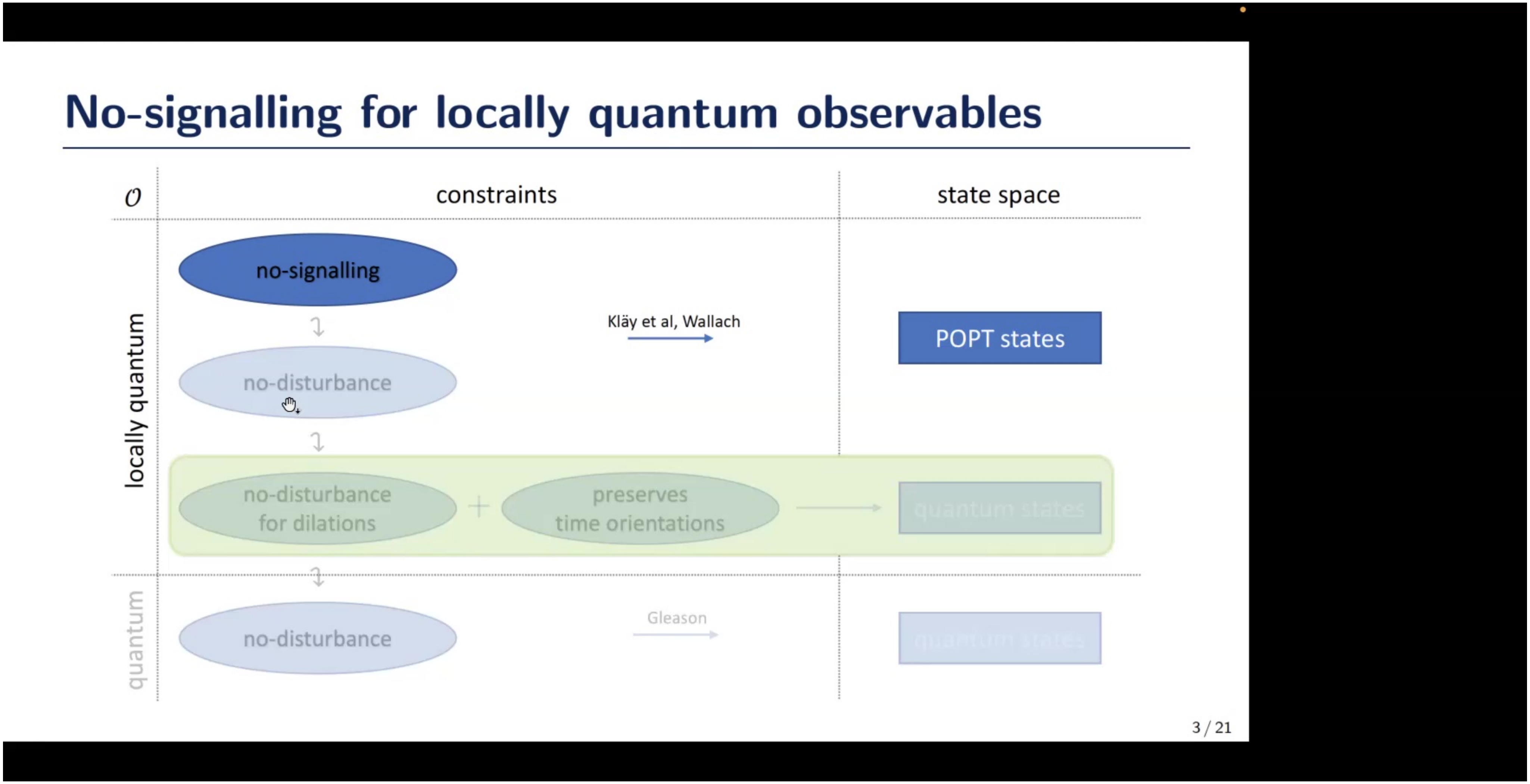
Theorem (Kläy et al. (1987), Wallach (2000))

Let $\mathcal{H}_\mathcal{A}$, $\mathcal{H}_\mathcal{B}$ be Hilbert spaces with $\dim(\mathcal{H}_\mathcal{A}), \dim(\mathcal{H}_\mathcal{B}) \geq 3$ finite. There is a one-to-one correspondence between non-signalling behaviours $(\mu(A, B \mid a, b))_{a \in \mathcal{L}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}}, b \in \mathcal{L}(\mathcal{H}_\mathcal{B})_{\mathrm{sa}}}$ and operators $\rho_\mu \in \mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$, $\mathrm{tr}[\rho_\mu] = 1$ and $\sigma_\mu(a \otimes b) := \mathrm{tr}[\rho_\mu(a \otimes b)] \geq 0$ for $a, b \geq 0$.

Remarks:

- ρ_μ is called a **POPT (positive on pure tensors)** or a **block positive state**
- POPT correlations are **no stronger than** quantum correlations, Barnum et al. (2010)
  $\rightarrow$ holds for bipartite case only: Acín et al. (2010); Lobo et al. (2021)

problem: POPT states are quantum states if and only if they are positive



No-signalling for locally quantum observables

$\mathcal{O}$

constraints

state space

locally quantum

no-signalling

no-disturbance

Kläy et al, Wallach

POPT states

no-disturbance for dilations

+

preserves time orientations

quantum states

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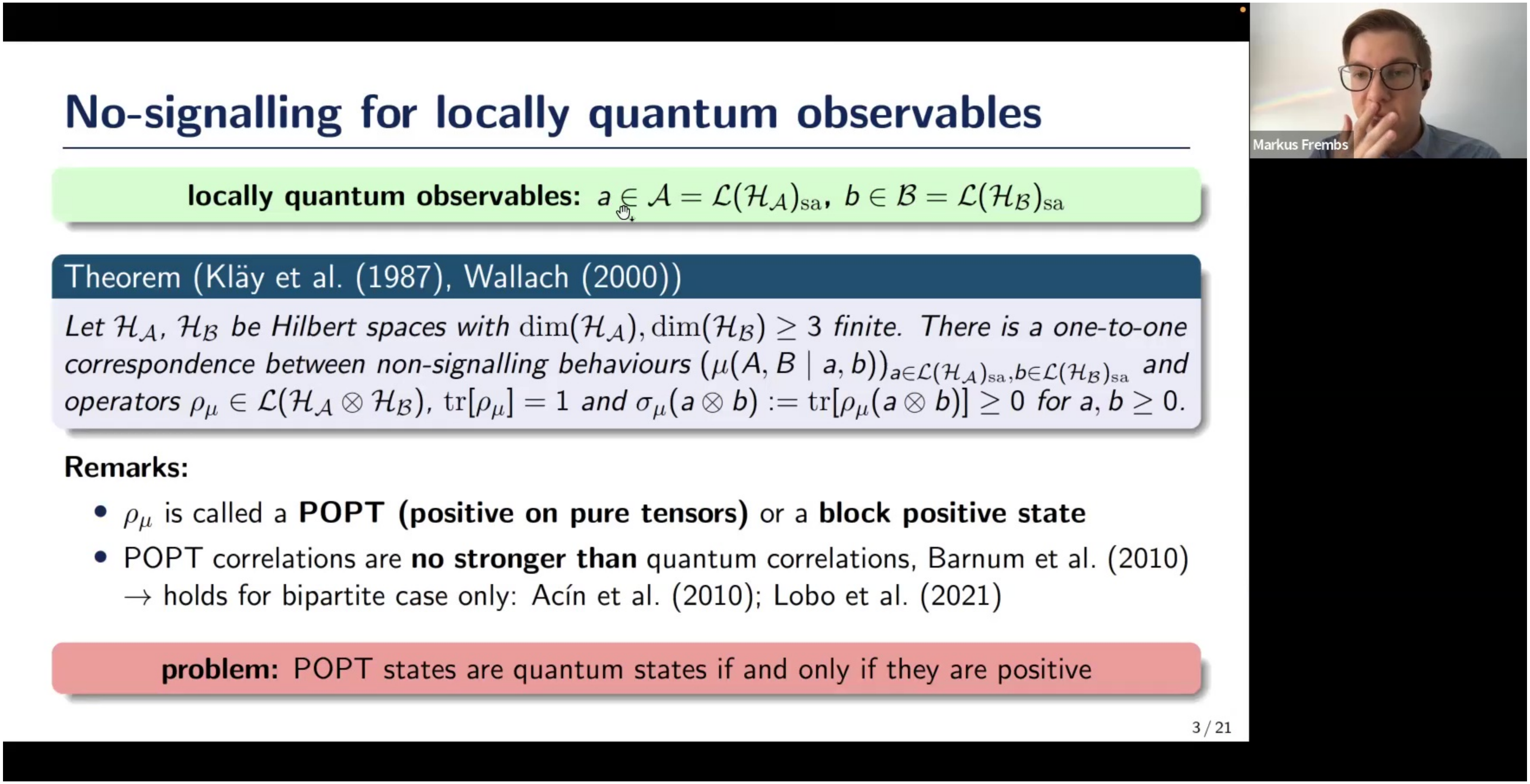
Gleason

quantum states



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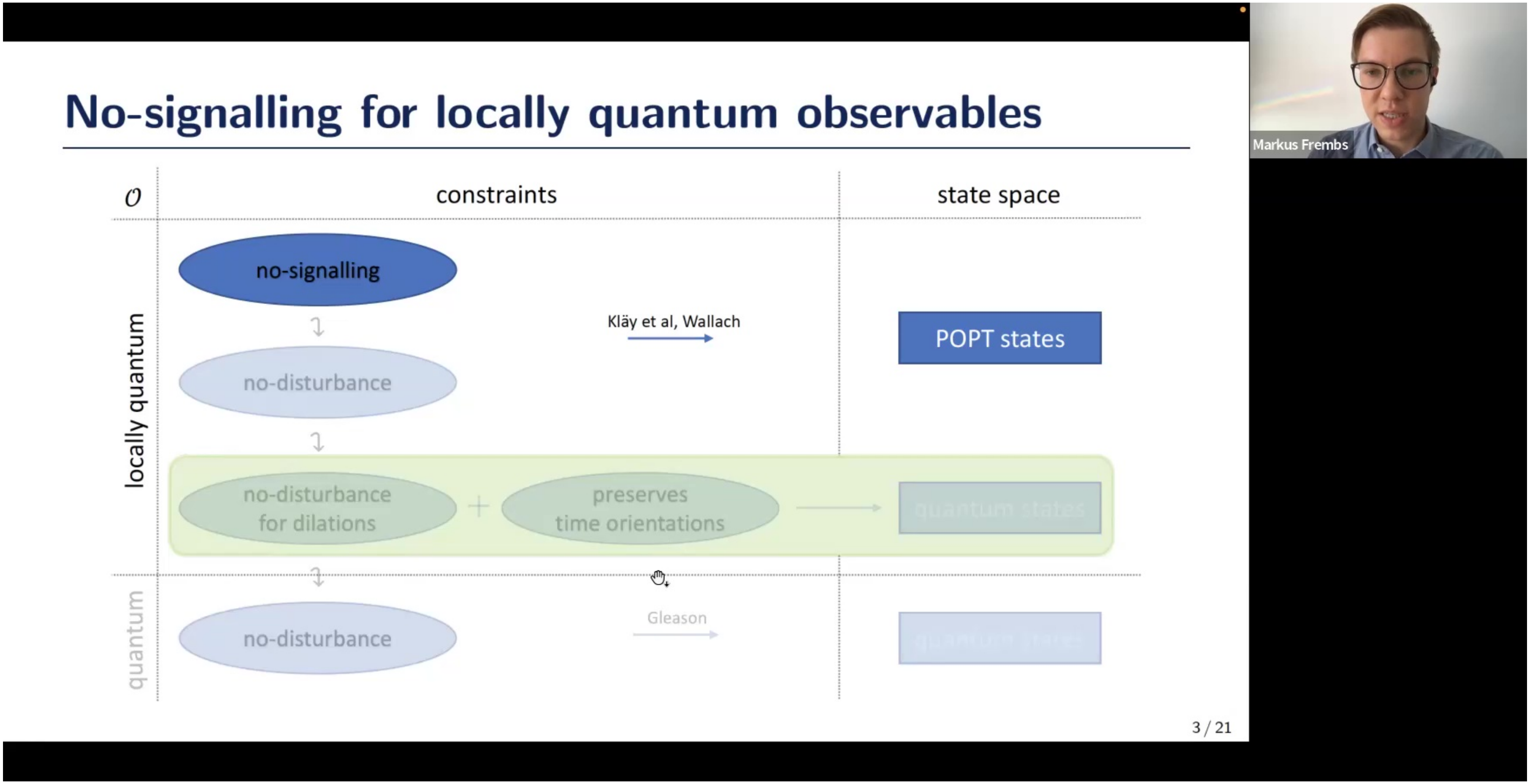

No-signalling for locally quantum observables

$\mathcal{O}$

constraints

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POPT states

quantum states

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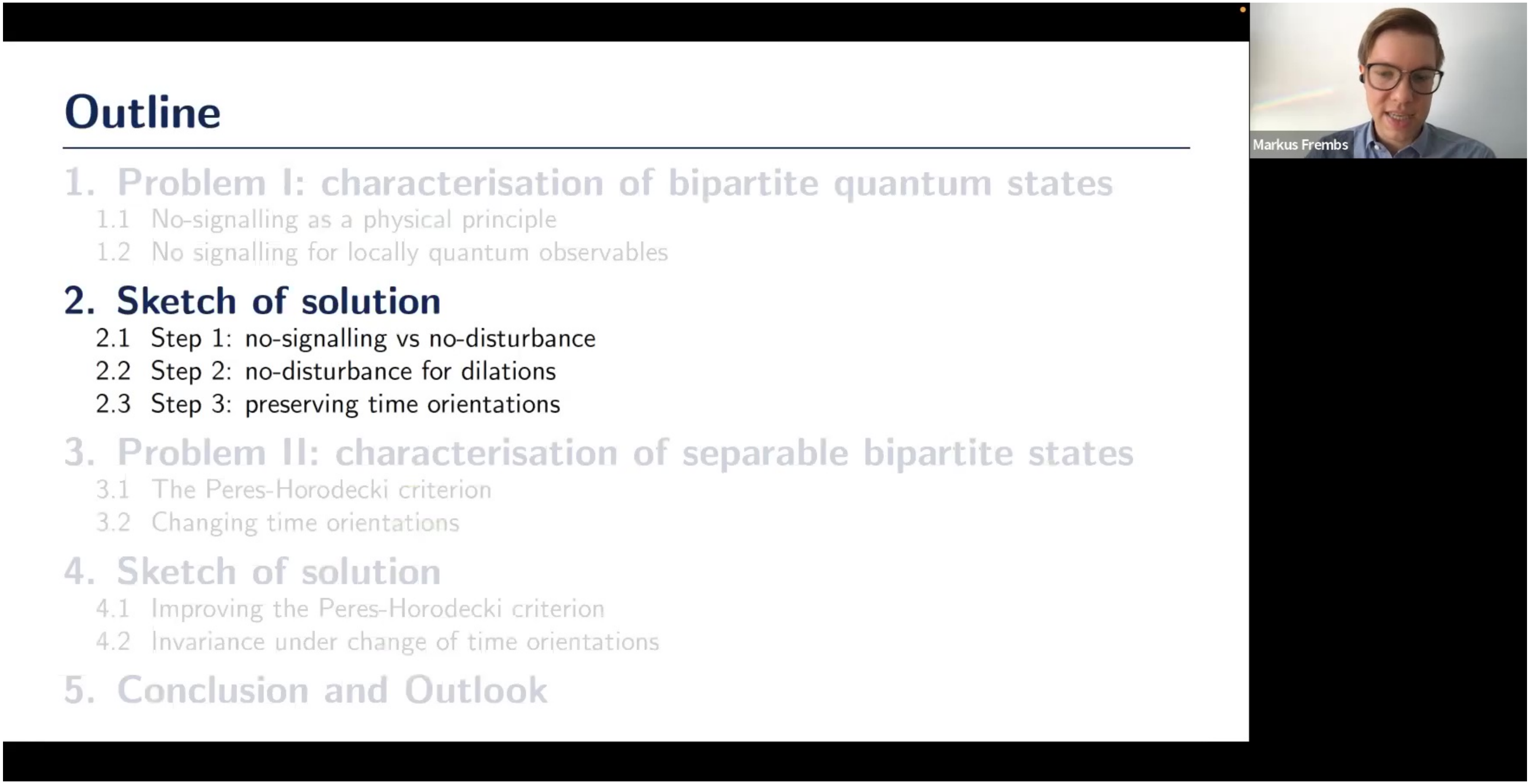

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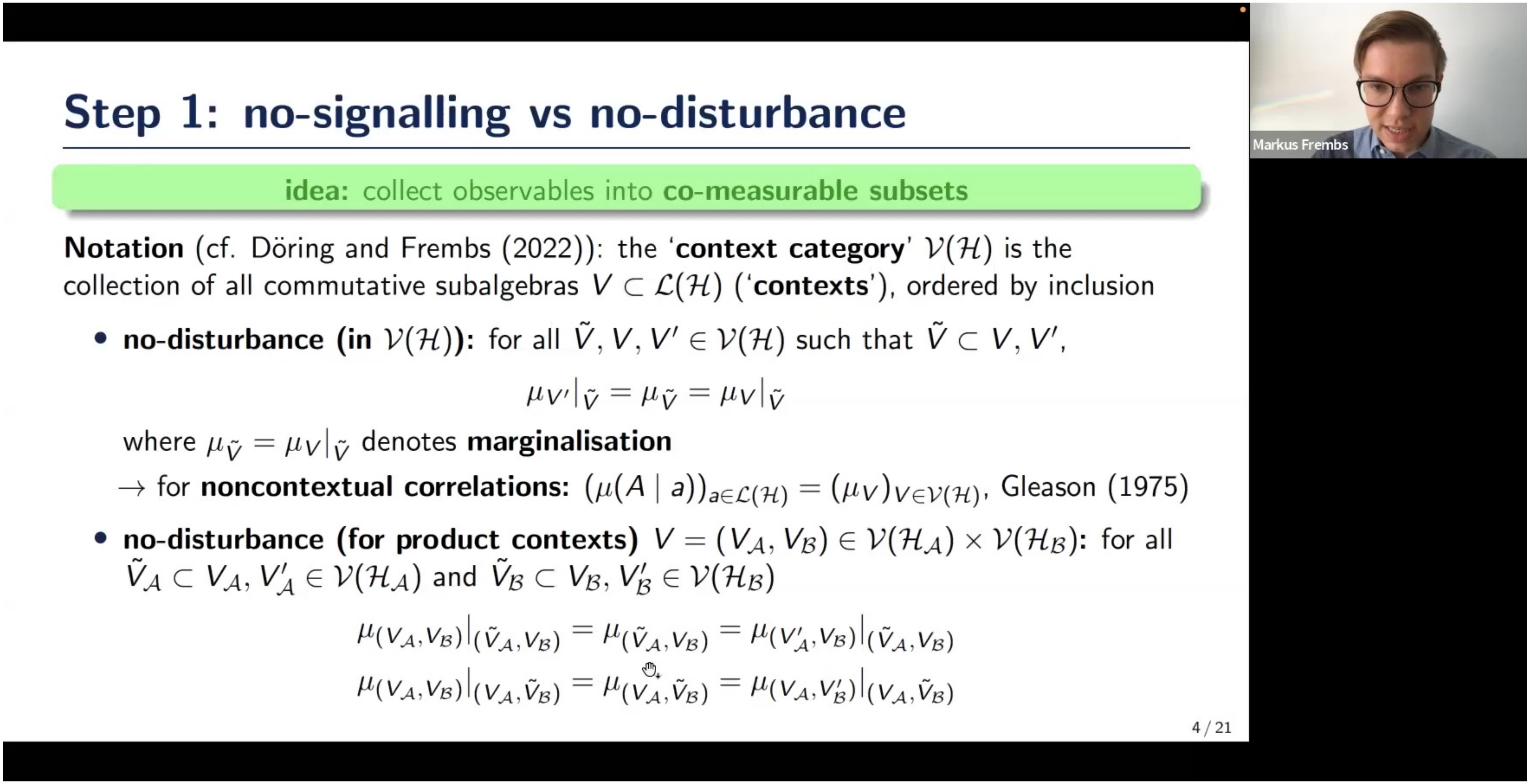


Step 1: no-signalling vs no-disturbance

idea: collect observables into **co-measurable subsets**

Notation (cf. Döring and Frembs (2022)): the '**context category**' $\mathcal{V}(\mathcal{H})$ is the collection of all commutative subalgebras $V \subset \mathcal{L}(\mathcal{H})$ ('**contexts**'), ordered by inclusion

- **no-disturbance (in $\mathcal{V}(\mathcal{H})$):** for all $\tilde{V}, V, V' \in \mathcal{V}(\mathcal{H})$ such that $\tilde{V} \subset V, V'$,

$$\mu_{V'}|_{\tilde{V}} = \mu_{\tilde{V}} = \mu_V|_{\tilde{V}}$$

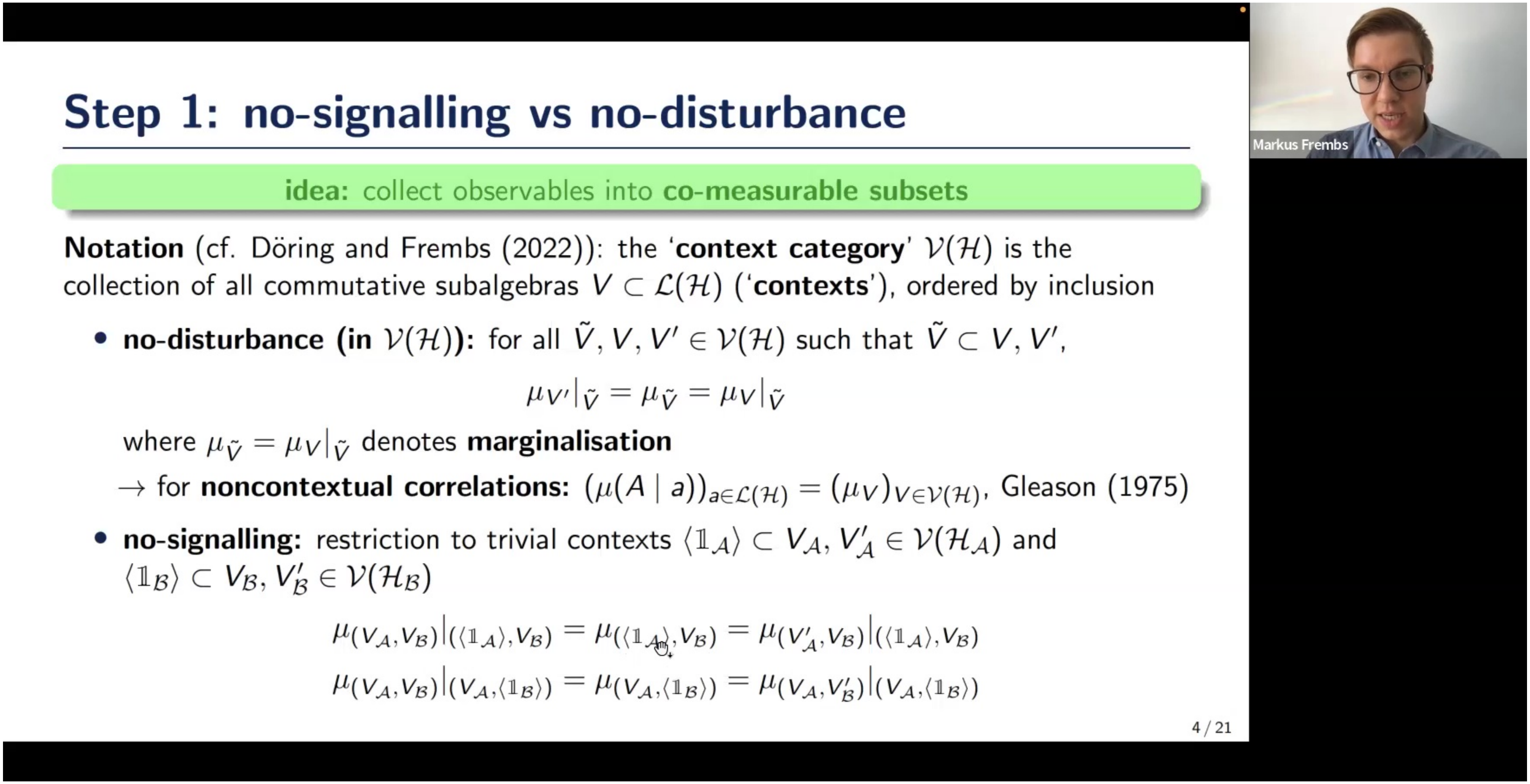
  where $\mu_{\tilde{V}} = \mu_V|_{\tilde{V}}$ denotes **marginalisation**

  → for **noncontextual correlations:** $(\mu(A \mid a))_{a \in \mathcal{L}(\mathcal{H})} = (\mu_V)_{V \in \mathcal{V}(\mathcal{H})}$, Gleason (1975)

- **no-disturbance (for product contexts)** $V = (V_\mathcal{A}, V_\mathcal{B}) \in \mathcal{V}(\mathcal{H}_\mathcal{A}) \times \mathcal{V}(\mathcal{H}_\mathcal{B})$**:** for all $\tilde{V}_\mathcal{A} \subset V_\mathcal{A}, V'_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A})$ and $\tilde{V}_\mathcal{B} \subset V_\mathcal{B}, V'_\mathcal{B} \in \mathcal{V}(\mathcal{H}_\mathcal{B})$

$$\mu_{(V_\mathcal{A}, V_\mathcal{B})}|_{(\tilde{V}_\mathcal{A}, V_\mathcal{B})} = \mu_{(\tilde{V}_\mathcal{A}, V_\mathcal{B})} = \mu_{(V'_\mathcal{A}, V_\mathcal{B})}|_{(\tilde{V}_\mathcal{A}, V_\mathcal{B})}$$

$$\mu_{(V_\mathcal{A}, V_\mathcal{B})}|_{(V_\mathcal{A}, \tilde{V}_\mathcal{B})} = \mu_{(V_\mathcal{A}, \tilde{V}_\mathcal{B})} = \mu_{(V_\mathcal{A}, V'_\mathcal{B})}|_{(V_\mathcal{A}, \tilde{V}_\mathcal{B})}$$



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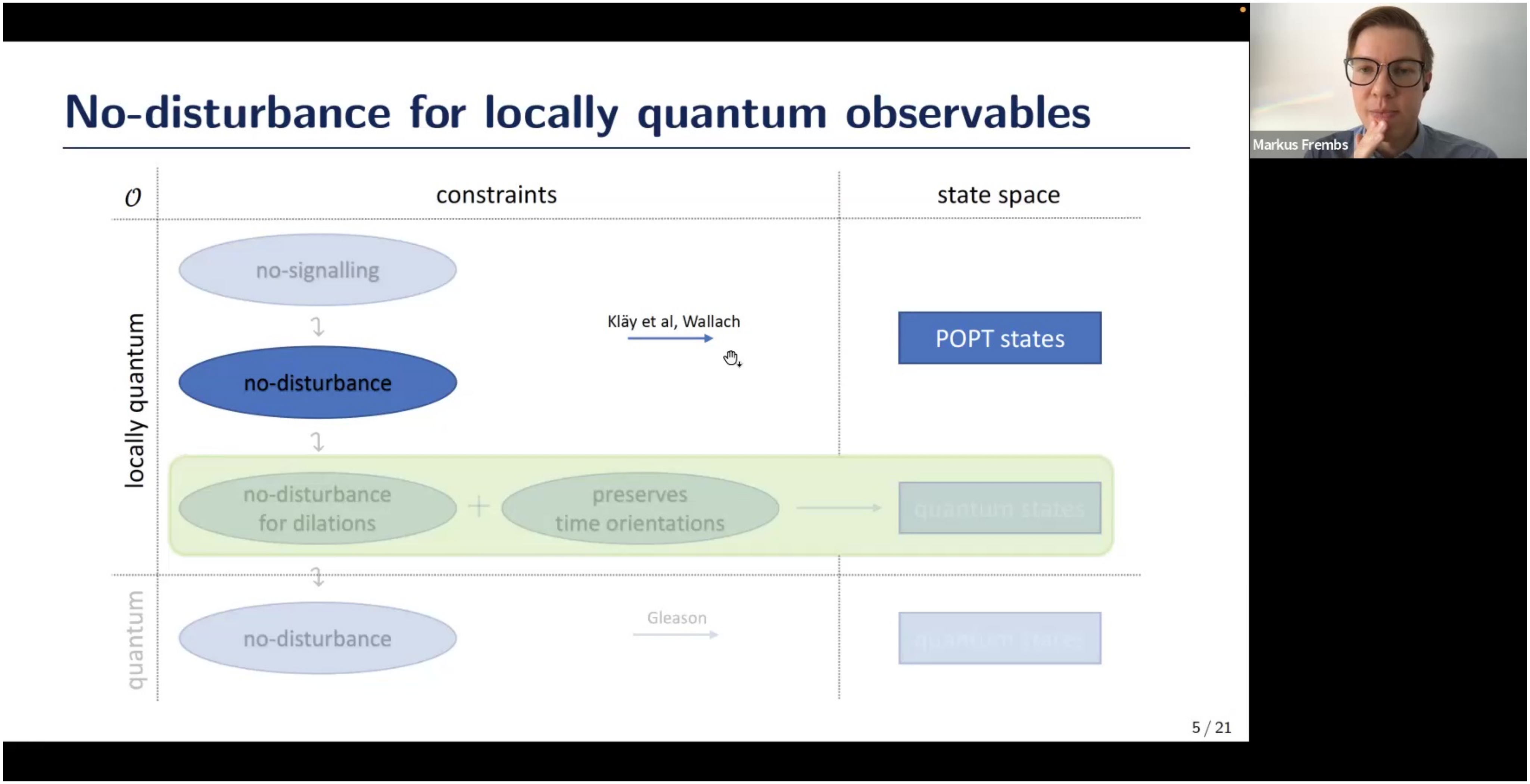
- **no-signalling:** restriction to trivial contexts $\langle \mathbb{1}_\mathcal{A} \rangle \subset V_\mathcal{A}, V'_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A})$ and $\langle \mathbb{1}_\mathcal{B} \rangle \subset V_\mathcal{B}, V'_\mathcal{B} \in \mathcal{V}(\mathcal{H}_\mathcal{B})$

$$\mu_{(V_\mathcal{A}, V_\mathcal{B})}|_{(\langle \mathbb{1}_\mathcal{A} \rangle, V_\mathcal{B})} = \mu_{(\langle \mathbb{1}_\mathcal{A} \rangle, V_\mathcal{B})} = \mu_{(V'_\mathcal{A}, V_\mathcal{B})}|_{(\langle \mathbb{1}_\mathcal{A} \rangle, V_\mathcal{B})}$$

$$\mu_{(V_\mathcal{A}, V_\mathcal{B})}|_{(V_\mathcal{A}, \langle \mathbb{1}_\mathcal{B} \rangle)} = \mu_{(V_\mathcal{A}, \langle \mathbb{1}_\mathcal{B} \rangle)} = \mu_{(V_\mathcal{A}, V'_\mathcal{B})}|_{(V_\mathcal{A}, \langle \mathbb{1}_\mathcal{B} \rangle)}$$



No-disturbance for locally quantum observables

$\mathcal{O}$

constraints

state space

locally quantum

no-signalling

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Kläy et al, Wallach

POPT states

no-disturbance for dilations + preserves time orientations

quantum states

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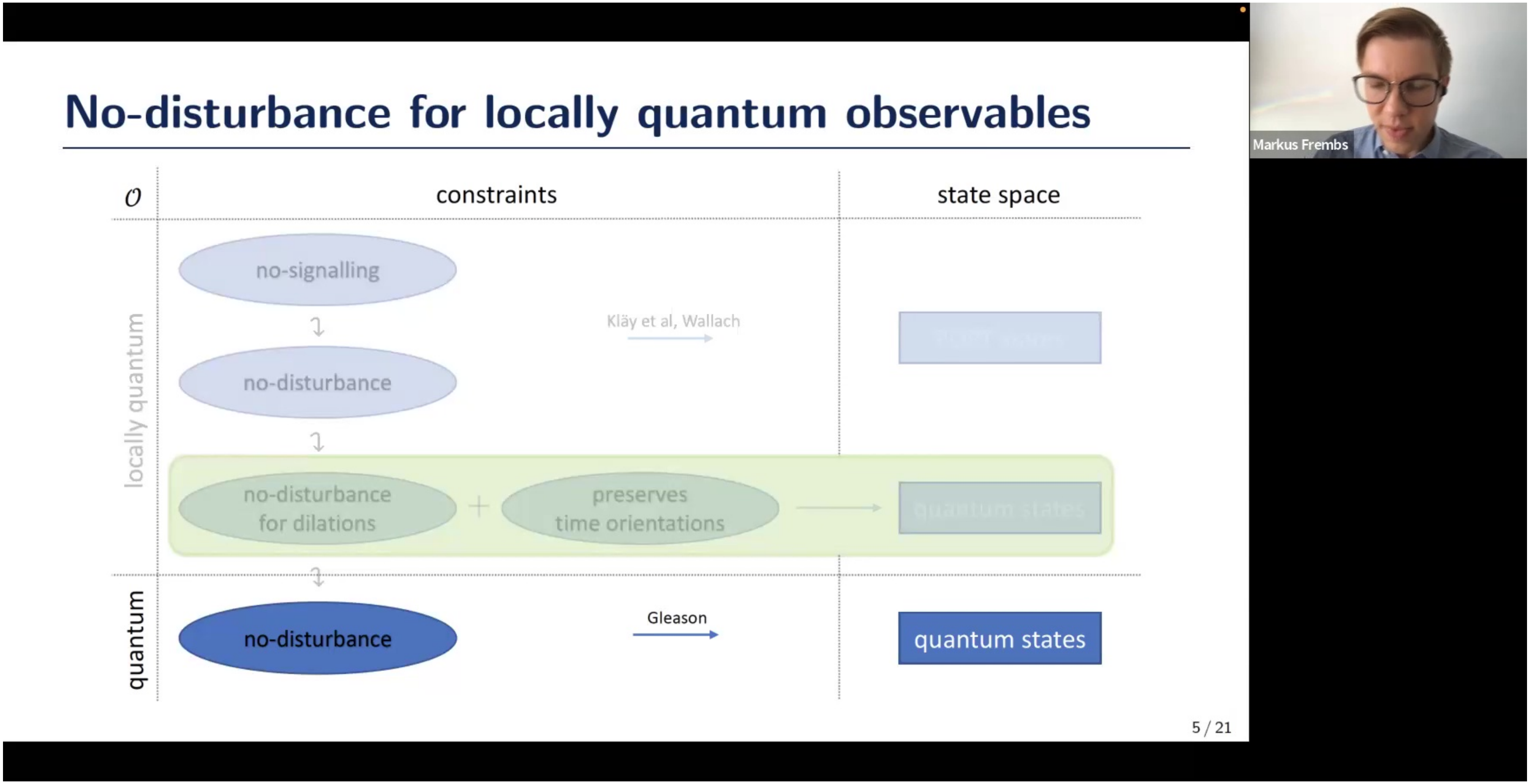

No-disturbance for locally quantum observables

$\mathcal{O}$

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↓

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no-disturbance for dilations + preserves time orientations → quantum states

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Kläy et al, Wallach

[illegible] states

quantum

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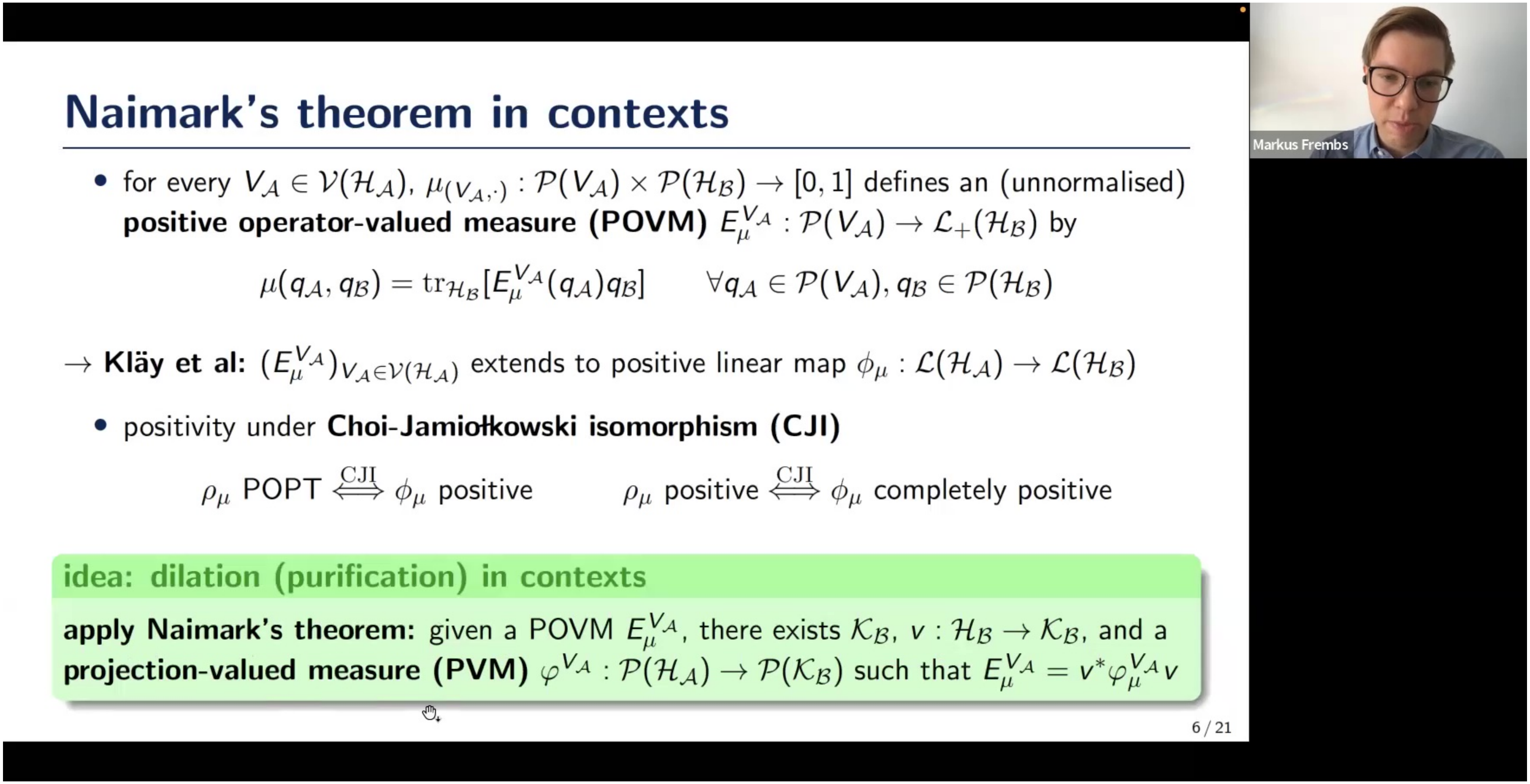


Naimark's theorem in contexts

- for every $V_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A})$, $\mu_{(V_\mathcal{A},\cdot)} : \mathcal{P}(V_\mathcal{A}) \times \mathcal{P}(\mathcal{H}_\mathcal{B}) \to [0,1]$ defines an (unnormalised) **positive operator-valued measure (POVM)** $E^{V_\mathcal{A}}_\mu : \mathcal{P}(V_\mathcal{A}) \to \mathcal{L}_+(\mathcal{H}_\mathcal{B})$ by

$$\mu(q_\mathcal{A}, q_\mathcal{B}) = \mathrm{tr}_{\mathcal{H}_\mathcal{B}}[E^{V_\mathcal{A}}_\mu(q_\mathcal{A})q_\mathcal{B}] \qquad \forall q_\mathcal{A} \in \mathcal{P}(V_\mathcal{A}), q_\mathcal{B} \in \mathcal{P}(\mathcal{H}_\mathcal{B})$$

→ **Kläy et al:** $(E^{V_\mathcal{A}}_\mu)_{V_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A})}$ extends to positive linear map $\phi_\mu : \mathcal{L}(\mathcal{H}_\mathcal{A}) \to \mathcal{L}(\mathcal{H}_\mathcal{B})$

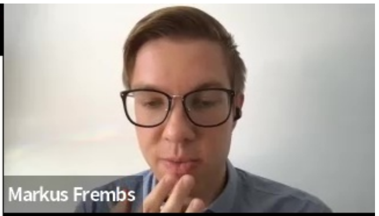
- positivity under **Choi-Jamiołkowski isomorphism (CJI)**

$$\rho_\mu \text{ POPT} \overset{\text{CJI}}{\Longleftrightarrow} \phi_\mu \text{ positive} \qquad\qquad \rho_\mu \text{ positive} \overset{\text{CJI}}{\Longleftrightarrow} \phi_\mu \text{ completely positive}$$

idea: dilation (purification) in contexts

apply Naimark's theorem: given a POVM $E^{V_\mathcal{A}}_\mu$, there exists $\mathcal{K}_\mathcal{B}$, $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}_\mathcal{B}$, and a **projection-valued measure (PVM)** $\varphi^{V_\mathcal{A}} : \mathcal{P}(\mathcal{H}_\mathcal{A}) \to \mathcal{P}(\mathcal{K}_\mathcal{B})$ such that $E^{V_\mathcal{A}}_\mu = v^*\varphi^{V_\mathcal{A}}_\mu v$

Step 2: no-disturbance for dilations



Definition

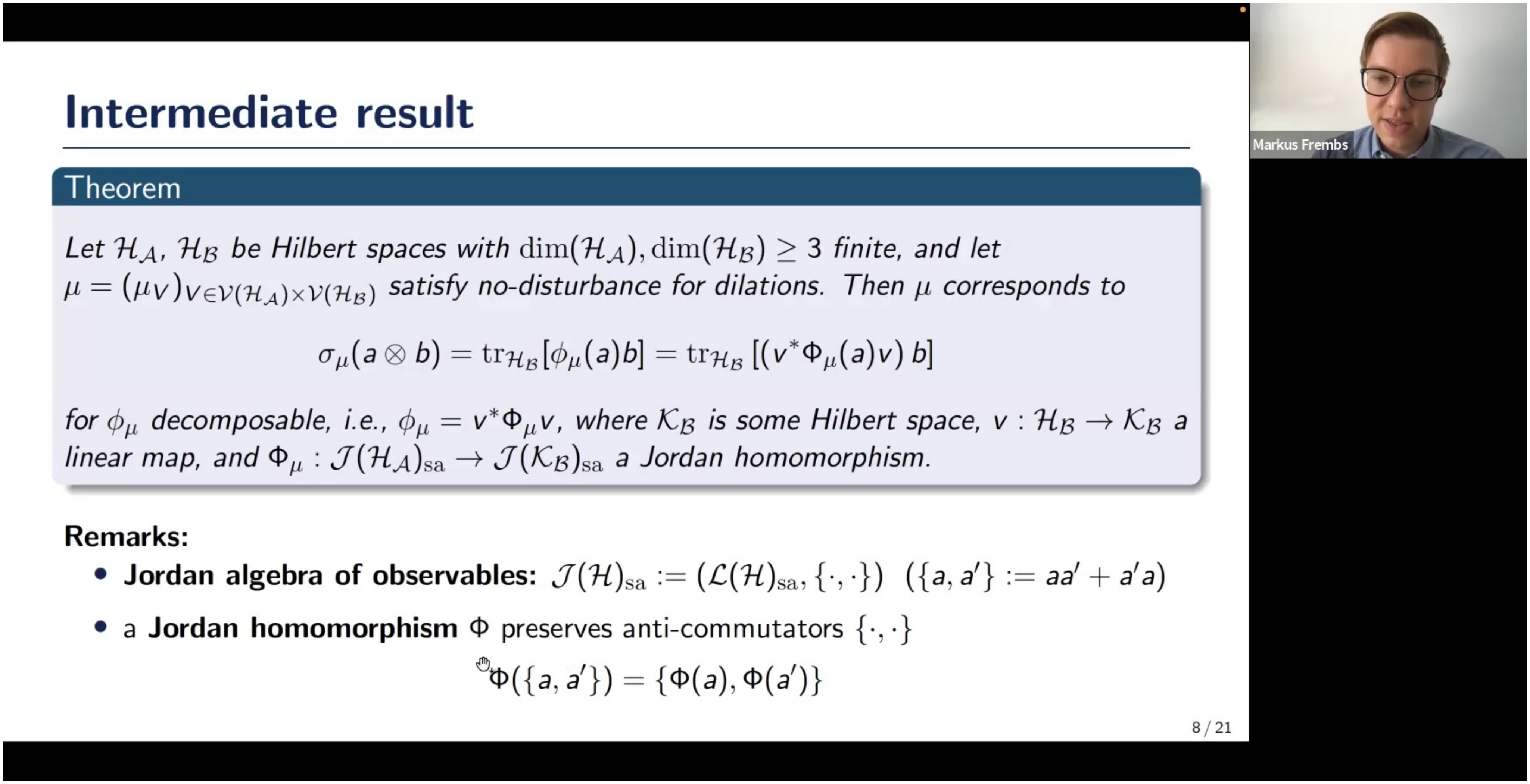
We say that $\mu = (\mu_V)_{V \in \mathcal{V}(\mathcal{H}_\mathcal{A}) \times \mathcal{V}(\mathcal{H}_\mathcal{B})}$ *satisfies no-disturbance for dilations if* $\mu(q_\mathcal{A}, q_\mathcal{B}) = \mathrm{tr}_{\mathcal{H}_\mathcal{B}}[(v^*\varphi_\mu^{V_\mathcal{A}}(q_\mathcal{A})v)\ q_\mathcal{B}]$ *for a Hilbert space* $\mathcal{K}_\mathcal{B}$*, linear map* $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}_\mathcal{B}$*, and projection-valued measures* $(\varphi_\mu^{V_\mathcal{A}})_{V_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A})}$*,* $\varphi_\mu^{V_\mathcal{A}} : \mathcal{P}(V_\mathcal{A}) \to \mathcal{P}(\mathcal{K}_\mathcal{B})$ *such that*

$$\forall q_\mathcal{A} \in \mathcal{P}(V_\mathcal{A}), V_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A}), q_{\mathcal{K}_\mathcal{B}} \in \mathcal{P}(\mathcal{K}_\mathcal{B}) : \quad \mu'(q_\mathcal{A}, q_{\mathcal{K}_\mathcal{B}}) := \mathrm{tr}_{\mathcal{H}_\mathcal{B}}\left[v^*\varphi_\mu^{V_\mathcal{A}}(q_\mathcal{A})q_{\mathcal{K}_\mathcal{B}}v\right]$$

satisfies the no-disturbance principle for all product contexts in $\mathcal{V}(\mathcal{H}_\mathcal{A}) \times \mathcal{V}(\mathcal{K}_\mathcal{B})$.

Remarks

- interpret $\mu_{(V_\mathcal{A},\cdot)}$ for every $V_\mathcal{A} \in \mathcal{V}(\mathcal{H}_\mathcal{A})$ as a **state of incomplete information**
  → arises from **coarse-graining of ancillary degrees of freedom**
- **noncontextuality condition**
  → stronger version of Gleason's theorem applies, J. Bunce and Wright (1993)



Intermediate result

Theorem

Let $\mathcal{H}_\mathcal{A}$, $\mathcal{H}_\mathcal{B}$ *be Hilbert spaces with* $\dim(\mathcal{H}_\mathcal{A}), \dim(\mathcal{H}_\mathcal{B}) \geq 3$ *finite, and let* $\mu = (\mu_V)_{V \in \mathcal{V}(\mathcal{H}_\mathcal{A}) \times \mathcal{V}(\mathcal{H}_\mathcal{B})}$ *satisfy no-disturbance for dilations. Then* μ *corresponds to*

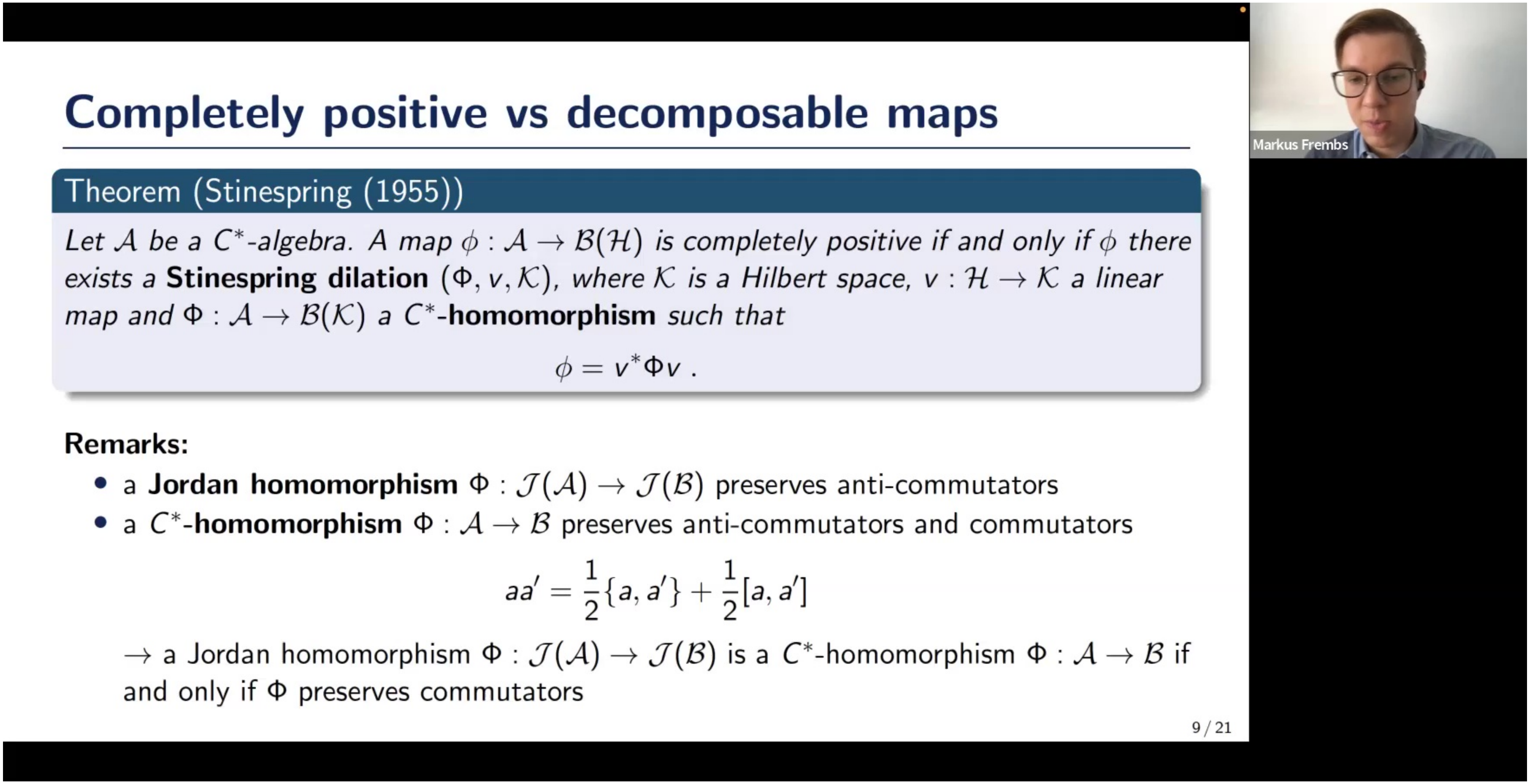
$$\sigma_\mu(a \otimes b) = \mathrm{tr}_{\mathcal{H}_\mathcal{B}}[\phi_\mu(a)b] = \mathrm{tr}_{\mathcal{H}_\mathcal{B}}\left[(v^*\Phi_\mu(a)v)\ b\right]$$

for ϕ_μ *decomposable, i.e.,* $\phi_\mu = v^*\Phi_\mu v$*, where* $\mathcal{K}_\mathcal{B}$ *is some Hilbert space,* $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}_\mathcal{B}$ *a linear map, and* $\Phi_\mu : \mathcal{J}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{K}_\mathcal{B})_{\mathrm{sa}}$ *a Jordan homomorphism.*

Remarks:

- **Jordan algebra of observables:** $\mathcal{J}(\mathcal{H})_{\mathrm{sa}} := (\mathcal{L}(\mathcal{H})_{\mathrm{sa}}, \{\cdot,\cdot\})$ $(\{a, a'\} := aa' + a'a)$
- a **Jordan homomorphism** Φ preserves anti-commutators $\{\cdot,\cdot\}$

$$\Phi(\{a, a'\}) = \{\Phi(a), \Phi(a')\}$$



Completely positive vs decomposable maps

Theorem (Stinespring (1955))

Let $\mathcal{A}$ be a C^*-algebra. A map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is completely positive if and only if ϕ there exists a **Stinespring dilation** $(\Phi, v, \mathcal{K})$, *where $\mathcal{K}$ is a Hilbert space, $v : \mathcal{H} \to \mathcal{K}$ a linear map and $\Phi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ a* C^*-**homomorphism** *such that*

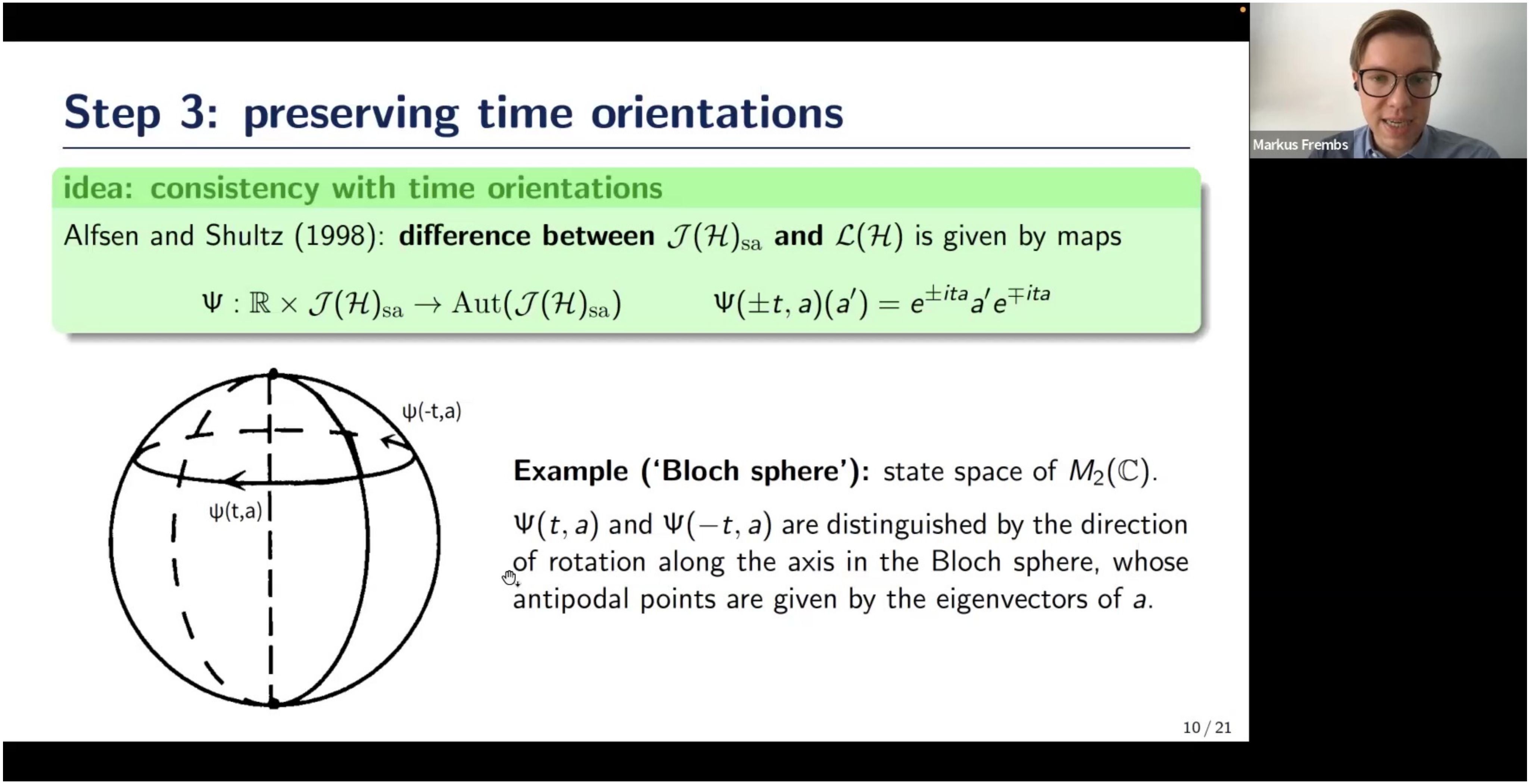
$$\phi = v^* \Phi v \ .$$

Remarks:

- a **Jordan homomorphism** $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ preserves anti-commutators
- a C^*-**homomorphism** $\Phi : \mathcal{A} \to \mathcal{B}$ preserves anti-commutators and commutators

$$aa' = \frac{1}{2}\{a, a'\} + \frac{1}{2}[a, a']$$

$\rightarrow$ a Jordan homomorphism $\Phi : \mathcal{J}(\mathcal{A}) \to \mathcal{J}(\mathcal{B})$ is a C^*-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ if and only if Φ preserves commutators



Step 3: preserving time orientations

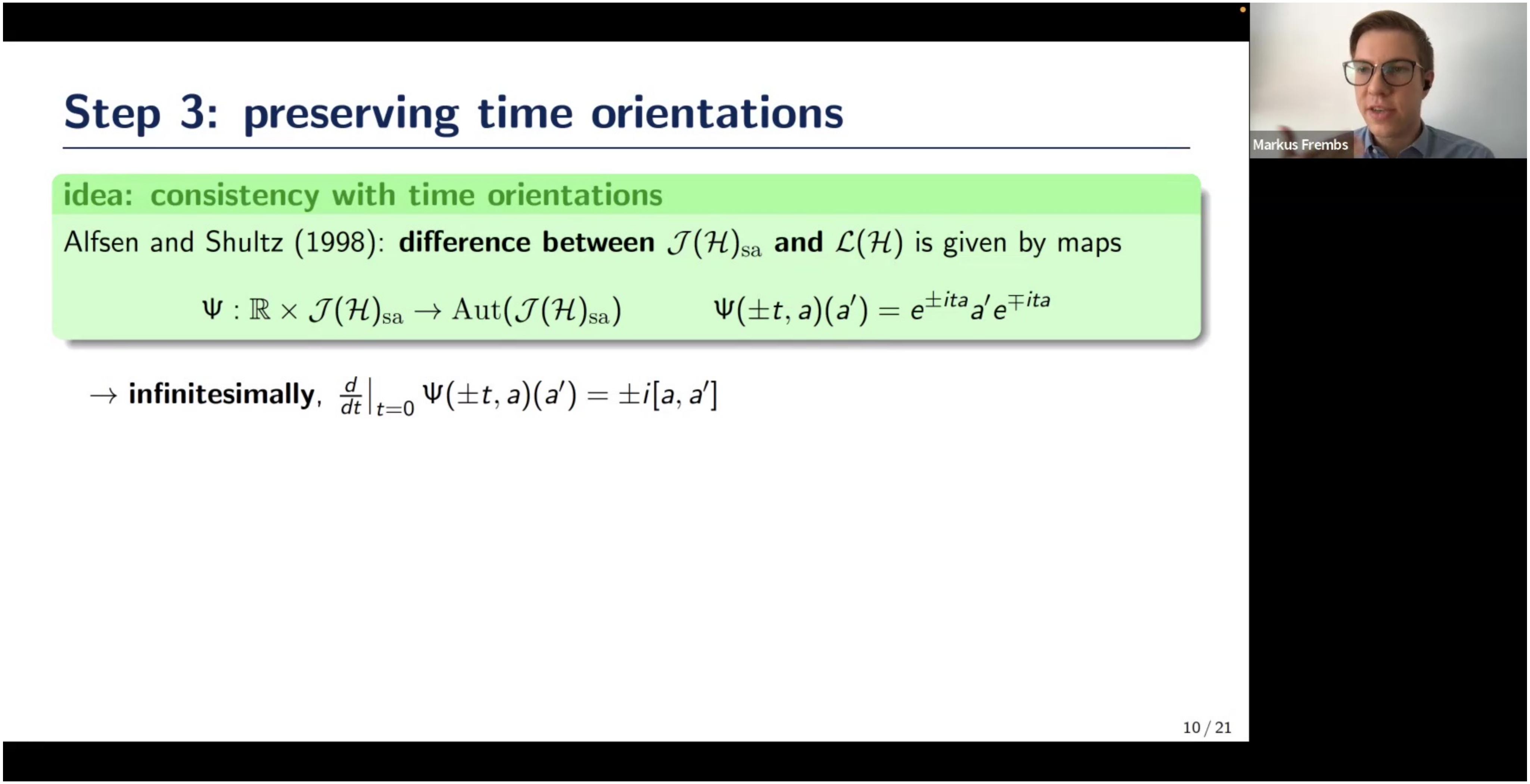
idea: consistency with time orientations

Alfsen and Shultz (1998): **difference between $\mathcal{J}(\mathcal{H})_{\mathrm{sa}}$ and $\mathcal{L}(\mathcal{H})$** is given by maps

$$\Psi : \mathbb{R} \times \mathcal{J}(\mathcal{H})_{\mathrm{sa}} \to \mathrm{Aut}(\mathcal{J}(\mathcal{H})_{\mathrm{sa}}) \qquad \Psi(\pm t, a)(a') = e^{\pm ita} a' e^{\mp ita}$$

Example ('Bloch sphere'): state space of $M_2(\mathbb{C})$.

$\Psi(t, a)$ and $\Psi(-t, a)$ are distinguished by the direction of rotation along the axis in the Bloch sphere, whose antipodal points are given by the eigenvectors of a.



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→ **infinitesimally**, $\frac{d}{dt}\big|_{t=0} \Psi(\pm t, a)(a') = \pm i[a, a']$

Intermediate result

Theorem

Let $\mathcal{H}_\mathcal{A}$, $\mathcal{H}_\mathcal{B}$ *be Hilbert spaces with* $\dim(\mathcal{H}_\mathcal{A}), \dim(\mathcal{H}_\mathcal{B}) \geq 3$ *finite, and let* $\mu = (\mu_V)_{V \in \mathcal{V}(\mathcal{H}_\mathcal{A}) \times \mathcal{V}(\mathcal{H}_\mathcal{B})}$ *satisfy no-disturbance for dilations. Then* μ *corresponds to*

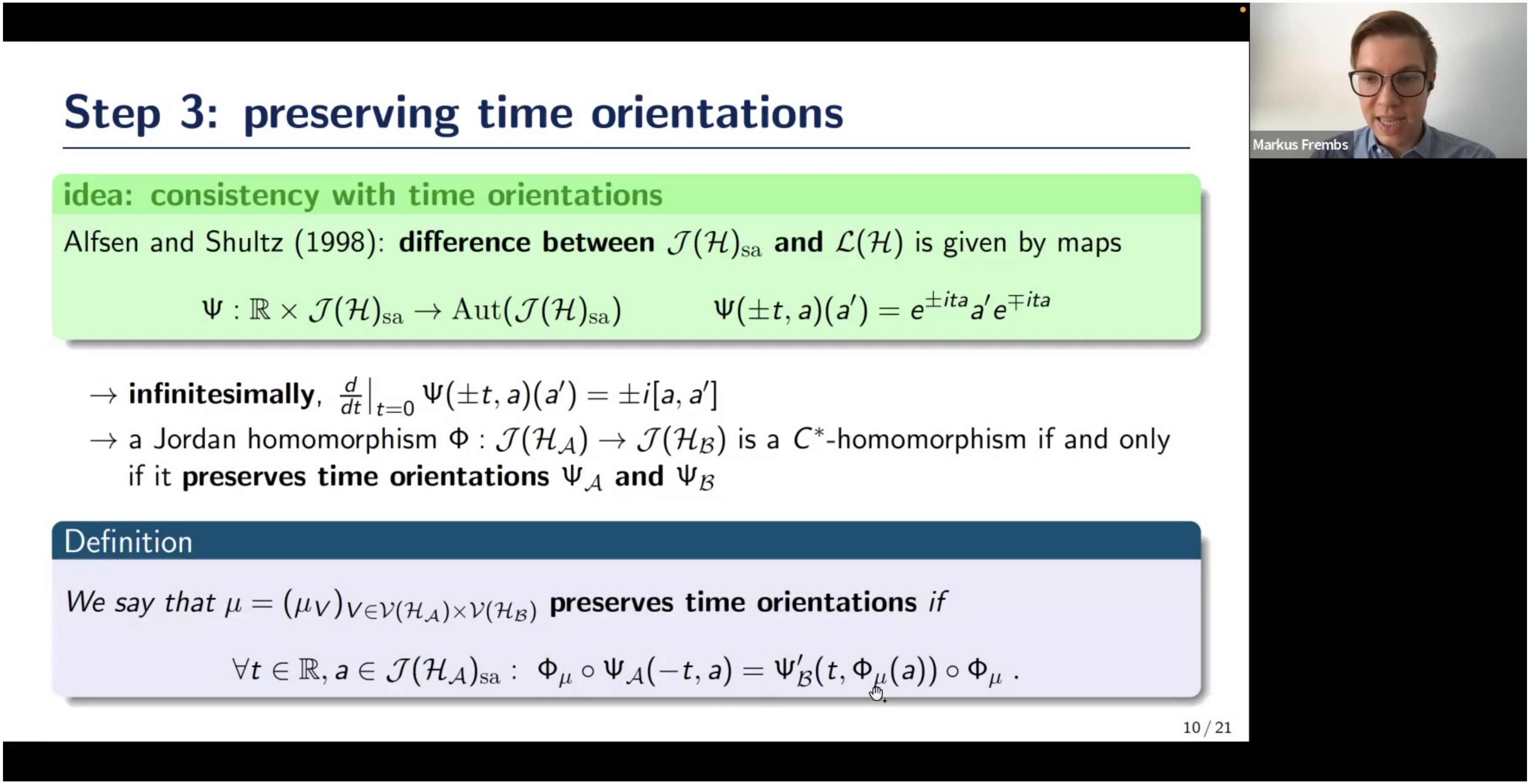
$$\sigma_\mu(a \otimes b) = \mathrm{tr}_{\mathcal{H}_\mathcal{B}}[\phi_\mu(a)b] = \mathrm{tr}_{\mathcal{H}_\mathcal{B}}\left[(v^*\Phi_\mu(a)v)\, b\right]$$

for ϕ_μ *decomposable, i.e.,* $\phi_\mu = v^*\Phi_\mu v$*, where* $\mathcal{K}_\mathcal{B}$ *is some Hilbert space,* $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}_\mathcal{B}$ *a linear map, and* $\Phi_\mu : \mathcal{J}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{K}_\mathcal{B})_{\mathrm{sa}}$ *a Jordan homomorphism.*

Remarks:

- **Jordan algebra of observables:** $\mathcal{J}(\mathcal{H})_{\mathrm{sa}} := (\mathcal{L}(\mathcal{H})_{\mathrm{sa}}, \{\cdot,\cdot\})$ $(\{a, a'\} := aa' + a'a)$
- a **Jordan homomorphism** Φ preserves anti-commutators $\{\cdot,\cdot\}$

$$\Phi(\{a, a'\}) = \{\Phi(a), \Phi(a')\}$$


Markus Frembs

Step 3: preserving time orientations

idea: consistency with time orientations

Alfsen and Shultz (1998): **difference between $\mathcal{J}(\mathcal{H})_{\mathrm{sa}}$ and $\mathcal{L}(\mathcal{H})$** is given by maps

$$\Psi : \mathbb{R} \times \mathcal{J}(\mathcal{H})_{\mathrm{sa}} \to \mathrm{Aut}(\mathcal{J}(\mathcal{H})_{\mathrm{sa}}) \qquad \Psi(\pm t, a)(a') = e^{\pm ita} a' e^{\mp ita}$$

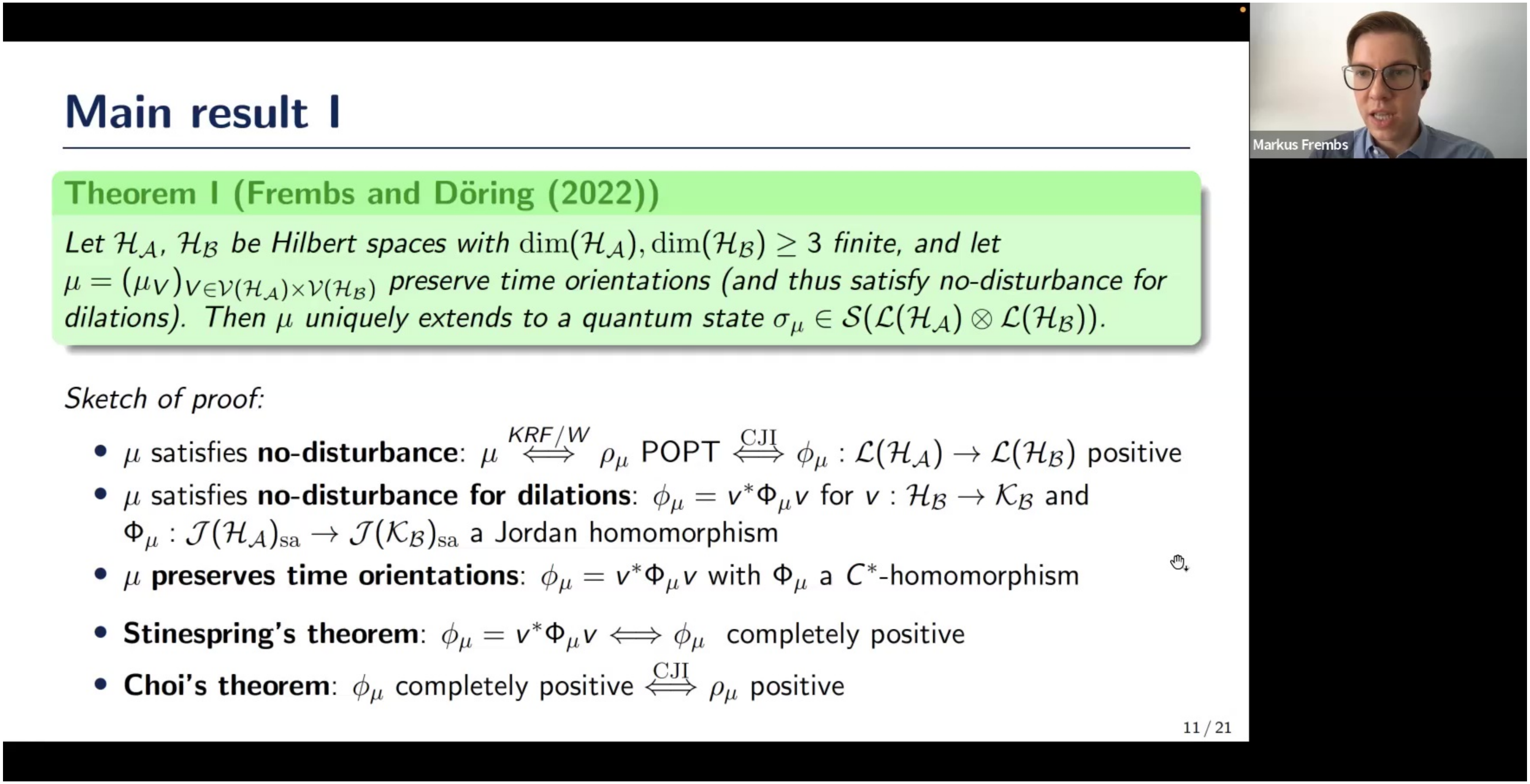
→ **infinitesimally**, $\frac{d}{dt}\big|_{t=0} \Psi(\pm t, a)(a') = \pm i[a, a']$

→ a Jordan homomorphism $\Phi : \mathcal{J}(\mathcal{H}_{\mathcal{A}}) \to \mathcal{J}(\mathcal{H}_{\mathcal{B}})$ is a C^*-homomorphism if and only if it **preserves time orientations** $\Psi_{\mathcal{A}}$ **and** $\Psi_{\mathcal{B}}$

Definition

We say that $\mu = (\mu_V)_{V \in \mathcal{V}(\mathcal{H}_{\mathcal{A}}) \times \mathcal{V}(\mathcal{H}_{\mathcal{B}})}$ **preserves time orientations** *if*

$$\forall t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{H}_{\mathcal{A}})_{\mathrm{sa}} : \ \Phi_\mu \circ \Psi_{\mathcal{A}}(-t, a) = \Psi'_{\mathcal{B}}(t, \Phi_\mu(a)) \circ \Phi_\mu \ .$$



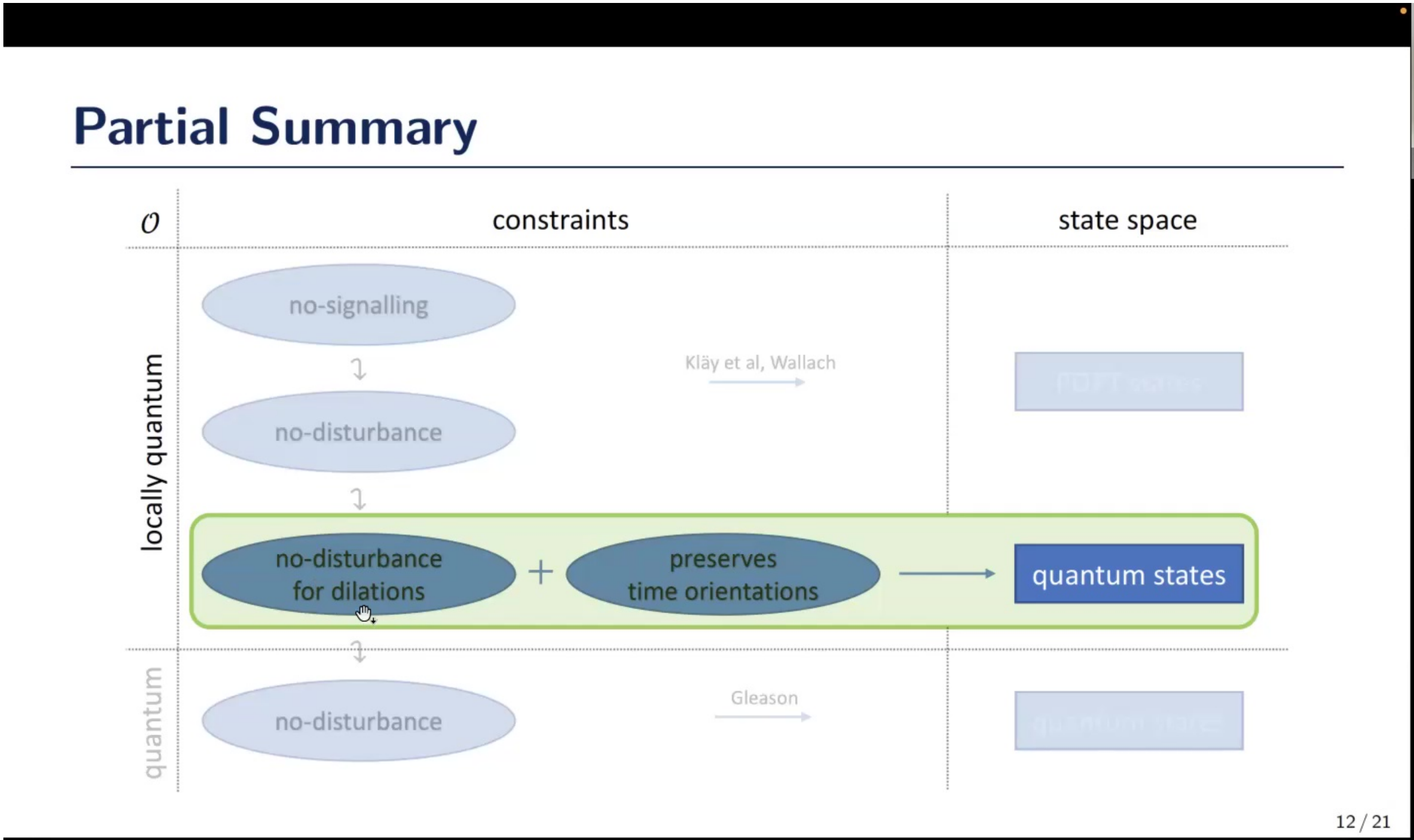
Main result I

Theorem I (Frembs and Döring (2022))

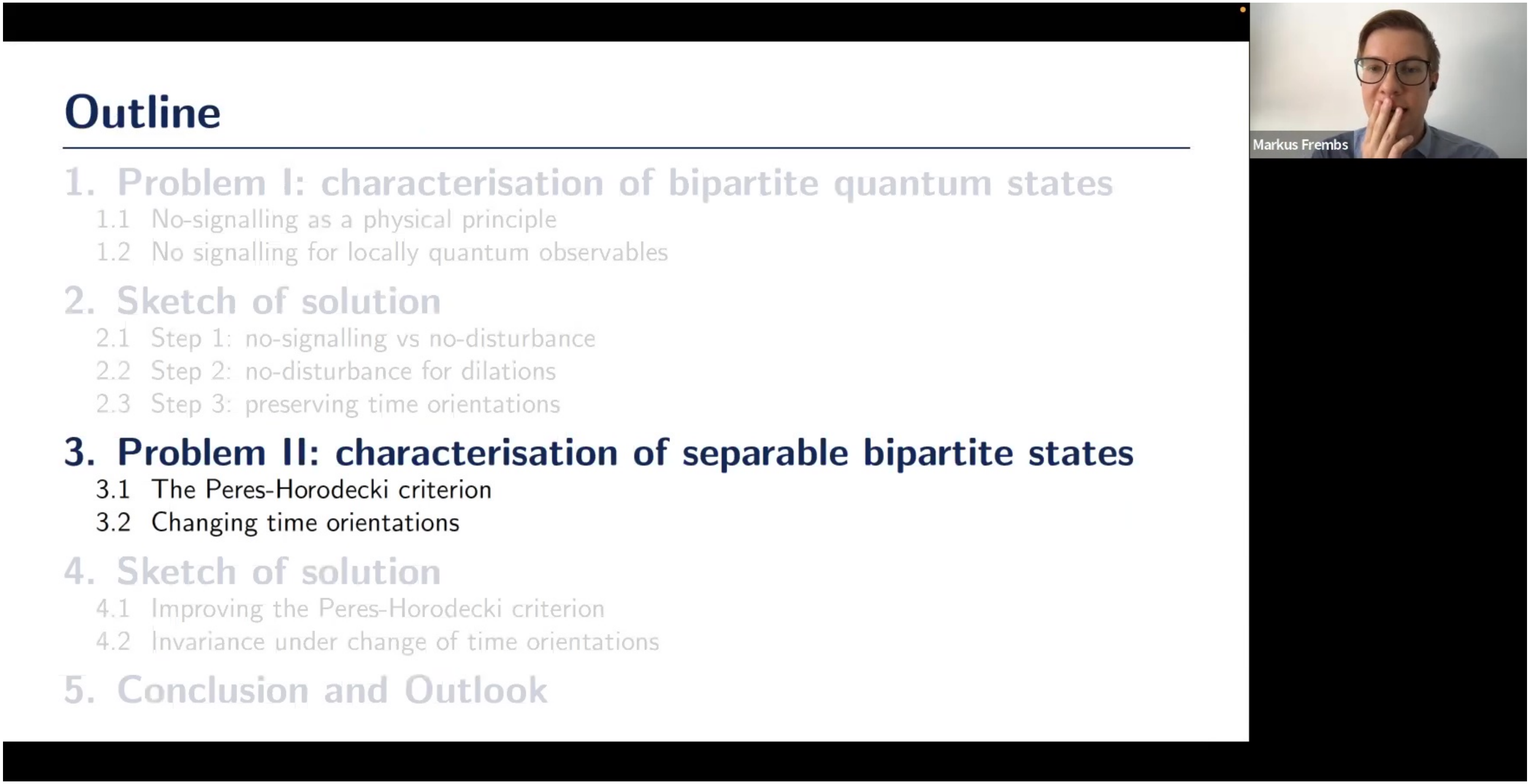
Let $\mathcal{H}_\mathcal{A}$, $\mathcal{H}_\mathcal{B}$ be Hilbert spaces with $\dim(\mathcal{H}_\mathcal{A}), \dim(\mathcal{H}_\mathcal{B}) \geq 3$ finite, and let $\mu = (\mu_V)_{V \in \mathcal{V}(\mathcal{H}_\mathcal{A}) \times \mathcal{V}(\mathcal{H}_\mathcal{B})}$ preserve time orientations (and thus satisfy no-disturbance for dilations). Then μ uniquely extends to a quantum state $\sigma_\mu \in \mathcal{S}(\mathcal{L}(\mathcal{H}_\mathcal{A}) \otimes \mathcal{L}(\mathcal{H}_\mathcal{B}))$.

Sketch of proof:

- μ satisfies **no-disturbance**: $\mu \overset{KRF/W}{\Longleftrightarrow} \rho_\mu$ POPT $\overset{\mathrm{CJI}}{\Longleftrightarrow} \phi_\mu : \mathcal{L}(\mathcal{H}_\mathcal{A}) \to \mathcal{L}(\mathcal{H}_\mathcal{B})$ positive
- μ satisfies **no-disturbance for dilations**: $\phi_\mu = v^* \Phi_\mu v$ for $v : \mathcal{H}_\mathcal{B} \to \mathcal{K}_\mathcal{B}$ and $\Phi_\mu : \mathcal{J}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}} \to \mathcal{J}(\mathcal{K}_\mathcal{B})_{\mathrm{sa}}$ a Jordan homomorphism
- μ **preserves time orientations**: $\phi_\mu = v^* \Phi_\mu v$ with Φ_μ a C^*-homomorphism
- **Stinespring's theorem**: $\phi_\mu = v^* \Phi_\mu v \Longleftrightarrow \phi_\mu$ completely positive
- **Choi's theorem**: ϕ_μ completely positive $\overset{\mathrm{CJI}}{\Longleftrightarrow} \rho_\mu$ positive



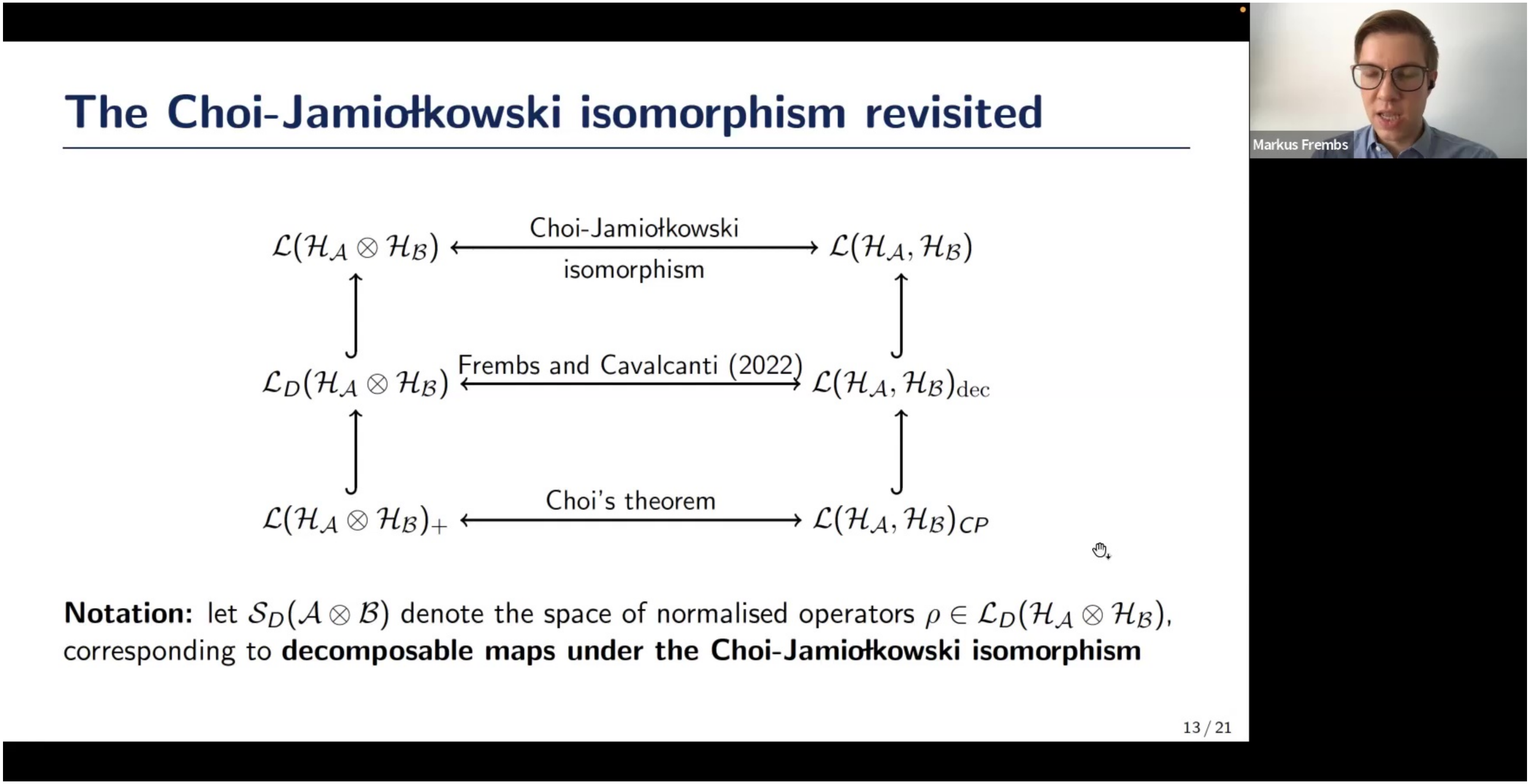
Partial Summary



Outline

The Choi-Jamiołkowski isomorphism revisited

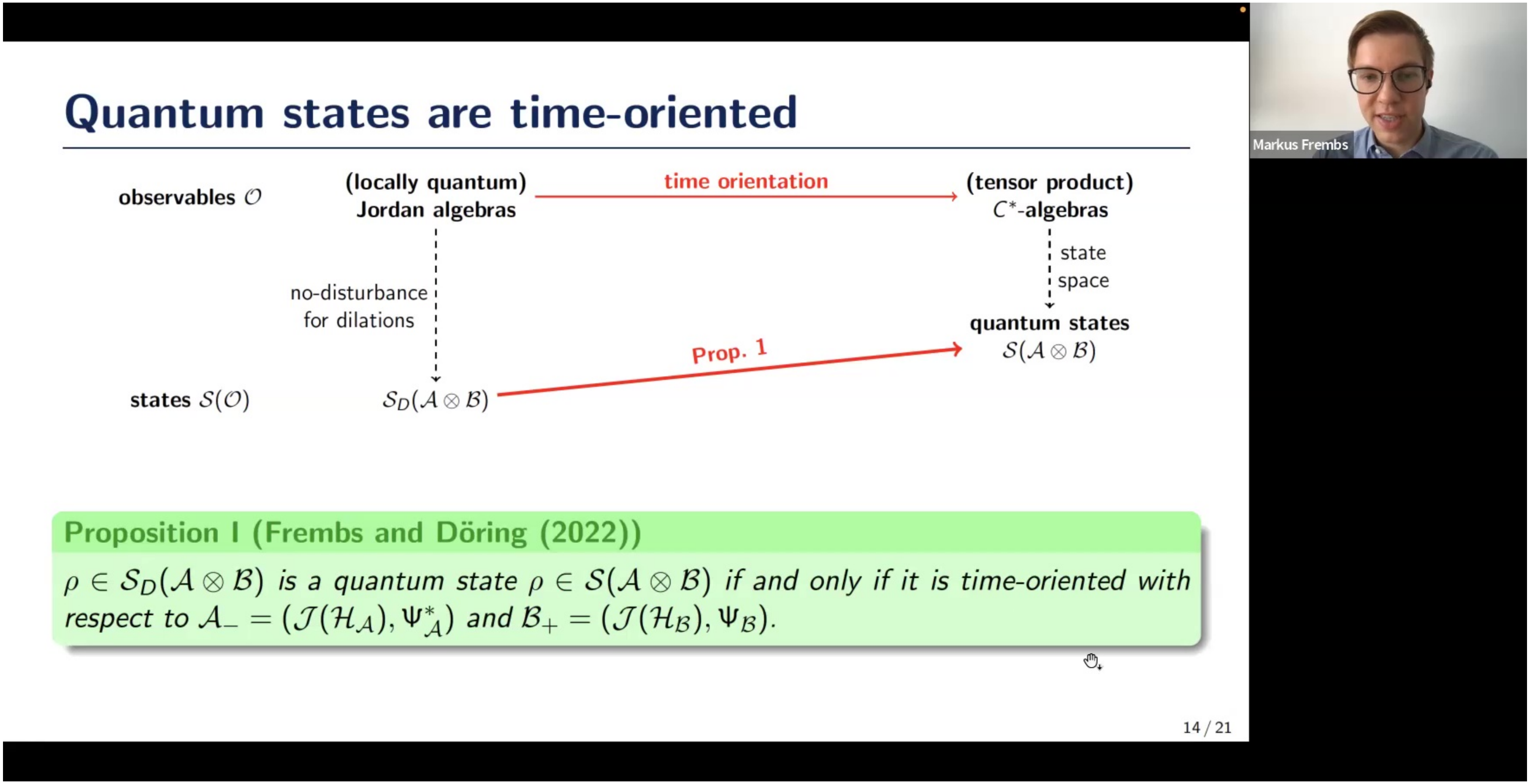
$$\begin{tikzcd}
\mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \arrow[r, leftrightarrow, "\text{Choi-Jamiołkowski}", "\text{isomorphism}"'] & \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B}) \\
\mathcal{L}_D(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \arrow[u, hook] \arrow[r, leftrightarrow, "\text{Frembs and Cavalcanti (2022)}"] & \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B})_{\mathrm{dec}} \arrow[u, hook] \\
\mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})_+ \arrow[u, hook] \arrow[r, leftrightarrow, "\text{Choi's theorem}"] & \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B})_{CP} \arrow[u, hook]
\end{tikzcd}$$



The Choi-Jamiołkowski isomorphism revisited

$$\begin{tikzcd}
\mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \arrow[rr, leftrightarrow, "\text{Choi-Jamiołkowski}", "\text{isomorphism}"'] & & \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B}) \\
\mathcal{L}_D(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \arrow[u, hook] \arrow[rr, leftrightarrow, "\text{Frembs and Cavalcanti (2022)}"] & & \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B})_{\mathrm{dec}} \arrow[u, hook] \\
\mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})_+ \arrow[u, hook] \arrow[rr, leftrightarrow, "\text{Choi's theorem}"] & & \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B})_{CP} \arrow[u, hook]
\end{tikzcd}$$

Notation: let $\mathcal{S}_D(\mathcal{A} \otimes \mathcal{B})$ denote the space of normalised operators $\rho \in \mathcal{L}_D(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B})$, corresponding to **decomposable maps under the Choi-Jamiołkowski isomorphism**

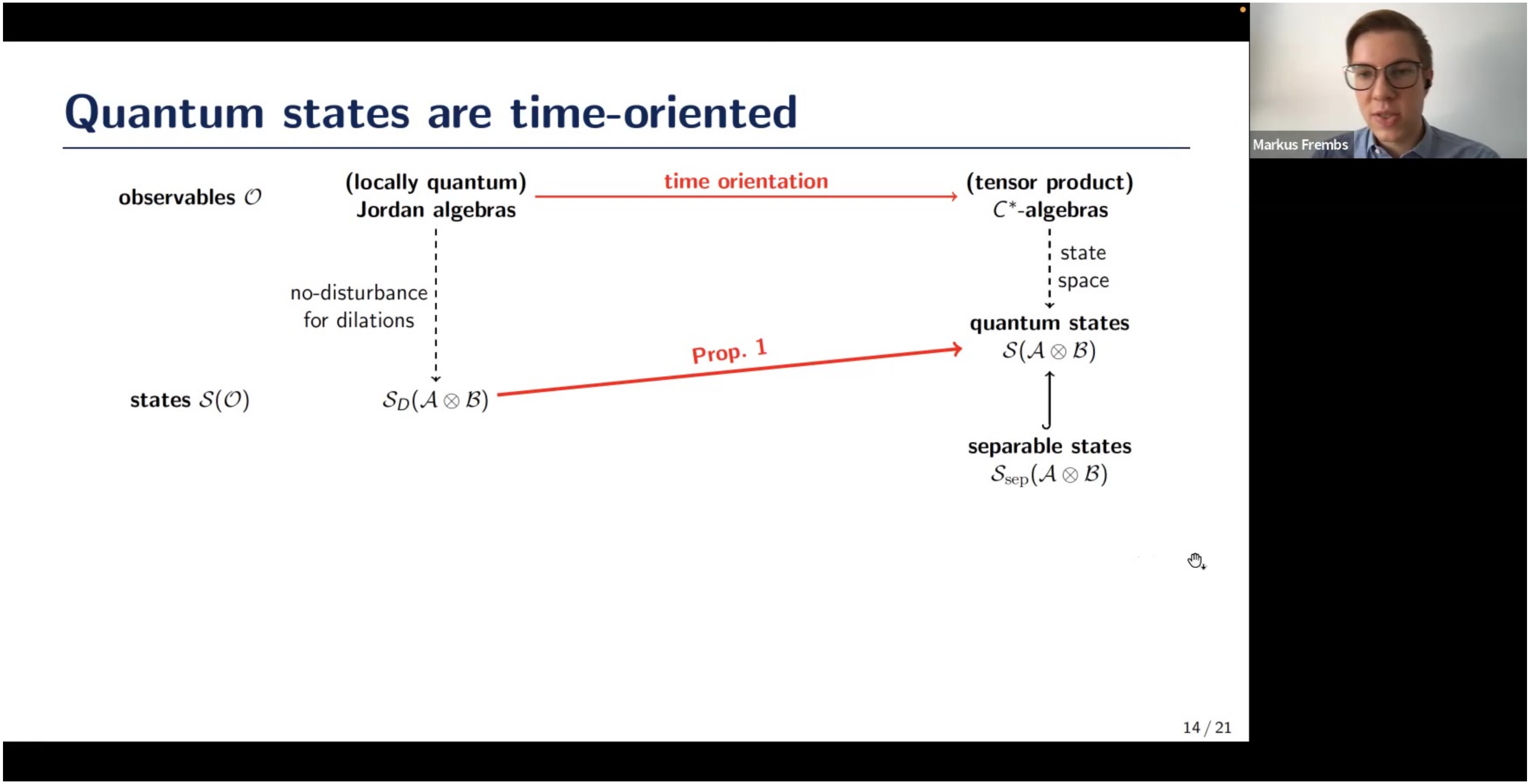


Quantum states are time-oriented

$$\begin{tikzcd}
\textbf{observables } \mathcal{O} & \textbf{(locally quantum) Jordan algebras} \arrow[r, "\text{time orientation}"] \arrow[dd, dashed, "\text{no-disturbance for dilations}"'] & \textbf{(tensor product) } C^*\textbf{-algebras} \arrow[d, dashed, "\text{state space}"] \\
 & & \textbf{quantum states } \mathcal{S}(\mathcal{A} \otimes \mathcal{B}) \\
\textbf{states } \mathcal{S}(\mathcal{O}) & \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B}) \arrow[ur, "\text{Prop. 1}"] &
\end{tikzcd}$$

Proposition I (Frembs and Döring (2022))

$\rho \in \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B})$ is a quantum state $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ if and only if it is time-oriented with respect to $\mathcal{A}_- = (\mathcal{J}(\mathcal{H}_\mathcal{A}), \Psi^*_\mathcal{A})$ and $\mathcal{B}_+ = (\mathcal{J}(\mathcal{H}_\mathcal{B}), \Psi_\mathcal{B})$.



Quantum states are time-oriented

observables $\mathcal{O}$

(locally quantum)
Jordan algebras

time orientation

(tensor product)
C^*-algebras

state
space

quantum states
$\mathcal{S}(\mathcal{A} \otimes \mathcal{B})$

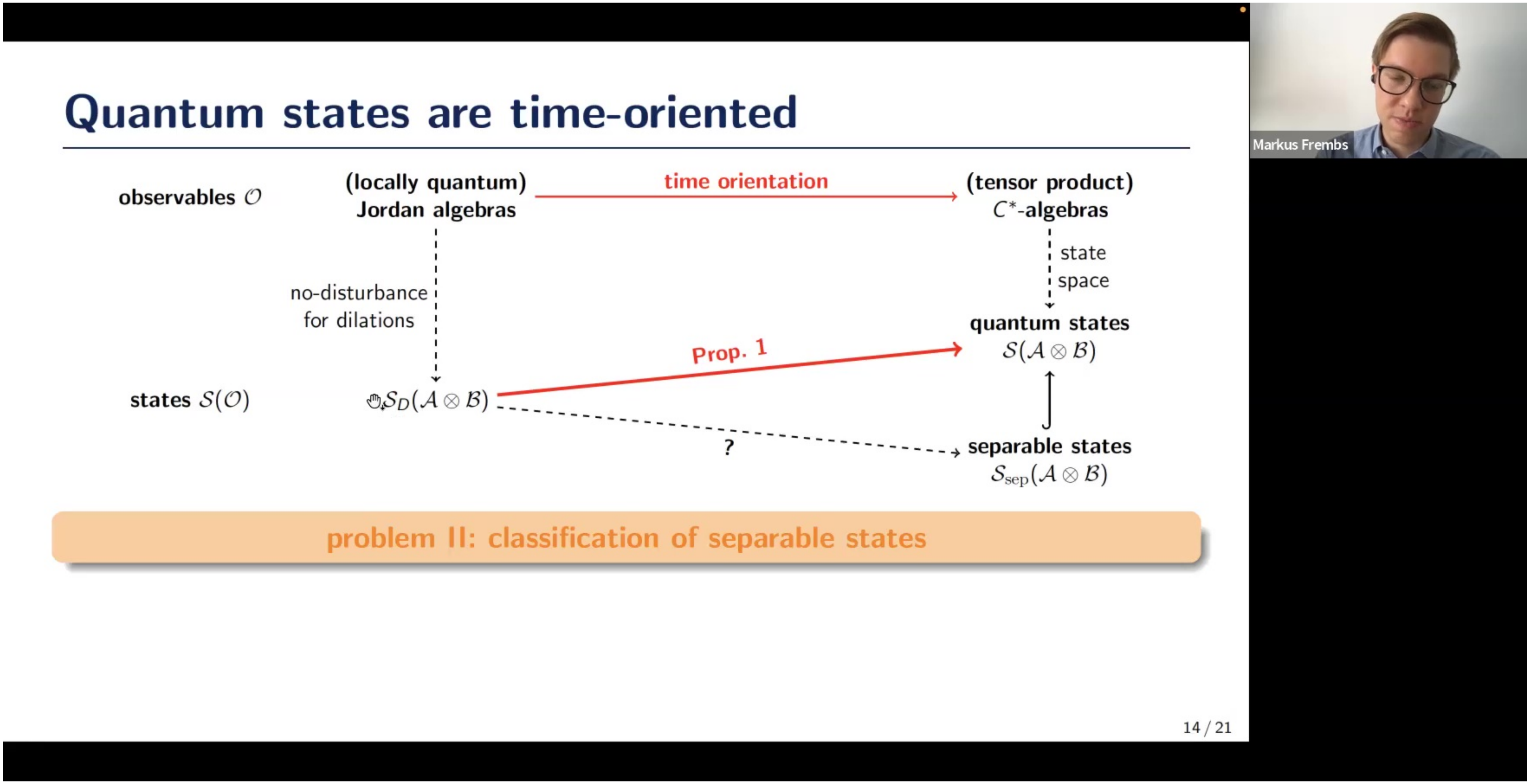
no-disturbance
for dilations

Prop. 1

states $\mathcal{S}(\mathcal{O})$

$\mathcal{S}_D(\mathcal{A} \otimes \mathcal{B})$

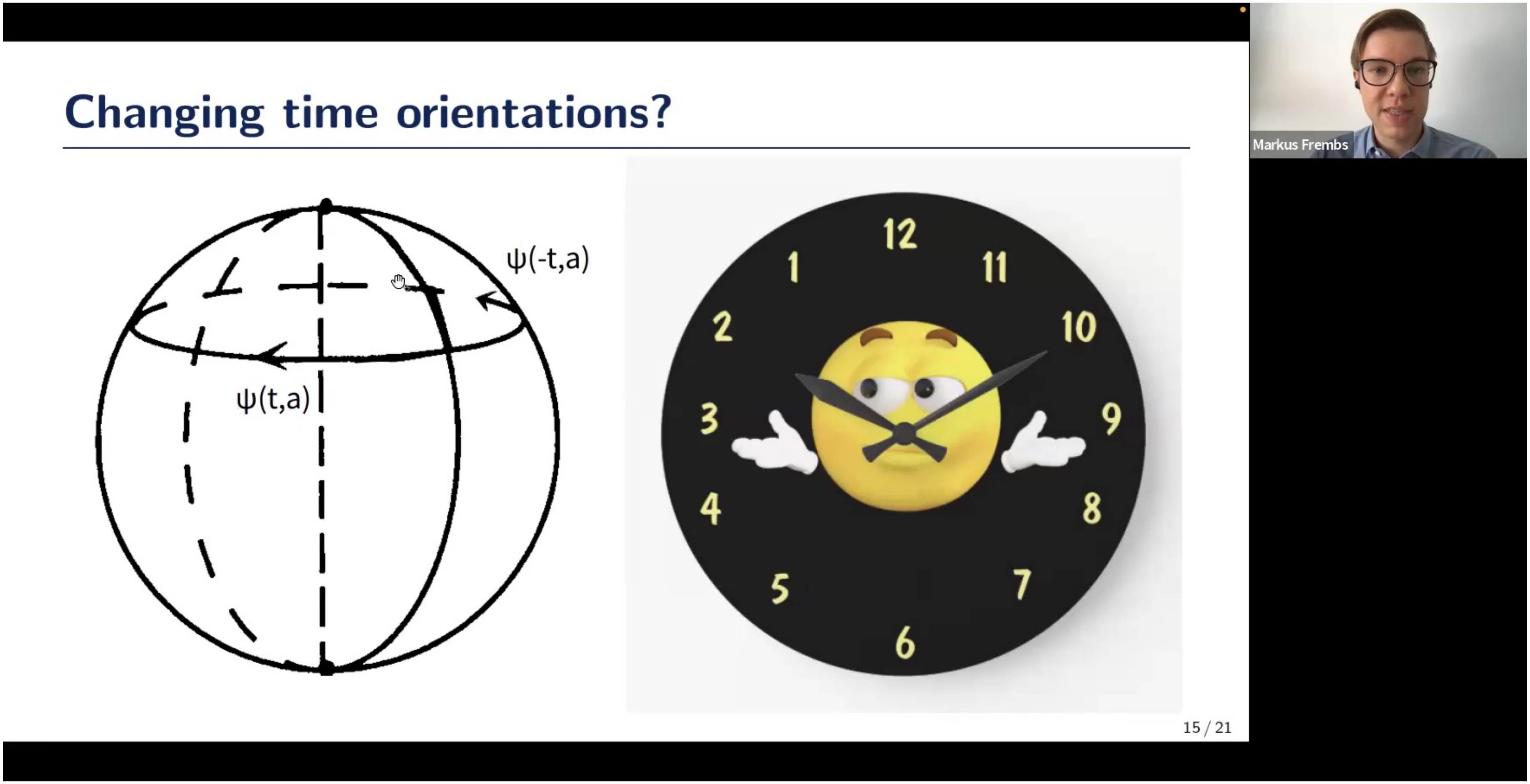
separable states
$\mathcal{S}_{\text{sep}}(\mathcal{A} \otimes \mathcal{B})$



Quantum states are time-oriented

$$\begin{tikzcd}
\textbf{observables } \mathcal{O} & \textbf{(locally quantum) Jordan algebras} \arrow[r, "\text{time orientation}"] \arrow[dd, dashed, "\text{no-disturbance for dilations}"'] & \textbf{(tensor product) } C^*\textbf{-algebras} \arrow[d, dashed, "\text{state space}"] \\
 & & \textbf{quantum states } \mathcal{S}(\mathcal{A} \otimes \mathcal{B}) \\
\textbf{states } \mathcal{S}(\mathcal{O}) & \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B}) \arrow[ur, "\text{Prop. 1}"] \arrow[dr, dashed, "?"'] & \\
 & & \textbf{separable states } \mathcal{S}_{\text{sep}}(\mathcal{A} \otimes \mathcal{B}) \arrow[uu, hook]
\end{tikzcd}$$

problem II: classification of separable states



Changing time orientations?

The Peres-Horodecki criterion - states and maps

- **Peres (1996):** $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$ separable $\Rightarrow$ **positive partial transpose (PPT)** $\rho^{T_\mathcal{A}}$
  $\rightarrow$ **Horodecki et al. (1996):** '$\Leftarrow$' only for $\dim(\mathcal{H}_\mathcal{A}) = 2$, $\dim(\mathcal{H}_\mathcal{B}) = 2, 3$
- **Choi(-Jamiołkowski) isomorphism:** $\mathcal{L}(\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}) \ni \rho_\phi \Longleftrightarrow \phi_\rho \in \mathcal{L}(\mathcal{H}_\mathcal{A}, \mathcal{H}_\mathcal{B})$

$$\rho_\phi = \sum_{ij} E_{ij} \otimes \phi(E_{ij}) \qquad \phi_\rho(a) = \mathrm{tr}_{\mathcal{H}_\mathcal{A}}[\rho(a^{T_\mathcal{A}} \otimes 1_\mathcal{B})]$$

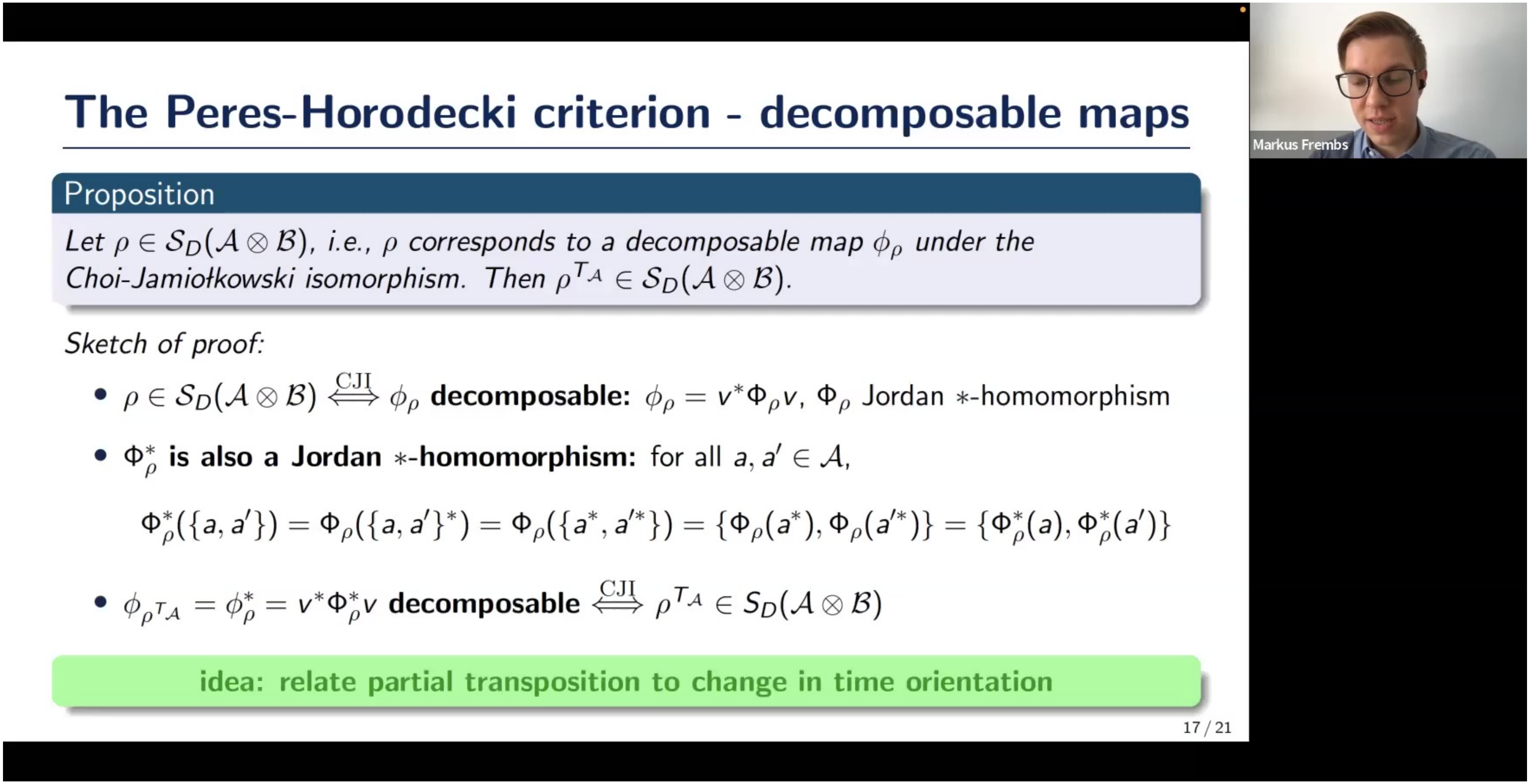
Lemma

Let $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$, let ϕ_ρ be the map under the Choi-Jamiołkowski isomorphism, and let $(\Phi_\rho, v, \mathcal{K}_\mathcal{B})$ be a Stinespring dilation of ϕ_ρ. Then

$$\phi_{\rho^{T_\mathcal{A}}} = \phi_\rho^* = v^* \Phi_\rho^* v \ .$$

Sketch of proof:

$$\sum_{ij} E_{ij} \otimes \phi_{\rho^{T_\mathcal{A}}}(E_{ij}) = \rho^{T_\mathcal{A}} = (\rho^*)^{T_\mathcal{A}} = \sum_{ij} E_{ij} \otimes \phi_\rho^*(E_{ij}) = \sum_{ij} E_{ij} \otimes v^* \Phi_\rho^*(E_{ij}) v$$



The Peres-Horodecki criterion - decomposable maps

Proposition

Let $\rho \in \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B})$, i.e., ρ corresponds to a decomposable map ϕ_ρ under the Choi-Jamiołkowski isomorphism. Then $\rho^{T_\mathcal{A}} \in \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B})$.

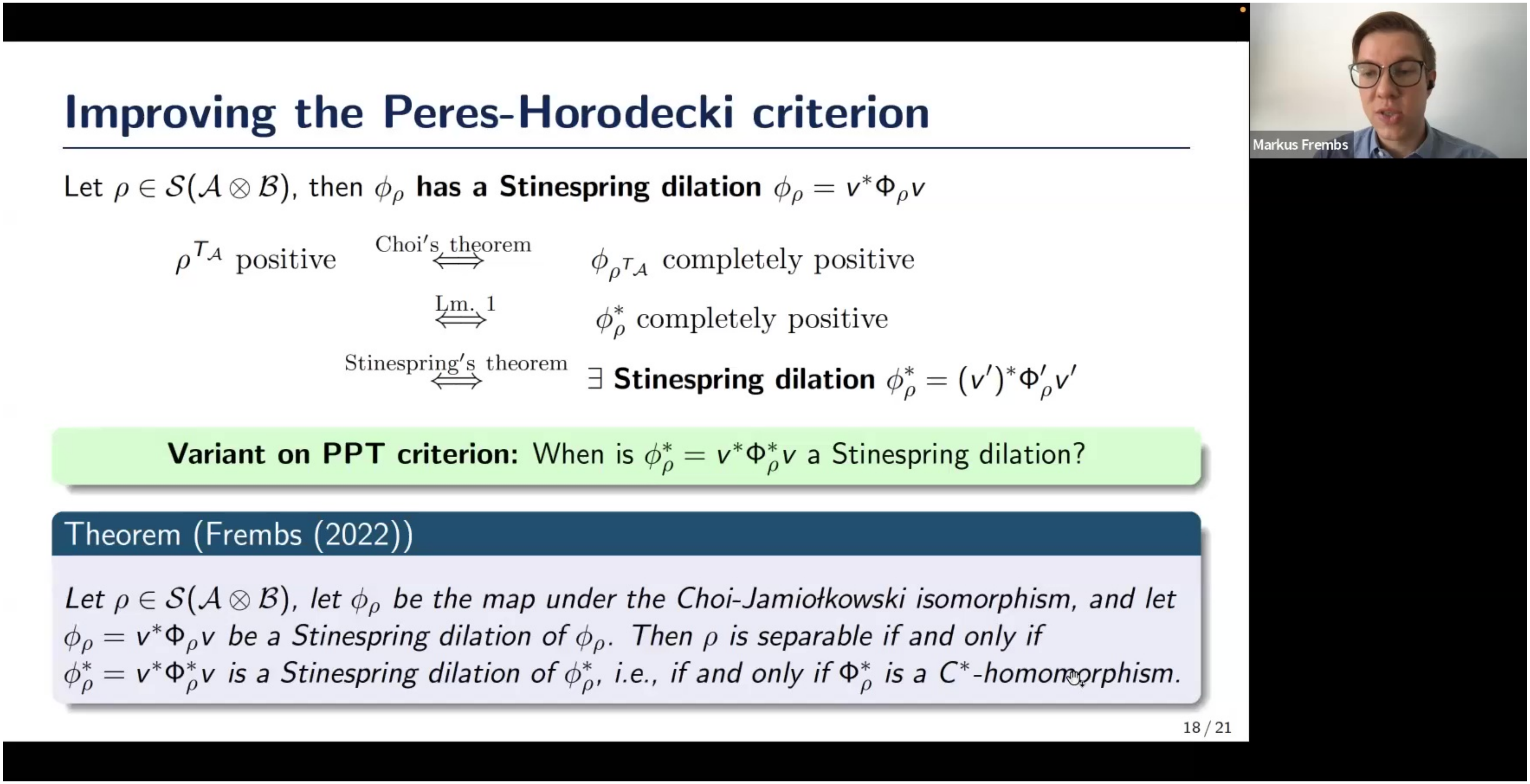
Sketch of proof:

- $\rho \in \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B}) \overset{\text{CJI}}{\Longleftrightarrow} \phi_\rho$ **decomposable:** $\phi_\rho = v^* \Phi_\rho v$, Φ_ρ Jordan $*$-homomorphism
- Φ^*_ρ **is also a Jordan $*$-homomorphism:** for all $a, a' \in \mathcal{A}$,

$$\Phi^*_\rho(\{a, a'\}) = \Phi_\rho(\{a, a'\}^*) = \Phi_\rho(\{a^*, a'^*\}) = \{\Phi_\rho(a^*), \Phi_\rho(a'^*)\} = \{\Phi^*_\rho(a), \Phi^*_\rho(a')\}$$

- $\phi_{\rho^{T_\mathcal{A}}} = \phi^*_\rho = v^* \Phi^*_\rho v$ **decomposable** $\overset{\text{CJI}}{\Longleftrightarrow} \rho^{T_\mathcal{A}} \in S_D(\mathcal{A} \otimes \mathcal{B})$

idea: relate partial transposition to change in time orientation



Improving the Peres-Horodecki criterion

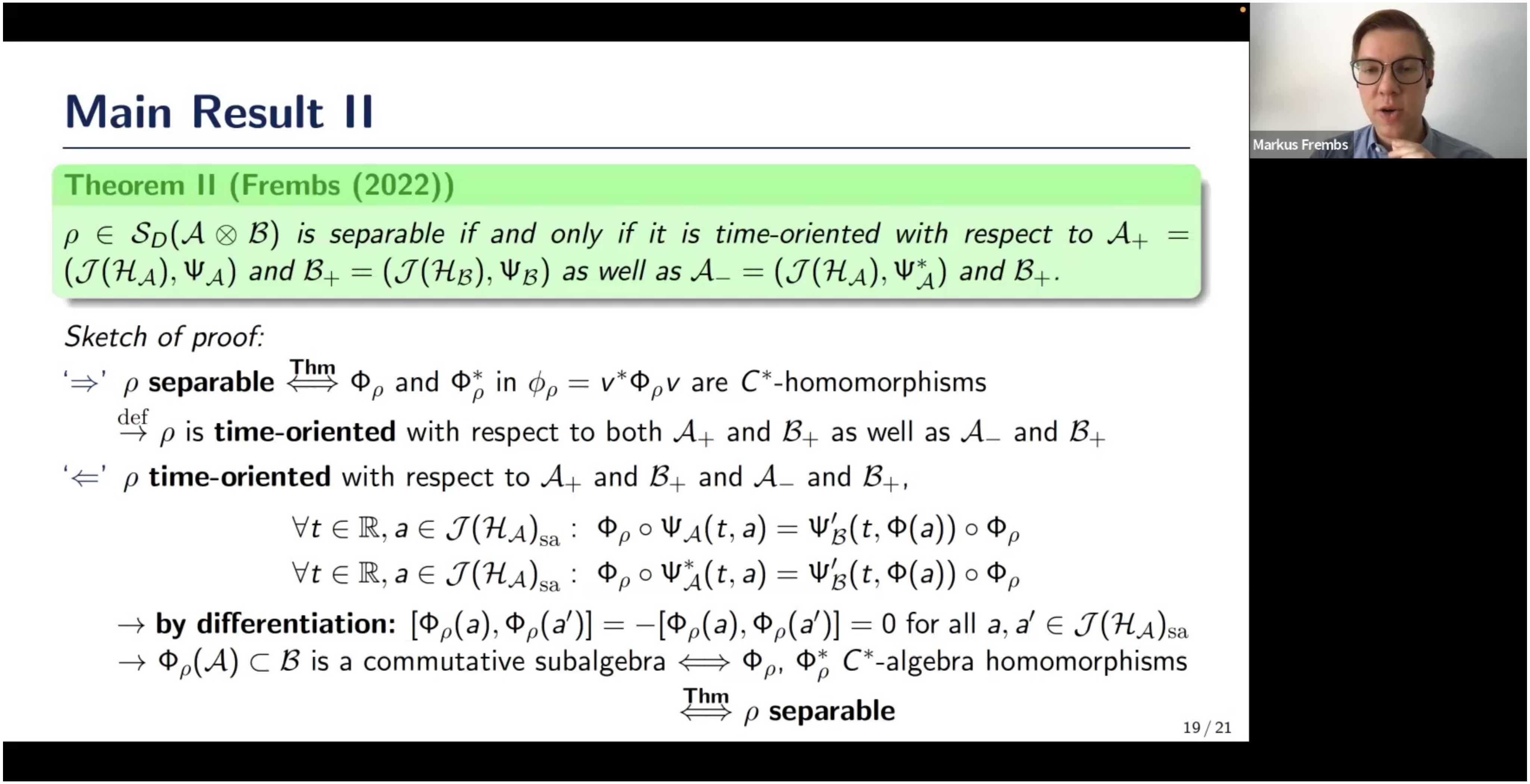
Let $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$, then ϕ_ρ **has a Stinespring dilation** $\phi_\rho = v^* \Phi_\rho v$

$$\begin{aligned} \rho^{T_\mathcal{A}} \text{ positive} \quad &\overset{\text{Choi's theorem}}{\Longleftrightarrow} \quad \phi_{\rho^{T_\mathcal{A}}} \text{ completely positive} \\ &\overset{\text{Lm. 1}}{\Longleftrightarrow} \quad \phi^*_\rho \text{ completely positive} \\ &\overset{\text{Stinespring's theorem}}{\Longleftrightarrow} \quad \exists \ \textbf{Stinespring dilation } \phi^*_\rho = (v')^* \Phi'_\rho v' \end{aligned}$$

Variant on PPT criterion: When is $\phi^*_\rho = v^* \Phi^*_\rho v$ a Stinespring dilation?

Theorem (Frembs (2022))

Let $\rho \in \mathcal{S}(\mathcal{A} \otimes \mathcal{B})$, let ϕ_ρ be the map under the Choi-Jamiołkowski isomorphism, and let $\phi_\rho = v^* \Phi_\rho v$ be a Stinespring dilation of ϕ_ρ. Then ρ is separable if and only if $\phi^*_\rho = v^* \Phi^*_\rho v$ is a Stinespring dilation of ϕ^*_ρ, i.e., if and only if Φ^*_ρ is a C^*-homomorphism.



Main Result II

Theorem II (Frembs (2022))

$\rho \in \mathcal{S}_D(\mathcal{A} \otimes \mathcal{B})$ is separable if and only if it is time-oriented with respect to $\mathcal{A}_+ = (\mathcal{J}(\mathcal{H}_\mathcal{A}), \Psi_\mathcal{A})$ and $\mathcal{B}_+ = (\mathcal{J}(\mathcal{H}_\mathcal{B}), \Psi_\mathcal{B})$ as well as $\mathcal{A}_- = (\mathcal{J}(\mathcal{H}_\mathcal{A}), \Psi^*_\mathcal{A})$ and $\mathcal{B}_+$.

Sketch of proof:

'$\Rightarrow$' ρ **separable** $\overset{\text{Thm}}{\Longleftrightarrow}$ Φ_ρ and Φ^*_ρ in $\phi_\rho = v^*\Phi_\rho v$ are C^*-homomorphisms

$\overset{\text{def}}{\rightarrow}$ ρ is **time-oriented** with respect to both $\mathcal{A}_+$ and $\mathcal{B}_+$ as well as $\mathcal{A}_-$ and $\mathcal{B}_+$

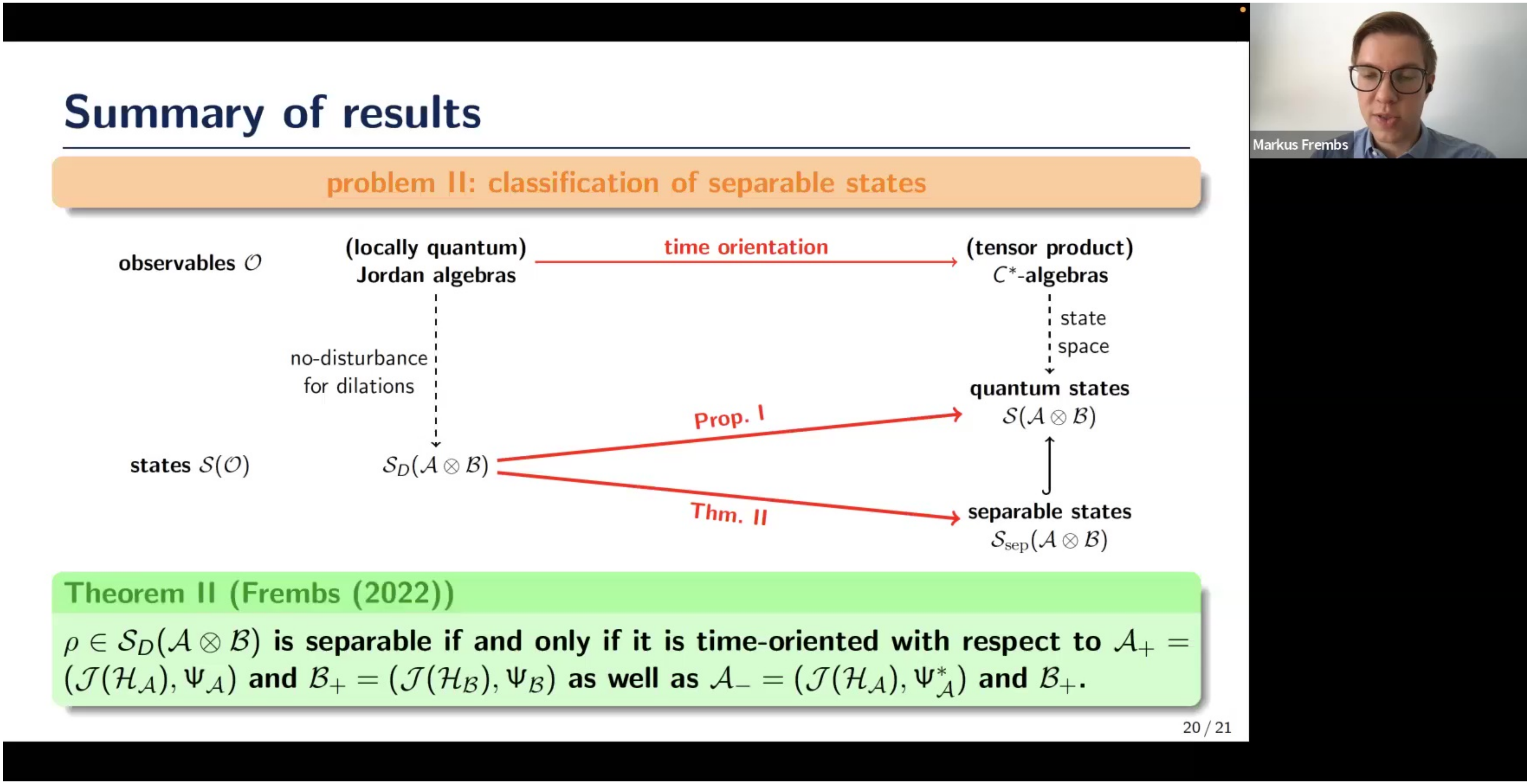
'$\Leftarrow$' ρ **time-oriented** with respect to $\mathcal{A}_+$ and $\mathcal{B}_+$ and $\mathcal{A}_-$ and $\mathcal{B}_+$,

$$\forall t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}} : \ \Phi_\rho \circ \Psi_\mathcal{A}(t,a) = \Psi'_\mathcal{B}(t, \Phi(a)) \circ \Phi_\rho$$
$$\forall t \in \mathbb{R}, a \in \mathcal{J}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}} : \ \Phi_\rho \circ \Psi^*_\mathcal{A}(t,a) = \Psi'_\mathcal{B}(t, \Phi(a)) \circ \Phi_\rho$$

$\rightarrow$ **by differentiation:** $[\Phi_\rho(a), \Phi_\rho(a')] = -[\Phi_\rho(a), \Phi_\rho(a')] = 0$ for all $a, a' \in \mathcal{J}(\mathcal{H}_\mathcal{A})_{\mathrm{sa}}$

$\rightarrow$ $\Phi_\rho(\mathcal{A}) \subset \mathcal{B}$ is a commutative subalgebra $\Longleftrightarrow$ Φ_ρ, Φ^*_ρ C^*-algebra homomorphisms

$\overset{\text{Thm}}{\Longleftrightarrow}$ ρ **separable**



Summary of results

problem II: classification of separable states

$$\begin{tikzcd}
\text{observables } \mathcal{O} & \text{(locally quantum) Jordan algebras} \arrow[rr, "\text{time orientation}"] \arrow[dd, dashed, "\text{no-disturbance for dilations}"'] & & \text{(tensor product) } C^*\text{-algebras} \arrow[d, dashed, "\text{state space}"] \\
 & & & \text{quantum states } \mathcal{S}(\mathcal{A}\otimes\mathcal{B}) \\
\text{states } \mathcal{S}(\mathcal{O}) & \mathcal{S}_D(\mathcal{A}\otimes\mathcal{B}) \arrow[rru, "\text{Prop. I}"] \arrow[rrd, "\text{Thm. II}"'] & & \\
 & & & \text{separable states } \mathcal{S}_{\mathrm{sep}}(\mathcal{A}\otimes\mathcal{B}) \arrow[uu, hook]
\end{tikzcd}$$

Theorem II (Frembs (2022))

$\rho \in \mathcal{S}_D(\mathcal{A}\otimes\mathcal{B})$ **is separable if and only if it is time-oriented with respect to** $\mathcal{A}_+ = (\mathcal{J}(\mathcal{H}_\mathcal{A}), \Psi_\mathcal{A})$ **and** $\mathcal{B}_+ = (\mathcal{J}(\mathcal{H}_\mathcal{B}), \Psi_\mathcal{B})$ **as well as** $\mathcal{A}_- = (\mathcal{J}(\mathcal{H}_\mathcal{A}), \Psi^*_\mathcal{A})$ **and** $\mathcal{B}_+$**.**

Conclusion

Summary

- (I) **classification of bipartite quantum states** within non-signalling correlations from:
  - (i) locally quantum observables,
  - (ii) no-disturbance for dilations
  - (iii) preservation of time orientations
- (II) **classification of separable states** from: invariance under change of time orientation

key insight: entanglement encodes intrinsic, relative notion of time direction

... towards a classification of entanglement in terms of time orientations ...

- time orientations in CJI, Frembs and Cavalcanti (2022)
- infinite dimensions (cf. Frembs and Döring (2022)), multipartite entanglement
- possible applications to quantum causal models, quantum ‘states over time’ and quantum Bayes’ theorem, (space)time from entanglement etc.

Thank you