Title: Effective algebro-homotopical constructions and a question of Kapustin

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Abstract: In recent years, the classification of fermionic symmetry protected topological phases has led to renewed interest in classical constructions of invariants in homotopy theory. In this talk, we focus on the description of Steenrod squares for triangulated spaces at the cochain level, introducing new formulas for the cup-i products and discussing their universality through an axiomatic approach. We also examine the interaction between Steenrod squares and the algebra structure in cohomology, providing a cochain level proof of the Cartan relation as requested by Kapustin. Time permitting, we will also study the Adem relation from this perspective.

Zoom link: https://pitp.zoom.us/j/98288876236?pwd=cHJVM3M1K3FsUmdtbENZenhKMnBkdz09

Effective algebro-homotopical constructions and a question of Kapustin

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January 13, 2023

Motivation & outline

1. Classification of SPT phases.
2. Definition on a triangulated space-time X using cochains as fields.
3. Passage from bosonic to fermionic phases brings $\mathbb{F}_2$.
4. Primary operations $\mathrm{Sq}^\bullet \colon H^\bullet(X;\mathbb{F}_2) \to H^\bullet(X;\mathbb{F}_2)$.
5. Represented at the cochain level by cup-i products.
6. Secondary operations from Cartan & Adem relations.
7. Represented at the cochain level by what?

Today's goal: Understand cup-i products better and use them to construct cochains enforcing the Cartan and Adem relations at the cochain level.

Important note: All cochain constructions are done locally on simplices $\mathbb{\Delta}^n$.

Product in cohomology

Consider the diagonal map

$$X \to X \times X.$$

Applying cohomology,

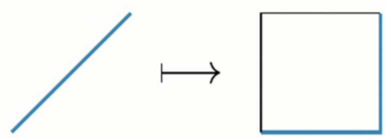
$$H^\bullet(X) \otimes H^\bullet(X) \to H^\bullet(X \times X) \to H^\bullet(X).$$

How to define it at the cochain level (locally)?

Dualize a natural chain approximation to the diagonal.

$$\Delta_0 \colon \mathrm{C}(\mathbb{\Delta}^n) \to \mathrm{C}(\mathbb{\Delta}^n) \otimes \mathrm{C}(\mathbb{\Delta}^n).$$

For example, if $d = 1$



$$[01] \ \mapsto \ [0] \otimes [01] + [01] \otimes [1]$$

Steenrod construction

Unlike the diagonal of spaces, chain approxs to it are not invariant under

$$x \otimes y \overset{T}{\mapsto} y \otimes x.$$

To correct homotopically the breaking of this symmetry, Steenrod introduced explicit "higher diagonals"

$$\Delta_i \colon \mathrm{C}(\mathbb{A}^n) \to \mathrm{C}(\mathbb{A}^n)^{\otimes 2} \quad \text{satisfying} \quad \partial\Delta_i = \big(1 \pm T\big)\Delta_{i-1}.$$

The cup-i products are their linear dual $\alpha \smile_i \beta = (\alpha \otimes \beta)\Delta_i(-)$ and represent at the cochain level both:

The product

$$H^\bullet(X;\mathbb{F}_2)^{\otimes 2} \longrightarrow H^\bullet(X;\mathbb{F}_2)$$
$$[\alpha] \otimes [\beta] \mapsto \big[\alpha \smile_0 \beta\big]$$

The Steenrod squares

$$H^d(X;\mathbb{F}_2) \xrightarrow{\mathrm{Sq}^k} H^d(X;\mathbb{F}_2)$$
$$[\alpha] \mapsto \big[(\alpha \smile_{d-k} \alpha)\big]$$

A new description of Steenrod's construction

Notation:

$$d_u[v_0,\dots,v_m] = [v_0,\dots,\widehat{v}_u,\dots,v_m]$$
$$\mathrm{P}^n_q = \big\{U \subseteq \{0,\dots,n\} : |U| = q\big\}$$
$$\forall\, U = \{u_1 < \cdots < u_q\} \in \mathrm{P}^n_q$$
$$d_U = d_{u_1} \cdots d_{u_q}$$
$$U^\varepsilon = \big\{u_i \in U \mid u_i + i \equiv \varepsilon \text{ mod } 2\big\}$$

Definition (Med.)

For a basis element $x \in \mathrm{C}_m(\mathbb{\Delta}^n, \mathbb{F}_2)$

$$\Delta_i(x) \;= \sum_{U \in \mathrm{P}^n_{m-i}} d_{U^0}(x) \otimes d_{U^1}(x)$$

Example:

$$\begin{aligned}\Delta_0[0,1,2] &= \Big(d_{12} \otimes \mathsf{id} + d_2 \otimes d_0 + \mathsf{id} \otimes d_{01}\Big)[0,1,2]^{\otimes 2} \\ &= [0] \otimes [0,1,2] + [0,1] \otimes [1,2] + [0,1,2] \otimes [2].\end{aligned}$$

Unicity

A cup-i construction is non-degenerate if for any simplex x

$$\boxed{x \smile_0 x \neq 0}$$

whenever $|x| = 0$. It is irreducible if for any proper face y of x

$$\boxed{\left(y^{(1)} \smile_i y^{(2)}\right)(x) = 0}$$

for any two faces $y^{(1)}$ and $y^{(2)}$ of y. It is free if for two simplices x and y

$$\boxed{x \smile_i y = y \smile_i x} \implies \boxed{x \smile_i y = 0}$$

whenever $|x| \neq i$ or $|y| \neq i$.

Theorem (Med.)
Two non-degenerate, irreducible and free cup-i constructions are equal up to a transposition for each $i \in \mathbb{N}$.

The Cartan relation

In cohomology:

$$0 = \mathrm{Sq}^k\left([\alpha][\beta]\right) + \sum_{i+j=k} \mathrm{Sq}^i\left([\alpha]\right)\mathrm{Sq}^j\left([\beta]\right).$$

At the cochain level

$$\delta\zeta_i(\alpha,\beta) = (\alpha \smile_0 \beta) \smile_i (\alpha \smile_0 \beta) + \sum_{i=j+k} (\alpha \smile_j \alpha) \smile_0 (\beta \smile_k \beta)$$

for some natural cochain $\zeta_i(\alpha,\beta)$.

EZ–AW contraction

For a pair of simplicial sets

$$\mathrm{C}_\bullet(X \times Y) \neq \mathrm{C}_\bullet(X) \otimes \mathrm{C}_\bullet(Y),$$

But there are explicit (locally defined) chain maps

$$\mathrm{AW}\colon\ \mathrm{C}_\bullet(X \times Y) \longleftrightarrow \mathrm{C}_\bullet(X) \otimes \mathrm{C}_\bullet(Y)\ : \mathrm{EZ}$$

such that

$$\mathrm{AW} \circ \mathrm{EZ} = \mathsf{id}, \qquad \mathrm{EZ} \circ \mathrm{AW} = \partial \circ \mathrm{SHI} + \mathrm{SHI} \circ \partial\,.$$

Group homology

$G \to \mathrm{E}G \to \mathrm{B}G$ universal bundle, $H_\bullet(G) \overset{\text{def}}{=} H_\bullet(BG)$.

A simplicial model for EG:

$$\begin{aligned} \mathrm{E}G_n &= \{(\sigma_0, \ldots, \sigma_n) \mid \sigma_i \in G\}, \\ d_i(\sigma_0, \ldots, \sigma_n) &= (\sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \sigma_n), \\ s_i(\sigma_0, \ldots, \sigma_n) &= (\sigma_0, \ldots, \sigma_i, \sigma_i, \ldots, \sigma_n). \end{aligned}$$

Let $x_i = (e, (12), e, \ldots, (12)^i) \in (\mathrm{E}\mathbb{S}_2)_i$.

Steenrod construction

$$\begin{aligned} \mathrm{C}_\bullet(\mathrm{E}\mathbb{S}_2) \otimes_{\mathbb{S}_2} (\mathrm{C}^\bullet)^{\otimes 2} &\to \mathrm{C}^\bullet \\ x_i \otimes \alpha \otimes \beta \quad &\mapsto \alpha \smile_i \beta. \end{aligned}$$

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Inducing

$$\begin{aligned} \mathrm{C}_\bullet(\mathrm{B}\mathbb{S}_2) \otimes \mathrm{D}(\mathrm{C}^\bullet) &\to \mathrm{C}^\bullet \\ x_i \otimes \alpha \otimes \alpha &\mapsto \alpha \smile_i \alpha. \end{aligned}$$

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Let $x_i = (e, (12), e, \dots, (12)^i) \in (\mathrm{E}\mathbb{S}_2)_i$.

Steenrod construction

$$\mathrm{C}_\bullet(\mathrm{E}\mathbb{S}_2) \otimes_{\mathbb{S}_2} (\mathrm{C}^\bullet)^{\otimes 2} \to \mathrm{C}^\bullet$$
$$x_i \otimes \alpha \otimes \beta \quad \mapsto \alpha \smile_i \beta.$$

Inducing

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$$x_i \otimes \alpha \otimes \alpha \quad \mapsto \alpha \smile_i \alpha.$$

Operations Sq^k controlled by $H_\bullet(\mathrm{B}\mathbb{S}_2; \mathbb{F}_2)$.

Composition of permutations

In general

$$\circ_{\mathbb{S}} \colon \mathbb{S}_r \times_{\mathbb{S}_r} (\mathbb{S}_{s_1} \times \cdots \times \mathbb{S}_{s_r}) \to \mathbb{S}_{s_1+\cdots+s_r}$$

For today

$$\circ_{\mathbb{S}} \colon \mathbb{S}_2 \times_{\mathbb{S}_2} (\mathbb{S}_2 \times \mathbb{S}_2) \to \mathbb{S}_4$$

Example

$$(12) \times (12) \times (12) \mapsto (14)(23)$$

Induced simplicial map

$$\mathrm{E}\mathbb{S}_2 \times_{\mathbb{S}_2} (\mathrm{E}\mathbb{S}_2 \times \mathrm{E}\mathbb{S}_2) \to \mathrm{E}\mathbb{S}_4$$

Induced chain map

$$\circ_{\mathcal{E}} \colon \mathcal{E}(2) \otimes_{\mathbb{S}_2} (\mathcal{E}(2) \otimes \mathcal{E}(2)) \xrightarrow{\mathrm{EZ}} \mathrm{C}_\bullet(\mathrm{E}\mathbb{S}_2 \times_{\mathbb{S}_2} (\mathrm{E}\mathbb{S}_2 \times \mathrm{E}\mathbb{S}_2)) \to \mathcal{E}(4)$$

where $\mathcal{E}(r) = \mathrm{C}_\bullet(\mathrm{E}\mathbb{S}_r)$.

E_∞-structure on simplicial chains

Theorem. (Berger–Fresse 2004)
Explicit $\mathbb{S}_r$-equivariant quasi-isomorphism

$$\mathcal{E}(r) \to \mathrm{Hom}((\mathrm{C}^\bullet)^{\otimes r}, \mathrm{C}^\bullet)$$

compatible with compositions and sending x_i in $\mathcal{E}(2)$ to $\smile_i$.

Reference: *Combinatorial operad actions on cochains.*

Notation abuse: We will identify elements in $\mathcal{E}(r)$ with their images.

Upshot

Theorem. (Med.)

Let $H\colon \mathcal{E}(2) \to \mathcal{E}(4)$ such that

$$\begin{aligned} \partial \circ H + H \circ \partial &= (23)F + G, \\ H(12) &= (12)(34)H. \end{aligned}$$

Then (using the Berger-Fresse map and linear duality),

$$\zeta_i(\alpha, \beta) = H(x_i)(\alpha \otimes \alpha \otimes \beta \otimes \beta)$$

is a Cartan coboundary for cocycles α, β, i.e.

$$\delta\zeta_i(\alpha, \beta) = (\alpha \smile_0 \beta) \smile_i (\alpha \smile_0 \beta) \; + \sum_{i=j+k} (\alpha \smile_j \alpha) \smile_0 (\beta \smile_k \beta).$$

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Question: How to construct such map H?

Big and little maps

Can be checked that

$$F + \mathrm{C}_\bullet(\mathrm{E}f) = 0.$$

Notice that

$$\begin{aligned} G(\sigma_0, \ldots, \sigma_n) &= \circ_{\mathcal{E}} \left(\tilde{x}_0 \otimes \mathrm{AW}(\sigma_0, \ldots, \sigma_n)^{\times 2} \right) \\ &= \mathrm{C}_\bullet(\circ_E)\big((e, \ldots, e) \times \mathrm{EZ} \circ \mathrm{AW}(\sigma_0, \ldots, \sigma_n)^{\times 2}\big) \\ &= \mathrm{C}_\bullet(\circ_E)\big((e, \ldots, e) \times (\sigma_0, \ldots, \sigma_n)^{\times 2}\big) \\ &+ \mathrm{C}_\bullet(\circ_E)\big((e, \ldots, e) \times (\partial\, \mathrm{SHI})(\sigma_0, \ldots, \sigma_n)^{\times 2}\big) \end{aligned}$$

So, if

$$H_2(\sigma_0, \ldots, \sigma_n) = \circ_{\mathcal{E}}\big((e, \ldots, e) \times SHI(\sigma_0, \ldots, \sigma_n)^{\times 2}\big)$$

then

$$G + \mathrm{C}_\bullet(\mathrm{E}g) = \partial \circ H_2 + H_2 \circ \partial\,.$$

Claim: $H_2(12) = (12)(34)H_2$.

Conjugate implies homotopic

Slogan: *Conjugate maps induce homotopic maps on classifying spaces.*

Recall that f and g conjugate.

Define $H_1 \colon \mathcal{E}(2) \to \mathcal{E}(4)$

$$H_1(\sigma_0, \ldots, \sigma_n) = \sum_{i=0}^{n} \big((23)f\sigma_0, \ldots, (23)f\sigma_i, g\sigma_i, \ldots, g\sigma_n\big).$$

Using a telescopic sum

$$(23)\,\mathrm{C}_\bullet(\mathrm{E}f) + \mathrm{C}_\bullet(\mathrm{E}g) = \partial \circ H_2 + H_2 \circ \partial\,.$$

Examples

For $i \in \mathbb{N}$ let $n(i) \in \mathbb{N}$ be the smallest such that

$$\zeta_i(\alpha, \beta)[0, \dots, n(i)]$$

is not identically 0 for all homogeneous cochains.

Its value is obtained by evaluating $(\alpha \otimes \alpha \otimes \beta \otimes \beta)$ on:

Open Jupyter notebook

cartan_coboundaries - Ju

https://hub.gke2.mybinder.org/user/ammedmar-simplicial_operators-e2o2dilj/notebooks/notebooks/cartan_coboundaries.ipynb#

No kernel | Trusted | Python 3 (ipykernel)

File Edit View Insert Cell Kernel Widgets Help

```python
        if cartan:
            print((f'z_{i}(0,...,{n}) =\n'
                   + Operator.display_action(cartan, (tuple(range(n+1,)),)
                   + '\n'))
            break
```

```
z_0(0,...,3) =
  (0, 1) x (1, 2) x (1, 2) x (2, 3)

z_1(0,...,4) =
  (0, 2, 3) x (0, 1, 2) x (3, 4) x (2, 3)
+ (0, 1) x (1, 2) x (1, 2, 4) x (2, 3, 4)

z_2(0,...,5) =
  (0, 1, 2, 3) x (0, 1, 3, 4) x (3, 4) x (4, 5)
+ (0, 1, 2, 3) x (1, 2, 3, 4) x (3, 4) x (4, 5)
+ (0, 1) x (1, 2) x (1, 2, 3, 4) x (2, 3, 4, 5)
+ (0, 1) x (1, 2) x (1, 2, 4, 5) x (2, 3, 4, 5)
+ (0, 2, 3) x (0, 1, 2) x (3, 4, 5) x (2, 3, 5)

z_3(0,...,6) =
  (0, 2, 3) x (0, 1, 2) x (3, 4, 5, 6) x (2, 3, 4, 5)
+ (0, 1, 2, 4, 5) x (0, 1, 2, 3, 4) x (5, 6) x (4, 5)
+ (0, 1, 2, 3) x (0, 1, 3, 4) x (3, 4, 6) x (4, 5, 6)
+ (0, 1, 2, 3) x (1, 2, 3, 4) x (3, 4, 6) x (4, 5, 6)
```