Title: Standard Model Lecture - 230111

Speakers: Latham Boyle

Collection: Standard Model (2022/2023)

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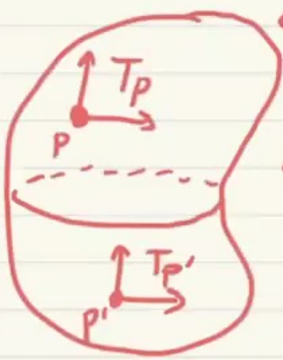
Standard Model

Lecture #2

Introduction to Non-Abelian Gauge Theory (a.k.a. "Yang-Mills Theory")

Lightning Review/Intro to GR: (1/2)



← 4-dimensional spacetime manifold M.

← Tangent bundle T

Consider $v \in T_p$

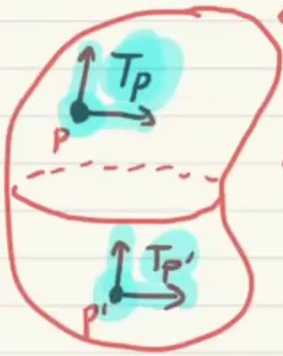
in x^μ coords: V^α

in $\tilde{x}^\mu$ coords: $\tilde{V}^\alpha$

$$\tilde{V}^\alpha = U^\alpha{}_\beta V^\beta \quad \left(U^\alpha{}_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\right)$$

- Since $U^\alpha{}_\beta = U^\alpha{}_\beta(x)$, $\partial_\mu V^\alpha$ isn't a tensor. $\quad \partial_\mu = \dfrac{\partial}{\partial x^\mu}$
- So introduce covariant deriv: $\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta} V^\beta$
- Demand $\Gamma^\alpha_{\mu\beta}$ transforms s.t. $\nabla_\mu V^\alpha$ <u>is</u> a tensor.
- Next note: $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta$ (Ricci identity)
- Implies $R^\alpha{}_{\beta\mu\nu} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$ is a tensor.
(the Riemann curvature tensor)

Lightning Review/Intro to GR: (1/2)



← 4-dimensional spacetime manifold M.

← Tangent bundle T

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} - \frac{\partial}{\partial x^\nu}\frac{\partial}{\partial x^\mu} = 0$$

Consider $v \in T_p$

in x^μ coords: v^α

in $\tilde{x}^\mu$ coords: $\tilde{v}^\alpha$

$$\tilde{v}^\alpha = U^\alpha{}_\beta v^\beta \quad \left(U^\alpha{}_\beta = \frac{\partial \tilde{x}^\alpha}{\partial x^\beta}\right)$$

- Since $U^\alpha{}_\beta = U^\alpha{}_\beta(x)$, $\partial_\mu v^\alpha$ isn't a tensor. $\quad \partial_\mu = \dfrac{\partial}{\partial x^\mu}$
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- Implies $R^\alpha{}_{\beta\mu\nu} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$ is a tensor. (the Riemann curvature tensor)

Lightning Review/Intro to GR: (2/2)

- Seek Lagrangian that is a scalar.
- Simplest possibility:
- $\delta^{\mu}_{\alpha} R^{\alpha}{}_{\beta\mu\nu} = R_{\beta\nu}$ (Ricci tensor)
- $g^{\beta\nu} R_{\beta\nu} = R$ (Ricci scalar)

Simplest non-trivial scalar.

- Add a dimensionful pre-factor to get the units right.
- Presto! The Einstein-Hilbert Lagrangian $\mathcal{L}_{EH}$:

$$\mathcal{L}_{EH} = \frac{1}{16\pi G_N} R \leftarrow$$ the action for gravity (General Relativity)

Lightning Review/Intro to GR: (2/2)

- Seek Lagrangian that is a scalar.
- Simplest possibility:
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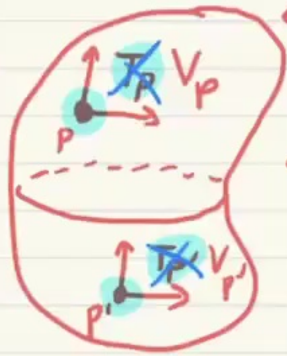
Simplest non-trivial scalar.

- Add a dimensionful pre-factor to get the units right.
- Presto! The Einstein-Hilbert Lagrangian L_{EH}:

$$L_{EH} = \frac{1}{16\pi G_N} R \leftarrow$$ the action for gravity (General Relativity)

Now Yang-Mills:



← 4-dimensional spacetime manifold M.

← ~~Tangent bundle T~~
Vector bundle V

$\psi^a \in V_p$

$\psi^a(x) \in V$ ← "matter field"

Yang-Mills theory is about curvature of this vector bundle

GR

$v^\alpha \in T_p$

$v^\alpha \to \tilde{v}^\alpha = U^\alpha{}_\beta v^\beta \quad \left(U^\alpha{}_\beta = \dfrac{\partial \tilde{x}^\alpha}{\partial x^\beta}\right)$

coordinate transformation

$\partial_\mu v^\alpha$ not a tensor

YM

$\psi^a \in V_p$ "gauge group"

$\psi^a \to \tilde{\psi}^a = U^a{}_b \psi^b \quad (U \in G)$

"gauge transformation"

$\partial_\mu \psi^a \to \partial_\mu \tilde{\psi}^a = \partial_\mu (U^a{}_b \psi^b)$

$= U^a{}_b (\partial_\mu \psi^b) + (\partial_\mu U^a{}_b) \psi^b$

not a tensor

9:41 AM Tue Jan 9

SM Lecture 2

GR

Introduce covariant deriv:

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta} V^\beta$$

"Christoffel Symbol"

"Levi-Civita connection"

Demand that $\Gamma^\alpha_{\mu\beta}$ transforms s.t. $\nabla_\mu V^\alpha$ is a tensor (under coord transformation)

YM

Introduce covariant deriv:

$$D_\mu \psi^a = \partial_\mu \psi^a + A_{\mu}{}^a{}_b \psi^b$$

"Yang-Mills gauge field"

"Connection on..."

Demand that $A_{\mu}{}^a{}_b$ transform s.t. $D_\mu \psi^a$ is a tensor (under gauge transformation)

9:41 AM Tue Jan 9

SM Lecture 2

So how does $A_{\mu}{}^{a}{}_{b}$ transform?

Want: $D_\mu \psi^a \to \tilde{D}_\mu \tilde{\psi}^a = U^a{}_b (D_\mu \psi^b)$

i.e.: $D_\mu \psi \to \tilde{D}_\mu \tilde{\psi} = U(D_\mu \psi)$

where $\tilde{\psi} = U\psi$, $D_\mu = \partial_\mu + A_\mu$, $\tilde{D} = \partial_\mu + \tilde{A}_\mu$

Exercise: Show that

$$A_\mu \to \tilde{A}_\mu = U A_\mu U^{-1} - (\partial_\mu U) U^{-1}$$
$$= U A_\mu U^{-1} + U(\partial_\mu U^{-1})$$

Or, with indices displayed:

$$A_{\mu}{}^{a}{}_{b} \to \tilde{A}_{\mu}{}^{a}{}_{b} = U^a{}_c A_{\mu}{}^{c}{}_{d} (U^{-1})^d{}_b - (\partial_\mu U^a{}_c)(U^{-1})^c{}_b$$
$$= U^a{}_c A_{\mu}{}^{c}{}_{d} (U^{-1})^d{}_b + U^a{}_c \partial_\mu (U^{-1})^c{}_b$$

GR

Introduce covariant deriv:

$$\nabla_\mu V^\alpha = \partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta} V^\beta$$

"Christoffel Symbol"

"Levi-Civita connection"

Demand that $\Gamma^\alpha_{\mu\beta}$ transforms s.t. $\nabla_\mu V^\alpha$ is a tensor (under coord transformation)

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta$$

$$\Rightarrow R^\alpha{}_{\beta\mu\nu} = \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$$

is "Riemann curvature" (curvature of tangent bundle T)

YM

Introduce covariant deriv:

$$D_\mu \psi^a = \partial_\mu \psi^a + A_{\mu}{}^a{}_b \psi^b$$

"Yang-Mills gauge field"

"Connection on..."

Demand that $A_{\mu}{}^a{}_b$ transform s.t. $D_\mu \psi^a$ is a tensor (under gauge transformation)

$$(D_\mu D_\nu - D_\nu D_\mu) \psi^a = F^a{}_{b\mu\nu} \psi^b$$

$$\Rightarrow F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

is "Yang-Mills field strength" (curvature of vector bundle V)

9:41 AM Tue Jan 9

SM Lecture 2

Disable "Erase Highlighter Only"? Open Options Ignore

GR

For Lagrangian, construct simplest scalar (coordinate invariant)

step 1: $R_{\beta\nu} = \delta^{\mu}_{\alpha} R^{\alpha}{}_{\beta\mu\nu}$

step 2: $R = g^{\beta\nu} R_{\beta\nu}$ (scalar!)

$$\Rightarrow \boxed{\mathcal{L}_{EH} = \mathcal{L}_{GR} = \frac{1}{16\pi G} R}$$

YM

For Lagrangian, construct simplest scalar: (coordinate + gauge invariant)

$\rightarrow$ $F_{b\nu} \overset{?}{=} \delta^{\mu}_{a} F^{a}{}_{b\mu\nu}$

doesn't make sense!

Instead

i) Note $F_{\mu\nu} \rightarrow U F_{\mu\nu} U^{-1}$ (exercise!)

ii) So $F_{\mu\nu} F^{\mu\nu} \rightarrow U F_{\mu\nu} F^{\mu\nu} U^{-1}$

iii) So $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \rightarrow \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$ (scalar!)

$$\Rightarrow \boxed{\mathcal{L}_{YM} = \frac{1}{4g^2} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}$$

So how does $A_{\mu\ b}^{\ a}$ transform?

Want: $D_\mu \psi^a \to \tilde{D}_\mu \tilde{\psi}^a = U^a{}_b (D_\mu \psi^b)$

i.e.: $D_\mu \psi \to \tilde{D}_\mu \tilde{\psi} = U(D_\mu \psi)$

where $\tilde{\psi} = U\psi$, $D_\mu = \partial_\mu + A_\mu$, $\tilde{D} = \partial_\mu + \tilde{A}_\mu$

Exercise: Show that

$$A_\mu \to \tilde{A}_\mu = U A_\mu U^{-1} - (\partial_\mu U) U^{-1}$$
$$= U A_\mu U^{-1} + U(\partial_\mu U^{-1})$$

Or, with indices displayed:

$$A_{\mu\ b}^{\ a} \to \tilde{A}_{\mu\ b}^{\ a} = U^a_c A_{\mu\ d}^{\ c} (U^{-1})^d_b - (\partial_\mu U^a_c)(U^{-1})^c_b$$
$$= U^a_c A_{\mu\ d}^{\ c} (U^{-1})^d_b + U^a_c \partial_\mu (U^{-1})^c_b$$

GR	YM
For Lagrangian, construct simplest scalar (coordinate invariant)	For Lagrangian, construct simplest scalar: (coordinate + gauge invariant)
step 1: $R_{\beta\nu} = \delta^{\mu}_{\alpha} R^{\alpha}{}_{\beta\mu\nu}$	$\longrightarrow$ $F_{b\nu} \stackrel{?}{=} \delta^{\mu}_{a} F^{a}{}_{b\mu\nu}$ $\uparrow$ doesn't make sense!
step 2: $R = g^{\beta\nu} R_{\beta\nu}$ (scalar!)	Instead
$\Rightarrow \boxed{\mathcal{L}_{EH} = \mathcal{L}_{GR} = \frac{1}{16\pi G} R}$	i) Note $F_{\mu\nu} \to U F_{\mu\nu} U^{-1}$ (exercise!)
	ii) So $F_{\mu\nu} F^{\mu\nu} \to U F_{\mu\nu} F^{\mu\nu} U^{-1}$
	iii) So $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}] \to \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$ (scalar!)
	$\Rightarrow \boxed{\mathcal{L}_{YM} = \frac{1}{4g^2} \mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]}$