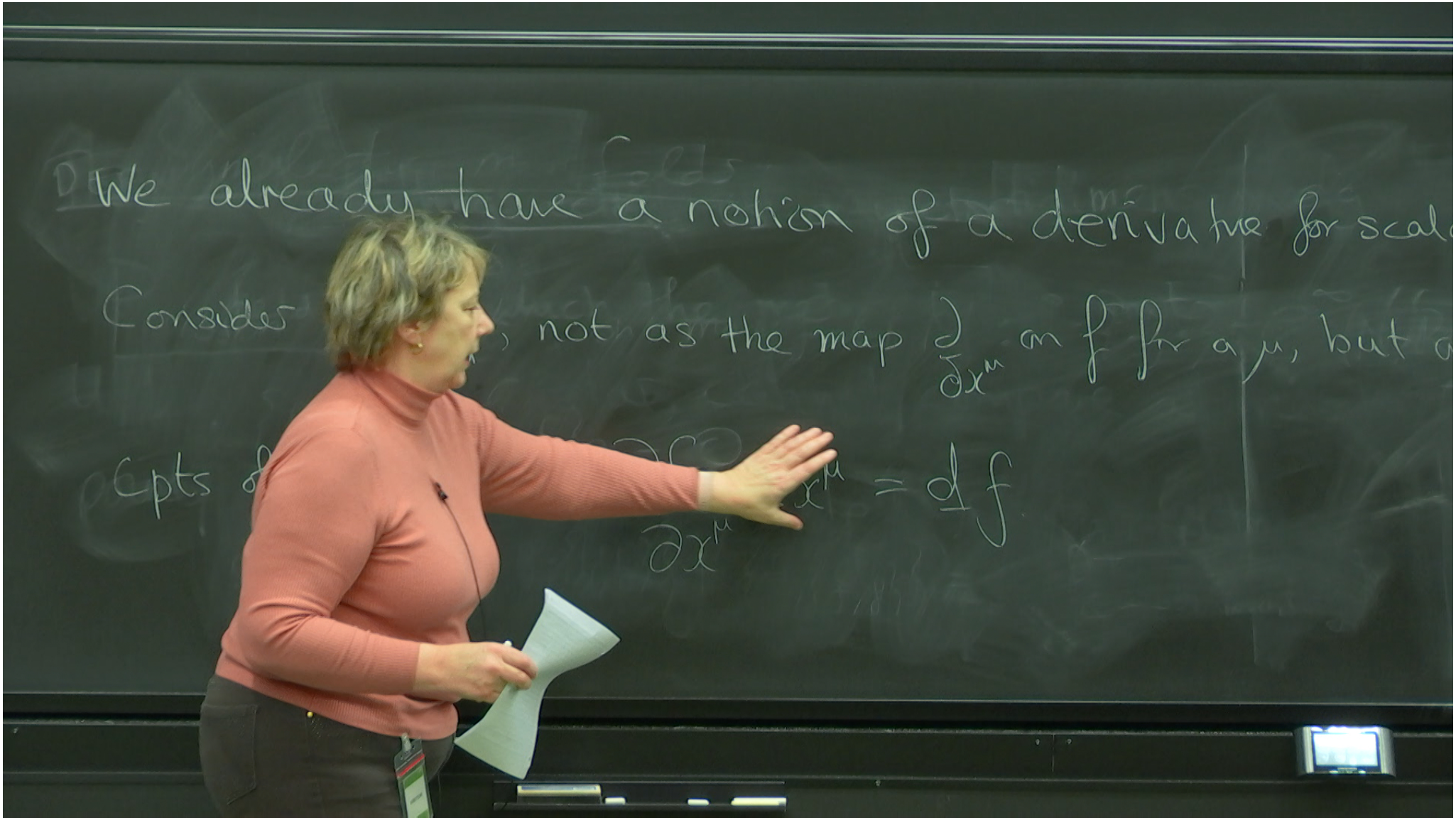
Title: Gravitational Physics Lecture - 230113

Speakers: Ruth Gregory

Collection: Gravitational Physics (2022/2023)

Date: January 13, 2023 - 3:45 PM

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We already have a notion of a derivative for scal
Consider
, not as the map $\frac{\partial}{\partial x^\mu}$ on f for a μ, but
pts
$\frac{\partial f}{\partial x^\mu} x^\mu = df$

scalar

out as

$\underline{d} : C^\infty(M) \longrightarrow T^*(M)$

s.t. at P $\quad \langle \underline{d}f | I \rangle = I f.$

scalar

out as

$\underline{d} : C^\infty(M) \longrightarrow T^*(M)$

s.t. at P $\quad \langle \underline{d}f | \underline{I} \rangle = \underline{I} f.$

$\forall \underline{I} \in T_P(M)$ $\quad$ any $f \in C^\infty(M)$

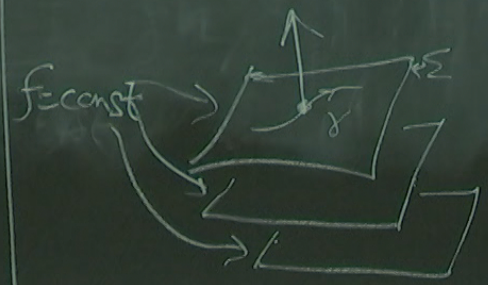
scalar

out as

$\underline{d} : C^\infty(M) \longrightarrow T^*(M)$

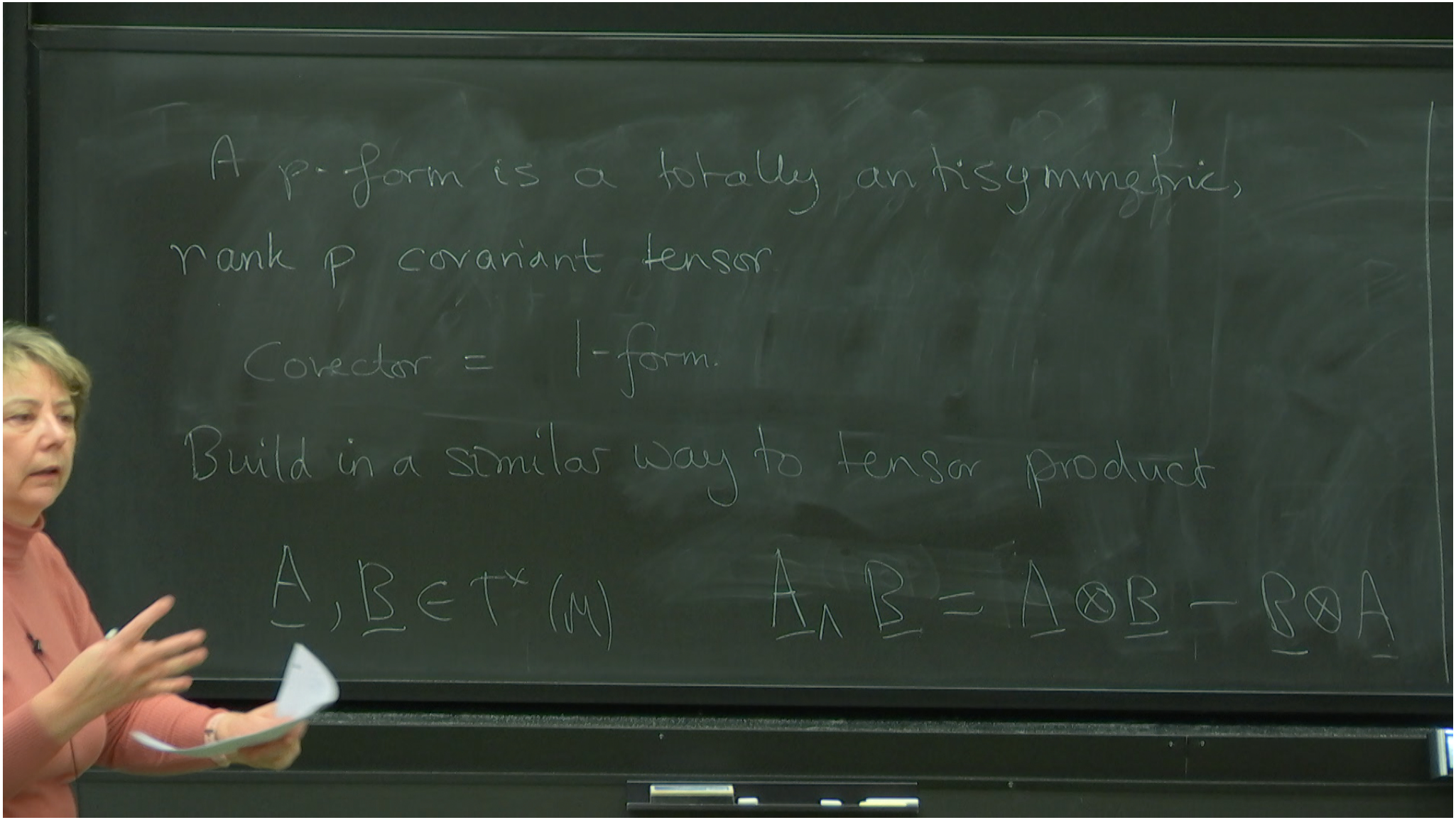
s.t. at P $\quad \langle \underline{d}f | I \rangle = If.$

$\forall\, I \in T_P(M)$ any $f \in C^\infty(M)$



$\underline{d}f$ is "normal" to surfaces.

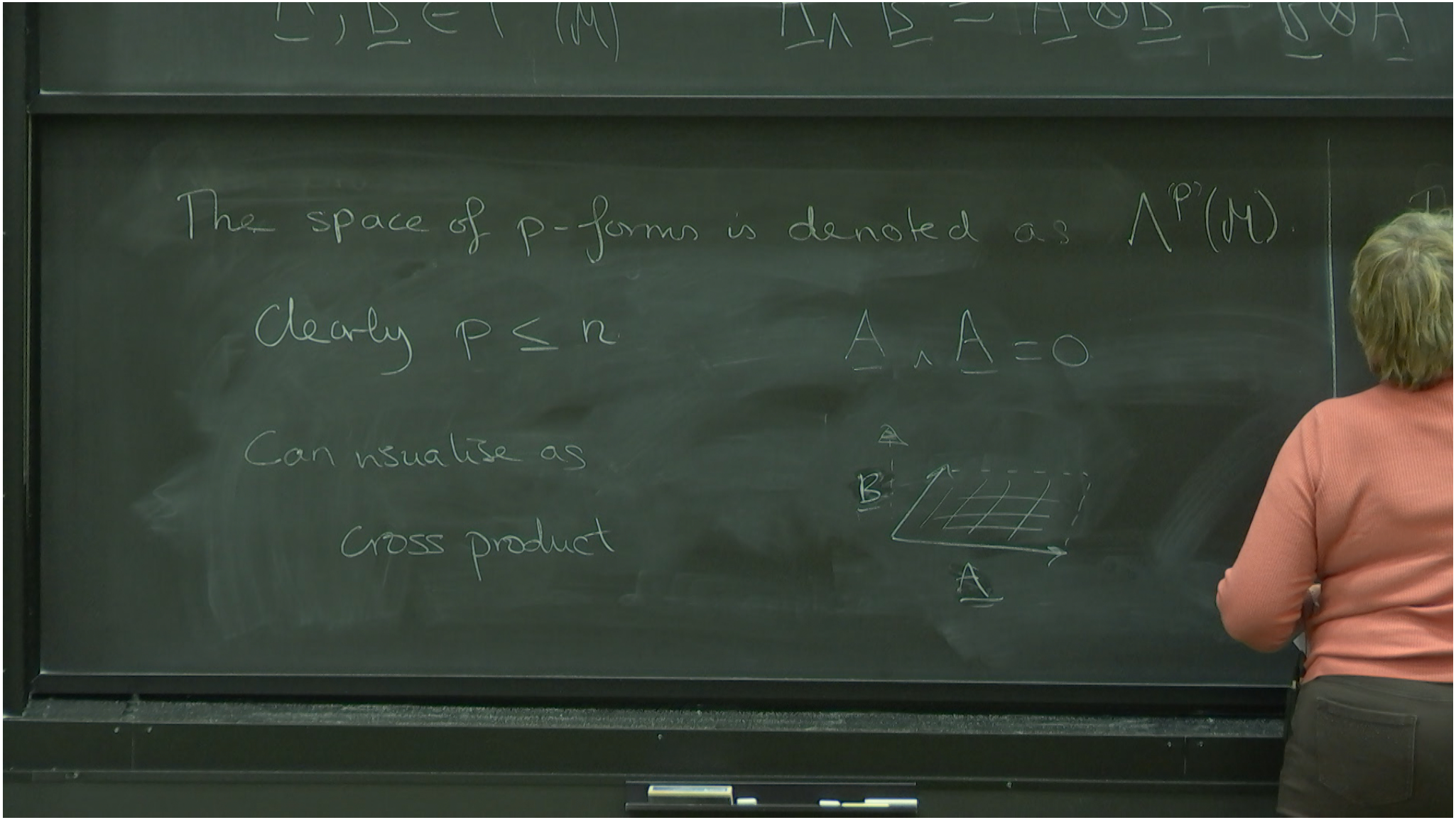
$\gamma \subset \Sigma \quad \langle \underline{d}f | I \rangle = If = 0.$


A p-form is a totally antisymmetric, rank p covariant tensor
Covector = 1-form.
Build in a similar way to tensor product
$\underline{A}, \underline{B} \in T^*(M)$
$\underline{A} \wedge \underline{B} = \underline{A} \otimes \underline{B} - \underline{B} \otimes \underline{A}$

Here, we have a 2-form

In general (in cpts)

$$[\underset{\text{p-form}}{\underline{A}^{(p)}} \wedge \underset{\text{2-form}}{\underline{B}^{(q)}}]_{a_1 \ldots a_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[a_1 \ldots a_p} B_{a_{p+1} \ldots a_{p+q}]}$$


The space of p-forms is denoted as $\Lambda^p(M)$.
Clearly $p \leq n$
$\underline{A} \wedge \underline{A} = 0$
Can visualise as
cross product
$\underline{B}$
$\underline{A}$

The n-form is unique up to a factor

$$\epsilon_{abcd} = \pm 1$$

The n-form is unique up to a factor

$\epsilon_{abcd} = \pm 1$ (sign of perm of a,b,c,d of 0,1,2,3)

Strictly speaking, ϵ is a tensor density.

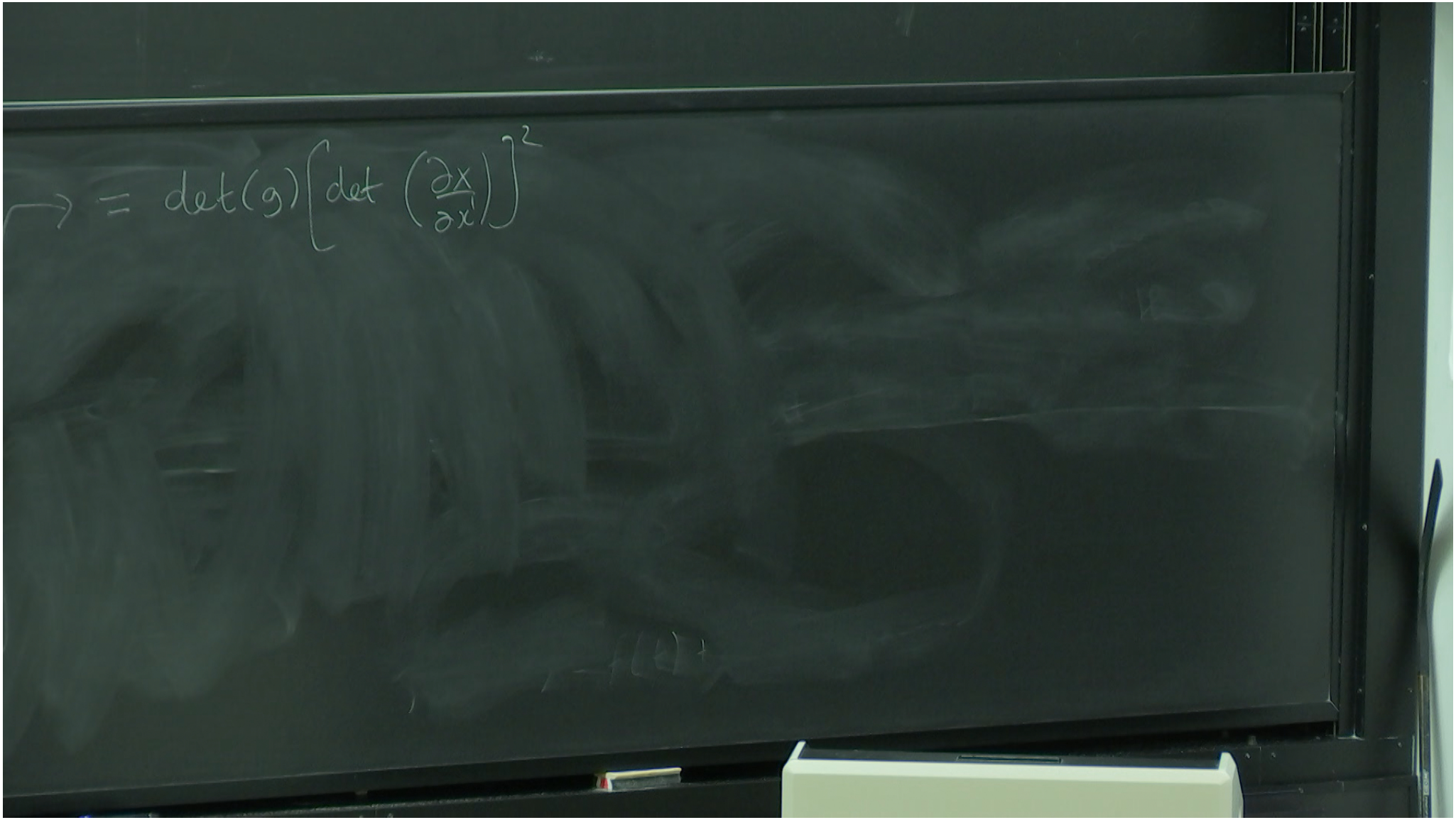
$$\underline{\epsilon} = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_n} = \frac{1}{n!}\det\left(\frac{\partial x}{\partial x'}\right)\epsilon_{\mu'_1\cdots\mu'_n} dx'^{\mu'_1}\wedge\cdots\wedge dx'^{\mu'_n}$$

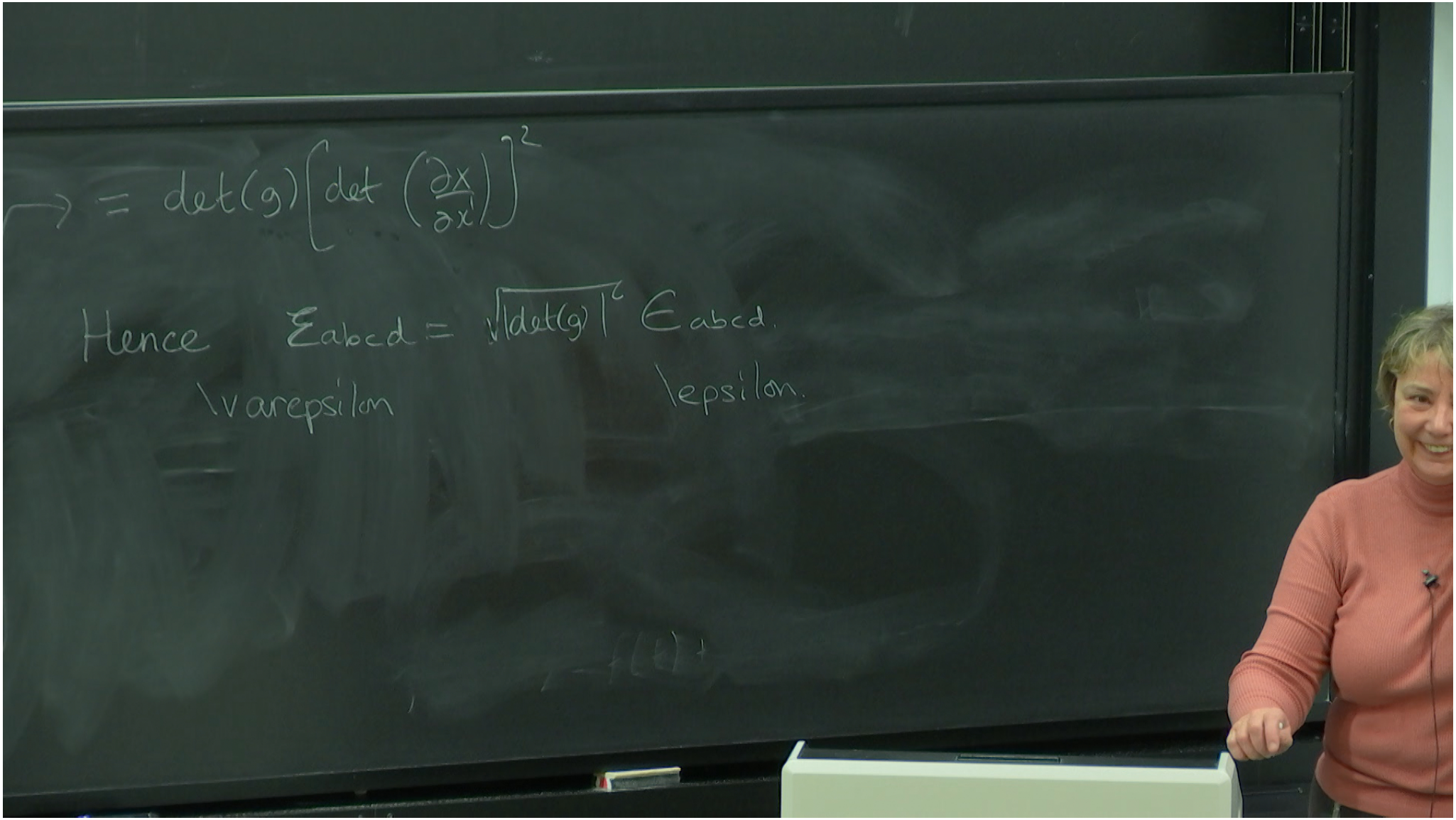
$\det M_{ab}$: $\epsilon_{ab} M^a{}_c M^b{}_d = (\det M)\epsilon_{cd}$.

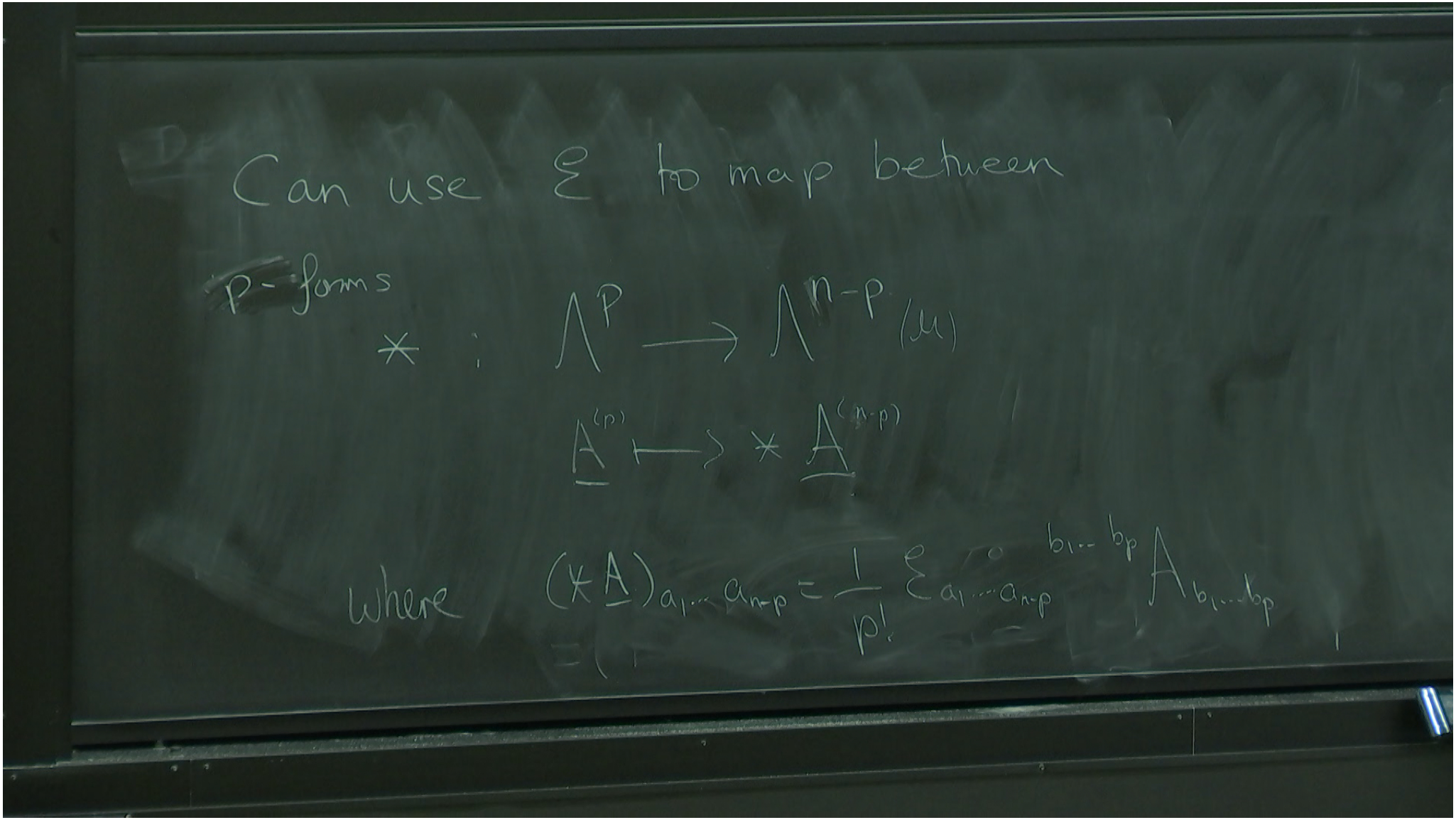
$\epsilon_{12} = 1 = -\epsilon_{21}$

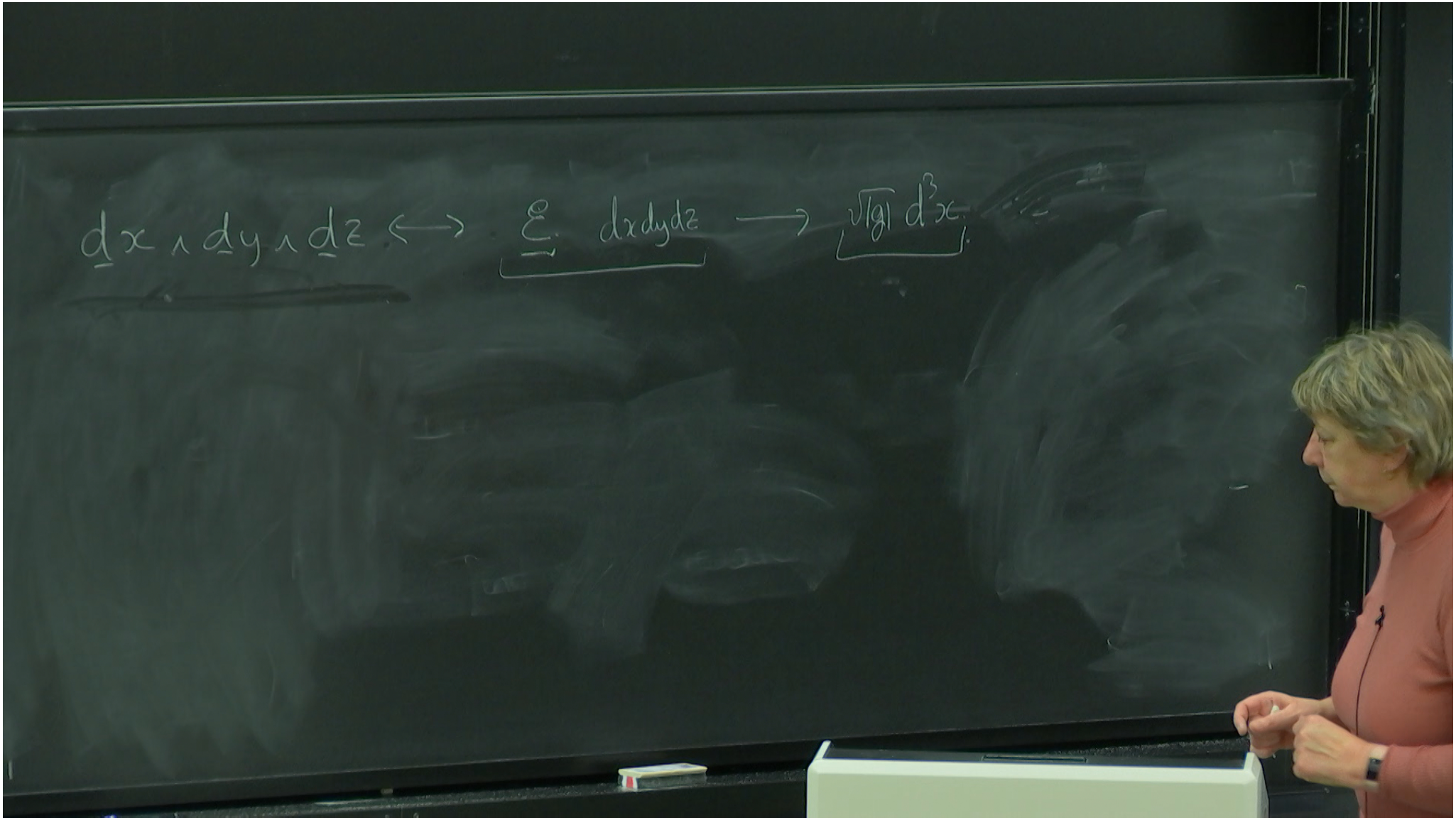
With a metric, we can make the ϵ-tensor covariant

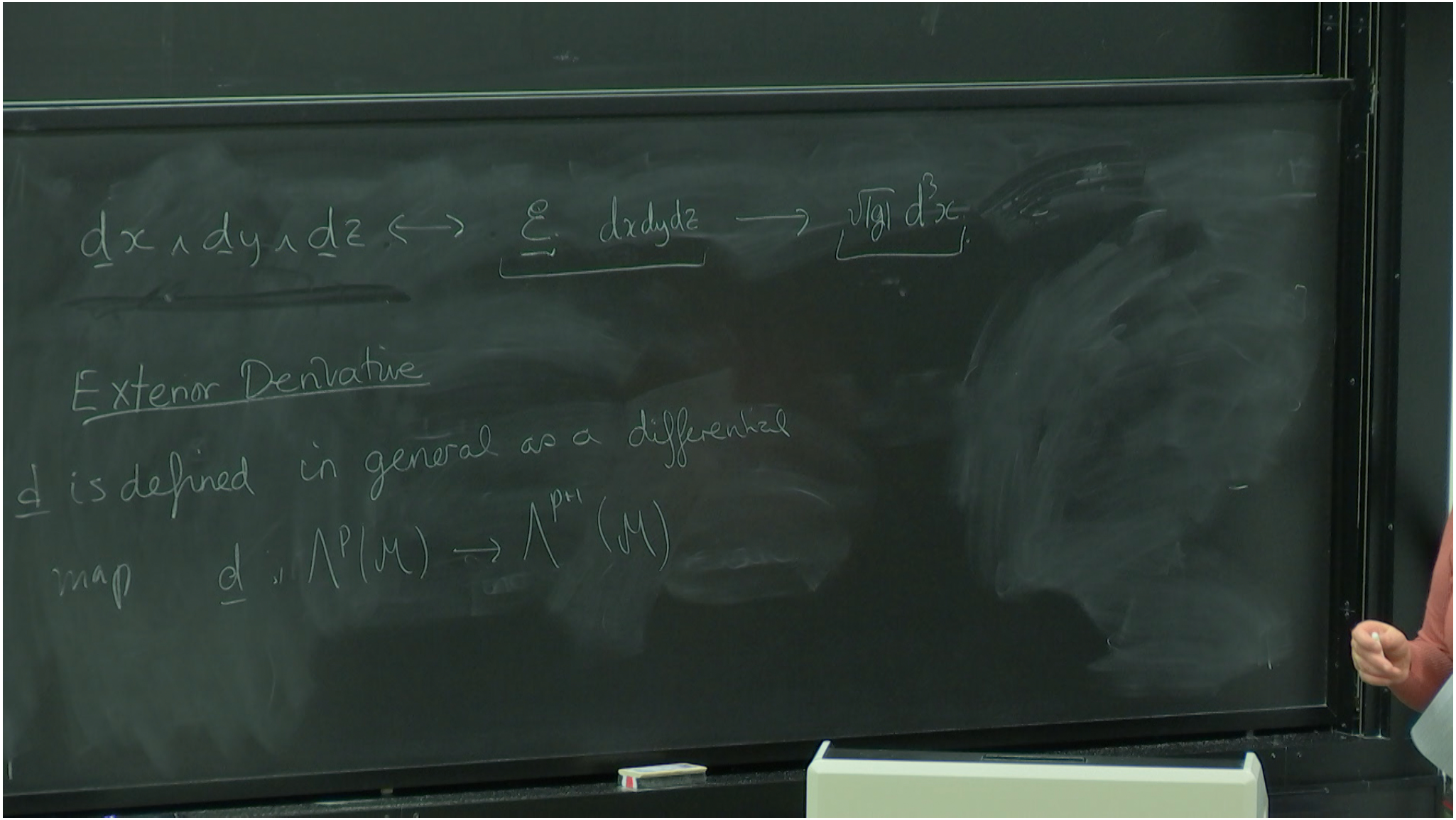
$$\det(g_{ab}) \rightarrow \det(g'_{a'b'}) = \det\left(g_{ab}\frac{\partial x^a}{\partial x'^{a'}}\frac{\partial x^b}{\partial x'^{b'}}\right)$$


$\rightarrow = \det(g)\left[\det\left(\frac{\partial x}{\partial x'}\right)\right]^2$


$\rightarrow = \det(g)\left[\det\left(\frac{\partial x}{\partial x'}\right)\right]^2$
Hence $\varepsilon_{abcd} = \sqrt{|\det(g)|}\,\epsilon_{abcd}$.
\varepsilon
\epsilon.

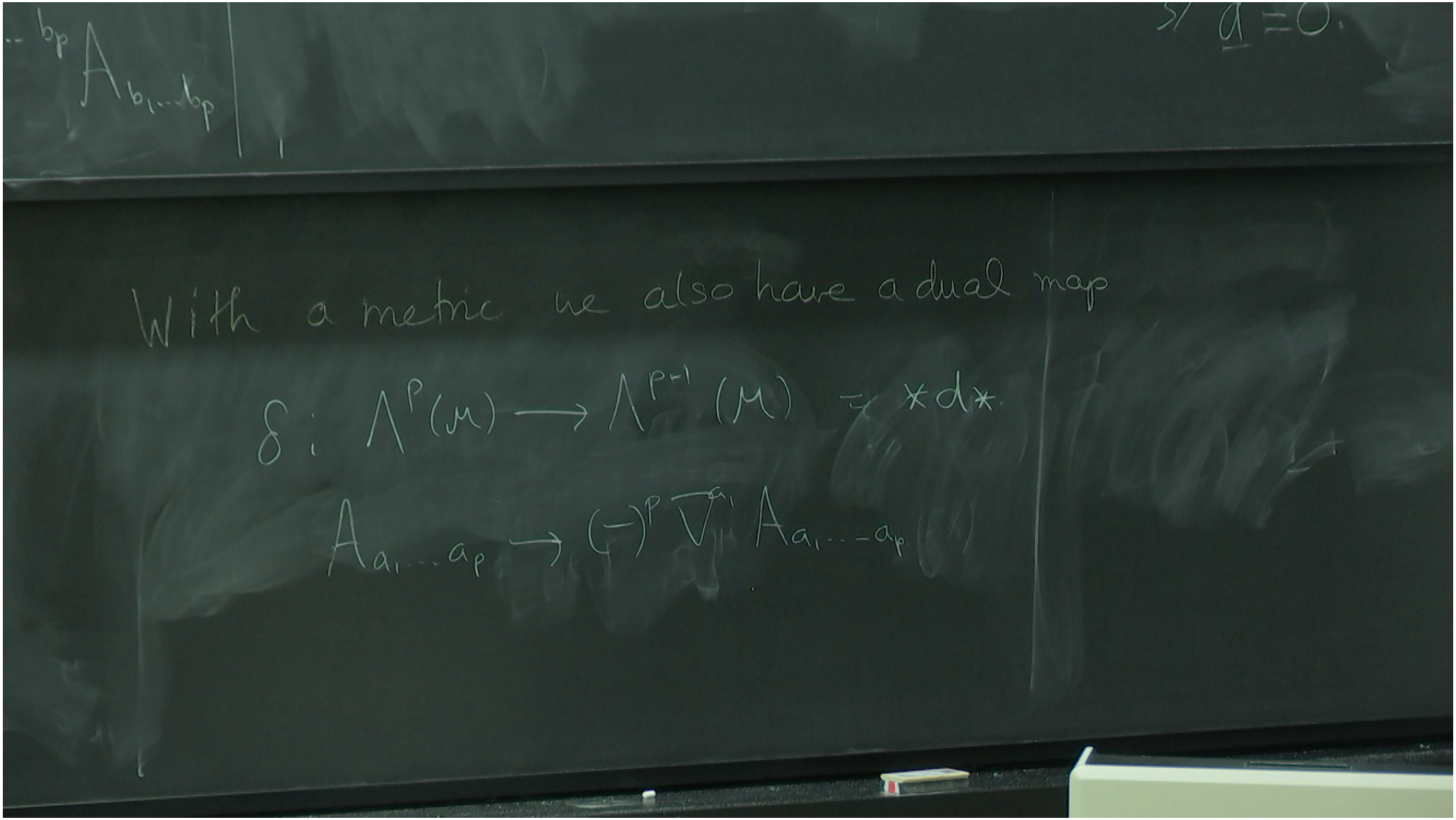

Can use ε to map between
p-forms
$* : \Lambda^p \longrightarrow \Lambda^{n-p}(\mathcal{M})$
$\underline{A}^{(p)} \longmapsto * \underline{A}^{(n-p)}$
where $(*\underline{A})_{a_1 \cdots a_{n-p}} = \frac{1}{p!} \varepsilon_{a_1 \cdots a_{n-p}}{}^{b_1 \cdots b_p} A_{b_1 \cdots b_p}$


$\underline{d}x \wedge \underline{d}y \wedge \underline{d}z \longleftrightarrow \underline{\underline{\varepsilon}}\, dxdydz \longrightarrow \sqrt{|g|}\, d^3x$


$\underline{d}x \wedge \underline{d}y \wedge \underline{d}z \longleftrightarrow \underline{\varepsilon}\, dxdydz \longrightarrow \sqrt{|g|}\, d^3x$
Exterior Derivative
$\underline{d}$ is defined in general as a differential
map $\underline{d} : \Lambda^p(M) \to \Lambda^{p+1}(M)$

In components:

$$(\underline{d}\underline{A})_{a_1 \ldots a_{p+1}} = \frac{(p+1)!}{p!}\, \partial_{[a_1} A_{a_2 \ldots a_{p+1}]}.$$

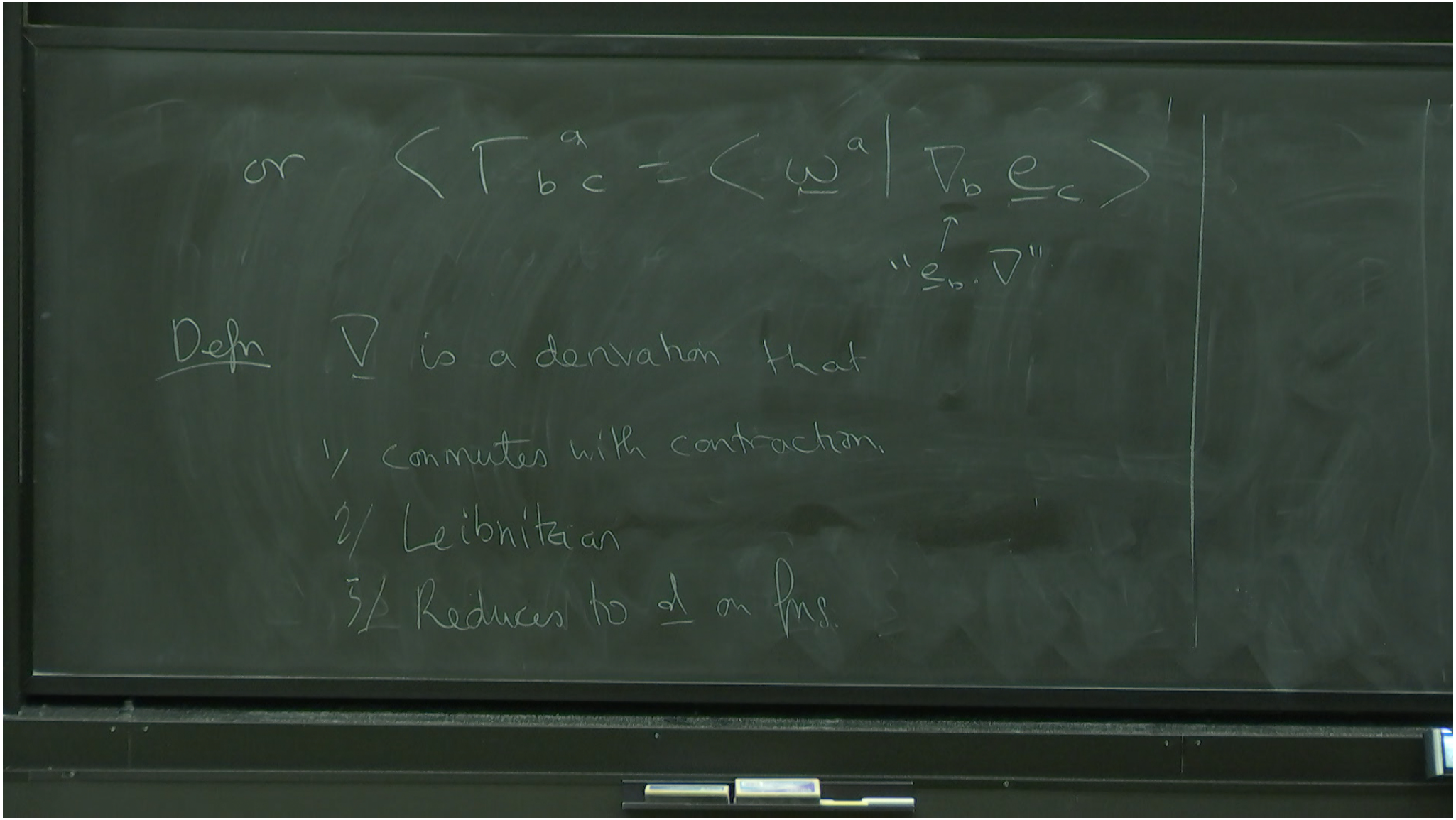

$\cdots b_p A_{b_1 \cdots b_p}$
$\nabla_a A^a = 0$.
With a metric we also have a dual map
$\delta : \Lambda^p(M) \longrightarrow \Lambda^{p-1}(M) = *d*$
$A_{a_1 \cdots a_p} \rightarrow (-)^p \nabla^{a_1} A_{a_1 \cdots a_p}$

The covariant derivative / connection

The covariant deriv works by introducing a connection between tangent spaces $\{\underline{e}_a\}$ of $T_p(M) \longrightarrow \{\underline{e}'_{b'}\}$ of $T_{p+\delta p}(M)$

$$\nabla \underline{e}_b = \Gamma^c_{\ a\ b}\, \underline{e}_c \otimes \underline{\omega}^a$$

derivation · basis vector · scalars · basis vector · basis covector


or $\langle \Gamma^{a}_{bc} = \langle \underline{\omega}^{a} | \nabla_b \underline{e}_c \rangle$
"$\underline{e}_b \cdot \nabla$"
Defn ∇ is a derivation that
1/ commutes with contraction
2/ Leibnitzian
3/ Reduces to $\underline{d}$ on fns.

For a general vector:

$$\nabla \underline{V} = \nabla(V^a \underline{e}_a)$$

$$= \nabla(\overset{\text{SCALARS}}{V^a})\,\underline{e}_a + V^a \nabla \underline{e}_a$$

$$= \underline{d}V^a \underline{e}_a + V^a \Gamma^c_{ba}\, \underline{e}_c \otimes \underline{\omega}^b$$

$$= (V^a{}_{,b} + \Gamma^a_{bc} V^c)\, \underline{e}_a \otimes \underline{\omega}^b .$$

$$\underline{\epsilon} = \frac{1}{n!} \epsilon_{\mu_1 \cdots \mu_n} \underbrace{dx^{\mu_1}}_{\frac{\partial x^{\mu_1}}{\partial x'^{\mu'_1}} dx'^{\mu'_1}} \wedge \cdots \wedge dx^{\mu_n} = \frac{1}{n!} \det\left(\frac{\partial x}{\partial x'}\right) \epsilon_{\mu'_1 \cdots \mu'_n} dx$$