Title: Gravitational Physics Lecture - 230113

Speakers: Ruth Gregory

Collection: Gravitational Physics (2022/2023)

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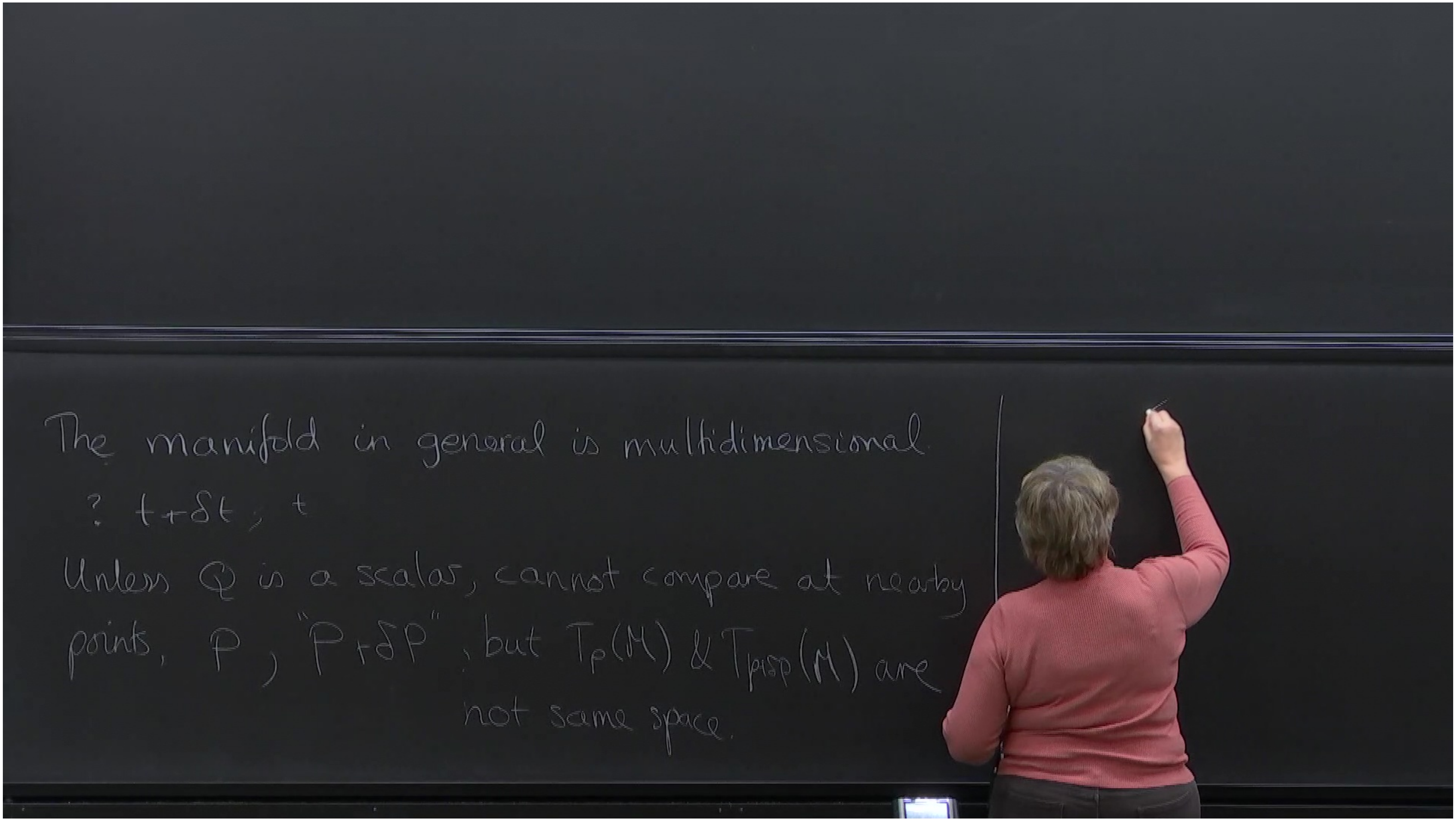
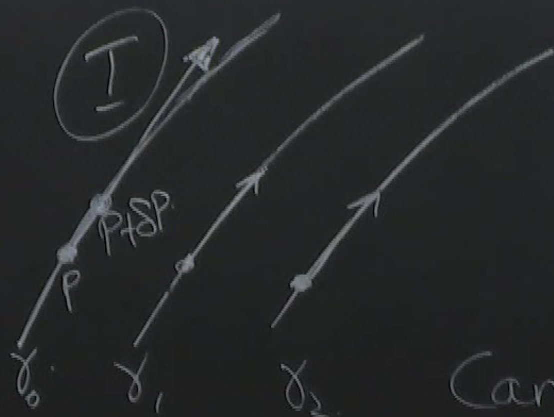

The manifold in general is multidimensional.
? t+δt ; t
Unless Q is a scalar, cannot compare at nearby
points, P , "P+δP" , but T_P(M) & T_{P+δP}(M) are
not same space.

nal.

nearby

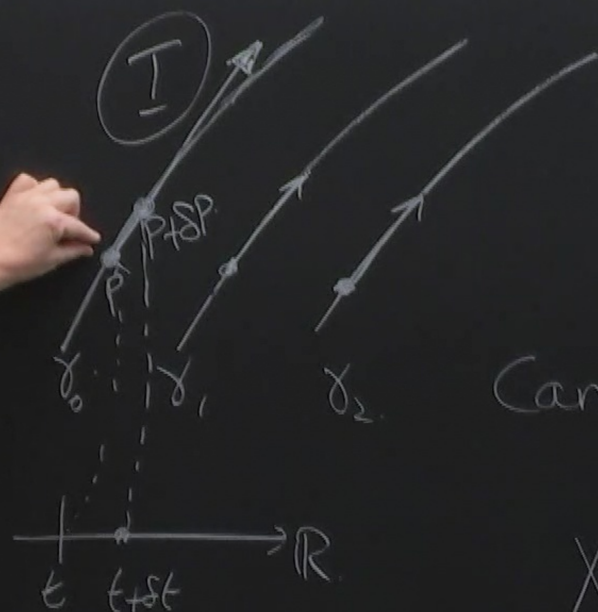
are



I is tangent to γ,

γ is the <u>integral curve</u> of I.

Can use I to link nearby points.



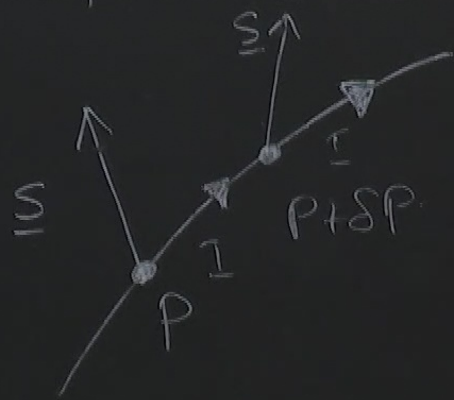
I is tangent to γ,

γ is the <u>integral curve</u> of I.

Can use I to link nearby points.

$$X^\mu_{p+\delta p} = X^\mu_p + \delta X^\mu = X^\mu_p + \underbrace{\varepsilon}_{\text{or } \delta t} T^\mu + \ldots\ldots$$

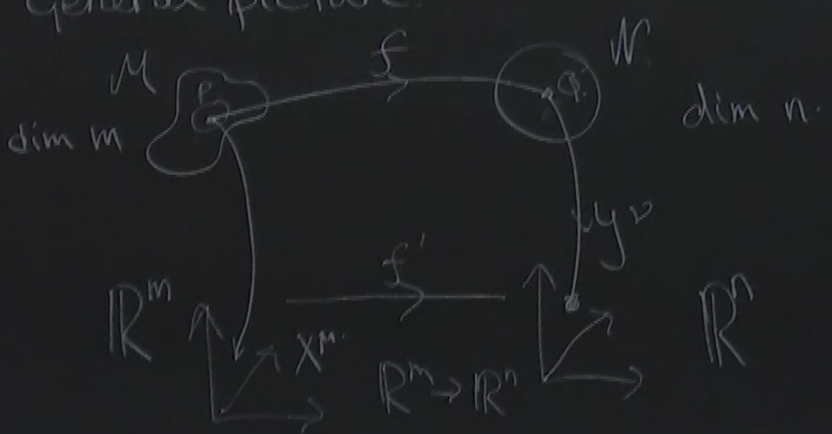
$\mathbb{I}$ defines a coord transfm locally.



By viewing moving along γ as a coord transfm, can identify an $\underline{S}(p+\delta p)$ from $\underline{S}(p)$

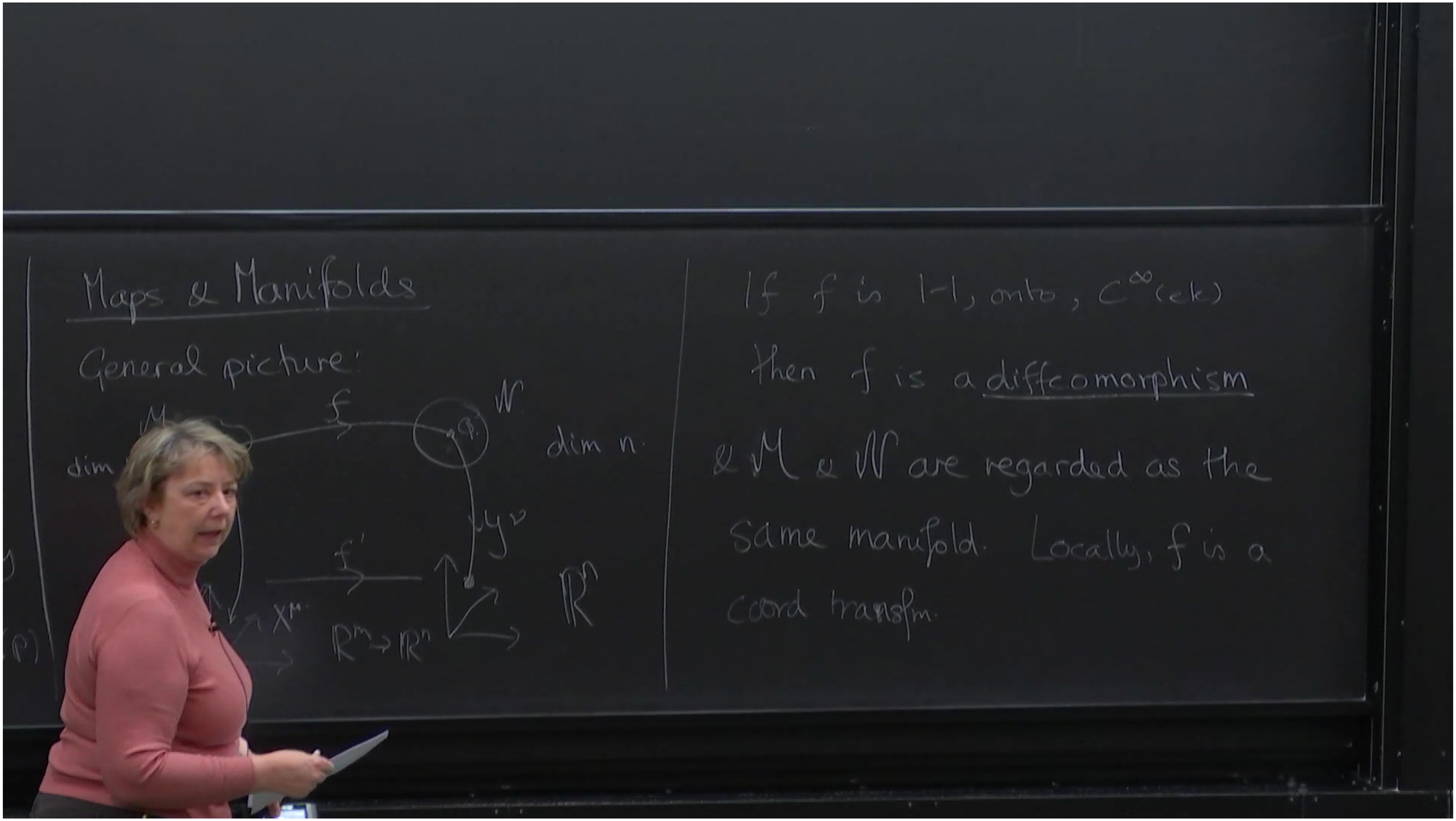
Maps & Manifolds

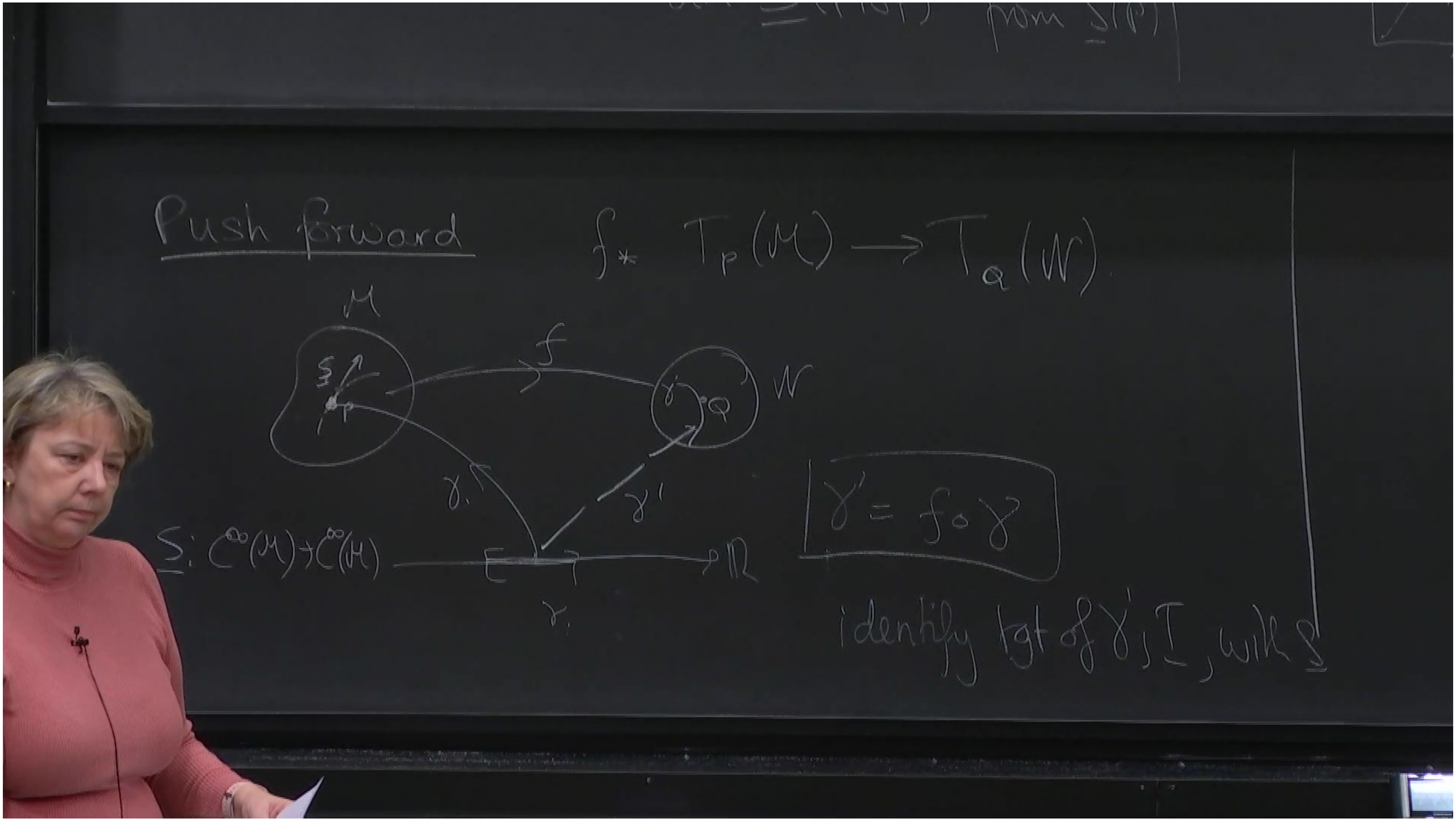
General picture:



If f is 1-1, onto, C^∞ (etc)

then f is a <u>diffeomorphism</u>


Maps & Manifolds
General picture:
f
N
dim n.
dim
ψ^ν
f'
x^μ
R^m → R^n
R^n
If f is 1-1, onto, C^∞ (etc)
then f is a diffeomorphism
& M & N are regarded as the
same manifold. Locally, f is a
coord transfm.


Push forward
f* Tp(M) → Tq(N)
M
f
N
ξ
p
q
γ
γ'
γ' = f∘γ
ξ: C∞(M)→C∞(M)
R
identify tgt of γ', I, with ξ

x^μ $\mathbb{R}^n \to \mathbb{R}^n$ $\mathbb{R}$

coord transfm.

Pull-back $f^*: T_Q^*(N) \longrightarrow T_p^*(M)$

defined via $\langle f^*(\underline{\omega}) \,|\, \underline{s} \rangle = \langle \underline{\omega} \,|\, f_*(\underline{s}) \rangle$

$$\begin{tikzcd} T_p(M) \arrow[r, "f_*"] \arrow[dr, dashed, "f_*(\underline{\omega})"'] & T_Q(N) \arrow[d, "\underline{\omega}"] \\ & \mathbb{R} \end{tikzcd}$$

Vector fields general coord transfmn

hence provide a (specific) map

between tangent & cotangent spaces.

$$\underline{S} = S^{\mu}\frac{\partial}{\partial x^{\mu}} = S'^{\nu}\frac{\partial}{\partial y^{\nu}}$$

$$S'^{\nu} = S^{\mu}\frac{\partial y^{\nu}}{\partial x^{\mu}} =$$

$$= S^{\nu} + \epsilon \, S$$

$$S'^{\nu}(\underbrace{P + \delta P}_{x^{\mu} + \epsilon T^{\mu}}) = S^{\nu}_{0}$$

$$S^{\nu} = S^{\mu} \frac{\partial y^{\nu}}{\partial x^{\mu}} = S^{\mu} \frac{\partial}{\partial x^{\mu}} \left(X^{\nu} + \epsilon T^{\nu} \right)$$

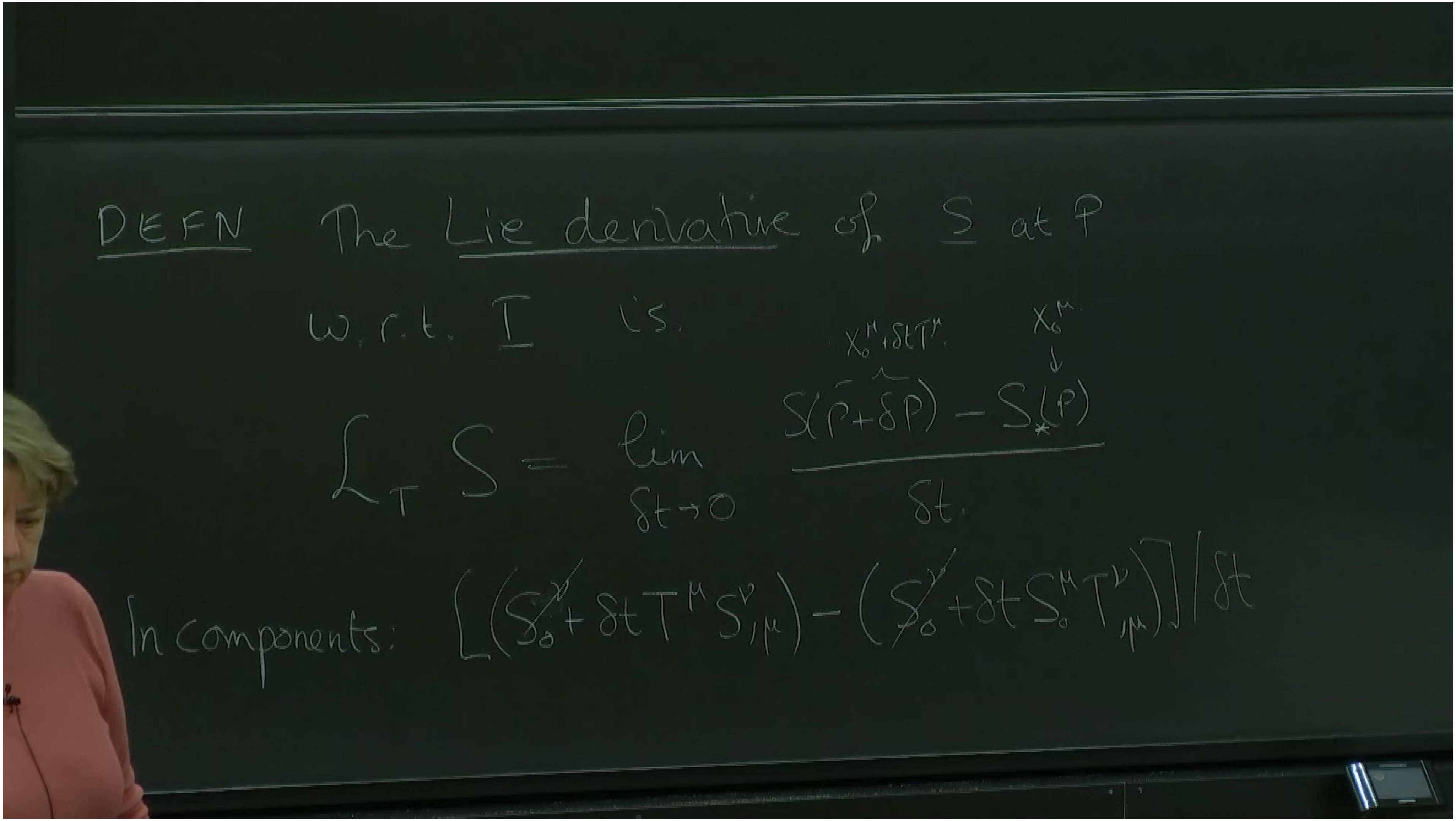
$$= S^{\nu} + \epsilon \, S^{\mu} T^{\nu}{}_{,\mu}$$

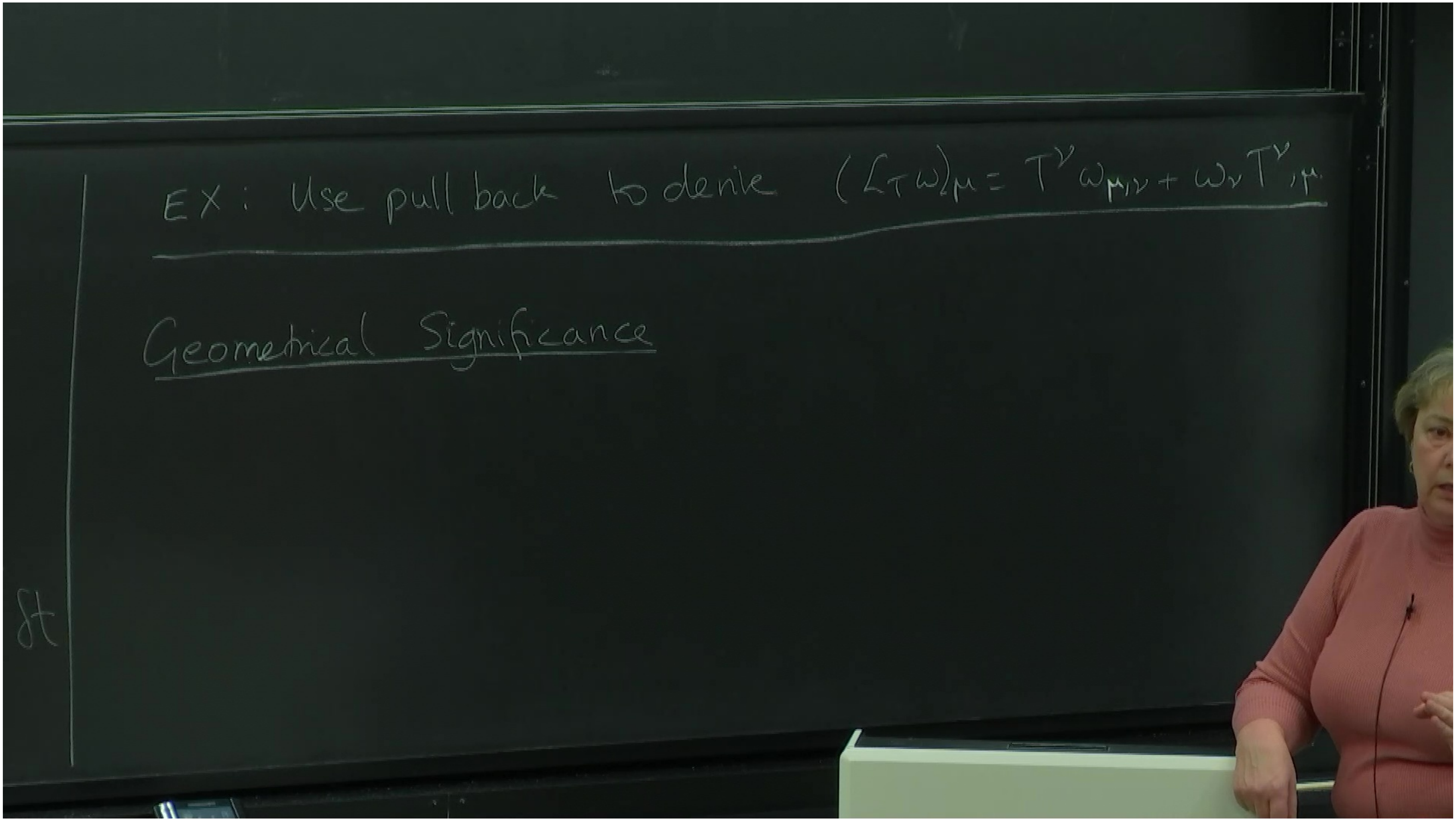
Use to compare with S at $P + \delta P$

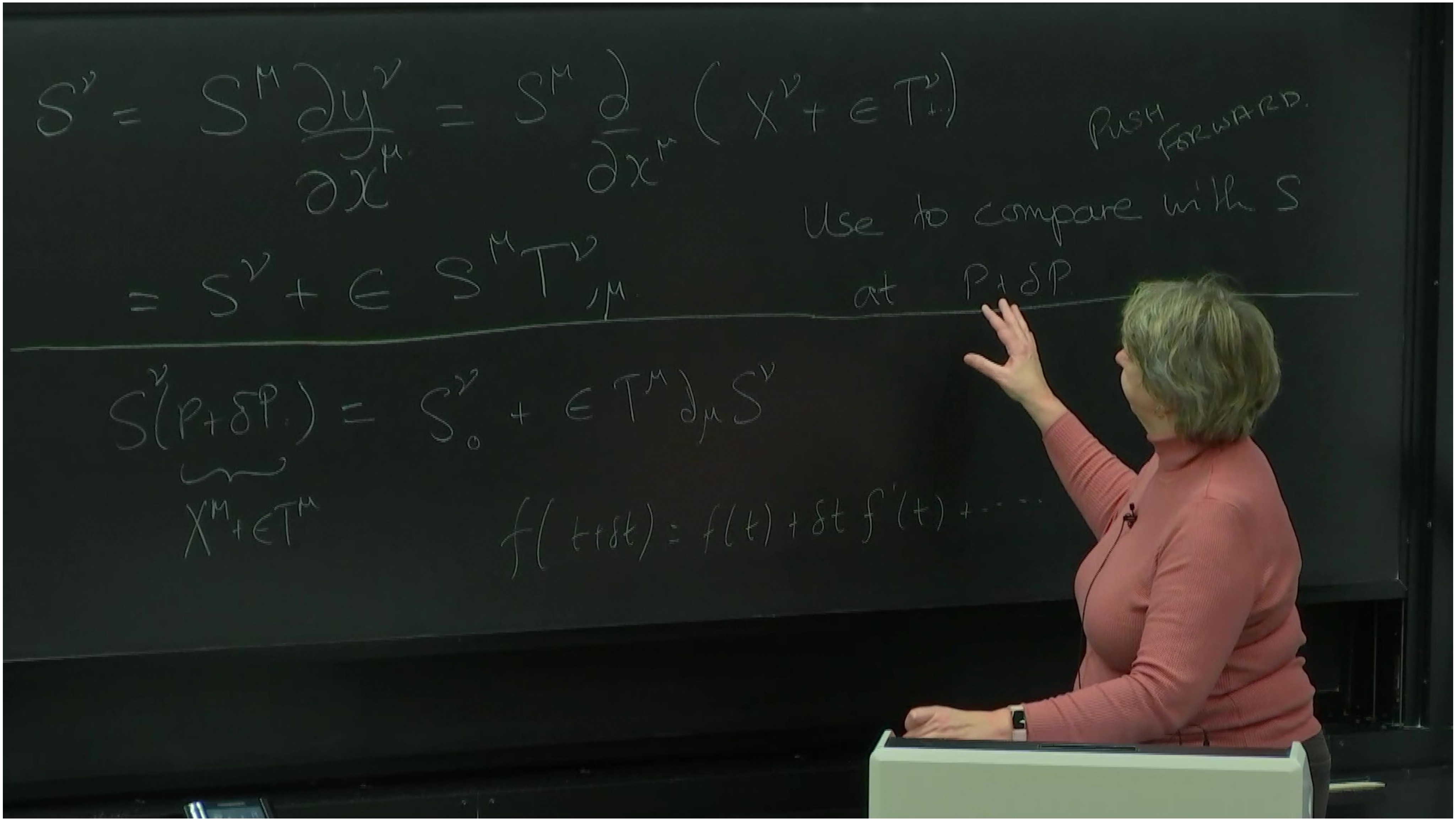
$$S^{\nu}(\underbrace{P + \delta P}_{X^{\mu} + \epsilon T^{\mu}}) = S^{\nu}_{0} + \epsilon \, T^{\mu} \partial_{\mu} S^{\nu}$$

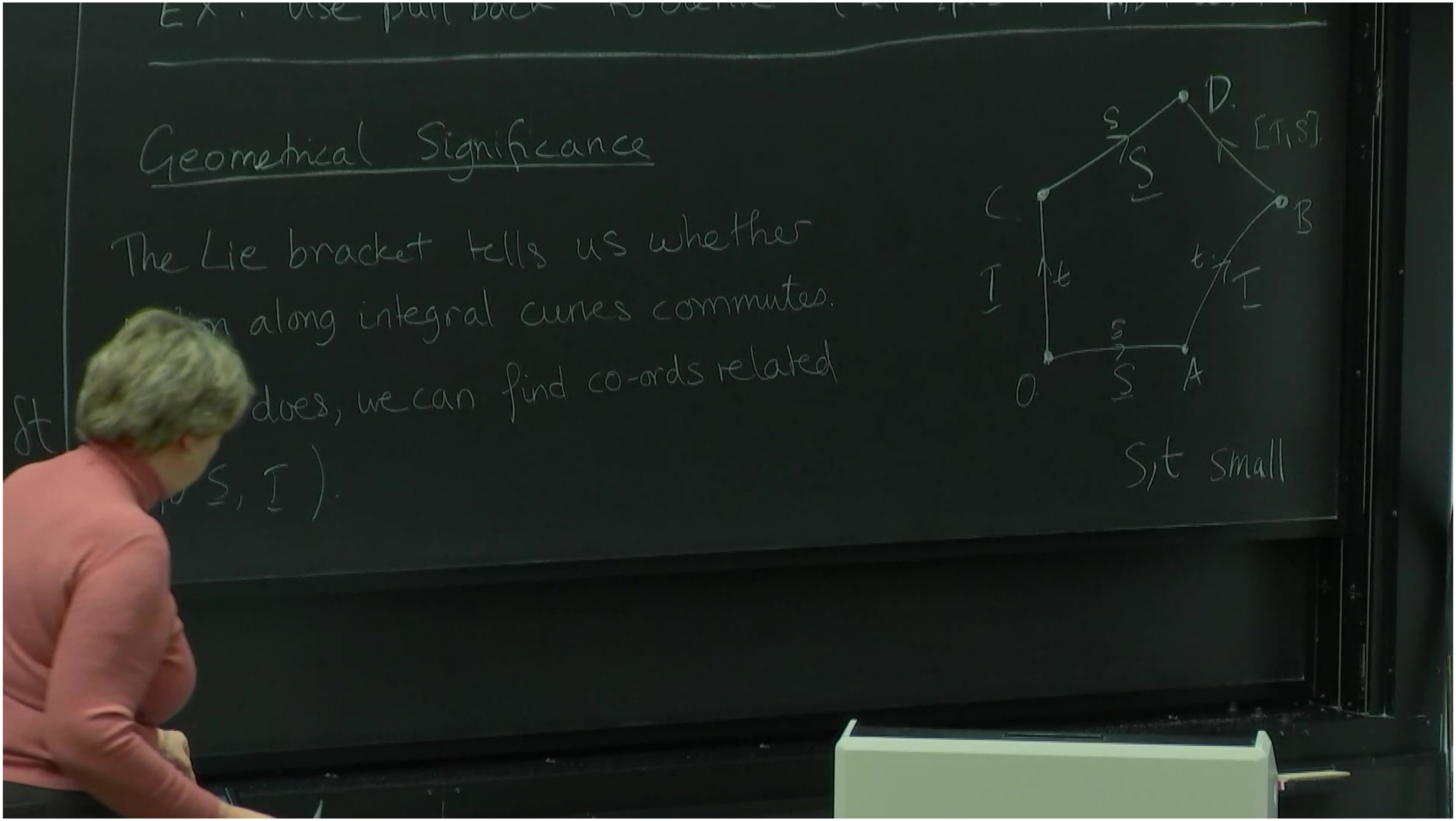
DEFN The **Lie derivative** of $\underline{S}$ at P w.r.t. $\underline{T}$ is

$$\mathcal{L}_T S = \lim_{\delta t \to 0} \frac{S(\overset{X_0^\mu + \delta t\, T^\mu}{\widetilde{P + \delta P}}) - S_*(\overset{X_0^\mu}{\downarrow}P)}{\delta t}$$


DEFN The Lie derivative of $\underline{S}$ at P
w.r.t. $\underline{T}$ is:
$X_0^\mu + \delta t T^\mu$
X_0^μ
$\mathcal{L}_T S = \lim_{\delta t \to 0} \frac{S(P+\delta P) - S_*(P)}{\delta t}$
In components: $\left[(\cancel{S_0^\nu} + \delta t\, T^\mu S^\nu_{,\mu}) - (\cancel{S_0^\nu} + \delta t\, S_0^\mu T^\nu_{,\mu}) \right] / \delta t$


EX: Use pull back to derive $(\mathcal{L}_T \omega)_\mu = T^\nu \omega_{\mu,\nu} + \omega_\nu T^\nu{}_{,\mu}$
Geometrical Significance
δt


$S'^{\nu} = S^{\mu}\frac{\partial y^{\nu}}{\partial x^{\mu}} = S^{\mu}\frac{\partial}{\partial x^{\mu}}\left(X^{\nu} + \epsilon T^{\nu}\right)$
PUSH FORWARD.
$= S^{\nu} + \epsilon S^{\mu} T^{\nu}{}_{,\mu}$
Use to compare with S at P+δP
$S^{\nu}(\underbrace{P+\delta P}_{X^{\mu}+\epsilon T^{\mu}}) = S^{\nu}_{0} + \epsilon T^{\mu}\partial_{\mu}S^{\nu}$
$f(t+\delta t) = f(t) + \delta t f'(t) + \dots$


Geometrical Significance
The Lie bracket tells us whether
along integral curves commutes.
does, we can find co-ords related
S, T).
D
C
B
O
A
s
S
[T, S]
t
T
S, t small

$$= \left(T^\mu S^\nu{}_{,\mu} - S^\mu T^\nu{}_{,\mu}\right) = [T, S]^\nu.$$

$$X_A{}^\mu = X_0{}^\mu + s\, S_0{}^\mu + \tfrac{1}{2} s^2 S_0{}^\nu S_0{}^\mu{}_{,\nu}.$$

$$X_B{}^\mu = X_A{}^\mu + t\, T_A{}^\mu + \tfrac{1}{2} t^2 T_A{}^\nu T_A{}^\mu{}_{,\nu}.$$

$$= X_0{}^\mu + s\, S_0{}^\mu + \tfrac{1}{2} s^2 S_0{}^\nu S_0{}^\mu{}_{,\nu}.$$

$$+\, t\left(T_0{}^\mu + s\, S_0{}^\nu T_0{}^\mu{}_{,\nu}\right)$$

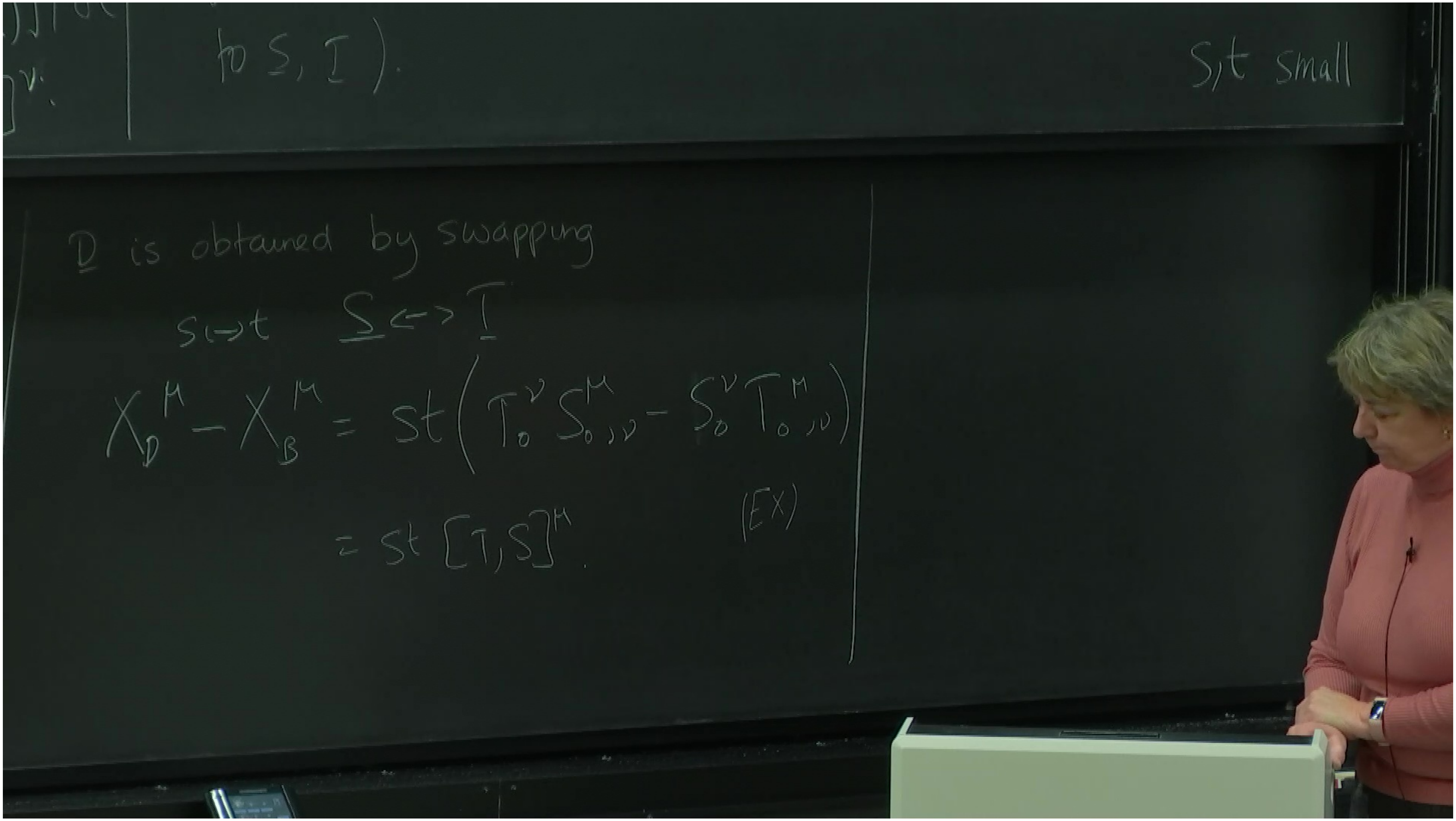
$$= \left(T^\mu S^\nu{}_{,\mu} - S^\mu T^\nu{}_{,\mu}\right) = [T, S]^\nu.$$

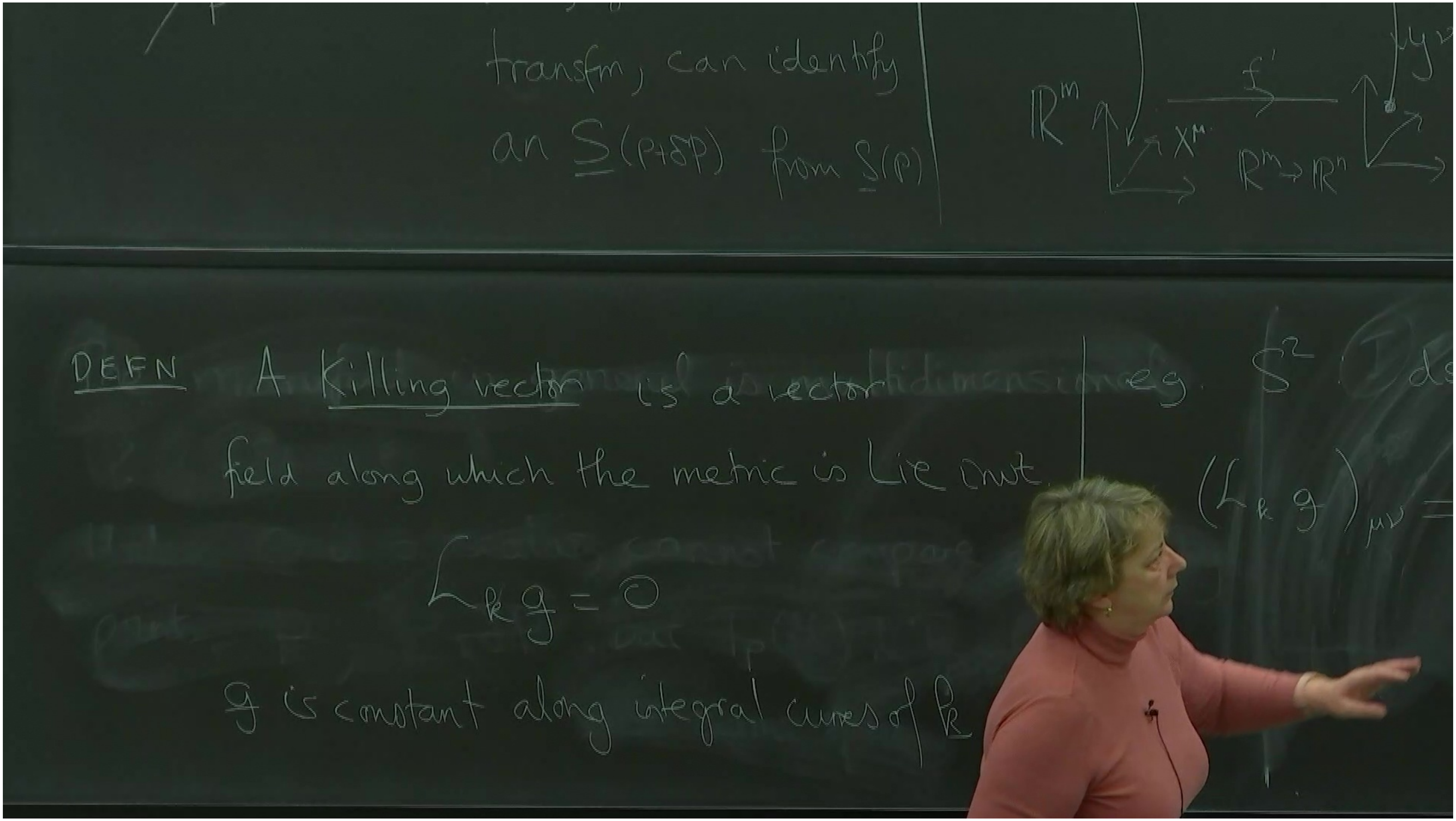
$$X_A{}^\mu = X_0{}^\mu + s\, S_0{}^\mu + \frac{1}{2} s^2 S_0{}^\nu S_0{}^\mu{}_{,\nu}.$$

$$X_B{}^\mu = X_A{}^\mu + t\, T_A{}^\mu + \frac{1}{2} t^2 T_A{}^\nu T_A{}^\mu{}_{,\nu}$$

$$= X_0{}^\mu + s\, S_0{}^\mu + \frac{1}{2} s^2 S_0{}^\nu S_0{}^\mu{}_{,\nu}$$

$$+ t\left(T_0{}^\mu + s\, S_0{}^\nu T_0{}^\mu{}_{,\nu}\right) + \frac{1}{2} t^2 T_0{}^\nu T_0{}^\mu{}_{,\nu}.$$


to $\underline{S}$, $\underline{T}$).
s,t small
$\underline{D}$ is obtained by swapping
$s \leftrightarrow t$ $\underline{S} \leftrightarrow \underline{T}$
$X_D^\mu - X_B^\mu = st\left(T_0^\nu S^\mu_{0,\nu} - S_0^\nu T^\mu_{0,\nu}\right)$
$= st\,[T,S]^\mu.$
(EX)


transfm, can identify
an S(p+δp) from S(p)
R^m
X^μ
f'
R^m → R^n
y^ν
DEFN A Killing vector is a vector
field along which the metric is Lie invt.
L_k g = 0
g is constant along integral curves of k
e.g. S^2
ds
(L_k g)_μν =

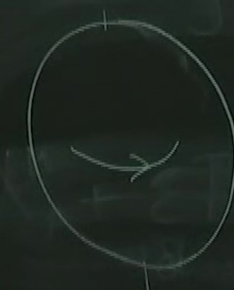
X^μ $\mathbb{R}^m \to \mathbb{R}^n$ $\mathbb{R}$

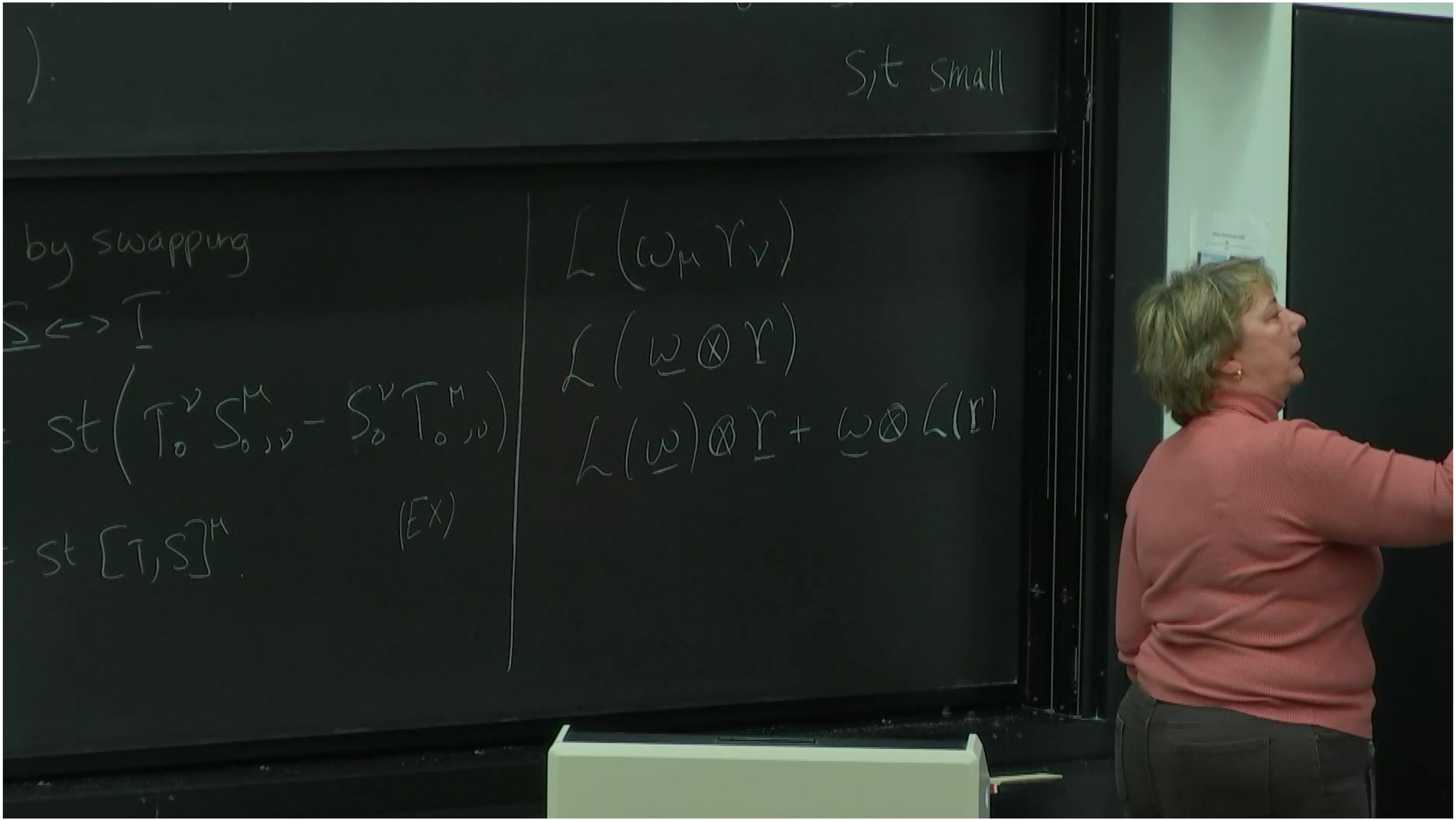
coord transfm.

e.g. S^2 (I) $ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ $\qquad \underline{k} = \dfrac{\partial}{\partial \varphi}$ $\qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$(\mathcal{L}_k g)_{\mu\nu} = k^\sigma g_{\mu\nu,\sigma} + g_{\sigma\nu} \cancel{k^\sigma{}_{,\mu}} + g_{\mu\sigma} \cancel{k^\sigma{}_{,\nu}}$$

$$= \frac{\partial}{\partial \varphi}(g_{\mu\nu}) = 0.$$




S,t small
by swapping
S <-> T
st(T₀^ν S₀^μ,ν − S₀^ν T₀^μ,ν)
st [T,S]^μ
(EX)
L(ω_μ Υ_ν)
L(ω ⊗ Υ)
L(ω) ⊗ Υ + ω ⊗ L(Υ)