Title: Gravitational Physics Lecture - 230127

Speakers: Ruth Gregory

Collection: Gravitational Physics (2022/2023)

Date: January 27, 2023 - 9:00 AM

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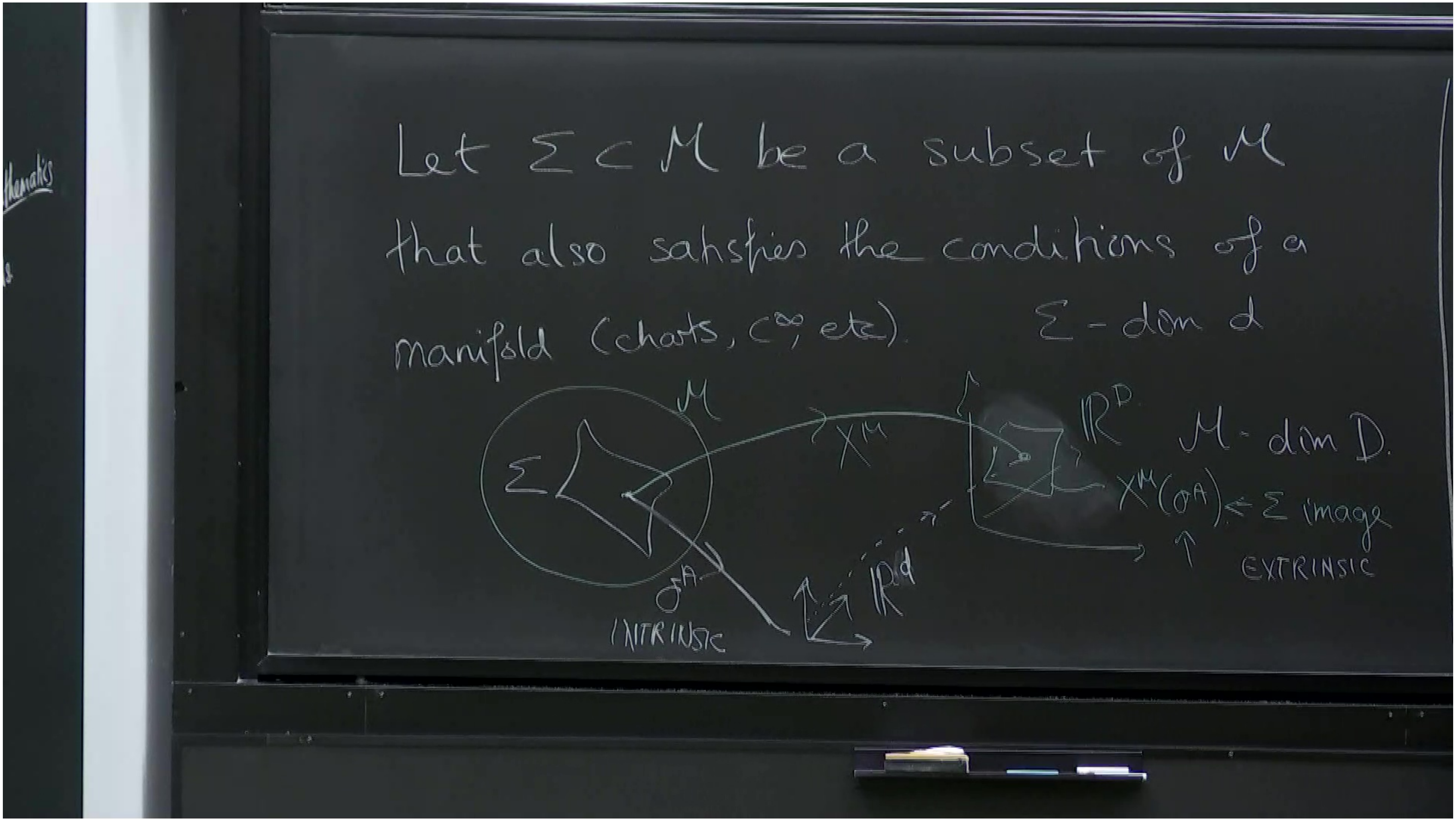

Let $\Sigma \subset \mathcal{M}$ be a subset of $\mathcal{M}$
that also satisfies the conditions of a
manifold (charts, C^∞, etc).
Σ - dim d
$\mathcal{M}$
X^μ
$\mathbb{R}^D$
$\mathcal{M}$ - dim D.
Σ
$X^\mu(\sigma^A) \Leftarrow \Sigma$ image
EXTRINSIC
σ^A
INTRINSIC
$\mathbb{R}^d$

Defn The <u>codimension</u> of Σ in $\mathcal{M}$ is $n = D - d$.

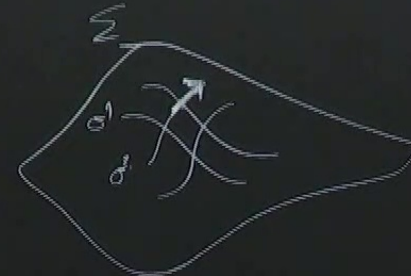
i.e. $\exists$ n lin. indep vectors

$$\underline{n}_i \in T_p(\mathcal{M})$$

s.t. $\underline{n}_i(\sigma^A) = 0 \;\; \forall \sigma^A$

coord fns on Σ.

i.e. $n_i^{\ \mu} \dfrac{\partial X^\nu}{\partial \sigma^A} g_{\mu\nu} = 0$.



...on of Σ

vectors

$= 0 \quad \forall \sigma^A$

coord fns on Σ.

i.e. $$n_i^{\mu} \frac{\partial X^{\nu}}{\partial \sigma^A} g_{\mu\nu} = 0.$$

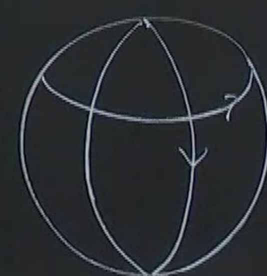
e.g. S^2 $\quad x^2 + y^2 + z^2 = 1.$

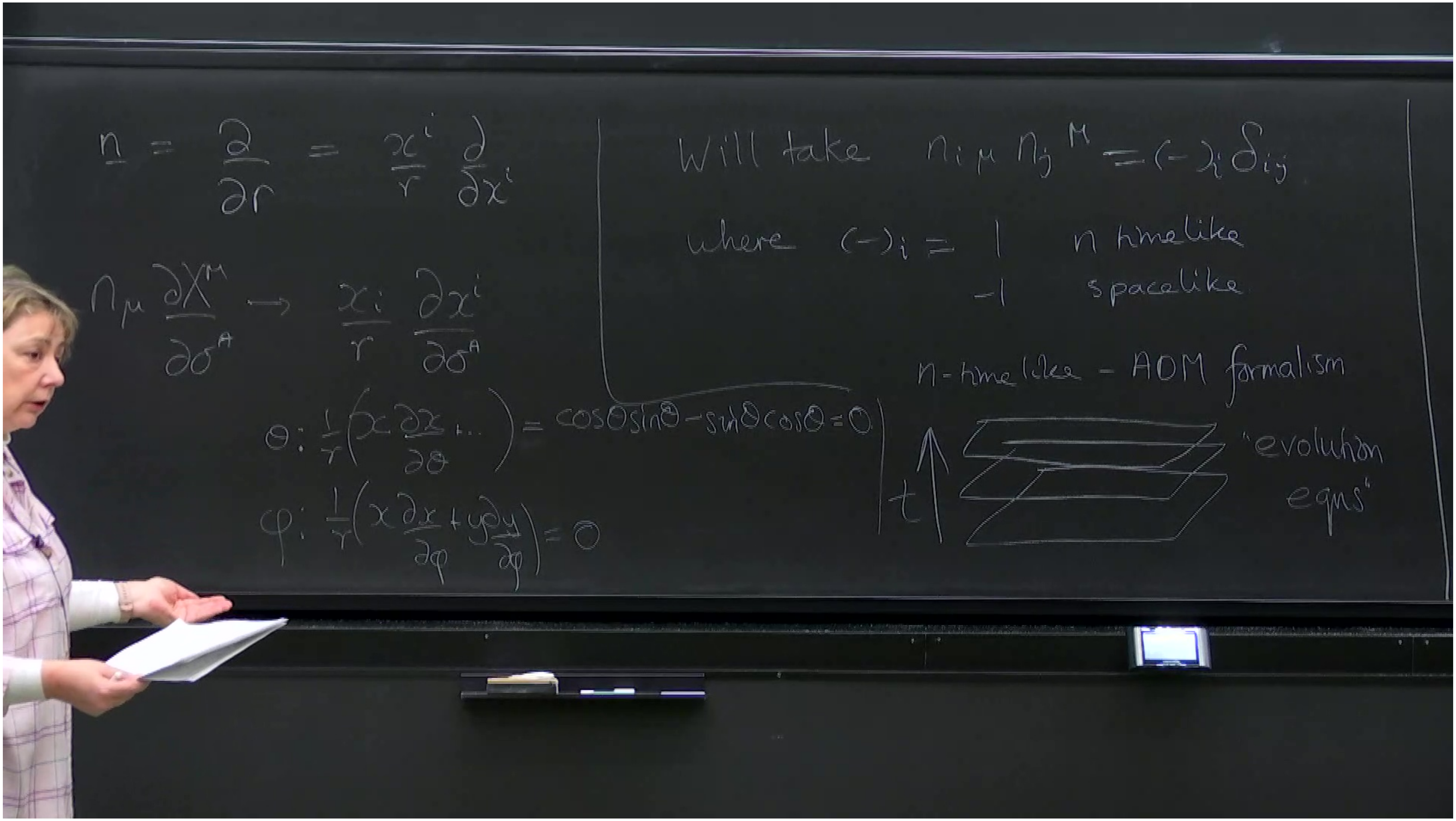
$X^{\mu} \leftrightarrow x, y, z. \qquad \sigma^A \leftrightarrow \theta, \varphi.$

$x = \sin\theta \cos\varphi$

$y = \sin\theta \sin\varphi$

$z = \cos\theta$




n = ∂/∂r = x^i/r ∂/∂x^i
n_μ ∂X^M/∂σ^A → x_i/r ∂x^i/∂σ^A
θ: 1/r (x ∂x/∂θ + ...) = cosθ sinθ − sinθ cosθ = 0
φ: 1/r (x ∂x/∂φ + y ∂y/∂φ) = 0
Will take n_{iμ} n_j^M = (−)_i δ_{ij}
where (−)_i = 1 n timelike
−1 spacelike
n - timelike - ADM formalism
t
"evolution eqns"

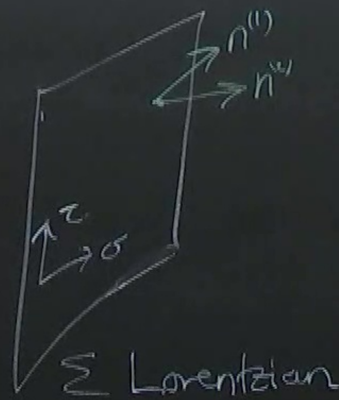
$i \, \delta_{ij}$

ke

ke

formalism

"evolution eqns"



$\underline{n}_i$ spacelike

- string worldsheet

WLOG take $\underline{n}$ spacelike

Arnowitt
Deser
Misner

<u>Defn</u> The 1st fundamental form, or induced metric, of Σ is

$$h_{ab} = g_{ab} + n_{ia} n_{ib}$$

it is the metric Σ inherits from M

Σ also has an <u>intrinsic</u> metric

$$\gamma_{AB} = X^{\mu}{}_{,A}\, X^{\nu}{}_{,B}\, g_{\mu\nu}.$$

$\dfrac{\partial X^{\nu}}{\partial \sigma^{B}}$ projects $T^*_p(M) \to T^*_p(\Sigma)$

inherits from M

trinsic metric

$g_{\mu\nu}$.

$T_p^*(M) \to T_p^*(\Sigma)$

$A, B \leftrightarrow$ intrinsic

a, b or $\mu, \nu \leftrightarrow M$, indices.

Defn The 2^{nd} fundamental form or extrinsic curvature measures how Σ curves in M.

$$K_{i\,\mu\nu} = h_{(\mu}{}^{\sigma} h_{\nu)}{}^{\lambda} \nabla_{(\mu} n_{i\,\lambda)}$$

...mental form

...easures how

$$\ldots \nabla_{(\mu} n_{i\lambda)}$$

or $K_{iAB} = X^{\mu}_{,A} X^{\nu}_{,B} \nabla_\mu n_{i\nu}$

$$= - n_{i\mu} D_A \left(\frac{\partial X^\mu}{\partial \sigma^B} \right)$$

where D is the connection inherited by Σ

$$D_\mu V_\nu = h_\mu{}^\lambda h_\nu{}^\sigma \nabla_\lambda V_\sigma . \quad \left(V_\nu n_i^\nu = 0 \right)$$

from $\mathcal{M}$

metric

$T_p^*(\Sigma)$

$A, B \longleftrightarrow$ intrinsic

a, b or $\mu, \nu \longleftrightarrow \mathcal{M}$, indices.

<u>Defn</u> The 2^{nd} fundamental form or extrinsic curvature measures how Σ curves in $\mathcal{M}$.

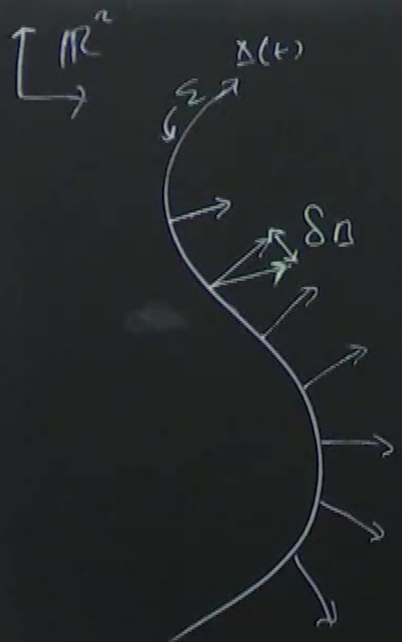
$$K_{i\,\mu\nu} = h_{\mu}{}^{\sigma} h_{\nu}{}^{\lambda} \nabla_{\sigma} n_{i\lambda}$$

or K_{i}

where

by Σ

D_{μ}



<u>Defn</u> The normal fundamental form

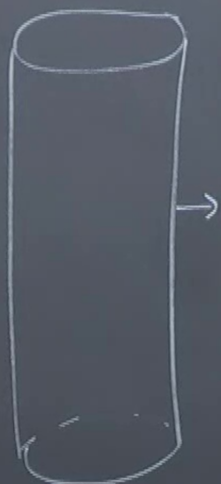
$$\beta_{\mu ij} = n_{i\nu} \nabla_\mu n_j^{\ \nu}.$$

is the connection on the normal bundle of Σ.

orm

ndle of Σ.

e.g. Cylinder. $x^2+y^2=a^2 \subset \mathbb{R}^3$.



$$n = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$h_{ab} = \delta_{ab} - n_a n_b$$

$$= \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

or in cyl polars.

$$\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & a^2 & 0 \\ 0 & 0 & 1 \end{array}\right).$$

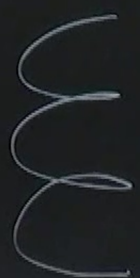
$$K_{ab} = \Gamma^r_{a\,b} \quad \text{(in polars)}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Eg <u>helix</u>



$$\underline{x}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

2 normals

$$\dot{x}(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix}$$

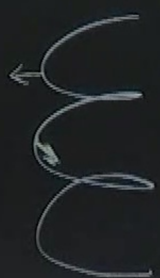
$$\underline{n}_1 = (\cos t, \sin t, 0)$$

$$\underline{n}_2 = (-\sin t, \cos t, -1)$$

$$\gamma_{AB} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$K_{AB} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

Eg <u>helix</u> $\underline{X}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$



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$\underline{n}_1 = \frac{\partial}{\partial \rho}$ in cyl polars

$$\underline{n}_2 = \frac{1}{\rho}\frac{\partial}{\partial \theta} - \frac{\partial}{\partial z}$$

$, 0)$

$t, -1)$

polars

Normal F. Form

$$\beta_a = n_{(1)b} \nabla_a n_{(2)}{}^b$$

$$= n_{(1)\rho} \Gamma_a{}^\rho{}_\theta n_{(2)}{}^\theta$$

$$= \delta_a^\theta (-\rho) \left(\frac{1}{\rho}\right)$$

i.e. $\beta = d\theta$.

Can decompose the curvature of Σ in terms of curvature of M & the embedding.

$$R_{(\Sigma)}{}^a{}_{bcd} = R_{(M)}{}^{a'}{}_{b'c'd'} h^a{}_{a'} h^{b'}{}_b h^{c'}{}_c h^{d'}{}_d + K_i{}^a{}_c K_{i\,bd} - K_i{}^a{}_d K_{i\,bc} \quad \text{(GAUSS)}$$

Normal F. Form

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