Title: Gravitational Physics Lecture - 230123

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Collection: Gravitational Physics (2022/2023)

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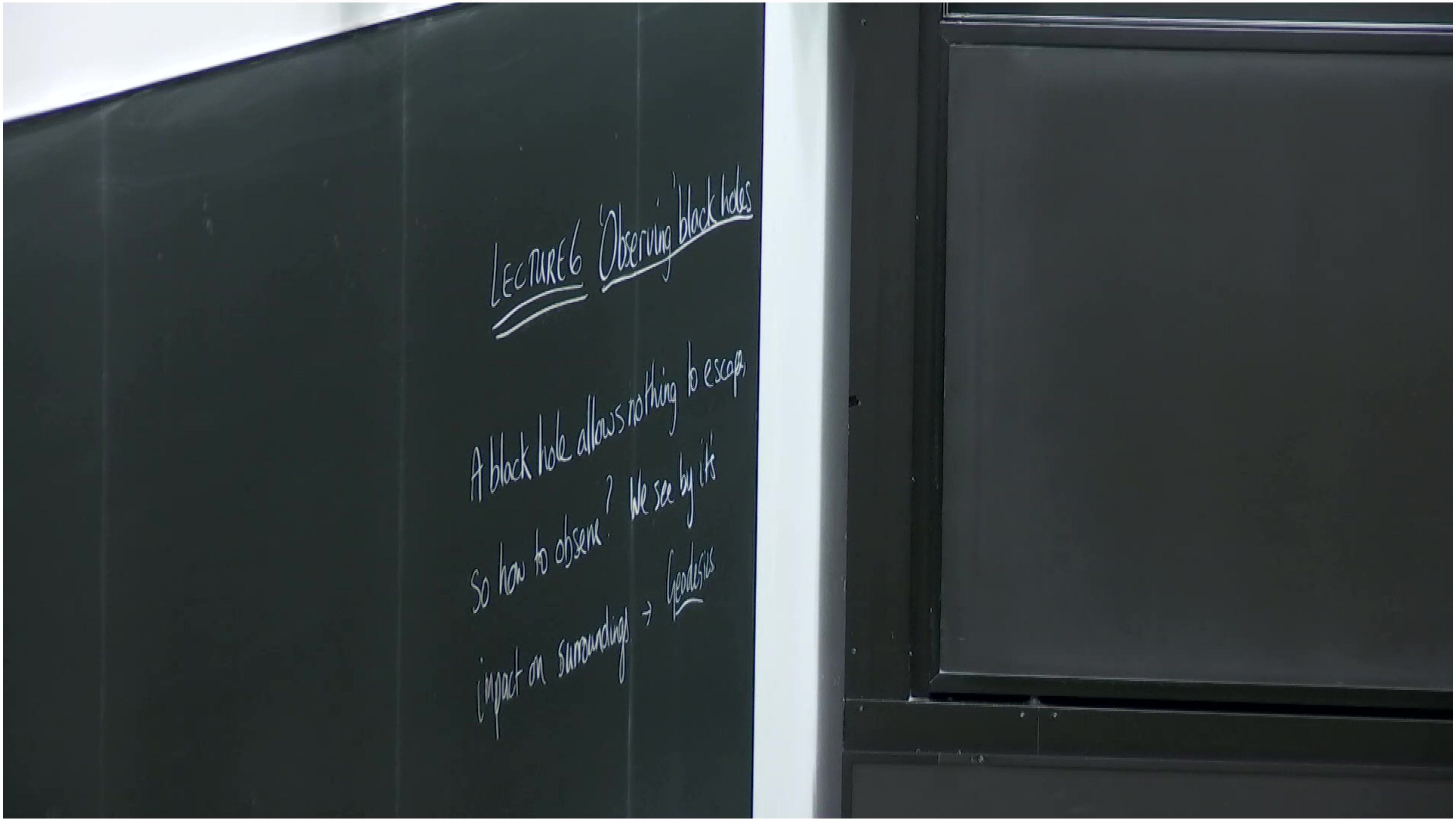
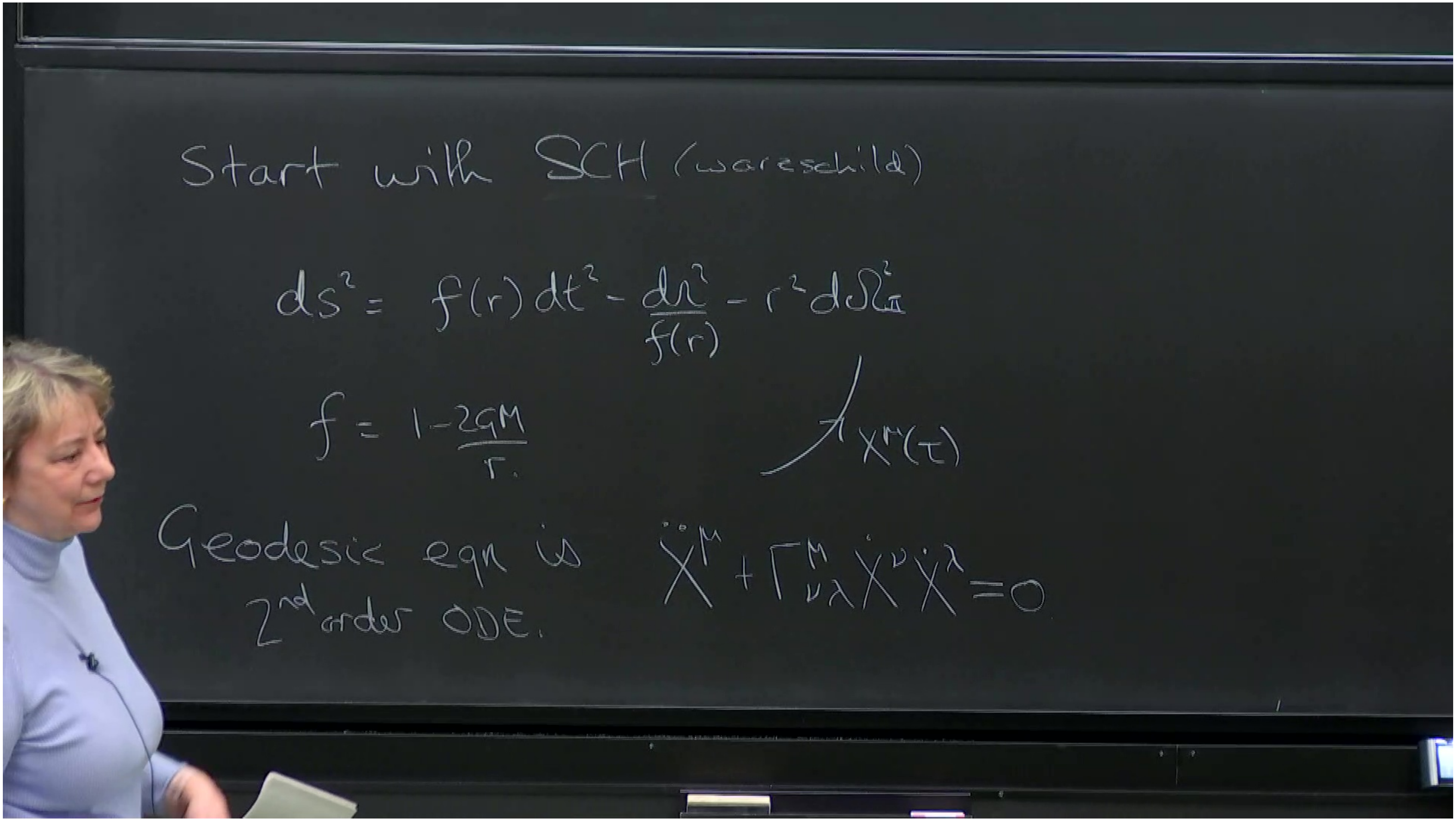

LECTURE 6 'Observing' black holes
A black hole allows nothing to escape,
so how to observe? We see by it's
impact on surroundings → Geodesics


Start with SCH (warzschild)
$ds^2 = f(r)\,dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega_{II}^2$
$f = 1 - \frac{2GM}{r}$.
$x^\mu(\tau)$
Geodesic eqn is $\ddot{x}^\mu + \Gamma^\mu_{\nu\lambda}\dot{x}^\nu \dot{x}^\lambda = 0$
2nd order ODE.

...arzschild)

$- r^2 d\Omega_{II}^2$

$+ \Gamma^\mu_{\nu\lambda} \dot{x}^\nu \dot{x}^\lambda = 0$

Can take short cuts

Let k^μ be a Killing vector

$$\nabla_{(\mu} k_{\nu)} = 0$$

Then

$$\frac{d}{d\tau}\left(k_\mu \dot{x}^\mu\right) = \dot{x}^\nu \nabla_\nu \left(k_\mu \dot{x}^\mu\right)$$

$$= \dot{x}^\nu \dot{x}^\mu \underbrace{\cancel{\nabla_\nu k_\mu}}_{\text{k.v.}} + k_\nu \underbrace{\cancel{\nabla_{\dot{x}} \dot{x}^\nu}}_{\text{geod.}}$$

hort cuts.

a Killing vector.

$$) = \dot{X}^\nu \nabla_\nu (k_\mu \dot{X}^\mu)$$

$$= \dot{X}^\nu \dot{X}^\mu \underbrace{\cancel{\nabla_\nu k_\mu}}_{k.v.} + k_\nu \underbrace{\cancel{\nabla_X \dot{X}^\nu}}_{geod.} = 0.$$

i.e. $k_\mu \dot{X}^\mu$ conserved along geodesic.

SCH: $k = \frac{\partial}{\partial t}$ $\quad g_{tt} \dot{X}^t = \text{const}$

or $f \dot{t} = E$. "

$k = \frac{\partial}{\partial \varphi} \rightarrow r^2 \dot{\varphi} = h$, const.

Choose "z-axis" to have geodesic in equatorial plane: $\theta = \pi/2$ (if $\dot{\theta} = 0$ initially, it remains so).

Now,

$$\frac{d}{d\tau}\left(g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}\right) = \dot{X}^{\sigma}\nabla_{\sigma}\left(g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu}\right)$$

$$= 0$$

i.e. $g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = \eta$ conserved

...ave geodesic

$\theta = \pi/2$ (if $\dot\theta = 0$

).

$\dot{X}^\sigma \nabla_\sigma (g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu)$

0

conserved

$\eta = 0 \longleftrightarrow$ null geodesic

$\eta = 1 \longleftrightarrow$ timelike "

Our geodesic problem reduces to

$$-g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = -\eta = \frac{\dot{r}^2}{f} - \frac{E^2}{f} + \frac{h^2}{r^2}.$$

or $\dot{r}^2 + V_{eff}(r) = 0$

geodesic

like

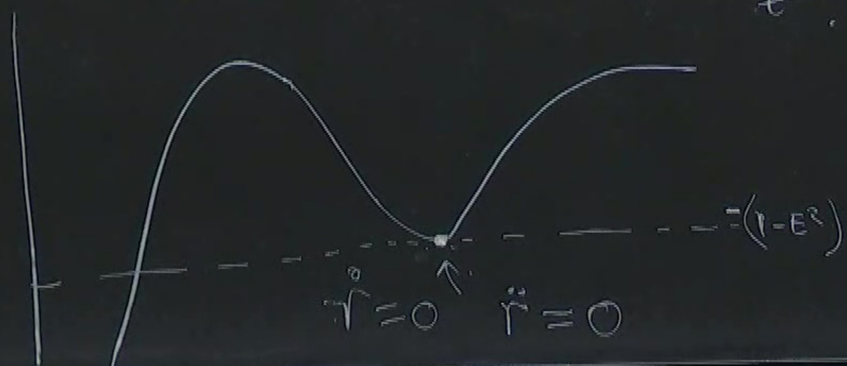
m reduces to

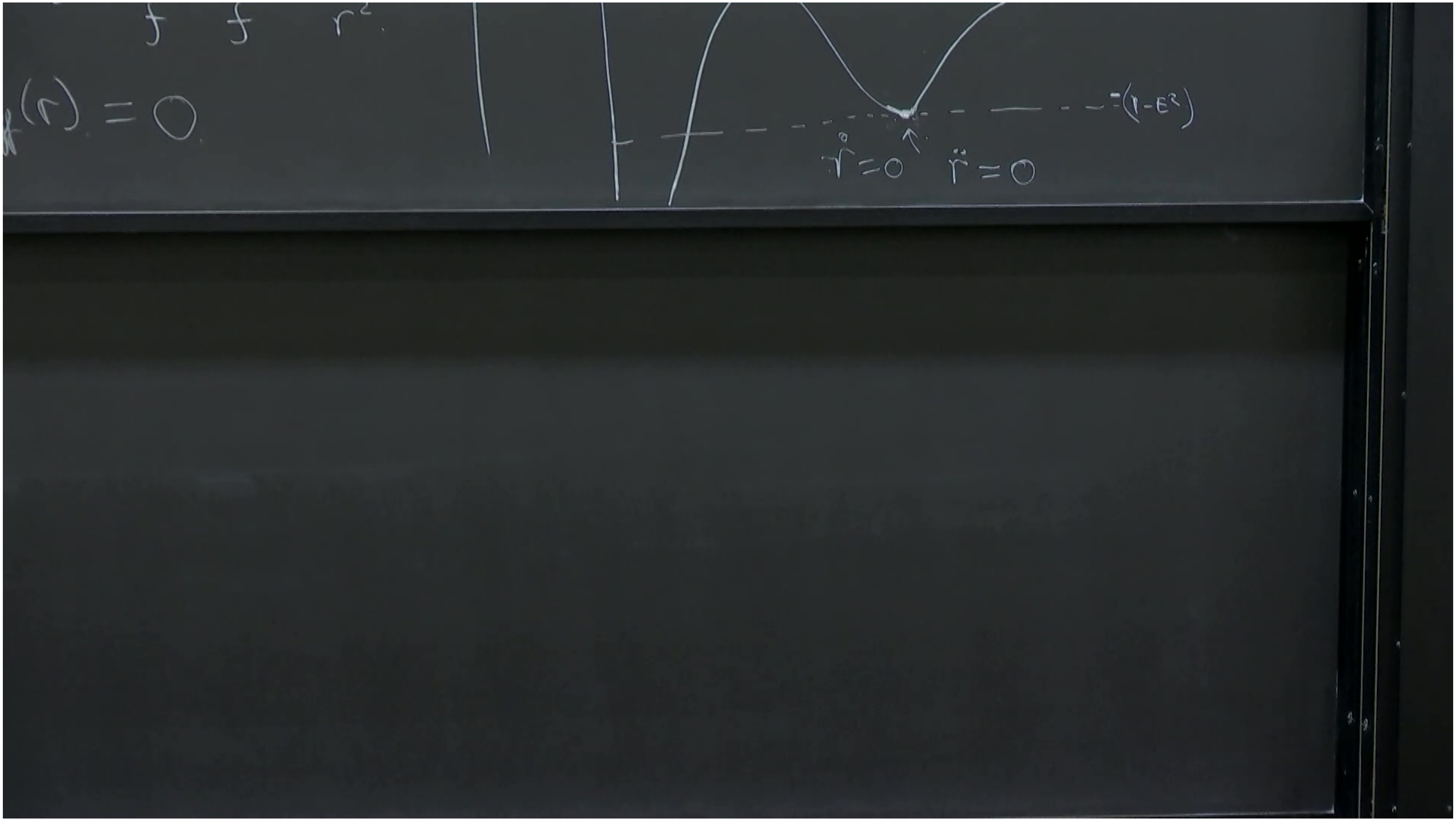
$$\frac{\dot{r}^2}{f} - \frac{E^2}{f} + \frac{h^2}{r^2} = 0$$

$$V_{eff}(r) = \underbrace{\eta - E^2}_{\text{sets zero}} \underbrace{- \frac{2GM\eta}{r}}_{\text{Newtonian}} + \underbrace{\frac{h^2}{r^2}}_{\text{centrifugal}} \underbrace{- \frac{2GMh^2}{r^3}}_{\text{G.R.}}$$

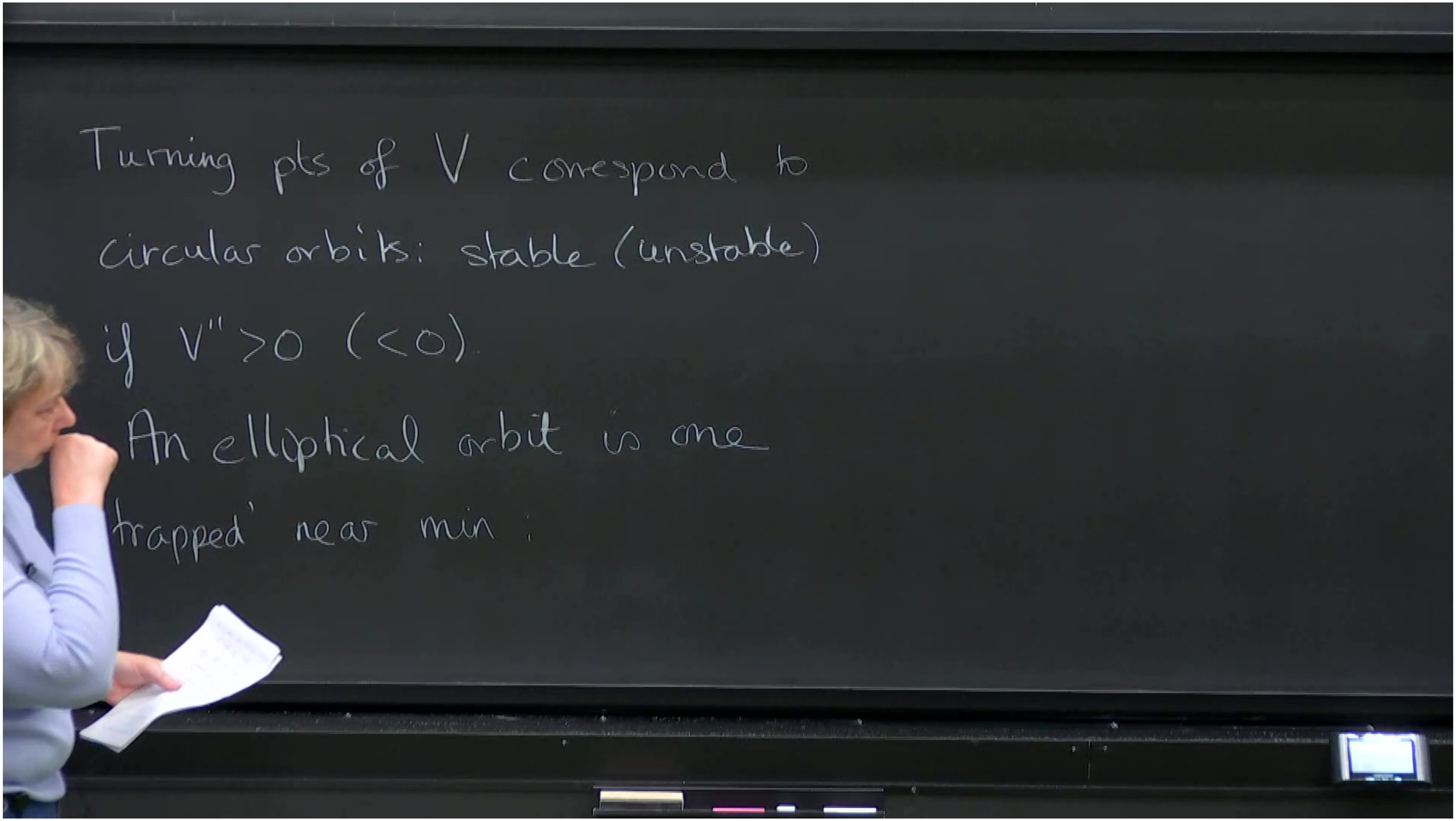
is an effective potential for the r-motion.

$\eta = 1$




f f r²
f(r) = 0
-(1-E²)
ṙ=0
r̈=0

Turning pts of V correspond to circular orbits: stable (unstable) if $V'' > 0$ (< 0).


Turning pts of V correspond to
circular orbits: stable (unstable)
if V'' > 0 (< 0)
An elliptical orbit is one
'trapped' near min :

f V correspond to
ks: stable (unstable)
<0)
al orbit is one
min :

For circular orbit

$$V = 1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3} = 0$$

$$V' = \underbrace{\frac{2GM}{r^2} - \frac{2h^2}{r^3} + \frac{6GM}{r^4}}_{h^2 = \frac{GMr^2}{(r-3GM)}} = 0$$

$$\ldots \frac{GMh^2}{r^3} = 0$$

$$\ldots = 0$$

2 constraints - give E & h in terms of r & M. Decreasing r - of circular orbit - decreases E & h and the max/min move closer together. At a critical value, they merge to a pt of inflection $V''=0$

$$V'' = -\frac{4GM}{r^3} + \frac{6h^2}{r^4} - \frac{24GMh^2}{r^5} = 0 \text{ at } r_{crit}$$

For circular orbit

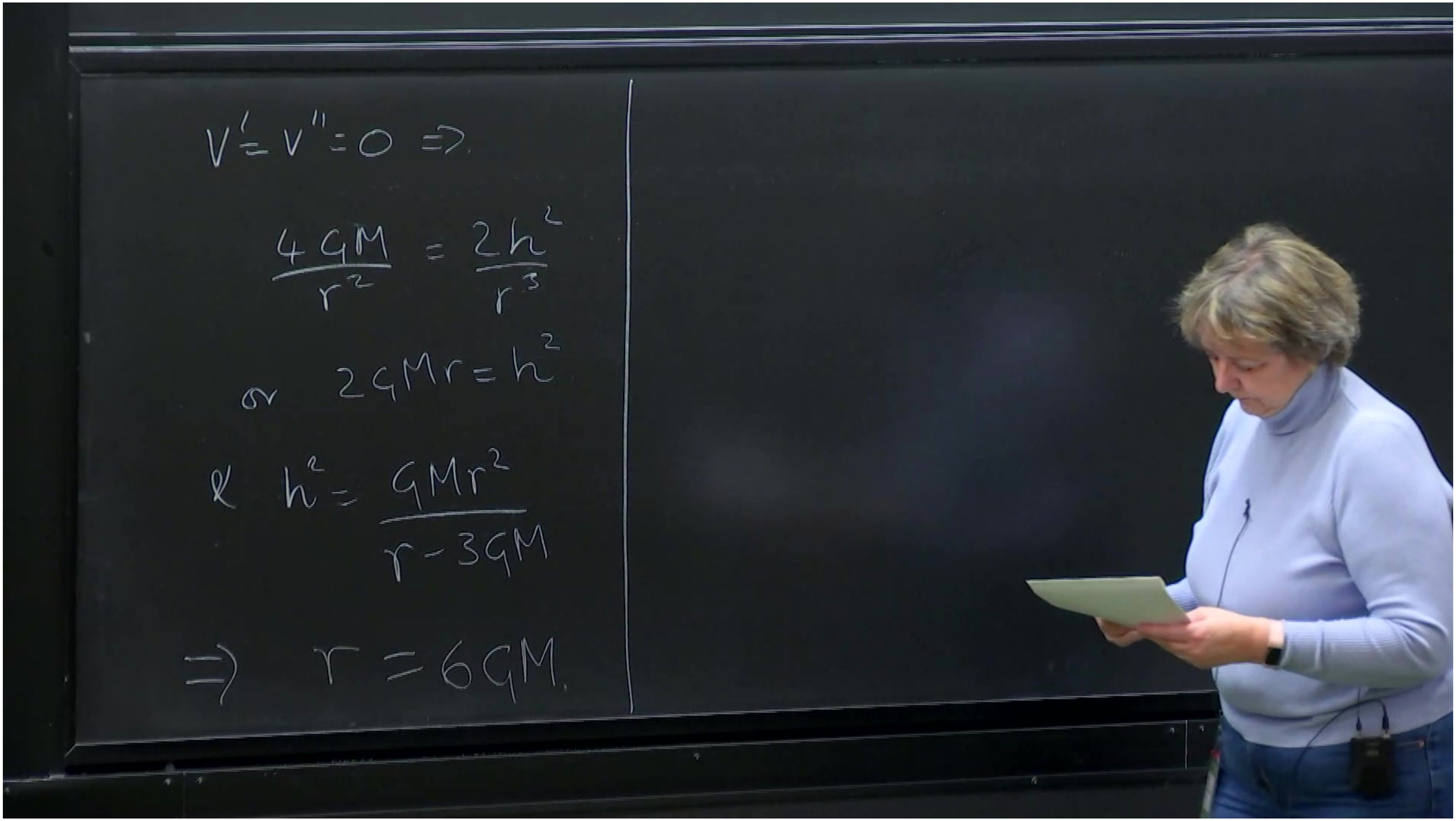
$$V = 1 - E^2 - \frac{2GM}{r} + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3} = 0$$

$$V' = \underbrace{\frac{2GM}{r^2} - \frac{2h^2}{r^3} + \frac{6GM}{r^4} = 0}$$

$$h^2 = \frac{GMr^2}{(r - 3GM)}$$

2 constraints - give
of r & M. Decreasing
E & h and the max
together. At a critical
to a pt of inflection

$$V'' = -\frac{4GM}{r^3} + \frac{6h^2}{r^4}$$



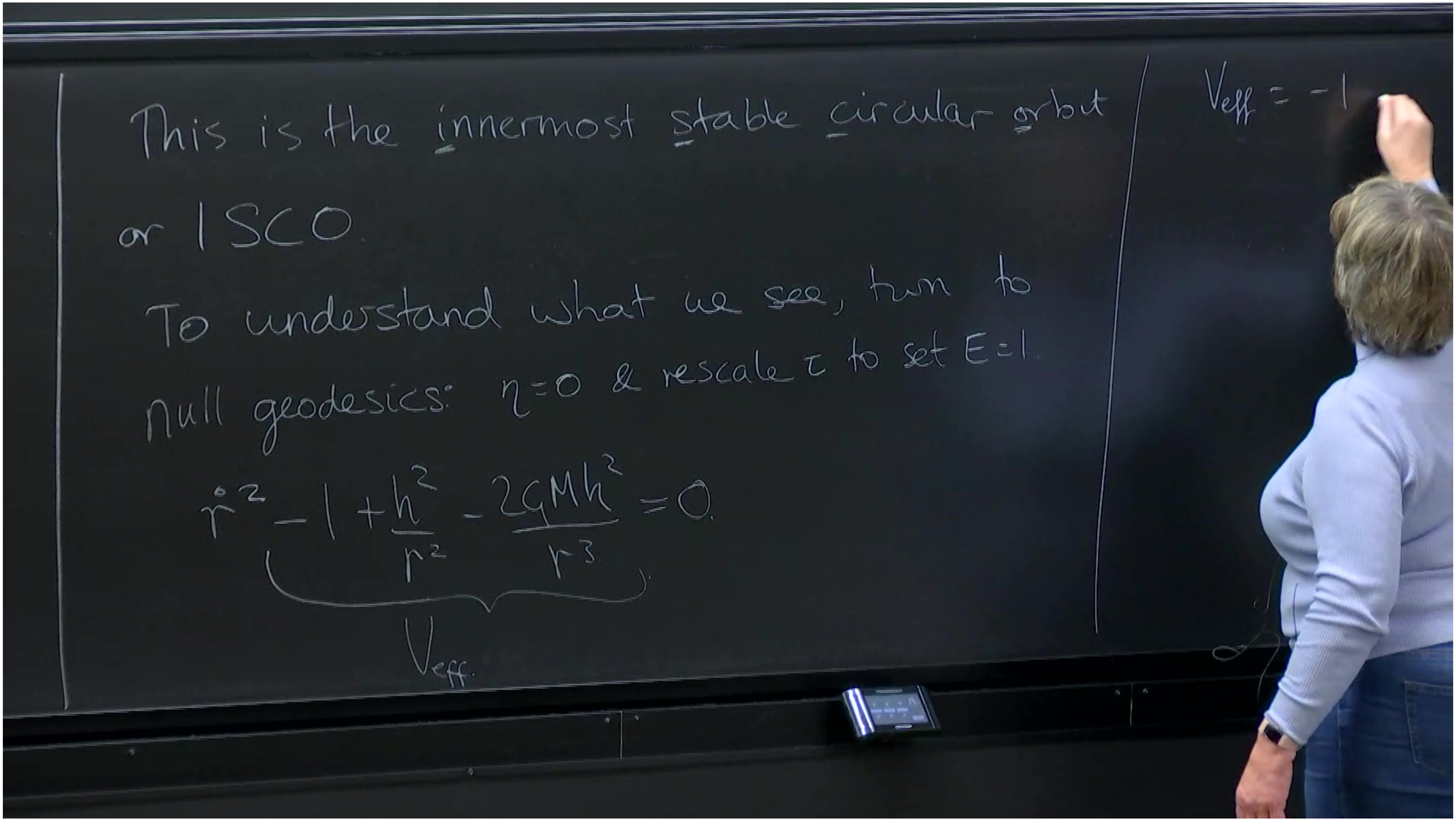
$V' = V'' = 0 \Rightarrow$

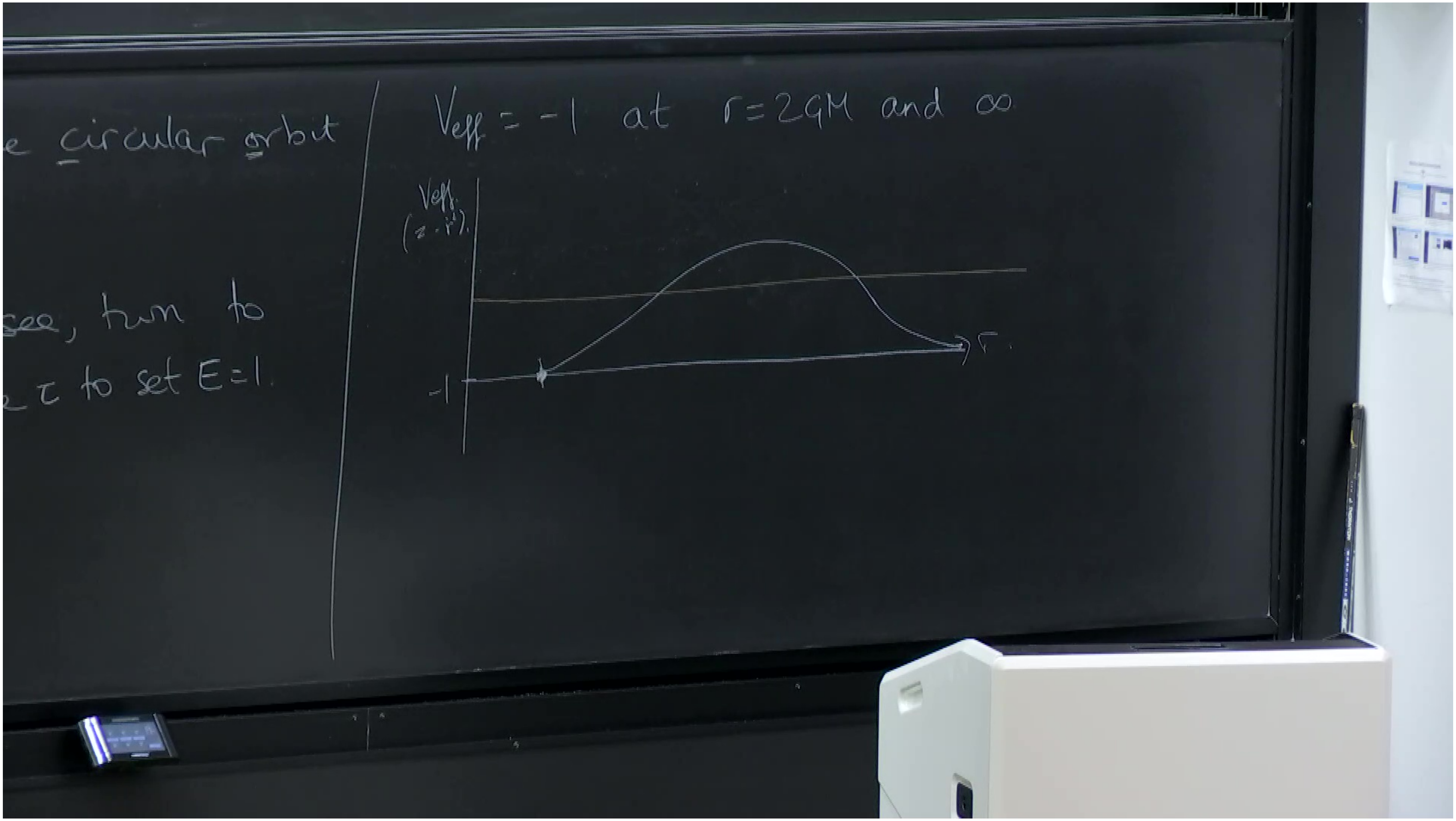
$\frac{4GM}{r^2} = \frac{2h^2}{r^3}$

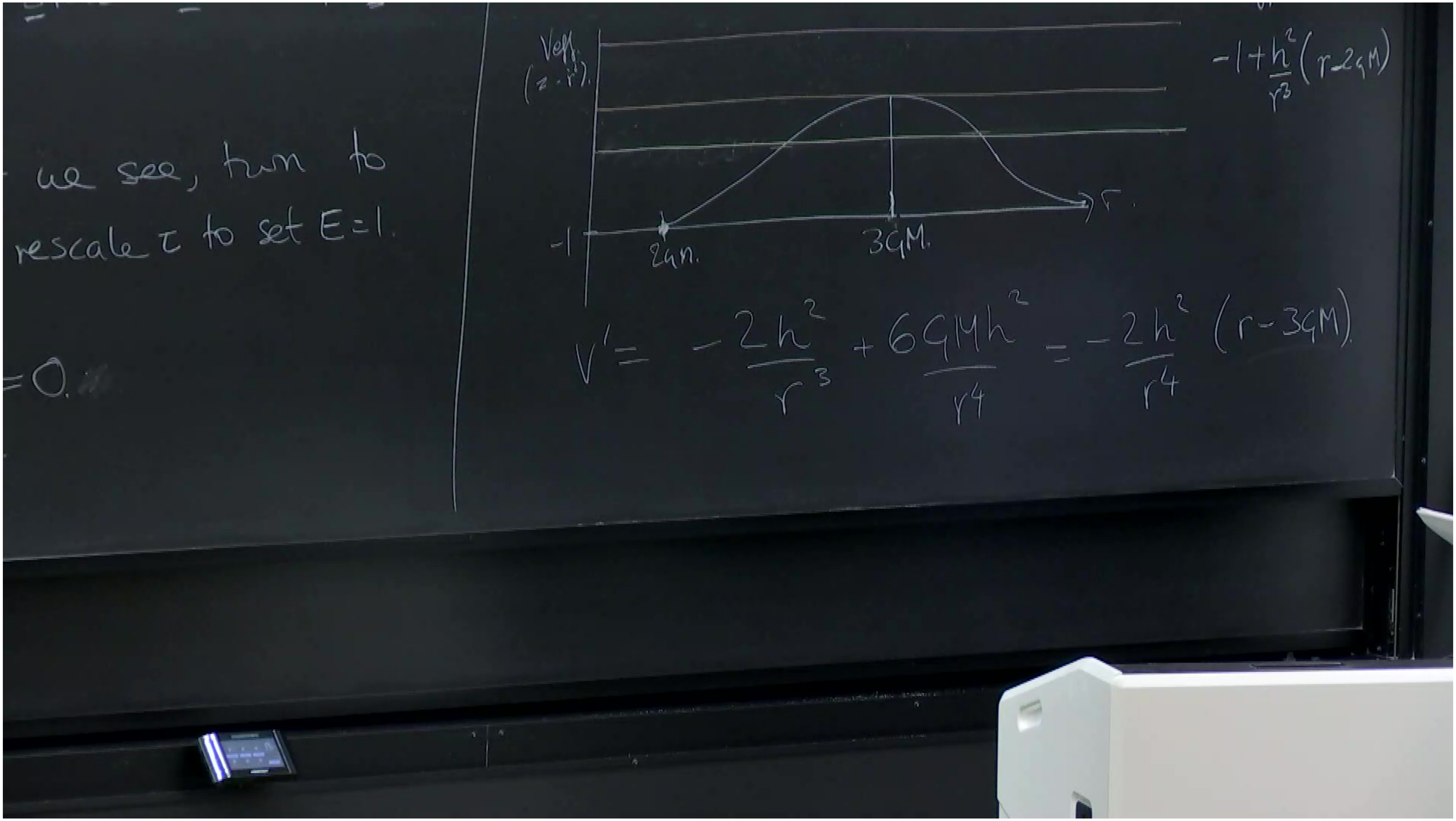
or $2GMr = h^2$

& $h^2 = \frac{GMr^2}{r-3GM}$

$\Rightarrow r = 6GM$


This is the innermost stable circular orbit
or ISCO.
To understand what we see, turn to
null geodesics: $\eta=0$ & rescale τ to set $E=1$.
$\underbrace{\dot{r}^2 - 1 + \frac{h^2}{r^2} - \frac{2GMh^2}{r^3}}_{V_{eff}} = 0.$
$V_{eff} = -1$


e circular orbit
see, turn to
e τ to set E=1.
Veff = -1 at r=2GM and ∞
Veff
-1
r


we see, turn to
rescale τ to set E=1
=0.
Veff
-1
2GM.
3GM.
r.
V' = -2h²/r³ + 6GMh²/r⁴ = -2h²/r⁴ (r-3GM)
-1 + h²/r³ (r-2GM)

$$V'' = \frac{6h^2}{r^4} - 24\frac{GMh^2}{r^5} = \frac{6h^2}{r^5}(r - 4GM)$$

< 0 at $3GM$.

Fixing $h^2 = 27G^2M^2$ gives

this unstable null orbit. (orange)

M)

Consider null ray from ∞

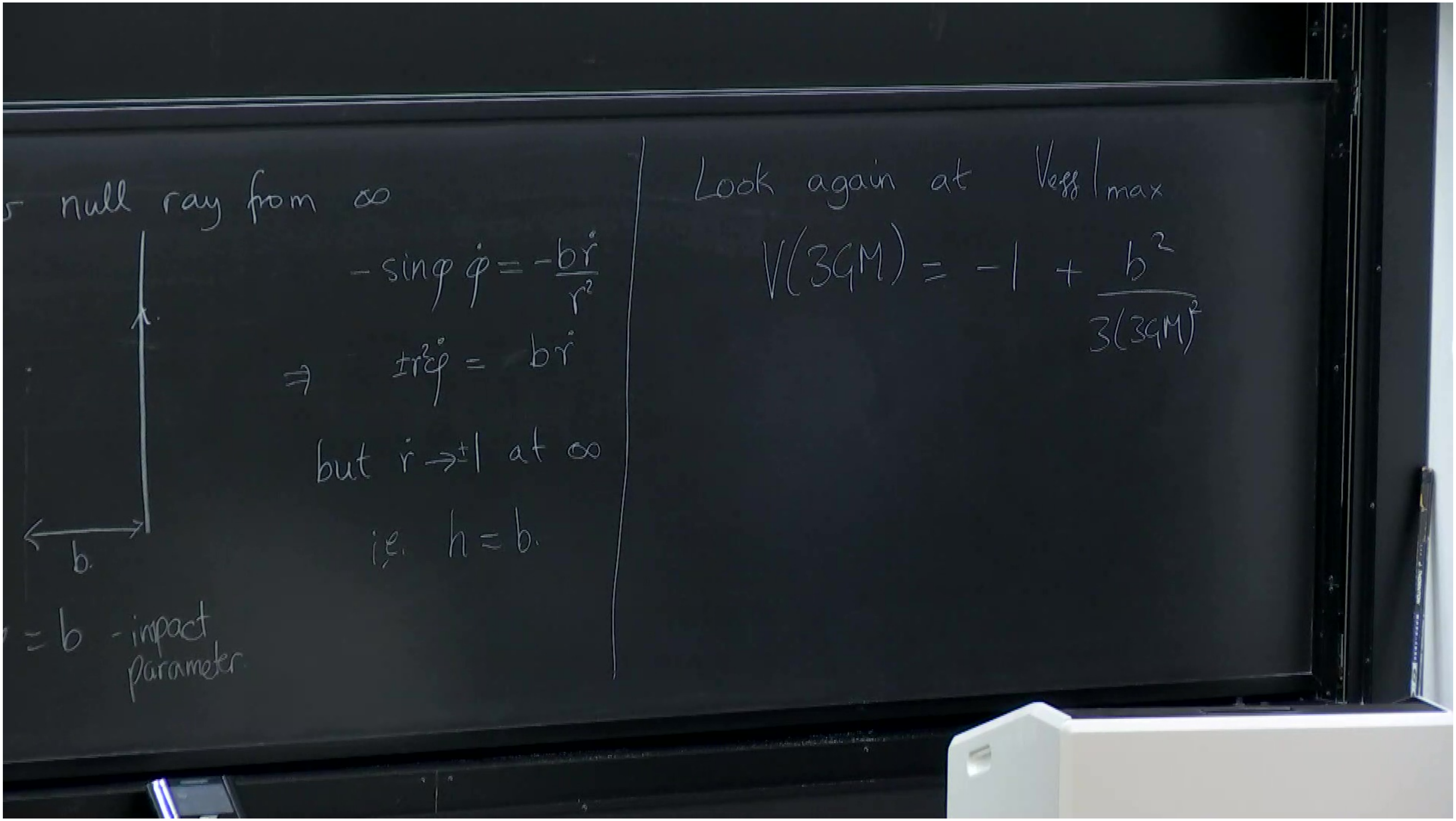
$$-\sin\varphi\,\dot{\varphi} = -\frac{b\dot{r}}{r^2}$$

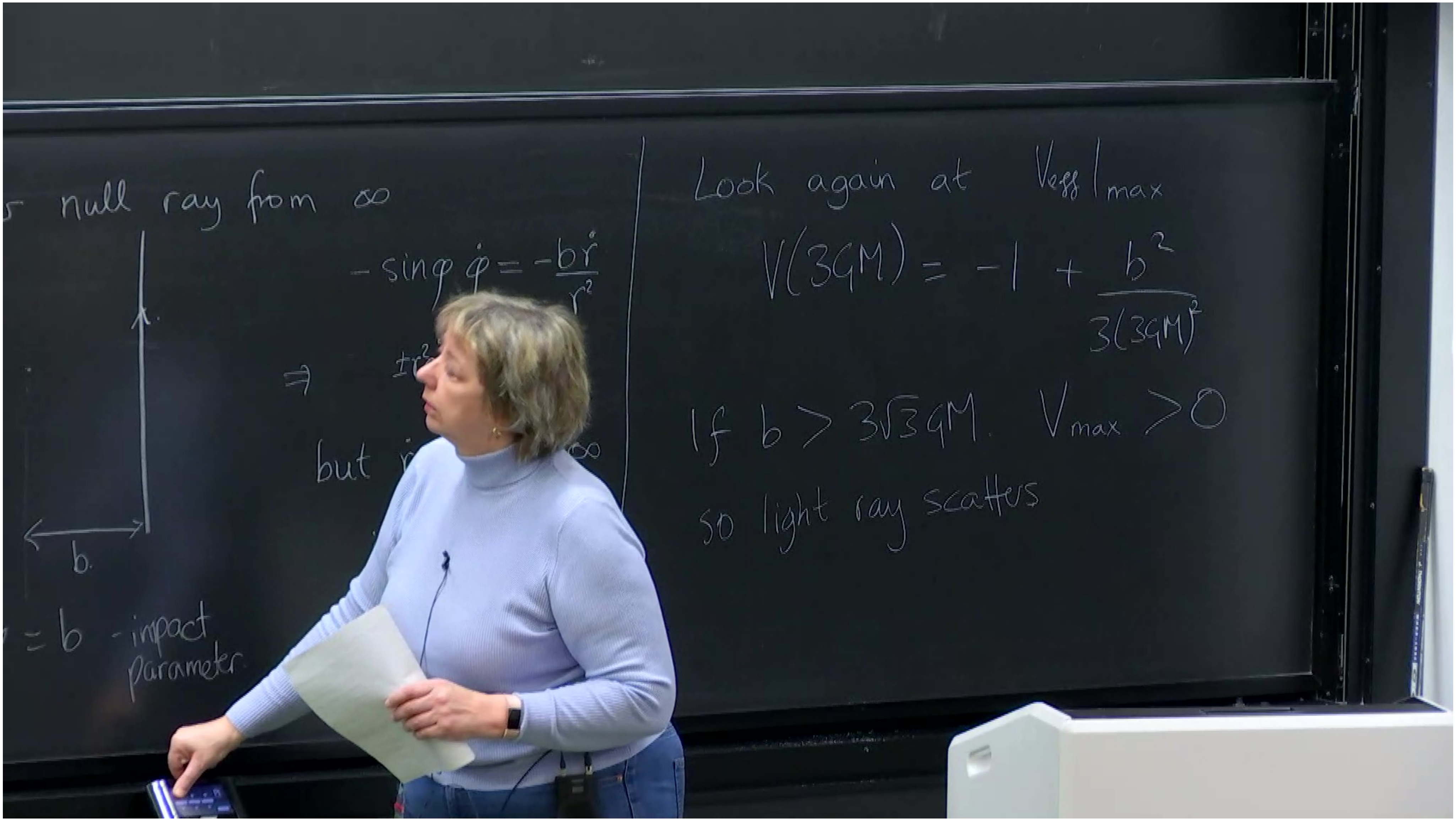
$$\Rightarrow \quad \pm r^2\dot{\varphi} = b\dot{r}$$

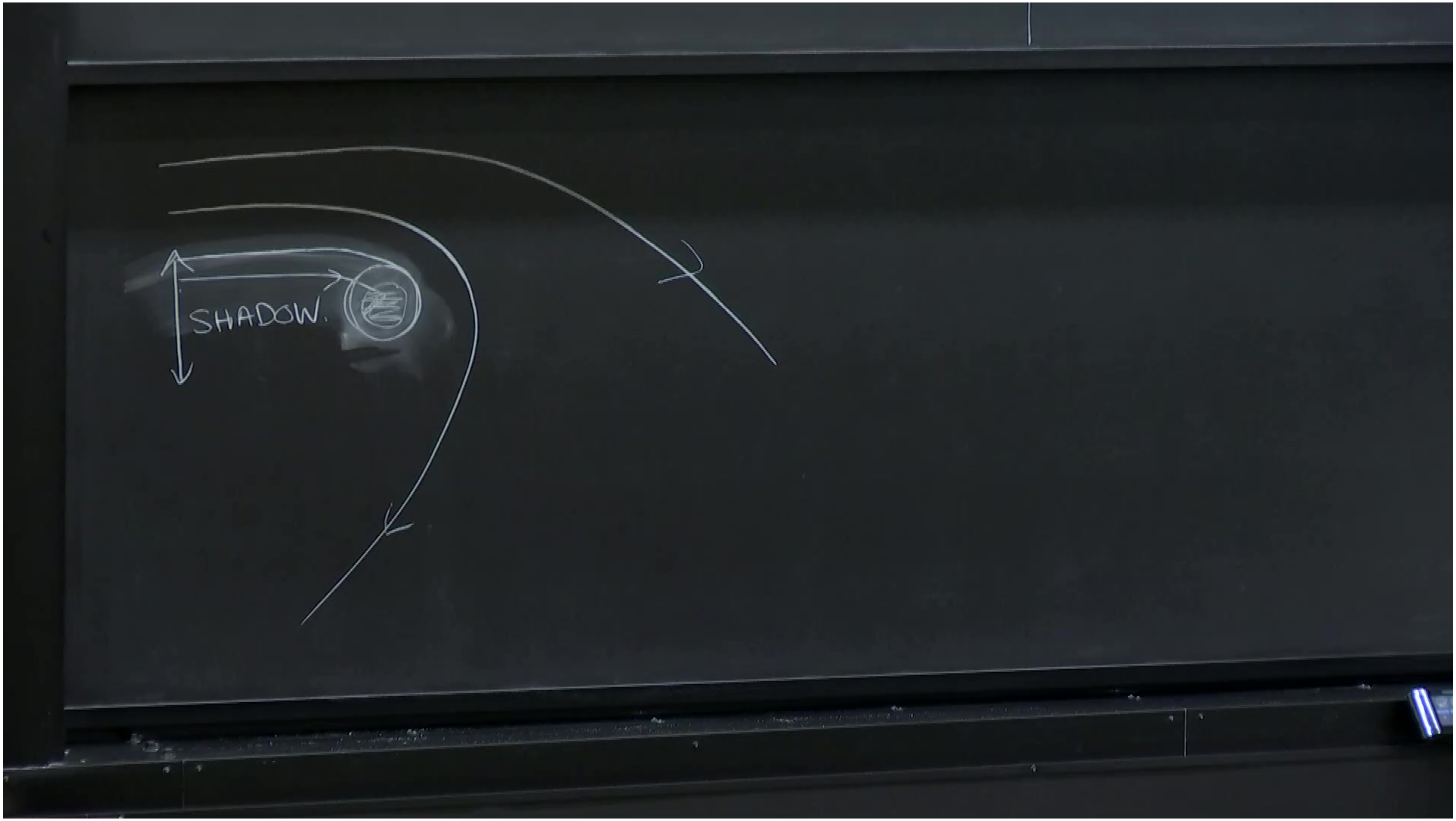
but $\dot{r} \to \pm 1$ at ∞

i.e. $h = b$.

$x = r\cos\varphi = b$ - impact parameter


null ray from ∞
$-\sin\varphi\,\dot{\varphi} = -\frac{b\dot{r}}{r^2}$
$\Rightarrow \pm r^2\dot{\varphi} = b\dot{r}$
but $\dot{r} \to \pm 1$ at ∞
i.e. $h = b$.
b.
$= b$ - impact parameter
Look again at $V_{eff}|_{max}$
$V(3GM) = -1 + \frac{b^2}{3(3GM)^2}$


null ray from ∞
- sin φ φ̇ = -b ṙ / r²
but
b
= b - impact parameter
Look again at V_eff |_max
V(3GM) = -1 + b² / 3(3GM)²
If b > 3√3 GM V_max > 0
so light ray scatters


SHADOW.

Kerr

$$ds^2 = \frac{\Delta}{\Sigma}\left[dt - a\sin^2\theta\, d\varphi\right]^2 - \frac{\sin^2\theta}{\Sigma}\left[a\,dt - (r^2+a^2)\,d\varphi\right]^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma\, d\theta^2$$

here $\Sigma = r^2 + a^2\cos^2\theta$ $\qquad a = J/M$

$\Delta = r^2 - 2GMr + a^2$