Title: Gravitational Physics Lecture - 230116

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Collection: Gravitational Physics (2022/2023)

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In G.R. we use the metric connection, w

& torsion free

- $\underline{\nabla}\, g = 0$

- $\underline{T}(\underline{u},\underline{v}) = \nabla_{\underline{u}} \underline{v} - \nabla_{\underline{v}} \underline{u} - [\underline{u},\underline{v}] = 0$

$$T^a_{\ bc} = \Gamma^a_{\ bc} - \Gamma^a_{\ cb} - C^a_{\ bc}$$

ion,

where $C^a_{\ bc} = \langle \underline{\omega}^a | [\underline{e}_b, \underline{e}_c] \rangle$

are the structure constants.

ion,

where $C^{a}_{\ bc} = \langle \underline{\omega}^{a} | [\underline{e}_b, \underline{e}_c] \rangle$

are the structure constants.

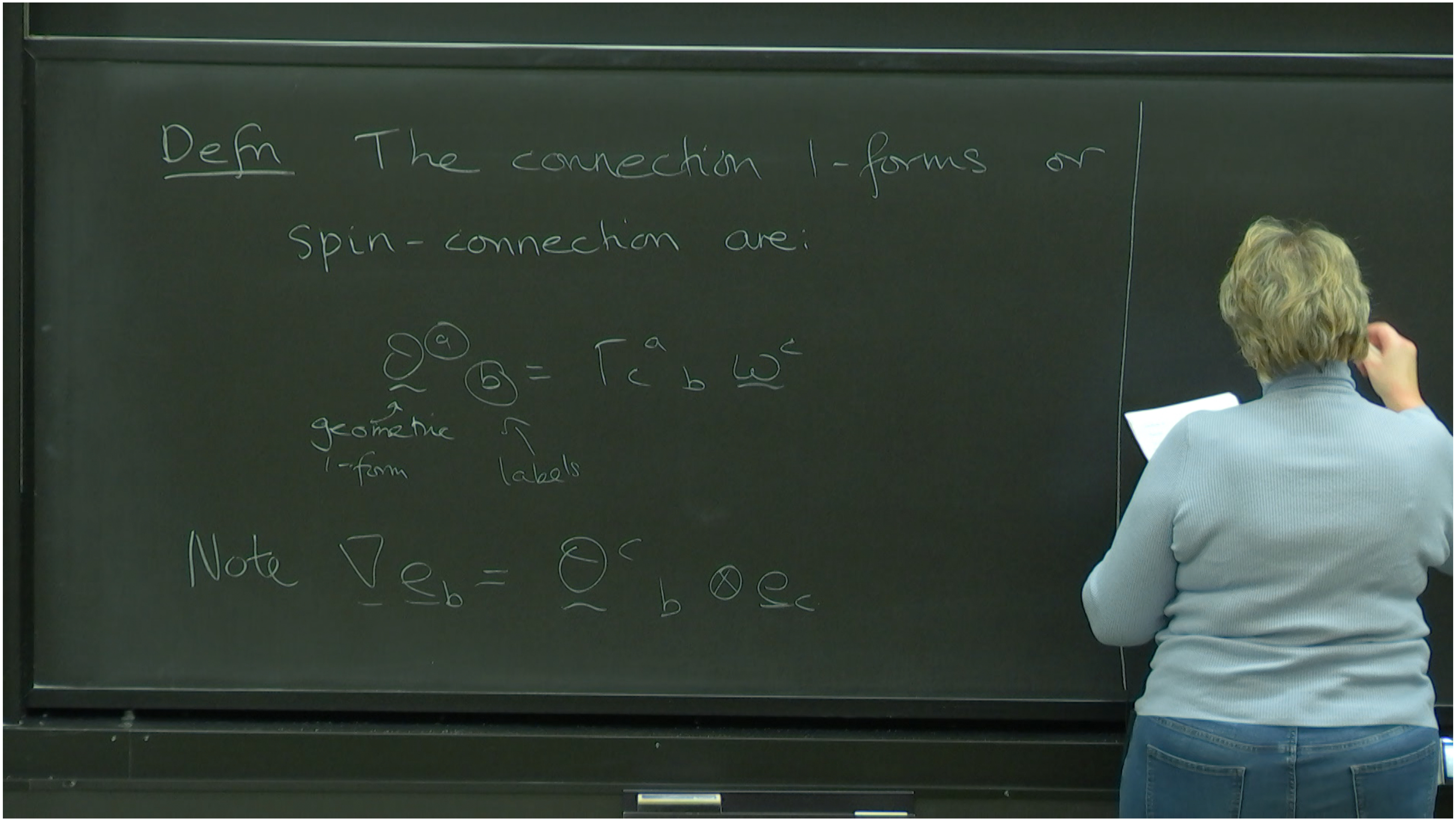
These vanish in a coord basis & get familiar

Levi-Civita connection / Christoffels.

$$\Gamma^{\mu}_{\ \nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma} \right)$$

<u>Defn</u> The connection 1-forms or spin-connection are:

$$\underset{\sim}{\Omega}^a{}_b = \Gamma_c{}^a{}_b \, \underset{\sim}{\omega}^c$$


Defn The connection 1-forms or
spin-connection are:
$\Theta^{a}{}_{b} = \Gamma_{c}{}^{a}{}_{b}\, \omega^{c}$
geometric
1-form
labels
Note $\nabla e_b = \Theta^{c}{}_{b} \otimes e_c$

For metric connection

$$\underline{\Theta}_{ac} + \underline{\Theta}_{ca} = \underline{d}\, g_{ac}$$

For metric connection

$$\underline{\omega}_{ac} + \underline{\omega}_{ca} = \underline{d}\, g_{ac}$$

<u>Proof</u>

$$\underline{d}(g_{ab}) = \underline{\nabla}(g_{ab})$$

$$= \underline{\nabla}(\langle \underline{g} | \underline{e}_a, \underline{e}_b \rangle) \quad =$$

nection

$\underline{d}\, g_{ac}$

$b)$

$g | \underline{e}_a, \underline{e}_b \rangle)$

$$= \langle \underline{g} | \nabla \underline{e}_a, \underline{e}_b \rangle + \langle \underline{g} | \underline{e}_a, \nabla \underline{e}_b \rangle$$

$$= \underline{\omega}^d \langle \underline{g} | \Gamma^c_{d\,a} \underline{e}_c, \underline{e}_b \rangle$$

$$+ \underline{\omega}^d \langle \underline{g} | \underline{e}_a, \Gamma^c_{d\,b} \underline{e}_c \rangle$$

nection

$\underline{d}\, g_{ac}$

$b)$

$g|\underline{e}_a, \underline{e}_b\rangle)$

$$= \langle \underline{g} | \nabla \underline{e}_a, \underline{e}_b \rangle + \langle \underline{g} | \underline{e}_a, \nabla \underline{e}_b \rangle$$

$$= \underset{\sim}{\omega}^d \langle \underline{g} | \Gamma^c{}_{da}\, \underline{e}_c, \underline{e}_b \rangle$$

$$+ \underset{\sim}{\omega}^d \langle \underline{g} | \underline{e}_a, \Gamma^c{}_{db}\, \underline{e}_c \rangle$$

$$= \underset{\sim}{\Theta}^c{}_a\, g_{cb} + \underset{\sim}{\Theta}^c{}_b\, g_{ac}$$

$$= \underset{\sim}{\Theta}_{ba} + \underset{\sim}{\Theta}_{ab}.$$

$$\underline{e}_a \to \underline{e}'_b = \Lambda_b{}^a \, \underline{e}_a$$

$$\downarrow$$

$$e_a{}^\mu \frac{\partial}{\partial x^\mu}$$

An o/n basis has

$$g_{ab} = \eta_{ab} \overset{\text{diag.}}{=} (+1, -1, -1, -1)$$

<u>Cartan's 1st eqn</u>

Let the torsion 2-form be

$$\underset{\sim}{T}^a = \frac{1}{2} T^a{}_{bc}\, \underset{\sim}{\omega}^b \wedge \underset{\sim}{\omega}^c$$

$$\underset{\sim}{T}^a =$$

$$= \nabla(\langle g | e_a, e_b \rangle)$$

$$= \Theta^c{}_a g_{cb} + \Theta^c{}_b g_{ac}$$

$$= \Theta_{ba} + \Theta_{ab}.$$

$$T^a = \frac{1}{2}\left(\Gamma_b{}^a{}_c - \Gamma_c{}^a{}_b - C^a_{bc}\right) \omega^b \wedge \omega^c$$

$$= \Gamma_b{}^a{}_c\, \omega^b \wedge \omega^c - \frac{1}{2} \underbrace{\langle \omega^a | [e_b, e_c] \rangle}\, \omega^b \wedge \omega^c$$

$$= \Theta^a{}_c \wedge \omega^c \qquad + \frac{1}{2} \langle d\omega^a | e_b, e_c \rangle\, \omega^b \wedge \omega^c$$

Recall: $\langle d\omega^a | e_b, e_c \rangle = e_b(\cancel{\underbrace{\langle \omega^a | e_c \rangle}_{\delta^a_c}}) - e_c(\cancel{\langle \omega^a | e_b \rangle}^{\delta^a_b}) - \langle \omega^a | [e_b, e_c] \rangle.$

i.e. $\boxed{\underline{T}^a = \underline{d}\,\underline{\omega}^a + \underline{\Theta}^a{}_c \wedge \underline{\omega}^c}$

i.e. $\boxed{\underline{T}^a = \underline{d}\,\underline{\omega}^a + \underline{\Theta}^a{}_c \wedge \underline{\omega}^c}$

In GR, $\underline{T}^a = 0$, $\underline{g}$ metric, choose o/n basis so $\underline{\Theta}_{ab}$ antisymmetric i.e. $\boxed{\underline{T}^a = d\underline{\omega}^a + \underline{\Theta}^a{}_c \wedge \underline{\omega}^c}$

In GR, $\underline{T}^a = 0$, g metric, choose o/n basis so Θ_{ab} antisymmetric.

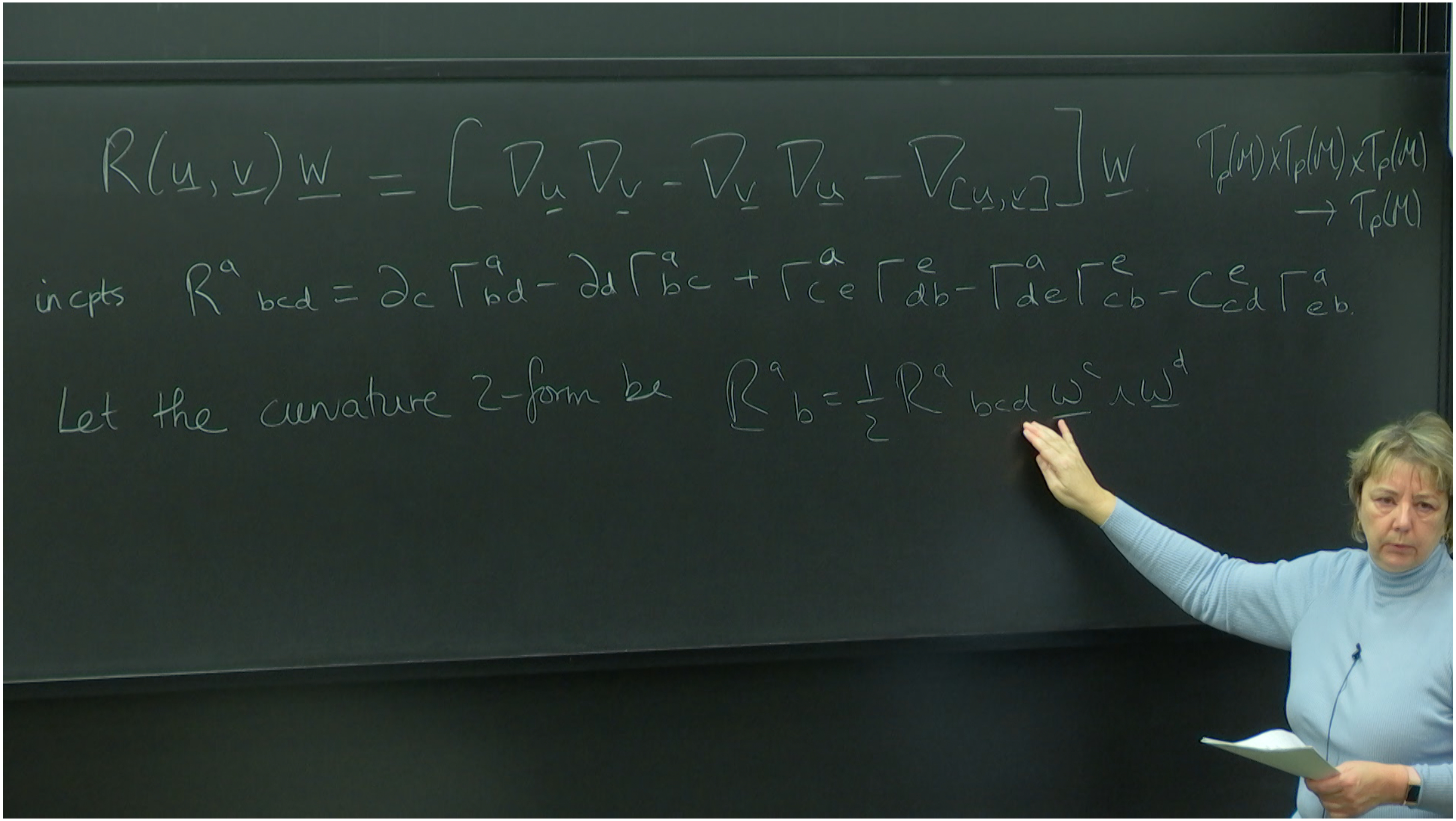
Cartan - 2

Curvature is defined via a commutator of the covariant derivs.

$$R(\underline{u},\underline{v})\underline{w} = \left[\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right]\underline{w}$$

$$T_p(M)\times T_p(M)\times T_p(M) \to T_p(M)$$

in cpts $$R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}.$$


$R(\underline{u},\underline{v})\underline{W} = [D_{\underline{u}} D_{\underline{v}} - D_{\underline{v}} D_{\underline{u}} - D_{[\underline{u},\underline{v}]}]\underline{W}$
$T_p(M) \times T_p(M) \times T_p(M) \to T_p(M)$
in cpts $R^a{}_{bcd} = \partial_c \Gamma^a{}_{bd} - \partial_d \Gamma^a{}_{bc} + \Gamma^a{}_{ce}\Gamma^e{}_{db} - \Gamma^a{}_{de}\Gamma^e{}_{cb} - C^e{}_{cd}\Gamma^a{}_{eb}$.
Let the curvature 2-form be $\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$

$$R(\underline{u},\underline{v})\underline{W} = \left[\nabla_{\underline{u}}\nabla_{\underline{v}} - \nabla_{\underline{v}}\nabla_{\underline{u}} - \nabla_{[\underline{u},\underline{v}]}\right]\underline{W} \qquad T_p(M)\times T_p(M)\times T_p(M) \to T_p(M)$$

in cpts $R^a{}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{db} - \Gamma^a_{de}\Gamma^e_{cb} - C^e_{cd}\Gamma^a_{eb}$.

Let the curvature 2-form be $\underline{R}^a{}_b = \frac{1}{2} R^a{}_{bcd}\, \underline{\omega}^c \wedge \underline{\omega}^d$

Cartan's 2nd structural eqn: $\boxed{\underline{R}^a{}_b = \underline{d}\,\underline{\Theta}^a{}_b + \underline{\Theta}^a{}_c \wedge \underline{\Theta}^c{}_b}$

Example S^2 : $ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

- $\underline{\omega}^\theta = \underline{d}\theta \qquad \underline{\omega}^\varphi = \sin\theta\, \underline{d}\varphi$
- $\underline{d}\underline{\omega}^\theta = 0 \qquad \underline{d}\underline{\omega}^\varphi = \cos\theta\, d\theta \wedge d\varphi$

$$= -\cot\theta\, \underline{\omega}^\varphi \wedge \underline{\omega}^\theta$$

$$= -\underline{\Theta}^\varphi{}_\theta \wedge \underline{\omega}^\theta$$

$\rightarrow \underline{\Theta}^\varphi{}_\theta = \cot\theta\, \underline{\omega}^\varphi = \cos\theta\, d\varphi.$

$$
\begin{aligned}
\underline{R}^{\varphi}{}_{\theta} &= \underline{d}\,\underline{\Theta}^{\varphi}{}_{\theta} \\
&= \underline{d}(\cos\theta \, d\varphi) \\
&= -\sin\theta \, d\theta \wedge d\varphi \\
&= -\underline{\omega}^{\theta} \wedge \underline{\omega}^{\varphi}
\end{aligned}
$$

$$
\Rightarrow R^{\hat\varphi}{}_{\hat\theta\hat\varphi\hat\theta} = 1.
$$

$$\underline{R}^{\varphi}{}_{\theta} = \underline{d}\,\underline{\Theta}^{\varphi}{}_{\theta}$$

$$= \underline{d}(\cos\theta\, d\varphi)/a$$

$$= -\sin\theta\, d\theta \wedge d\varphi/a$$

$$= -\underline{\omega}^{\theta} \wedge \underline{\omega}^{\varphi}/a^2$$

$$\Rightarrow R^{\hat{\varphi}}{}_{\hat{\theta}\hat{\varphi}\hat{\theta}} = 1/a^2$$

Example S^2 : $ds^2 = (d\theta^2 + \sin^2\theta\, d\varphi^2)\,a^2$

- $\underline{\omega}^\theta = a\,\underline{d}\theta \qquad \underline{\omega}^\varphi = a\sin\theta\, d\varphi$

- $\underline{d}\,\underline{\omega}^\theta = 0 \qquad \underline{d}\,\underline{\omega}^\varphi = a\cos\theta\, d\theta\wedge d\varphi$

$$= -\cot\theta\,\underline{\omega}^\varphi\wedge\underline{\omega}^\theta/a$$

$$= -\underline{\Theta}^\varphi{}_\theta\wedge\underline{\omega}^\theta$$

$$\rightarrow \underline{\Theta}^\varphi{}_\theta = \cot\theta\,\underline{\omega}^\varphi = \cos\theta\, d\varphi\,/a$$

$$\underline{R}^\varphi{}_\theta = \underline{d}\,\underline{\Theta}^\varphi{}_\theta$$

$$= \underline{d}(\cos\theta\, d\varphi)/a$$

$$= -\sin\theta\, d\theta\wedge d\varphi/a$$

$$= -\underline{\omega}^\theta\wedge\underline{\omega}^\varphi/a^2$$

$$\Rightarrow R^{\hat\varphi}{}_{\hat\theta\hat\varphi\hat\theta} = 1/a^2$$

$\Theta^{\varphi}{}_{\theta}$

$(\cos\theta\, d\varphi)/a$

$\sin\theta\, d\theta \wedge d\varphi / a$

$\underline{\omega}^{\theta} \wedge \underline{\omega}^{\varphi} / a^2$

$\hat{\varphi}\hat{\theta} = 1/a^2$

coord basis

$$R^{\mu}{}_{\nu\lambda\sigma} = e^{\mu}{}_{a}\, \omega_{\nu}{}^{b}\, \omega_{\lambda}{}^{c}\, \omega_{\sigma}{}^{d}\, R^{a}{}_{bcd}$$

$$= \frac{1}{a^2}\left(\delta^{\mu}{}_{\lambda}\, g_{\nu\sigma} - \delta^{\mu}{}_{\sigma}\, g_{\nu\lambda} \right)$$

Example $S^2 : \; ds^2 = (d\theta^2 + \sin^2\theta \, d\varphi^2)\, a^2$

- $\underset{\sim}{\omega}^{\theta} = a\, \underset{\sim}{d}\theta \qquad \underset{\sim}{\omega}^{\varphi} = a \sin\theta \, \underset{\sim}{d}\varphi$

- $\underset{\sim}{d}\underset{\sim}{\omega}^{\theta} = 0 \qquad \underset{\sim}{d}\underset{\sim}{\omega}^{\varphi} = a \cos\theta \, d\theta \wedge d\varphi$

$$= -\cot\theta \, \underset{\sim}{\omega}^{\varphi} \wedge \underset{\sim}{\omega}^{\theta} / a$$

$$= -\underset{\sim}{\Theta}^{\varphi}{}_{\theta} \wedge \underset{\sim}{\omega}^{\theta} \cancel{/a}$$

$$\rightarrow \underset{\sim}{\Theta}^{\varphi}{}_{\theta} = \frac{\cot\theta}{a} \underset{\sim}{\omega}^{\varphi} = \cos\theta \, d\varphi$$

$$\underset{\sim}{R}^{\varphi}{}_{\theta} = \underset{\sim}{d}\, \underset{\sim}{\Theta}^{\varphi}{}_{\theta}$$

$$= \underset{\sim}{d}(\cos\theta \ldots$$

$$= -\sin\theta \, d \ldots$$

$$= -\underset{\sim}{\omega}^{\theta} \wedge \ldots$$

$$\Rightarrow R^{\hat{\varphi}}{}_{\hat{\theta}} \ldots$$

$d\theta^2 + \sin^2\theta\, d\varphi^2)/a^2$

$\wedge \sin\theta\, d\varphi$

$\cos\theta\, d\theta \wedge d\varphi$

$\cot\theta\, \underline{\omega}^\varphi \wedge \underline{\omega}^\theta / a$

$\underline{\Theta}^\varphi{}_\theta \wedge \underline{\omega}^\theta / a$

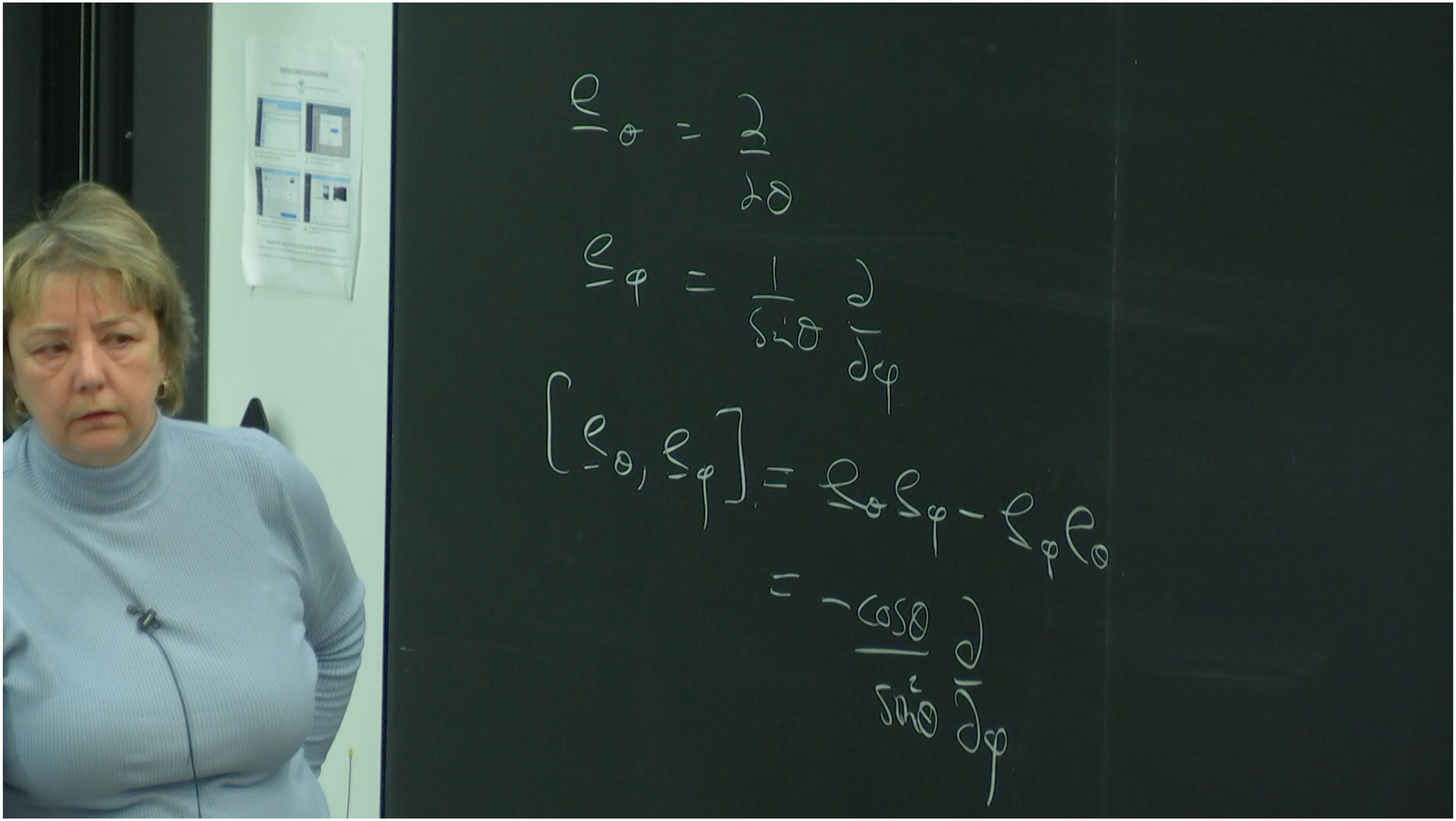
$\cos\theta\, d\varphi.$

$$\underline{R}^\varphi{}_\theta = \underline{d}\,\underline{\Theta}^\varphi{}_\theta$$
$$= \underline{d}(\cos\theta\, d\varphi)$$
$$= -\sin\theta\, d\theta \wedge d\varphi$$
$$= -\underline{\omega}^\theta \wedge \underline{\omega}^\varphi / a^2$$

$$\Rightarrow R^{\hat\varphi}{}_{\hat\theta\hat\varphi\hat\theta} = 1/a^2$$

coord basis

$$R^\mu{}_{\nu\lambda\sigma} = e^\mu{}_a\, \omega_\nu{}^b$$
$$= \frac{1}{a^2}(\delta^\mu{}_\lambda\, g_{\nu\sigma}$$


$\underline{e}_\theta = \frac{\partial}{\partial\theta}$
$\underline{e}_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}$
$[\underline{e}_\theta, \underline{e}_\varphi] = \underline{e}_\theta\underline{e}_\varphi - \underline{e}_\varphi\underline{e}_\theta$
$= -\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}$