Title: Mathematical Physics Lecture - 230118

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Collection: Mathematical Physics (2022/2023)

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# Last Time

Lagrangian $\longrightarrow$ Hamiltonians

Symp. manifold $M = \{\text{sol'ns to EL eqns}\}$

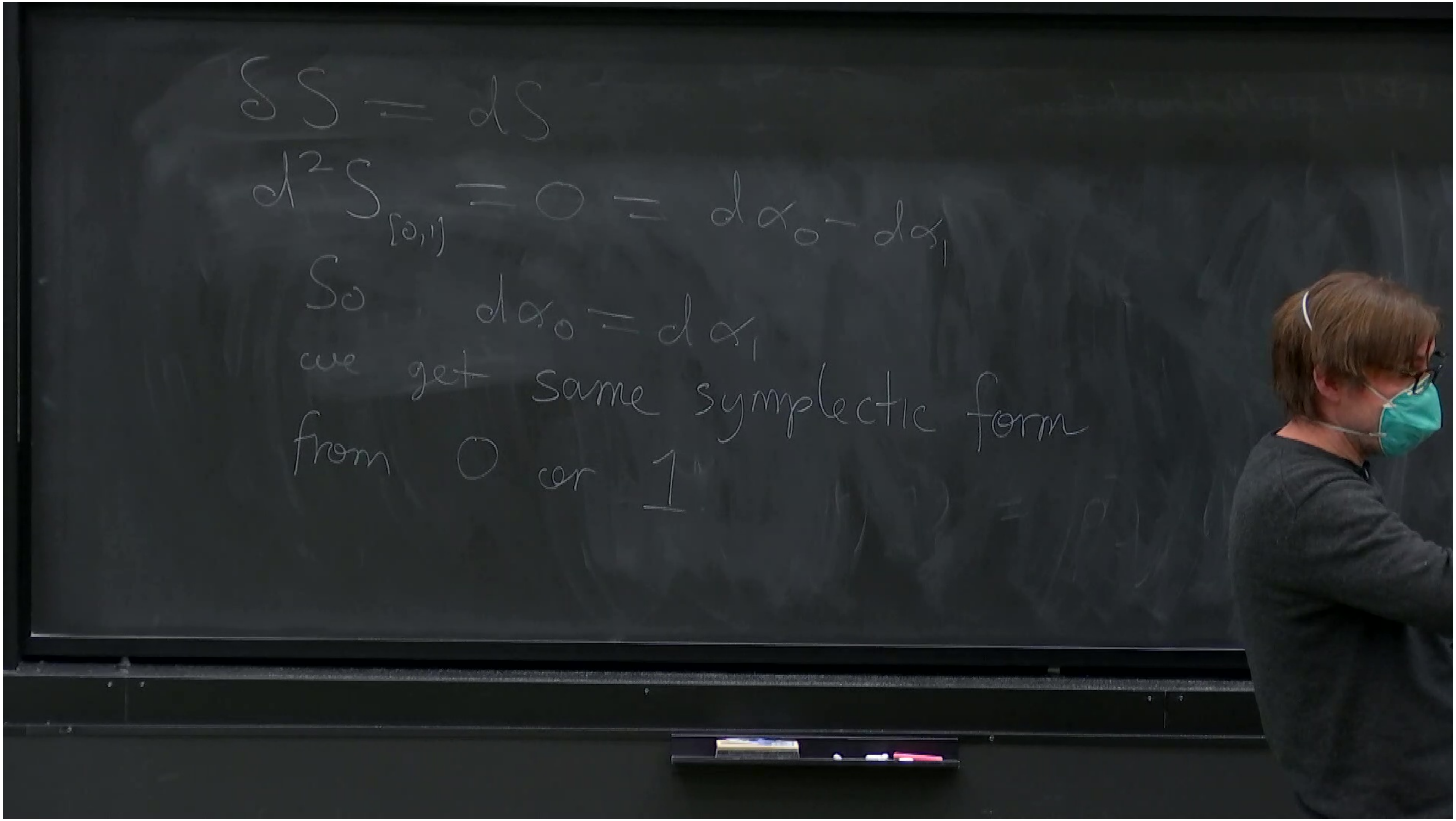
$\varphi$ field

$$S_{[0,1]}(\varphi) = \int_0^1 \mathcal{L}(\varphi)$$

$\delta S_{[0,1]}(\varphi)$ vanishes for variations which are $0$ near $0,1$

$$\delta S_{[0,1]}(\varphi) = \underbrace{\alpha_0}_{\text{coming from } 0} - \underbrace{\alpha_1}_{\text{coming from } 1}$$

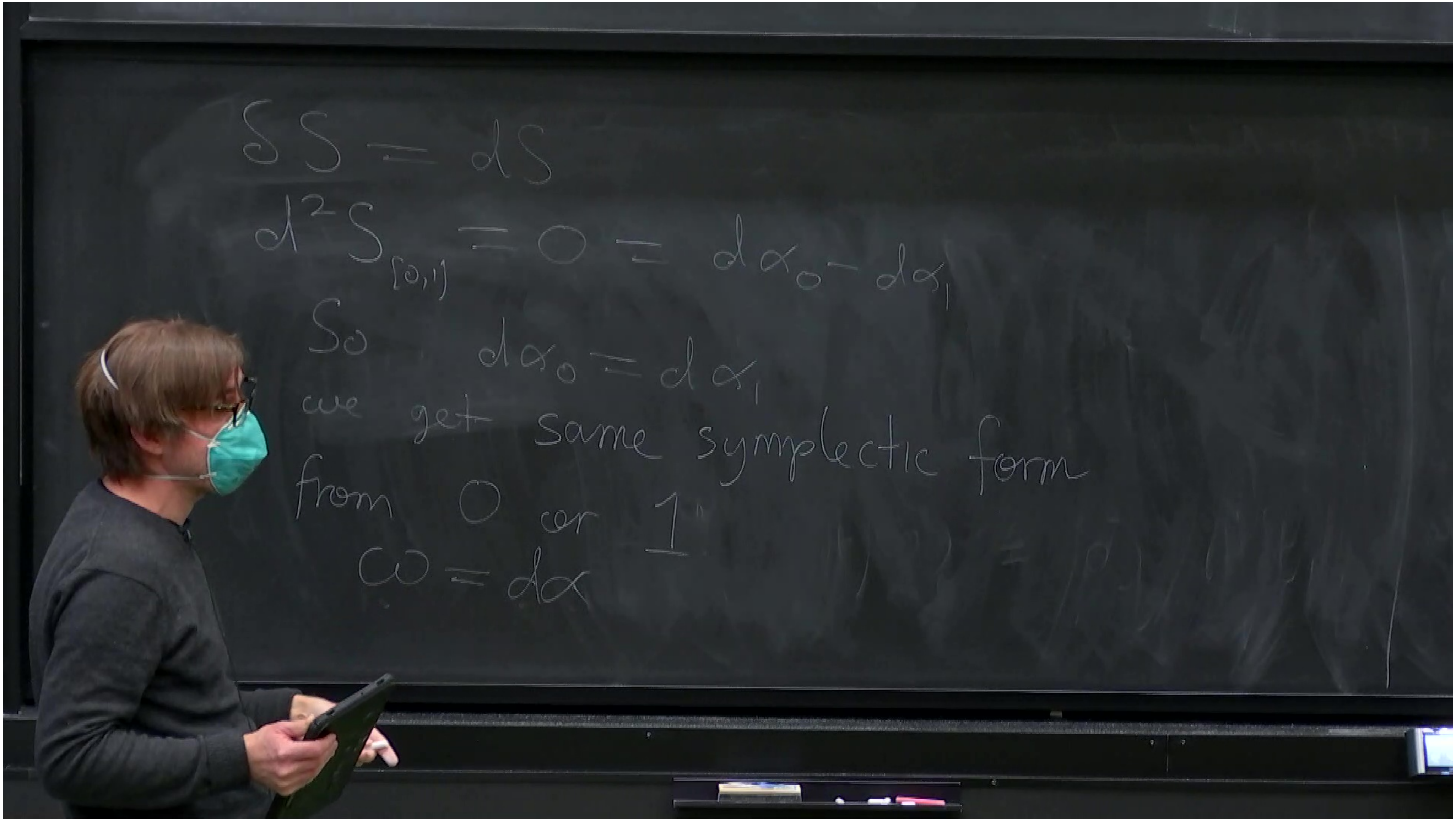
$\alpha := \alpha_0$ is variational 1-form.


$\delta S = dS$
$d^2 S_{[0,1]} = 0 = d\alpha_0 - d\alpha_1$
$S_0$ $d\alpha_0 = d\alpha_1$
we get same symplectic form
from 0 or 1

Today:

1) Do a field theory example

2) Go backwards: Hamiltonian $\longrightarrow$ Lagrangian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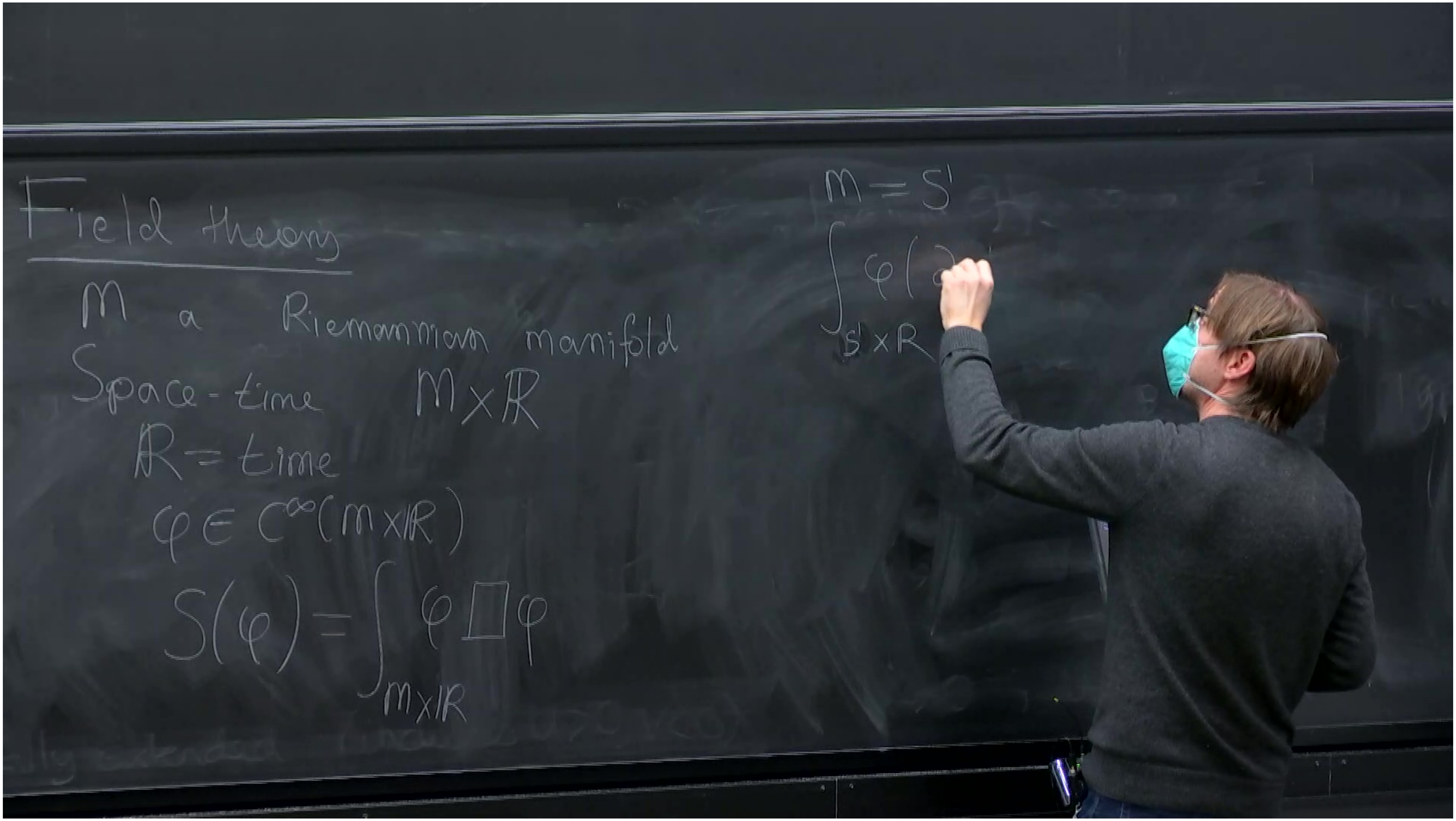
$\omega = d\alpha$


Field theory
M a Riemannian manifold
Space-time M × ℝ
ℝ = time
φ ∈ C^∞(M × ℝ)
S(φ) = ∫_{M×ℝ} φ □ φ
M = S¹
∫_{S¹×R} φ(

## Field theory

$M$ a Riemannian manifold

Space-time $M \times \mathbb{R}$

$\mathbb{R} = $ time

$\varphi \in C^\infty(M \times \mathbb{R})$

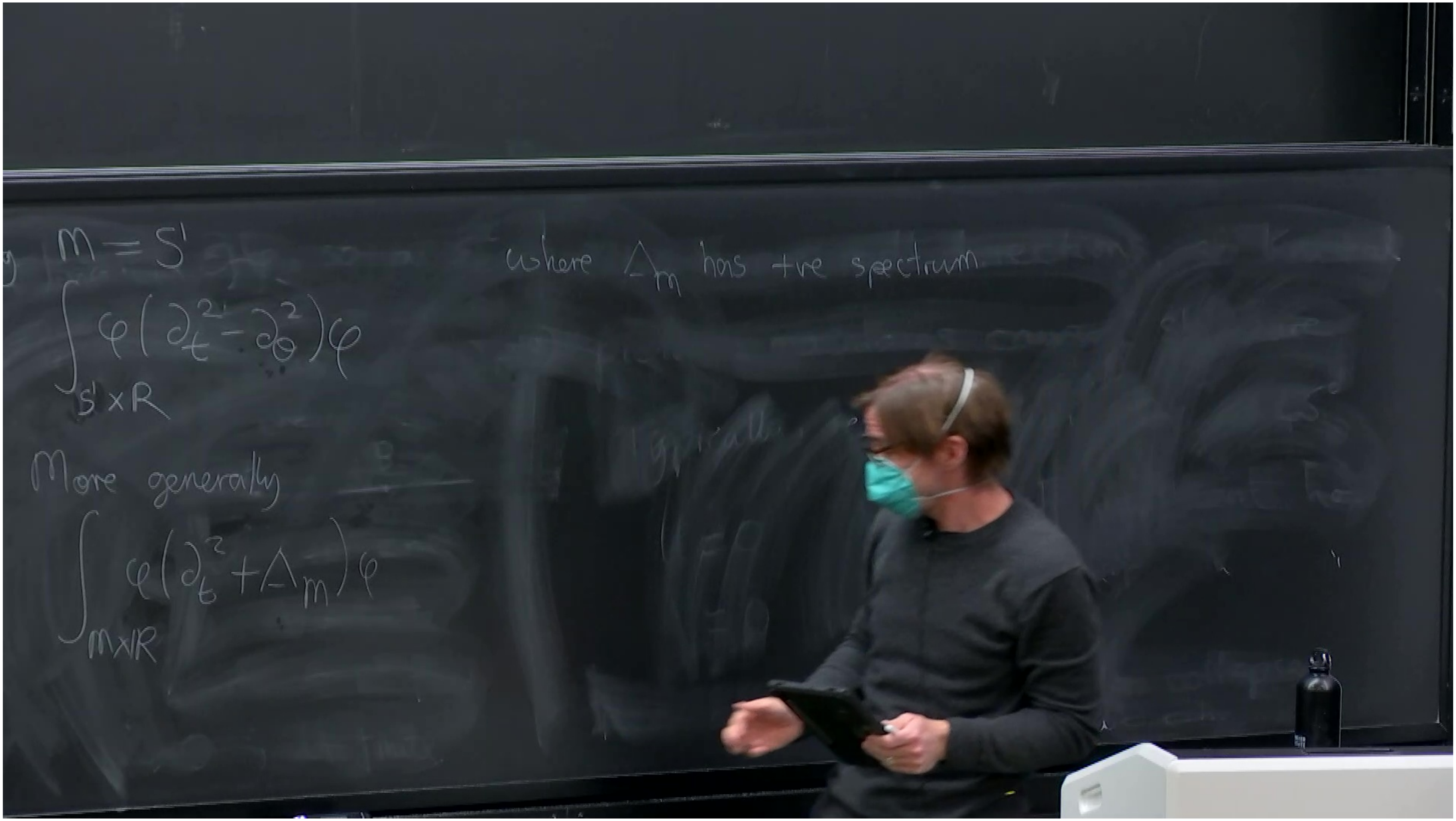
$$S(\varphi) = \int_{M\times\mathbb{R}} \varphi \, \Box \, \varphi$$

e.g. $M = S^1$

$$\int_{S^1 \times R} \varphi(\partial_t^2 - \partial_\theta^2)\varphi$$

More generally

$$\int_{M\times\mathbb{R}} \varphi(\partial_t^2 + \Delta_M)\varphi$$


M = S¹
∫_{S¹×R} φ(∂_t² − ∂_θ²)φ
More generally
∫_{M×R} φ(∂_t² + Δ_M)φ
where Δ_M has the spectrum

Phase space $= \{ \text{sol}^{\text{n}}\text{s to EL eq}^{\text{n}}\text{s} \}$

$$= \{ \varphi ,\ \Box \varphi = 0 \}$$

$\varphi.$

Phase space = { sol$^{n}$s to EL eq$^{n}$s }

$$= \{ \varphi ,\ \Box \varphi = 0 \}$$

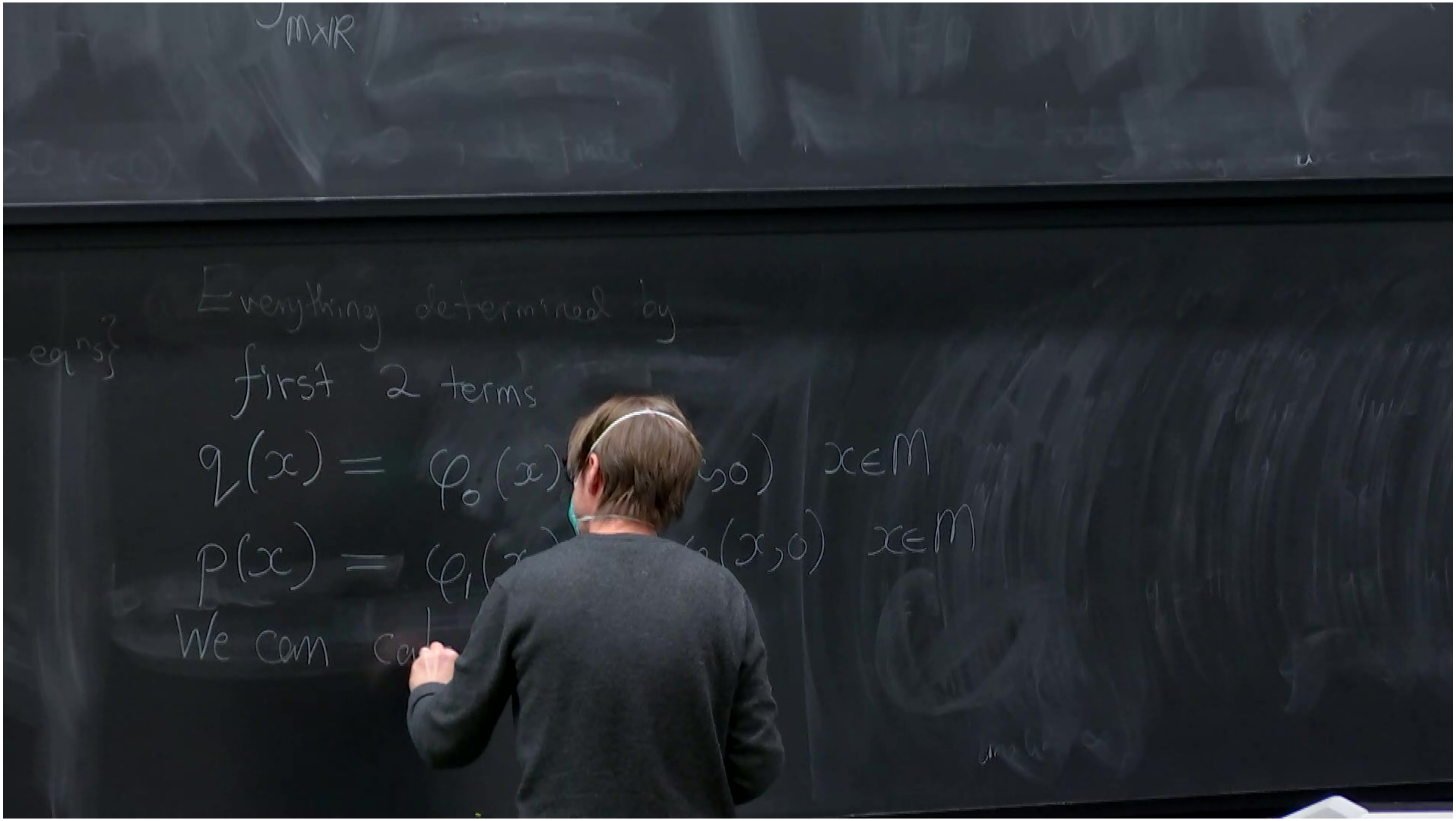
$$\varphi = \varphi_0 + t\varphi_1 + t^2 \varphi_2 + \dots$$

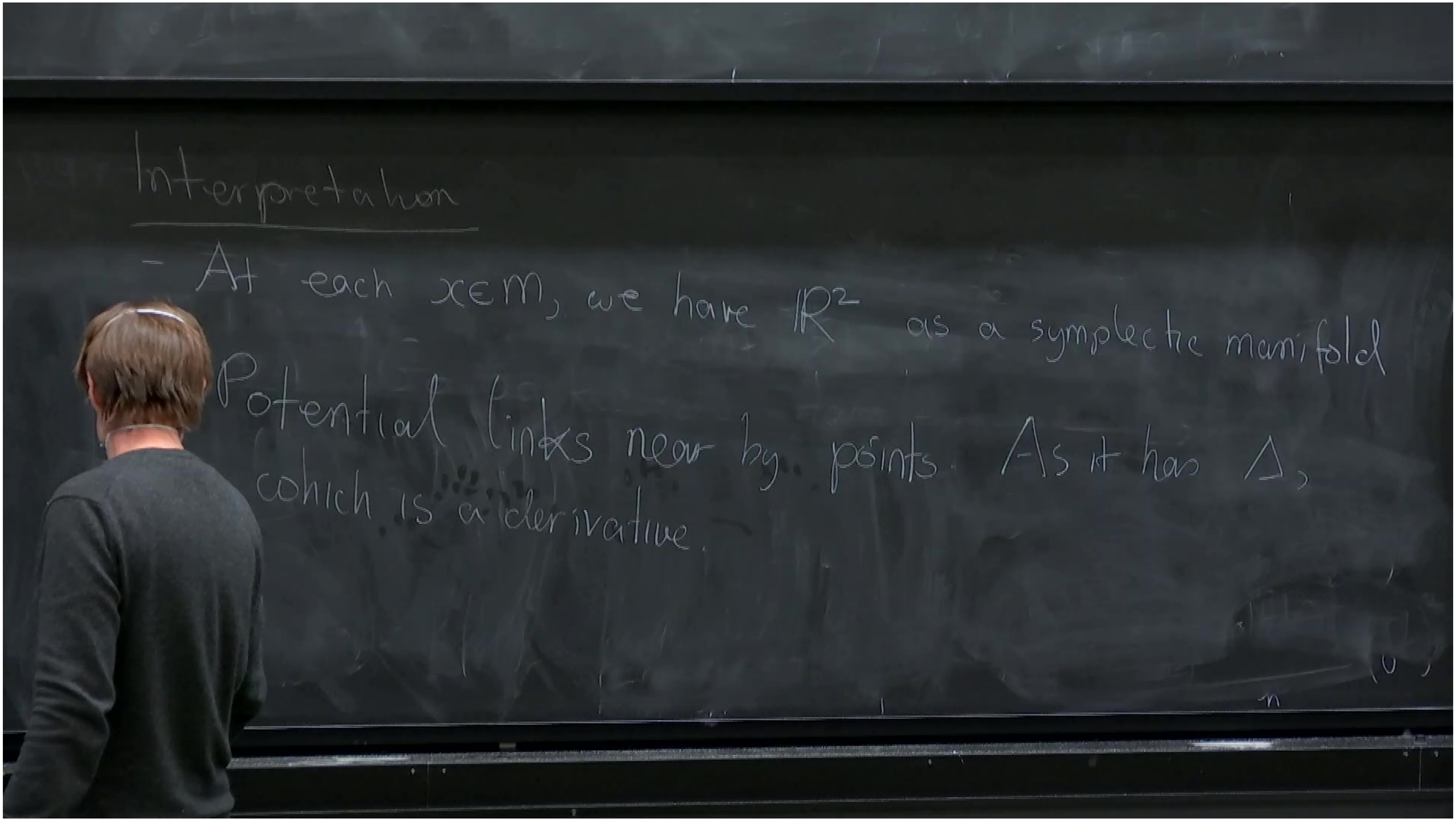
$\Box \varphi = 0$ says

$$2\varphi_2 + \Delta \varphi_0 = 0$$

$$6\varphi_3 + \Delta \varphi_1 = 0$$

$$\vdots$$


Everything determined by
first 2 terms
$q(x) = \varphi_0(x)$
$x \in M$
$p(x) = \varphi_1($
$(x,0) \quad x \in M$
We can


Interpretation
- At each $x \in M$, we have $\mathbb{R}^2$ as a symplectic manifold
- Potential links near by points. As it has $\Delta$,
which is a derivative.

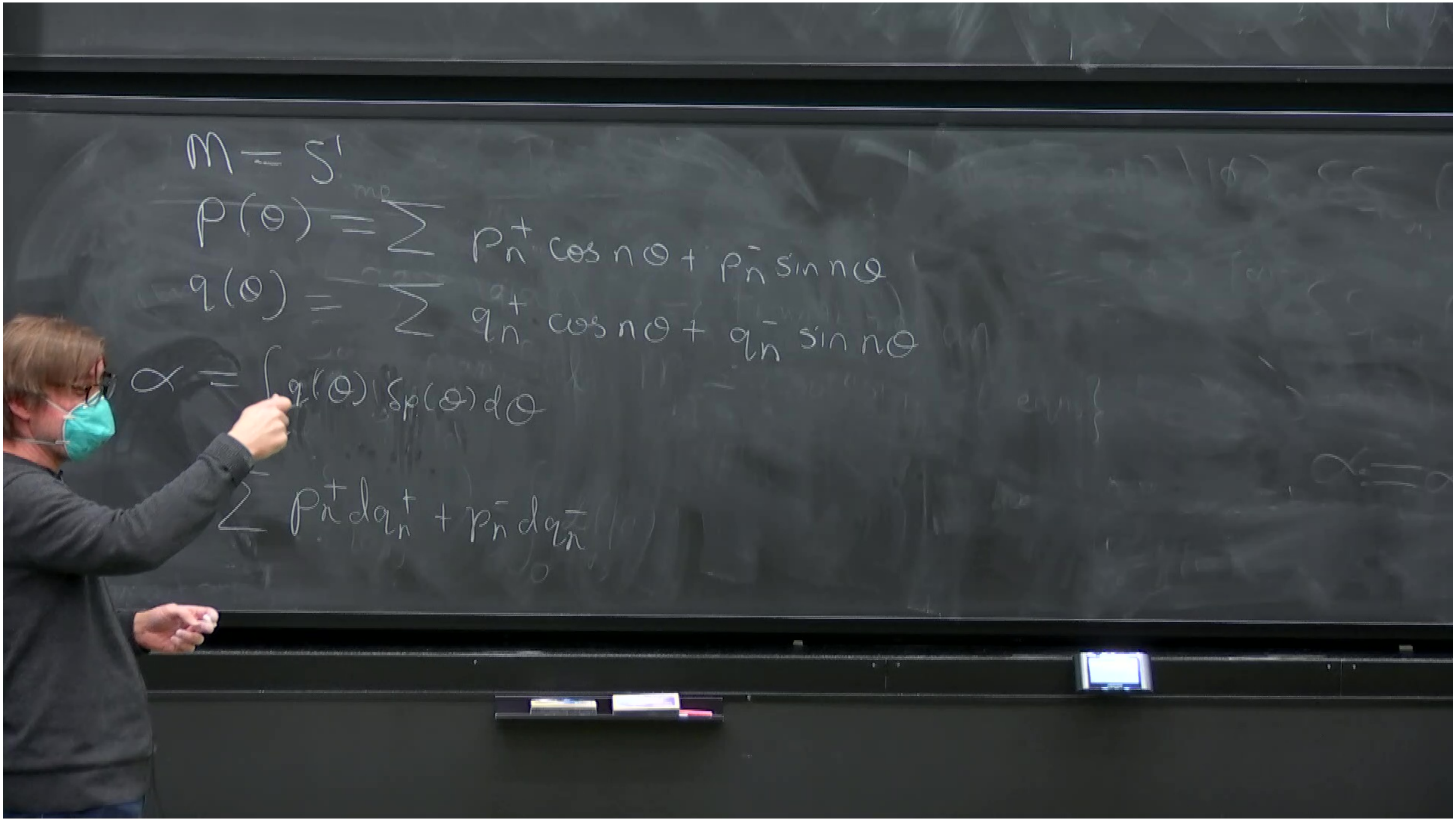
$$M = S^1$$

$$p(\theta) = \sum_{n\geq 0} p_n^+ \cos n\theta + p_n^- \sin n\theta$$

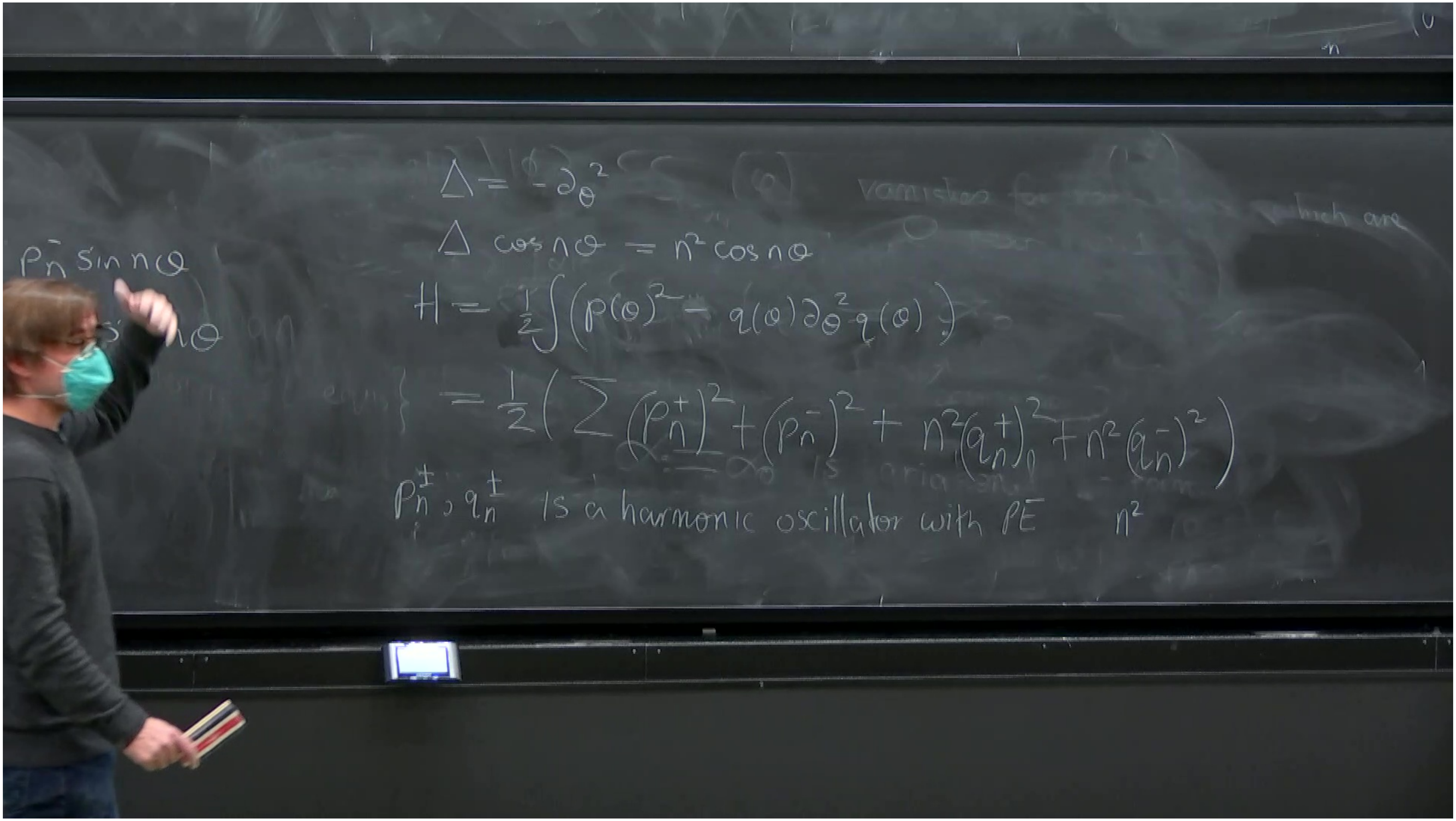
$$q(\theta) = \sum q_n^+ \cos n\theta + q_n^- \sin n\theta$$

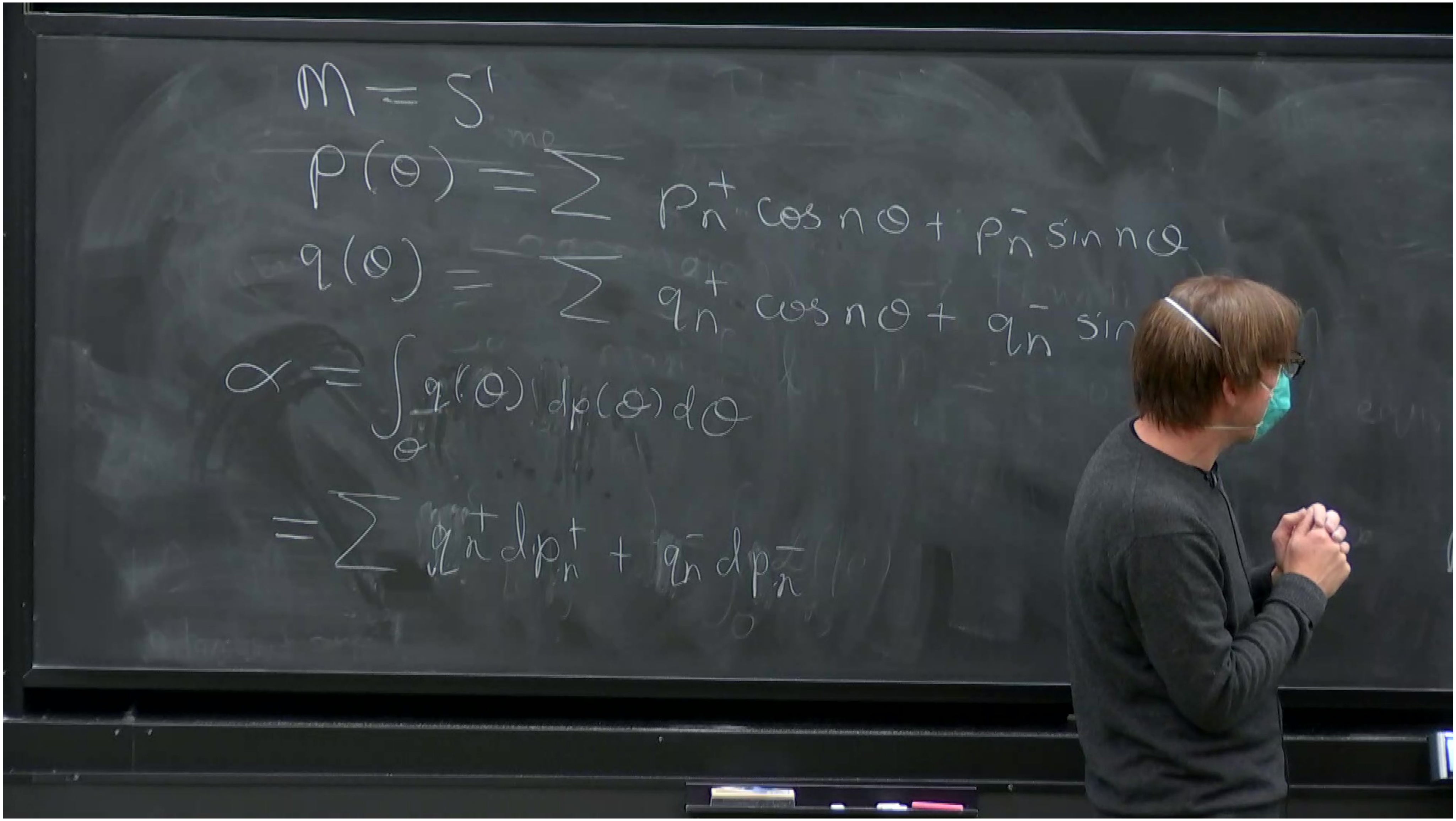
$$\alpha = \int_\theta q(\theta)\, d\, p(\theta)\, d\theta$$

$$= \sum p_n^+ dq_n^+ + p_n^- dq_n^-$$



$M = S^1$
$p(\theta) = \sum_{n\geq 0} p_n^+ \cos n\theta + p_n^- \sin n\theta$
$q(\theta) = \sum q_n^+ \cos n\theta + q_n^- \sin n\theta$
$\alpha = \int q(\theta)\, \delta p(\theta)\, d\theta$
$\sum p_n^+ dq_n^+ + p_n^- dq_n^-$


$p_n^- \sin n\theta$
$\Delta = -\partial_\theta^2$
$\Delta \cos n\theta = n^2 \cos n\theta$
$H = \frac{1}{2}\int \left(p(\theta)^2 - q(\theta)\partial_\theta^2 q(\theta)\right)$
$= \frac{1}{2}\left(\sum (p_n^+)^2 + (p_n^-)^2 + n^2(q_n^+)^2 + n^2(q_n^-)^2\right)$
$p_n^\pm, q_n^\pm$ is a harmonic oscillator with PE $n^2$


$M = S^1$
$p(\theta) = \sum p_n^+ \cos n\theta + p_n^- \sin n\theta$
$q(\theta) = \sum q_n^+ \cos n\theta + q_n^- \sin$
$\alpha = \int_\theta q(\theta)\, dp(\theta)\, d\theta$
$= \sum q_n^+ dp_n^+ + q_n^- dp_n^-$

$$\Delta = -\partial_\theta^2$$

$$\Delta \cos n\theta = n^2 \cos n\theta$$

$$H = \frac{1}{2}\int \left( p(\theta)^2 - q(\theta)\partial_\theta^2 q(\theta) \right)$$

$$= \frac{1}{2}\left( \sum (p_n^+)^2 + (p_n^-)^2 + n^2 (q_n^+)^2 + n^2 (q_n^-)^2 \right)$$

$p_n^\pm, q_n^\pm$ is a harmonic oscillator with PE $n^2$

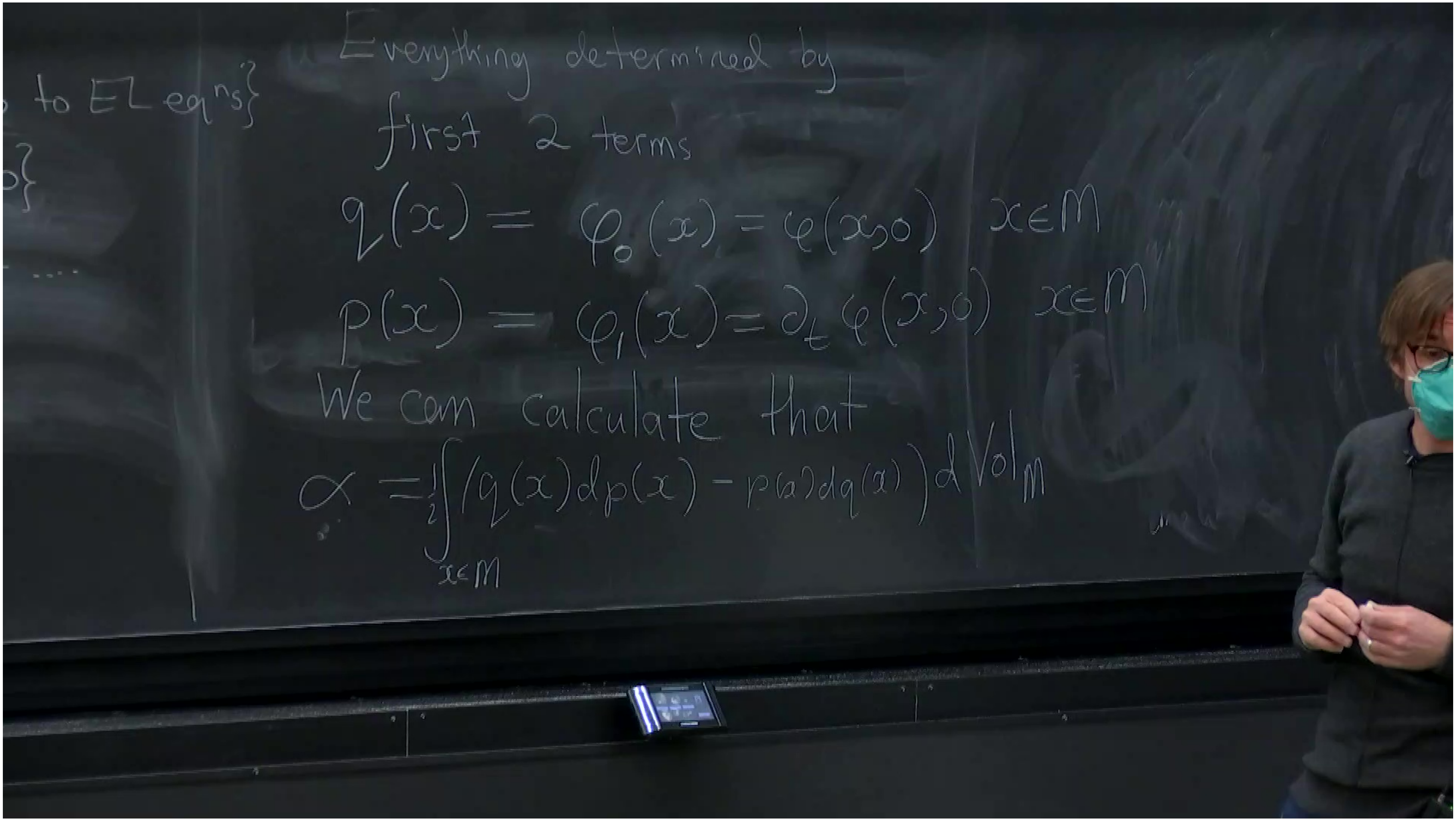
$dp^i \wedge dq_i \rightsquigarrow [p^i, q_j] = \hbar \delta^i_j$

Hamiltonian $\rightleftarrows$ Lagrangian

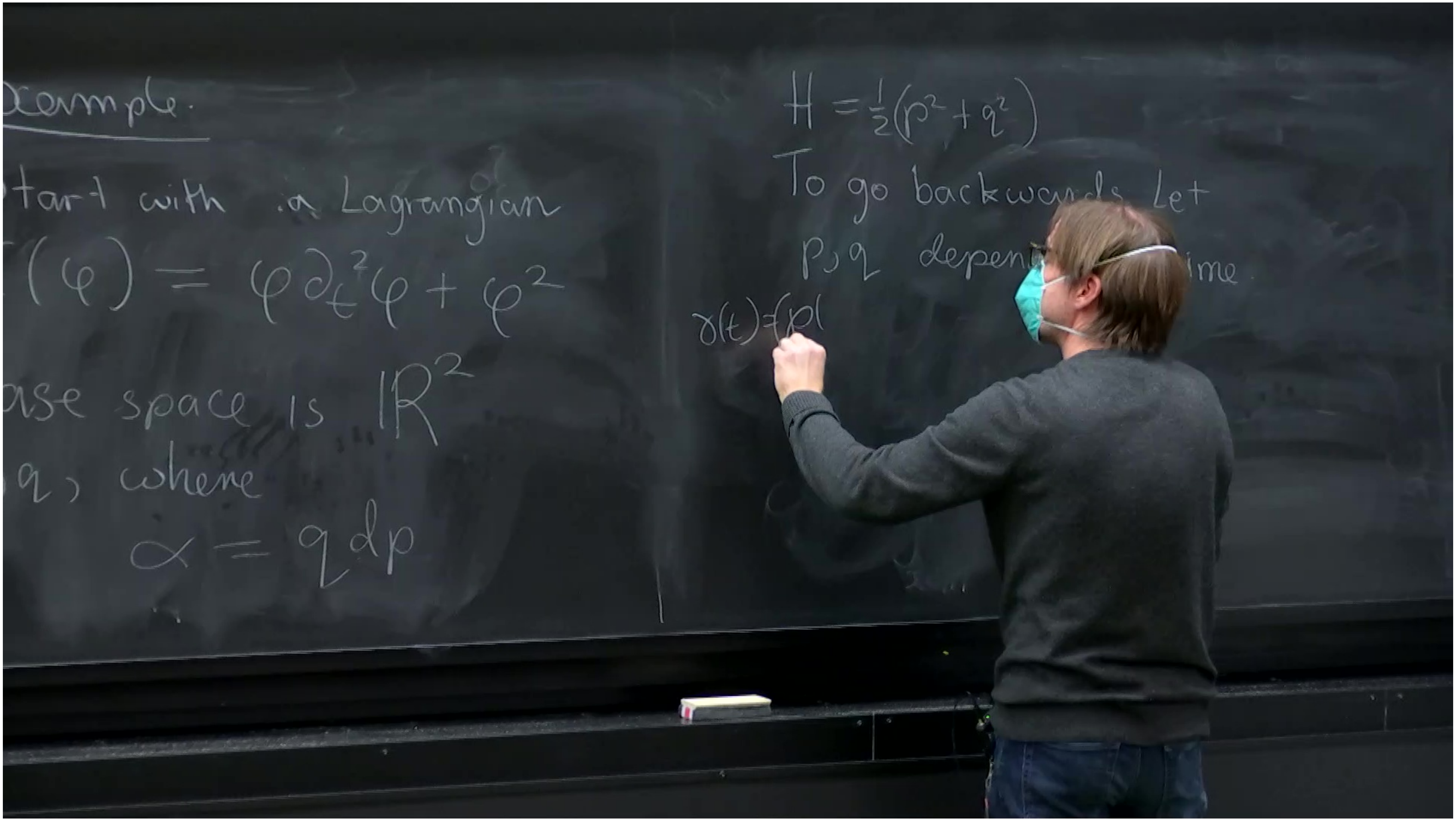
$M$ symplectic manifold

Assume $\omega = d\alpha$

$H : M \to \mathbb{R}$ is Hamiltonian



to EL eqns}
Everything determined by first 2 terms
$q(x) = \varphi_0(x) = \varphi(x,0) \quad x \in M$
$p(x) = \varphi_1(x) = \partial_t \varphi(x,0) \quad x \in M$
We can calculate that
$\alpha = \frac{1}{2}\int_{x\in M} \left( q(x)\, dp(x) - p(x)\, dq(x) \right) d\mathrm{Vol}_M$


xample.
tart with a Lagrangian
$(\varphi) = \varphi \partial_t^2 \varphi + \varphi^2$
ase space is $\mathbb{R}^2$
q, where
$\alpha = q\,dp$
$H = \frac{1}{2}(p^2 + q^2)$
To go backwards let
p, q depen ime
$\gamma(t) = (p($

th a Lagrangian

$$= \varphi \partial_t^2 \varphi + \varphi^2$$

ace is $\mathbb{R}^2$

here

$$= q\,dp$$

$$H = \frac{1}{2}(p^2 + q^2)$$

To go backwards: Let

$p, q$ depend on time

$$\gamma(t) = (p(t), q(t))$$

$$S(p,q) = \int q(t)\,dp(t) + \frac{1}{2}\left(p(t)^2 + q(t)^2\right)$$

$$S = -\int p(t)\,dq(t) + \frac{1}{2}\left(p(t)^2 + q(t)^2\right)$$

$p$ satisfies algebraic EOM

$$p = \partial_t q$$

$$S(q) = -\int (\partial_t q)^2 + \frac{1}{2}(\partial_t q)^2 + \frac{1}{2} q^2 )$$

$$= \int -\frac{1}{2}(\partial_t q)^2 + \frac{1}{2} q^2$$

$$= \int \frac{1}{2}\left(q\,\partial_t^2 q + q^2\right)$$

what we started with

So far:

Lagrangian $\longrightarrow$ Hamiltonian

$\|$

Lagrangian $\longleftarrow$ Hamiltonian

"in 1st order"

If $(M, \omega =$

If $(M, \omega = d\alpha, H)$

build Lagrangian for $\gamma : \mathbb{R} \to M$

Is the phase space $= M$?

In coords $p, q$, $\omega = dq^i \wedge dp_i$ $\alpha = q^i \wedge dp_i$

$\gamma(t) = (q^i(t), p_i(t))$

$$S(p_i(t), q^i(t)) = \int q^i(t)\,\partial_t p_i(t) + H(p,q)\,dt$$

contains no t-derivatives.

EOM are:

$$\partial_t p_i = -\frac{\partial H}{\partial q^i}$$

$$\partial_t q^i = \frac{\partial H}{\partial p_i}$$

$\Longrightarrow$ Phase space has coords $p, q$. ✓

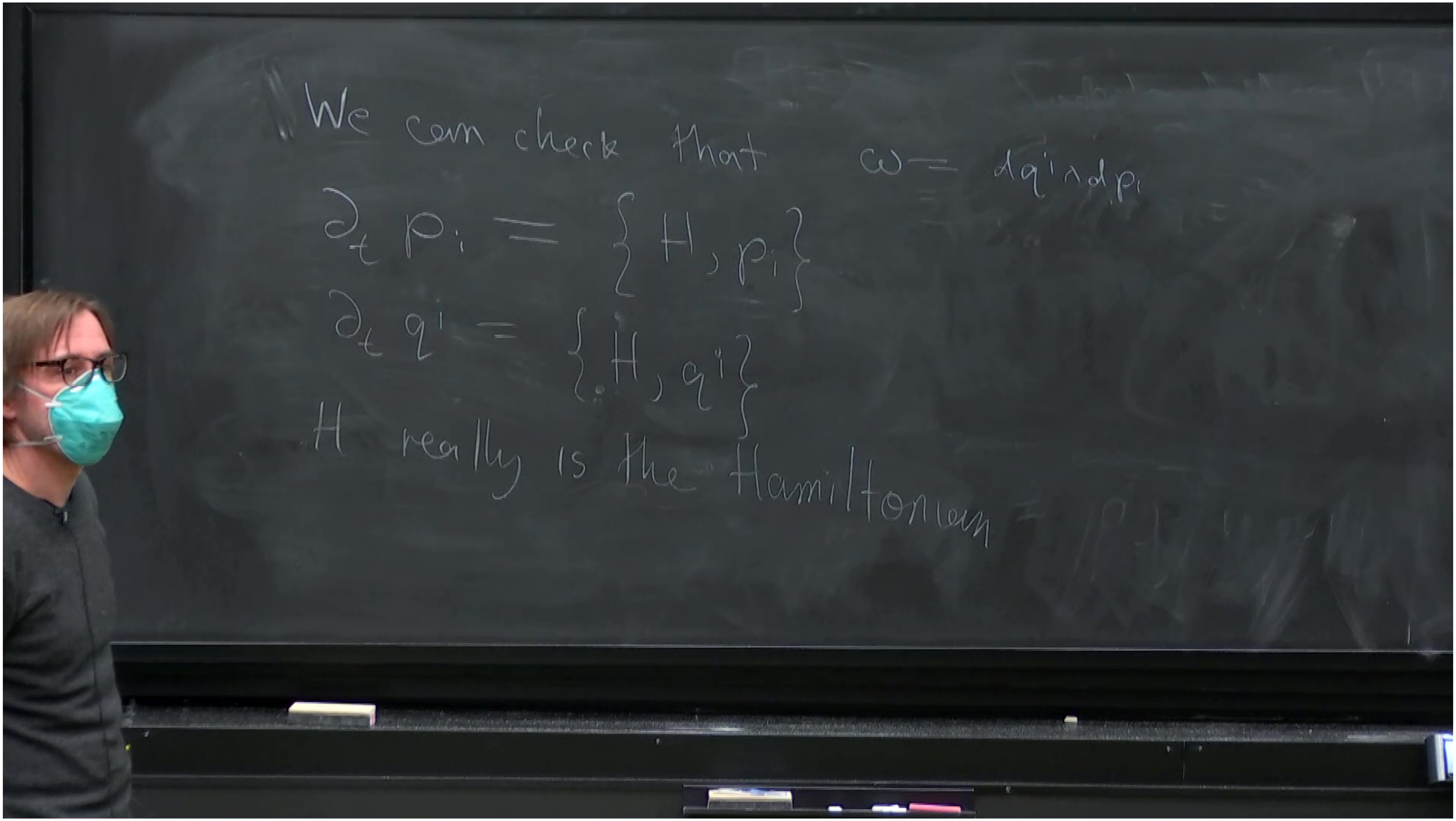
$$S(p_i(t), q^i(t)) = \int q^i(t)\,\partial_t p_i(t) + H(p,q)\,dt$$

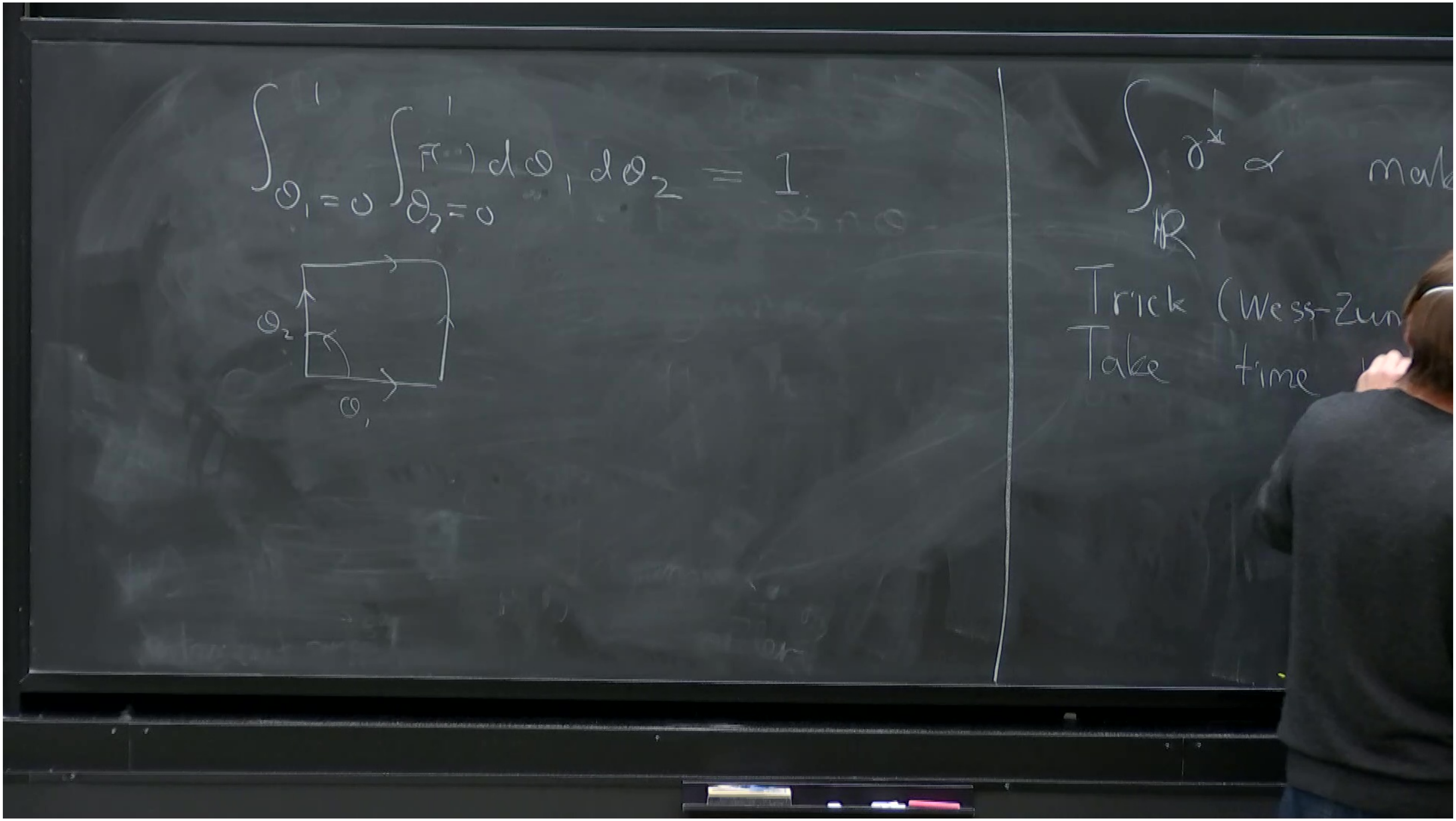
contains no t-derivat

EOM are:

$$\partial_t p_i = -\frac{\partial H}{\partial q^i} \implies \text{Phase space has coords } p, q \ \checkmark$$

$$\partial_t q^i = \frac{\partial H}{\partial p_i}$$


We can check that $\omega = dq^i \wedge dp_i$
$\partial_t p_i = \{H, p_i\}$
$\partial_t q^i = \{H, q^i\}$
H really is the Hamiltonian


$\int_{\theta_1=0}^{1}\int_{\theta_2=0}^{1} F(\cdot\cdot)\, d\theta_1\, d\theta_2 = 1$
$\theta_2$
$\theta_1$
$\int_{\mathbb{R}} \gamma^* \alpha$
mak
Trick (Wess-Zun
Take time

...kes no, sense

no)

...e $S^1$

Extend $\gamma: S^1 \to M$
to $\hat{\gamma}: D^2 \to M$

$$\int_{D^2} \hat{\gamma}^* \omega$$ = makes sense!

...which are

would like this to be 0

Or, we need

$$e^{i\int_{S^2}\gamma^*\omega}$$

to be 1.

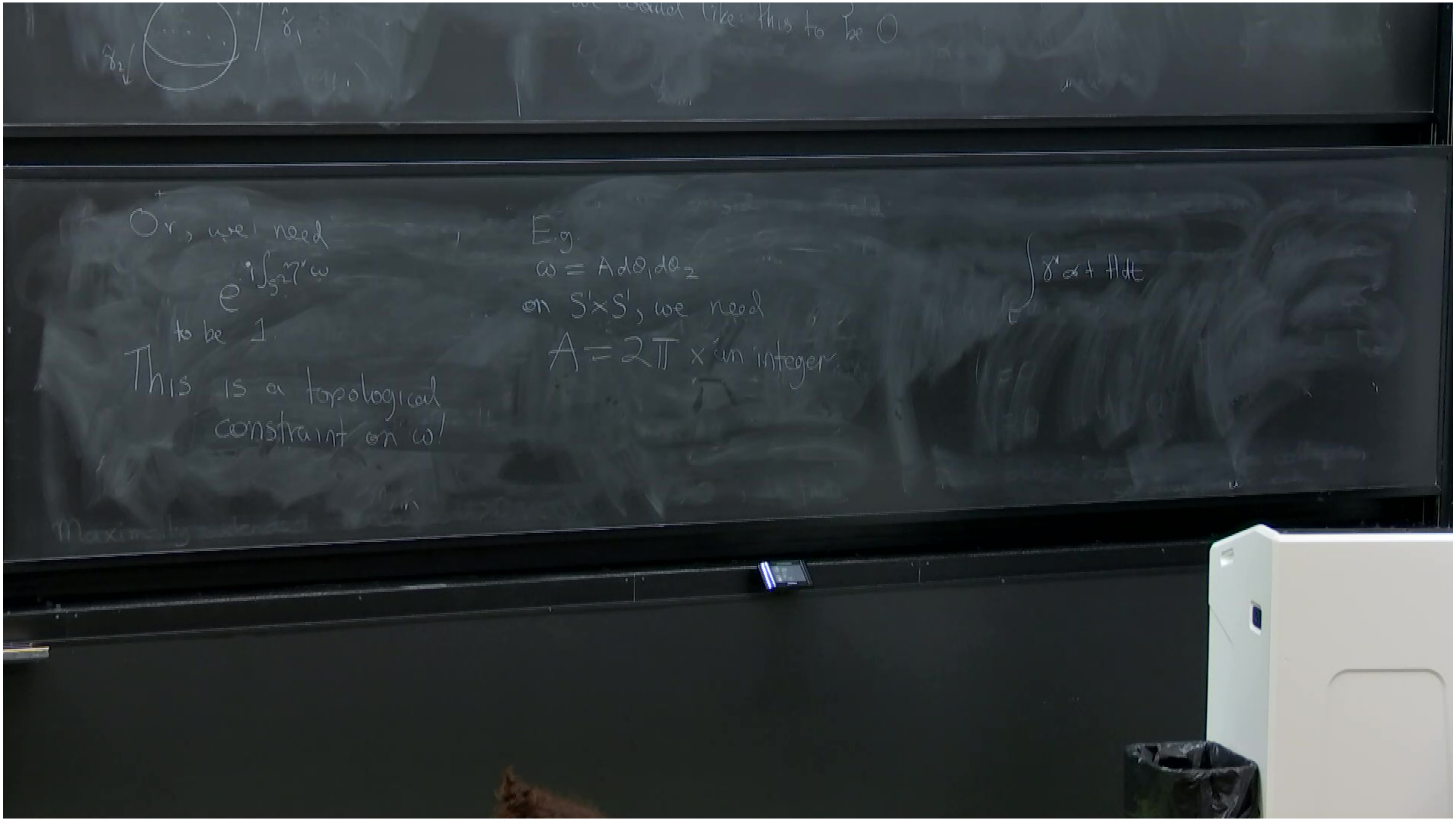
This is a topological constraint on $\omega$!

E.g.

$\omega = A\,d\theta_1 d\theta_2$

on $S^1\times S^1$, we need

$A = 2\pi \times$ an integer.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on $S^1\times S^1$, we need
$A = 2\pi \times$ an integer.
$\int \gamma^*\alpha + H dt$