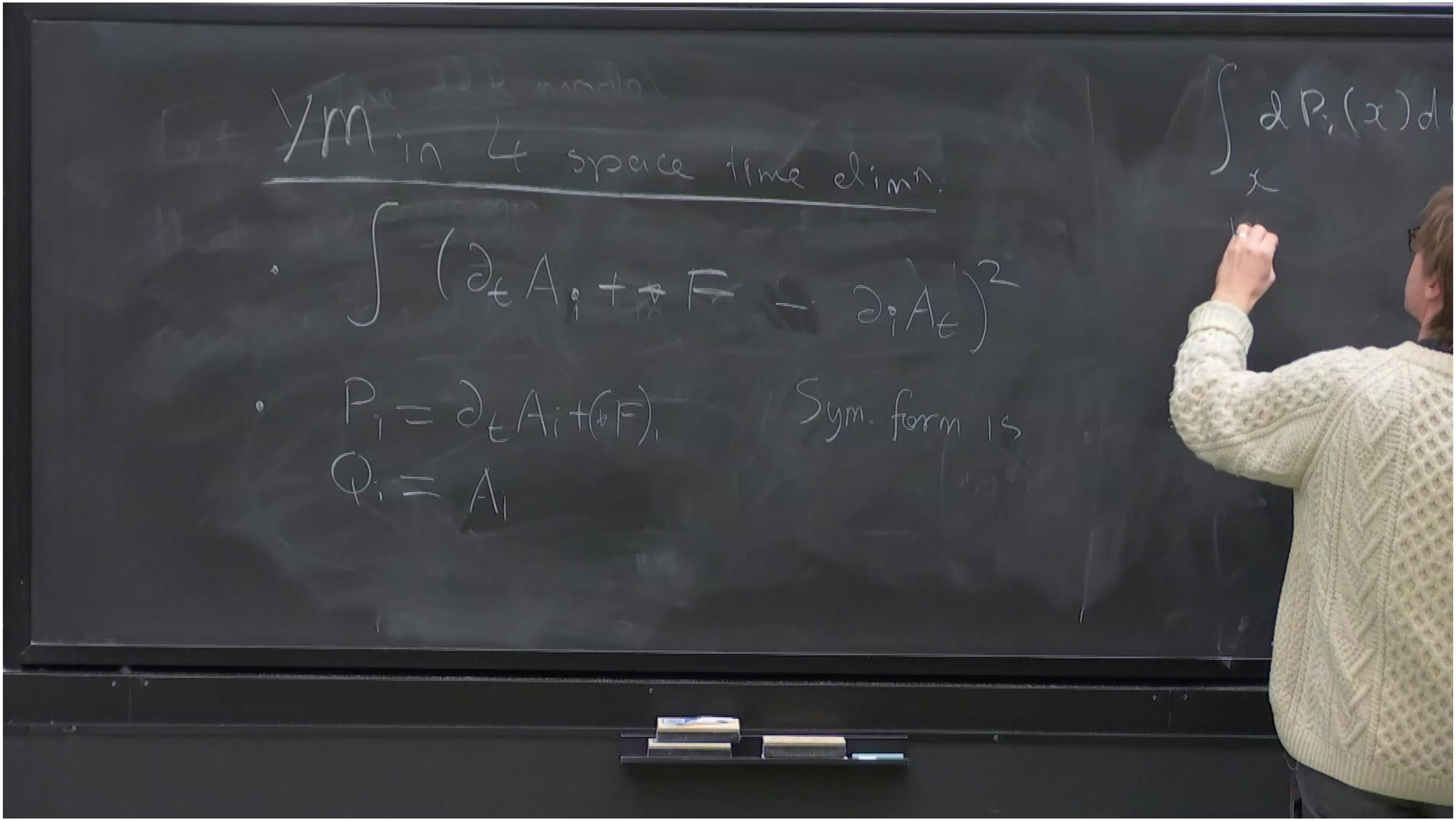
Title: Mathematical Physics Lecture - 230127

Speakers: Kevin Costello

Collection: Mathematical Physics (2022/2023)

Date: January 27, 2023 - 2:00 PM

URL: https://pirsa.org/23010021


YM in 4 space time dimn.
$\int (\partial_t A_i + *F - \partial_i A_t)^2$
$P_i = \partial_t A_i + (*F)_i$
$Q_i = A_i$
Sym. form is
$\int_x \partial P_i(x)\, d$

YM in 4 space time dimn.

- $\int (\partial_t A_i + {*F} - \partial_i A_t)^2$

- $P_i = \partial_t A_i + (*F)_i$

  $Q_i = A_i$

Sym. form is

$\int_x \delta P_i(x) \, \delta Q_i(x)$

$F = \partial_i Q_j - \partial_j Q_i$

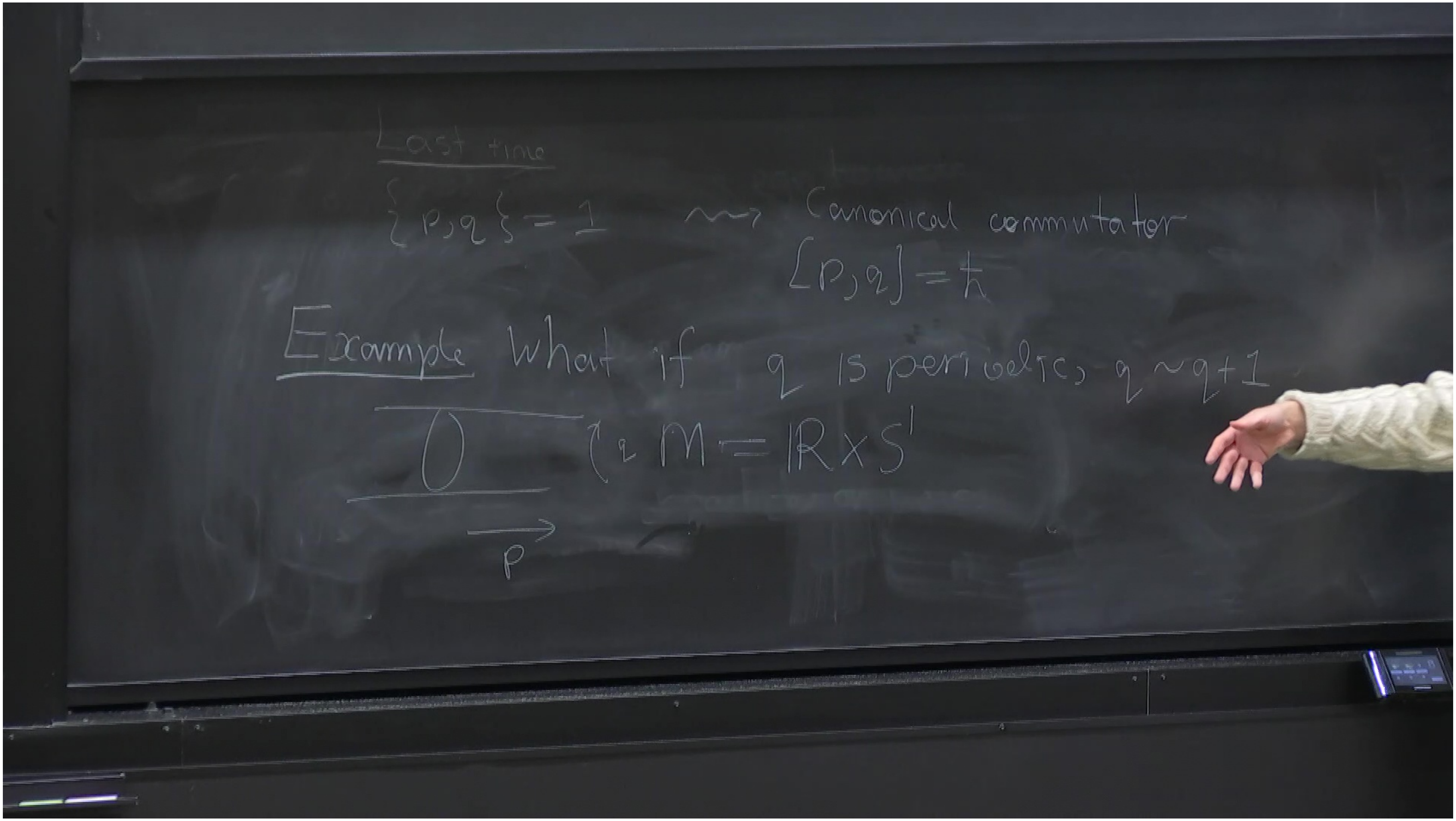
Hamiltonian is

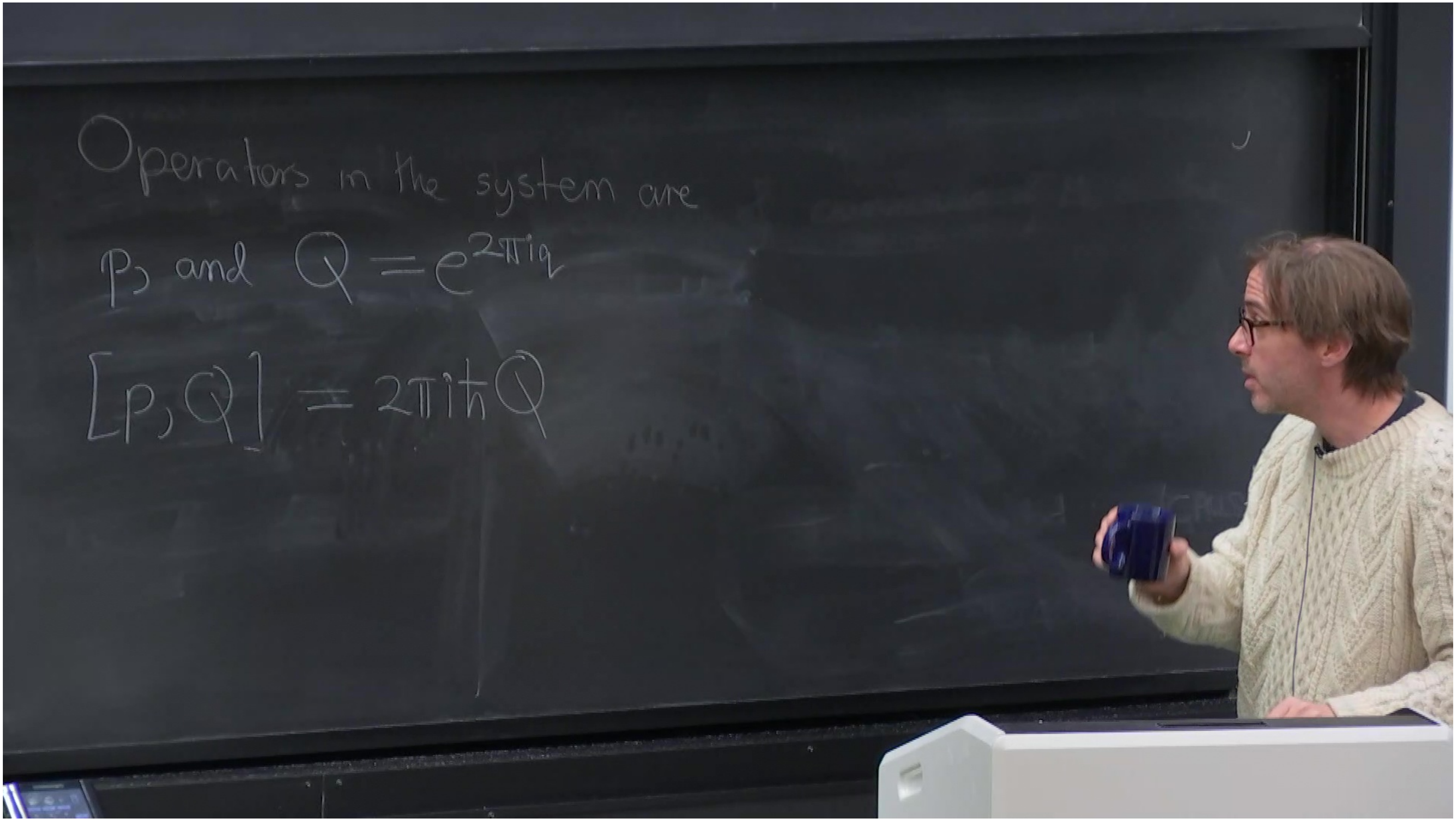
$P_i^2 + P_i(*F)_i$

$P_i \partial_t Q_i + \frac{1}{2}(P_i^2 + P_i(*F)_i$

$P_i = -\partial_t Q_i - P_i(*F)_i$

Reinsert


Last time
$\{p,q\} = 1 \rightsquigarrow$ Canonical commutator
$[p,q] = \hbar$
Example What if q is periodic, $q \sim q+1$
$Q\colon M = \mathbb{R} \times S^1$
p


Operators in the system are
p, and $Q = e^{2\pi i q}$
$[p, Q] = 2\pi i \hbar Q$

What if both p and q are periodic?

$$P = e^{2\pi i p}$$

$$Q = e^{2\pi i q}$$

Relation (follows from $[p, q] = \hbar$)

$$PQ = QPe^{2\pi i \hbar}$$

Let's try to build a Hilbert space where these algebras act.

$M = \mathbb{R}^2$

p, q

Hilbert space = functions of q

$$P \rightsquigarrow \hbar \frac{\partial}{\partial q}$$

Or

funct

Let's try to build a Hilbert space where these algebras act

$M = \underset{p,q}{\mathbb{R}^2}$

Hilbert space = functions of q

$$p \rightsquigarrow \hbar \frac{\partial}{\partial q}$$

Or Hilbert space is functions of p

$$q \rightsquigarrow -\hbar \frac{\partial}{\partial p}$$

These are the same: Fourier transform sends one to the other.

$$\mathcal{H} = \left\{ f : \mathbb{R}^2 \longrightarrow \mathbb{C} \text{ where } \frac{\partial f}{\partial p} = 0 \right\}$$

A (real) polarization is a manifold Y, $\dim Y = \frac{1}{2} \dim Y$

and to a m

$\mathcal{H} = \{ f : \mathbb{R}^2 \longrightarrow \mathbb{C}$
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A (real) polarization is
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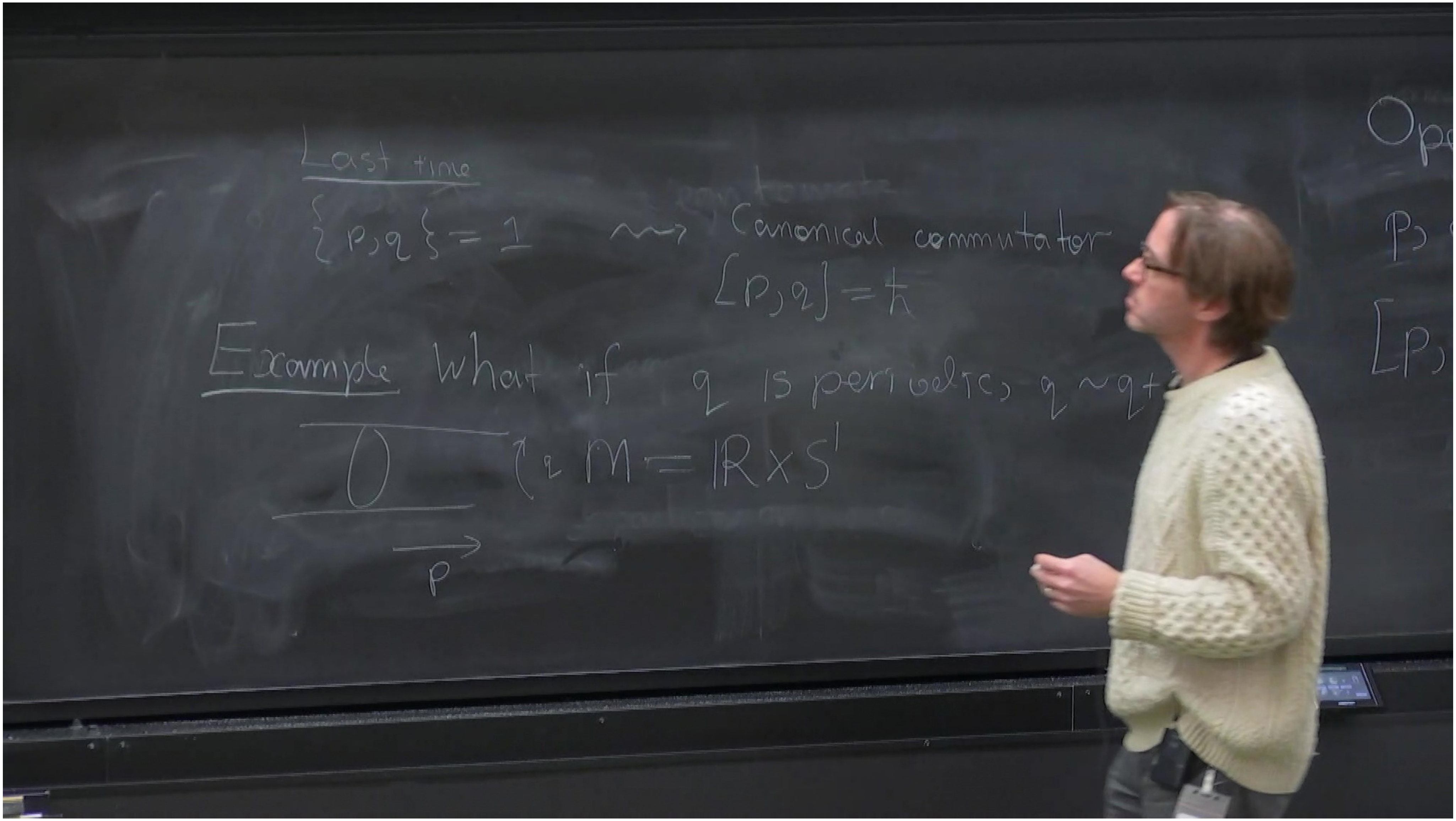

Last time
{p,q} = 1
Canonical commutator
[p,q] = ħ
Example What if q is periodic, q ~ q+
q M = ℝ×S¹
p

A basis is $\delta_{p=c}$

Q sends $\delta_{p=c} \to \delta_{p=c-2\pi\hbar}$

Subspace spanned by

$\ldots \delta_{-2\pi\hbar}, \delta_{0}, \delta_{2\pi\hbar}, \delta_{4\pi\hbar}, \ldots$

is closed under hitting it with operators.

Claim:

Hilbert space 1), $f(\varphi)$, is the same as the subspace spanned by $\delta_0, \delta_{2\pi\hbar}, \ldots$

A function of φ is a sum of

$$\varphi^n = e^{2\pi i n q}$$

$$P \to -i\hbar \frac{\partial}{\partial q}$$

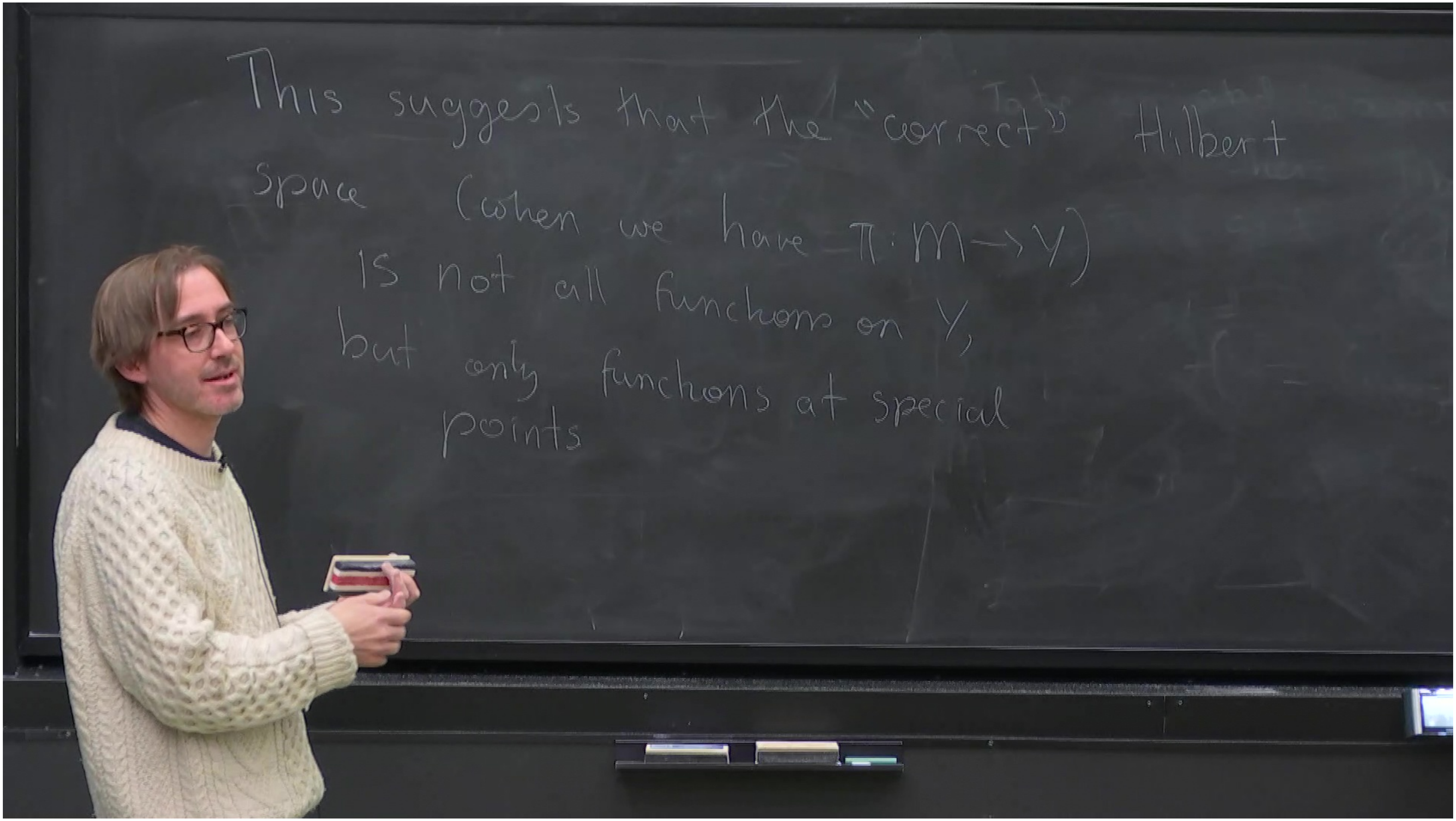
$$Q \cdot \varphi^n = \varphi^{n+1}$$

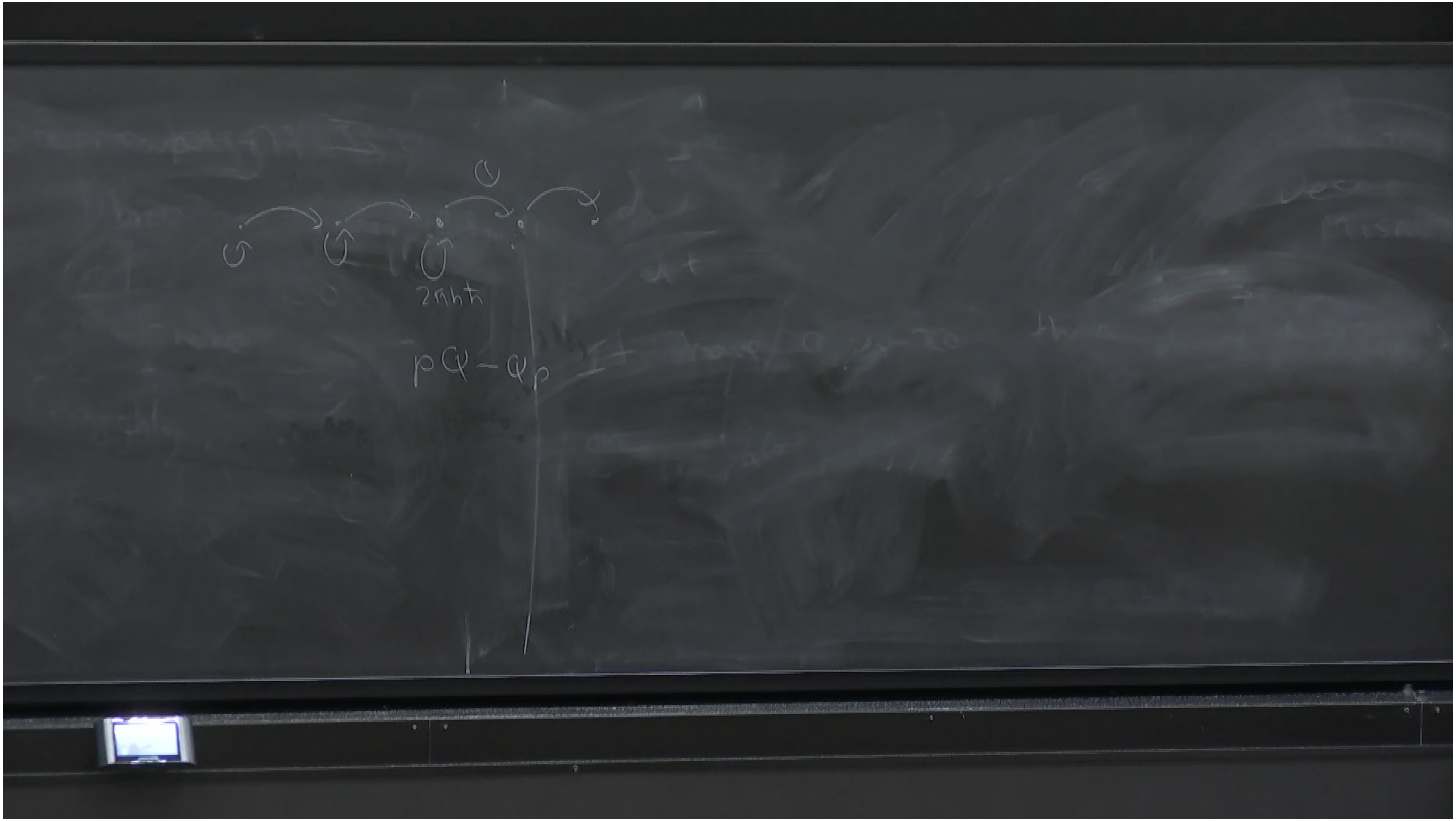
$$p \cdot \varphi^n = 2\pi n \hbar \, \varphi^n$$

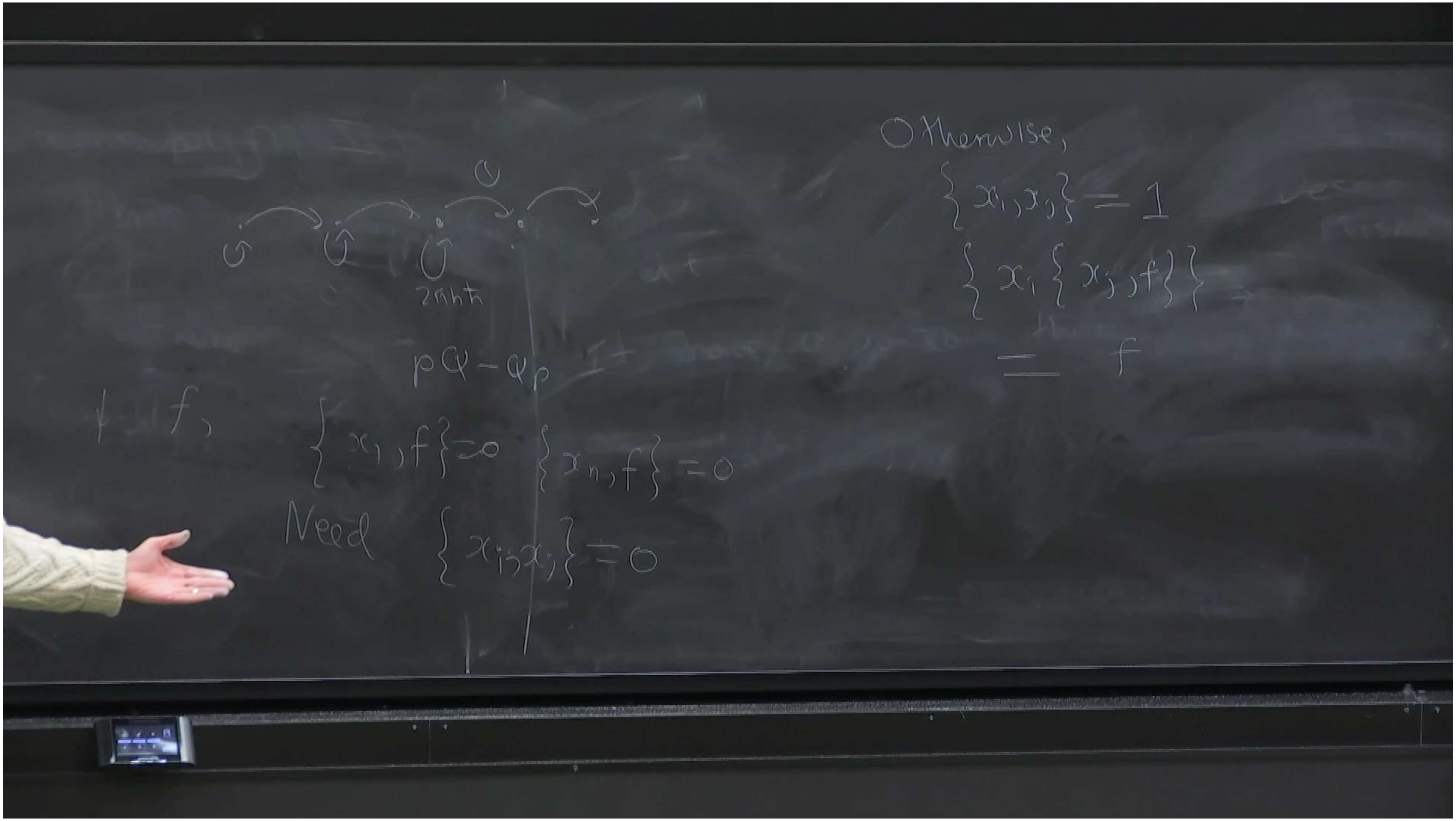
$$p \cdot \delta_{2\pi n \hbar} = 2\pi n \hbar \, \delta_{2\pi n \hbar}$$

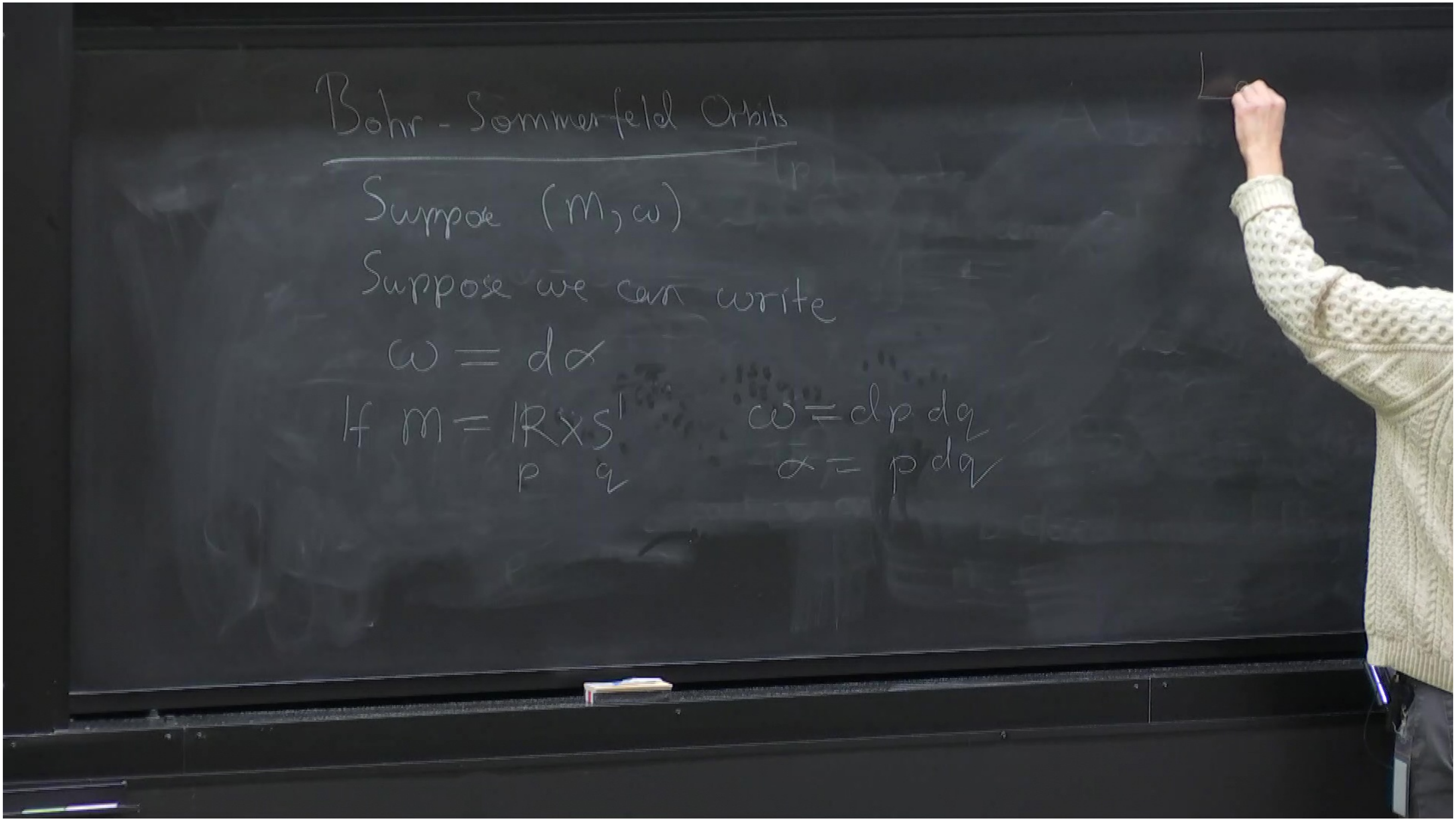
$$Q = e^{2\pi \hbar \partial_p} \delta_{2\pi n \hbar} = \delta_{2\pi (n+1) \hbar}$$

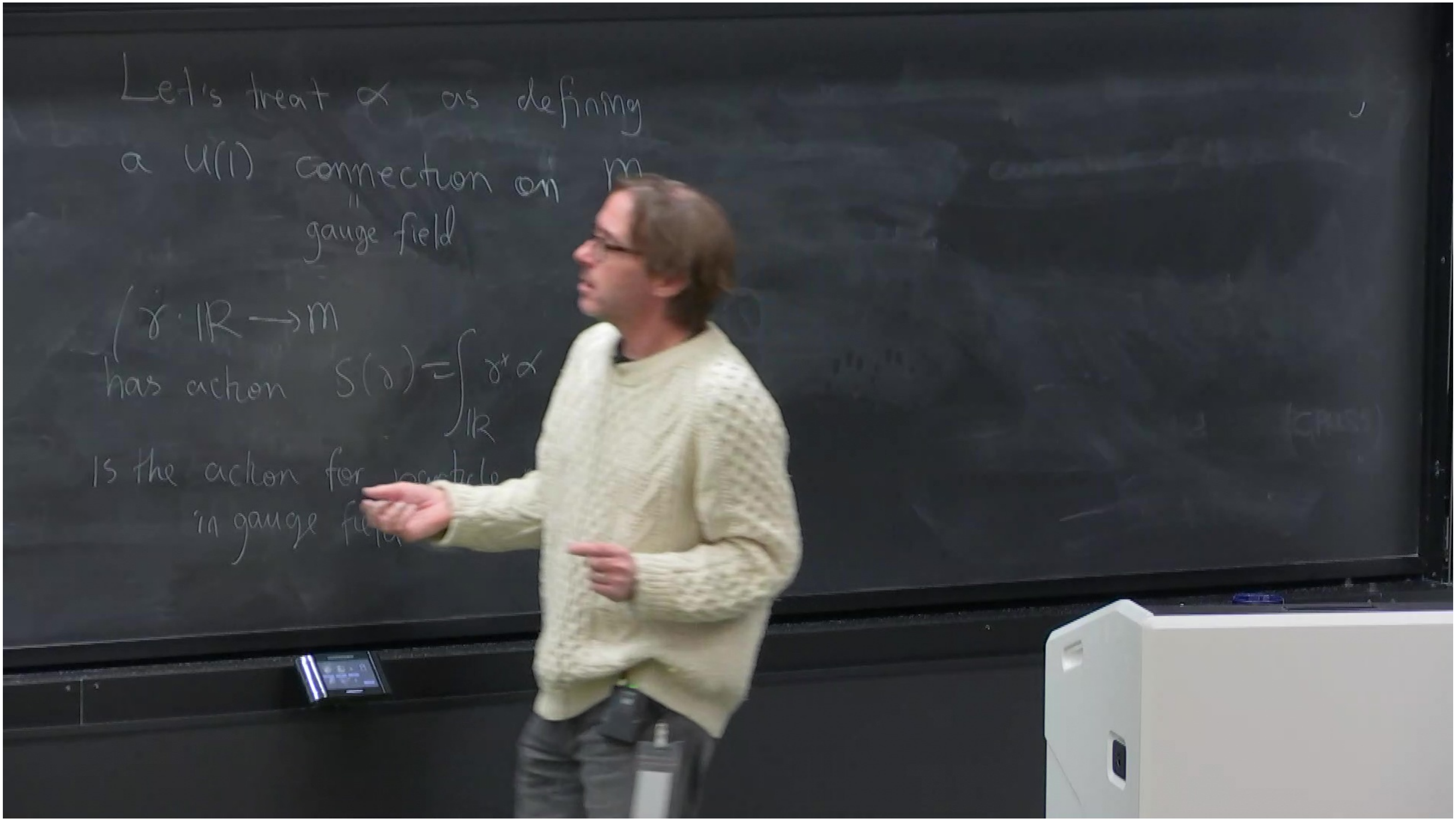
$e^{2\pi i n \hbar} \longleftrightarrow \delta_{p = 2\pi n \hbar}$ is compatible with p, Q action.


This suggests that the "correct" Hilbert
space (when we have $\pi: M \to Y$)
is not all functions on Y,
but only functions at special
points


2πnħ
pφ − φp


2πnħ
pq − qp
∀ f,
{x_1, f} = 0
{x_n, f} = 0
Need {x_i, x_j} = 0
Otherwise,
{x_i, x_j} = 1
{x_i {x_j, f}}
= f


Bohr-Sommerfeld Orbits
Suppose (M, ω)
Suppose we can write
ω = dα
If M = ℝ × S¹   ω = dp dq
p   q
α = p dq


Let's treat α as defining
a U(1) connection on M
gauge field
γ: ℝ → M
has action S(γ) = ∫ℝ γ*α
is the action for
in gauge

If we have a gauge field α,

γ is a loop in M,

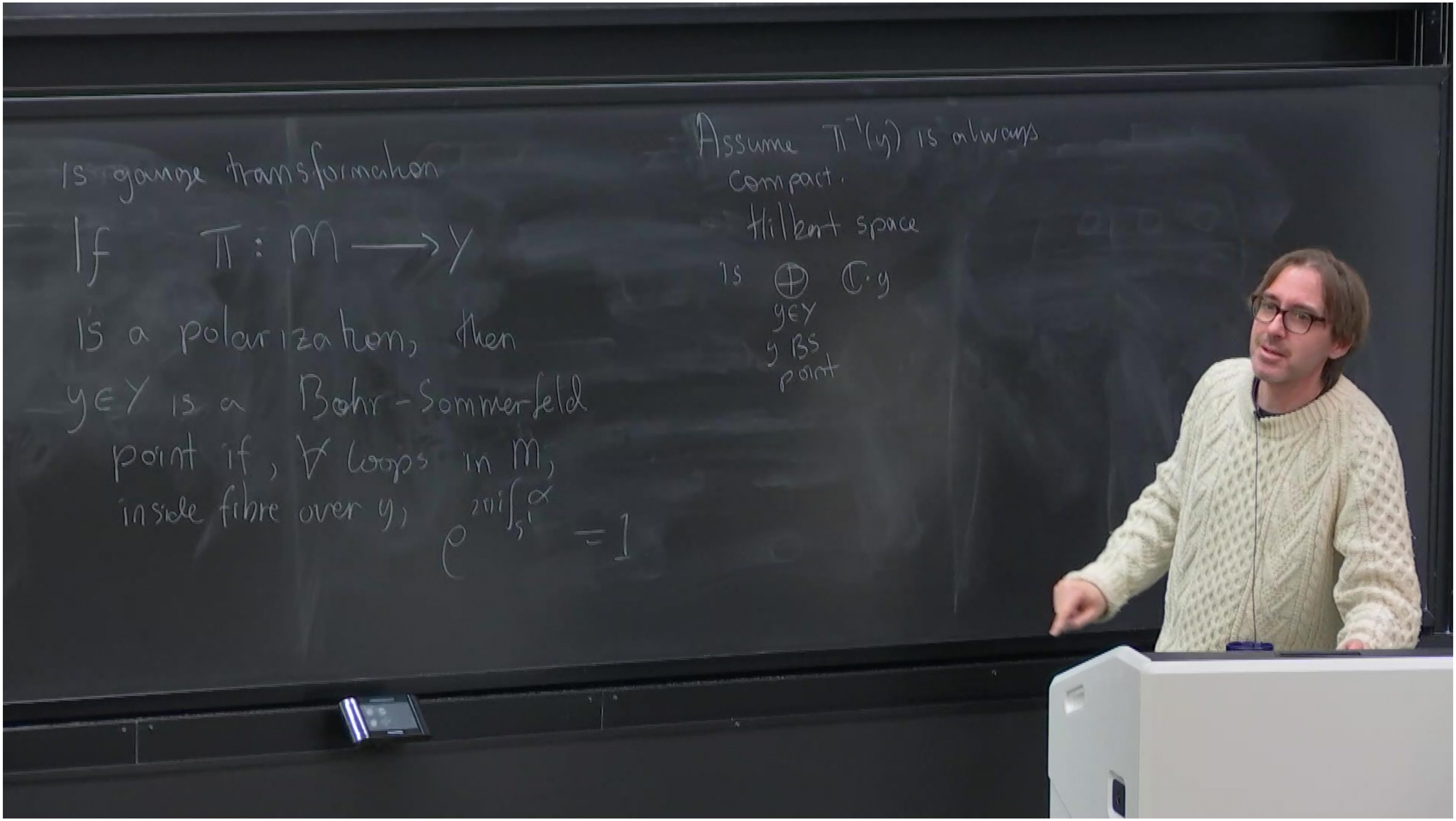
$e^{2\pi i \int_{S^1} \alpha}$ is a gauge invariant quantity

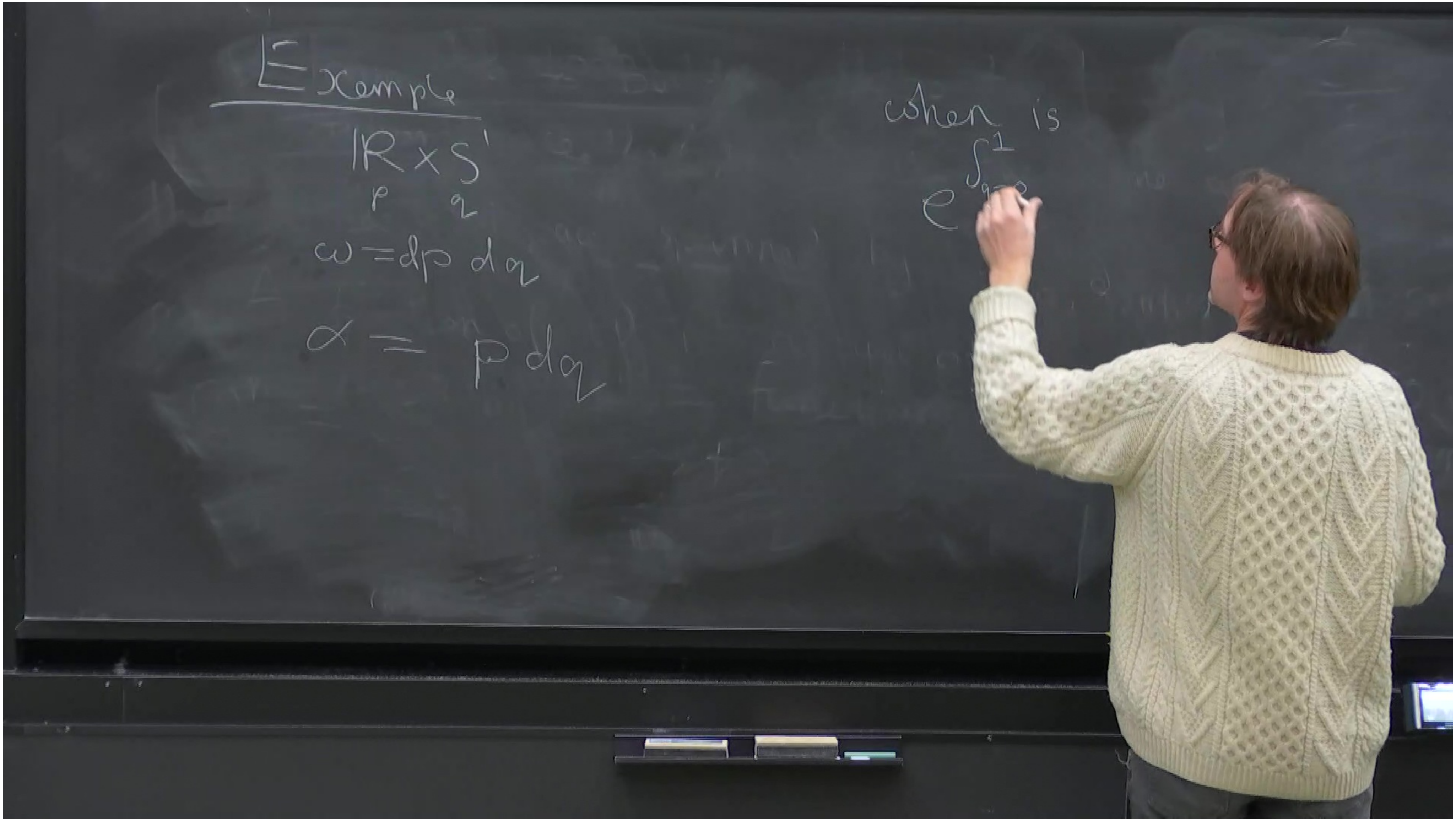
$$2\pi i \alpha \longrightarrow 2\pi i \alpha + X^{-1} dX$$

$$X : M \longrightarrow U(1)$$

is gauge transformation

If $\pi : M \longrightarrow$


is gauge transformation
If $\pi : M \longrightarrow Y$
is a polarization, then
$y \in Y$ is a Bohr-Sommerfeld
point if, $\forall$ loops in M,
inside fibre over y, $e^{2\pi i \int_{S^1} \alpha} = 1$
Assume $\pi^{-1}(y)$ is always
compact.
Hilbert space
is $\bigoplus_{\substack{y \in Y \\ y \text{ BS} \\ \text{point}}} \mathbb{C} \cdot y$


Example
$\mathbb{R} \times S^1$
$p \quad q$
$\omega = dp\, dq$
$\alpha = p\, dq$
when is

Example

$$\underset{p}{\mathbb{R}} \times \underset{q}{S^1}$$

$$\omega = dp\, dq$$

$$\alpha = p\, dq$$

when is

$$e^{\int_{q=0}^{1} (p\, dq)\, 2\pi i} = 1\,?$$

This is

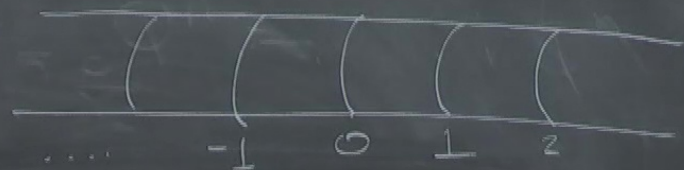
$$e^{2\pi i p} = 1,$$

$p \in \mathbb{Z}$, as before.

up to normalization

This differ by 1.

So, $\int_{\bigcirc} \omega = 1$

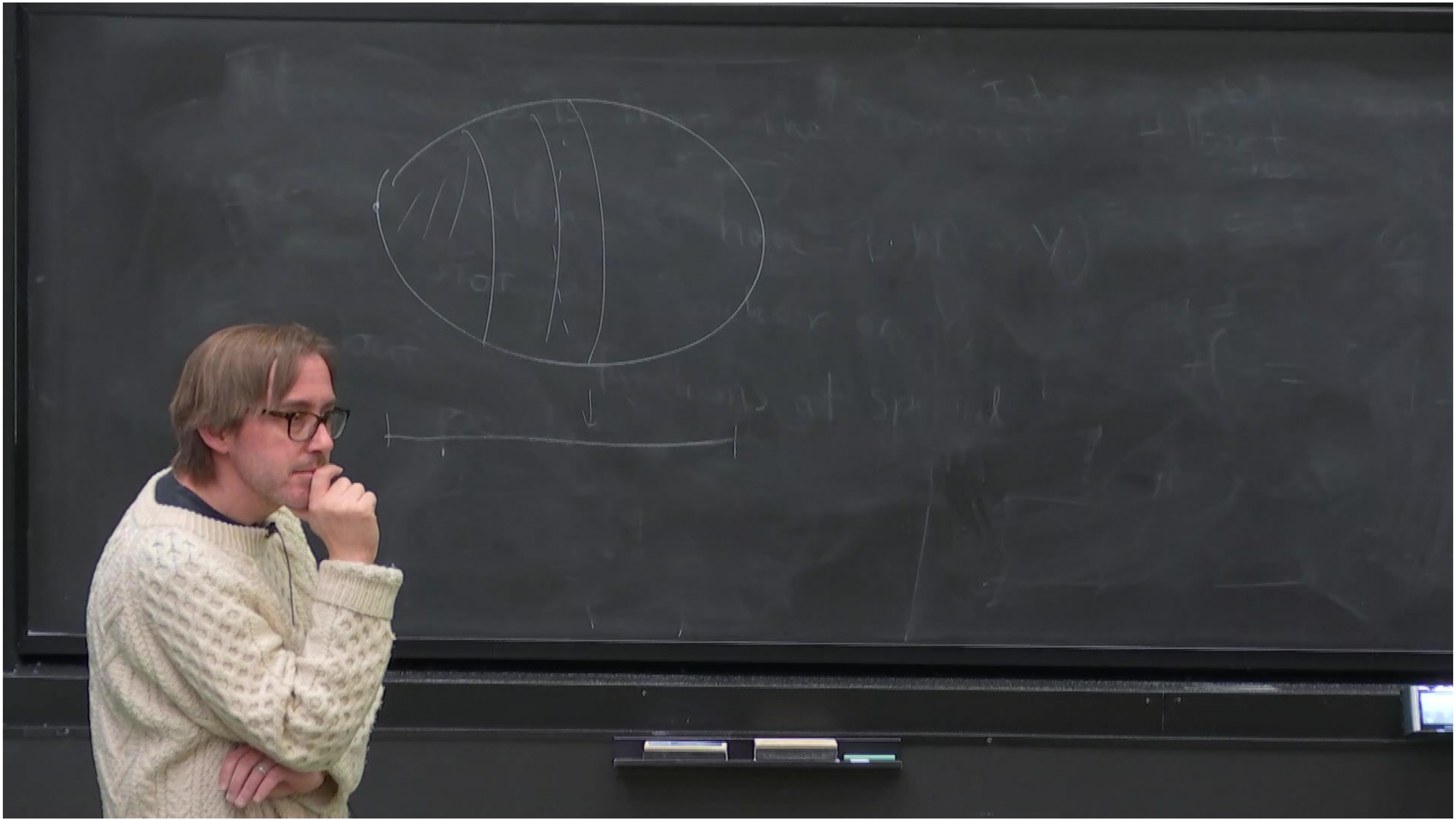


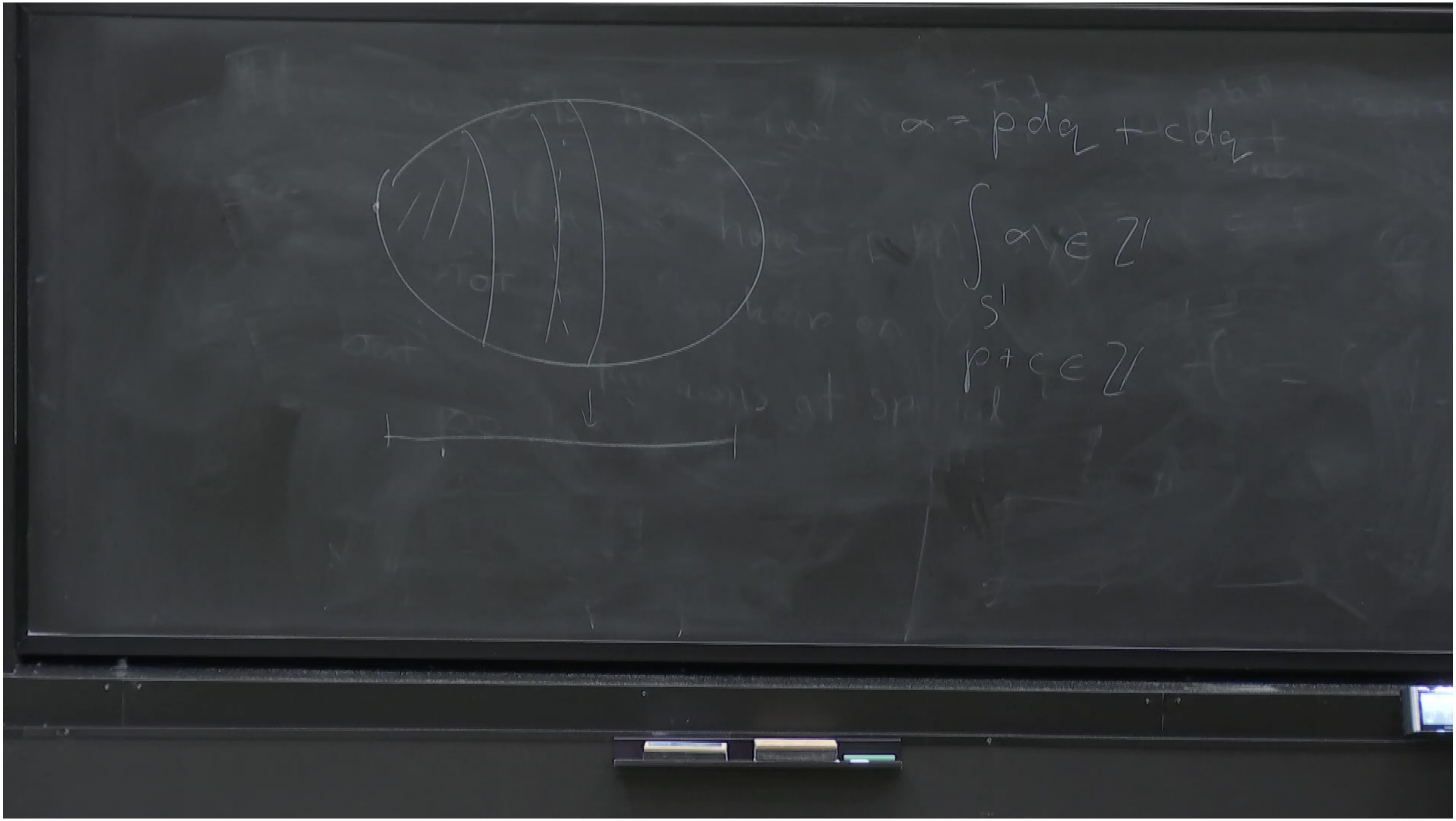
Area between 2 orbits = 1.

$$d\alpha = \omega$$

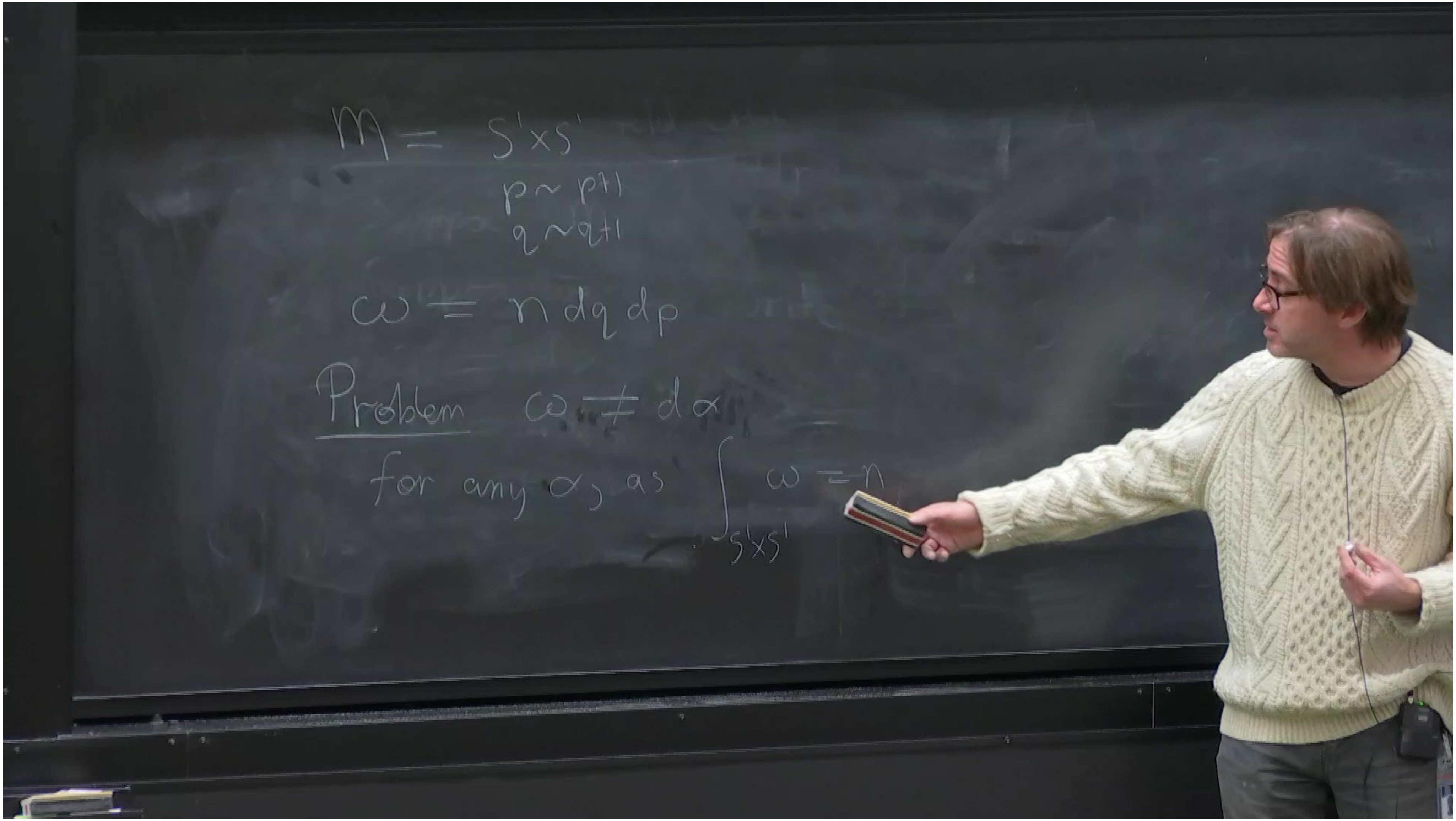
cylinder between orbits

$$\int \omega = \int d\alpha = -\int_{0} \alpha + \int_{0} \alpha$$





$\alpha = p\,dq_1 + c\,dq_2 +$
$\int_{S^1} \alpha_\gamma \in \mathbb{Z}$
$p + q \in \mathbb{Z}$


$M = S^1 \times S^1$
$p \sim p+1$
$q \sim q+1$
$\omega = n\, dq\, dp$
Problem $\omega \neq d\alpha$
for any α, as $\int_{S^1 \times S^1} \omega = n$

$M = S^1 \times S^1$

$p \sim p+1$

$q \sim q+1$

$\omega = n\, dq\, dp$

<u>Problem</u> $\omega \neq d\alpha$

for any α, as $\displaystyle\int_{S^1 \times S^1} \omega = n$

Better:

$\omega =$ Field strength of a $U(1)$ connection on a non-trivial $U(1)$ bundle.

Bundle is trivial away from $q = 0$

of

on

d $U(1)$

way from $\mathbb{R}$

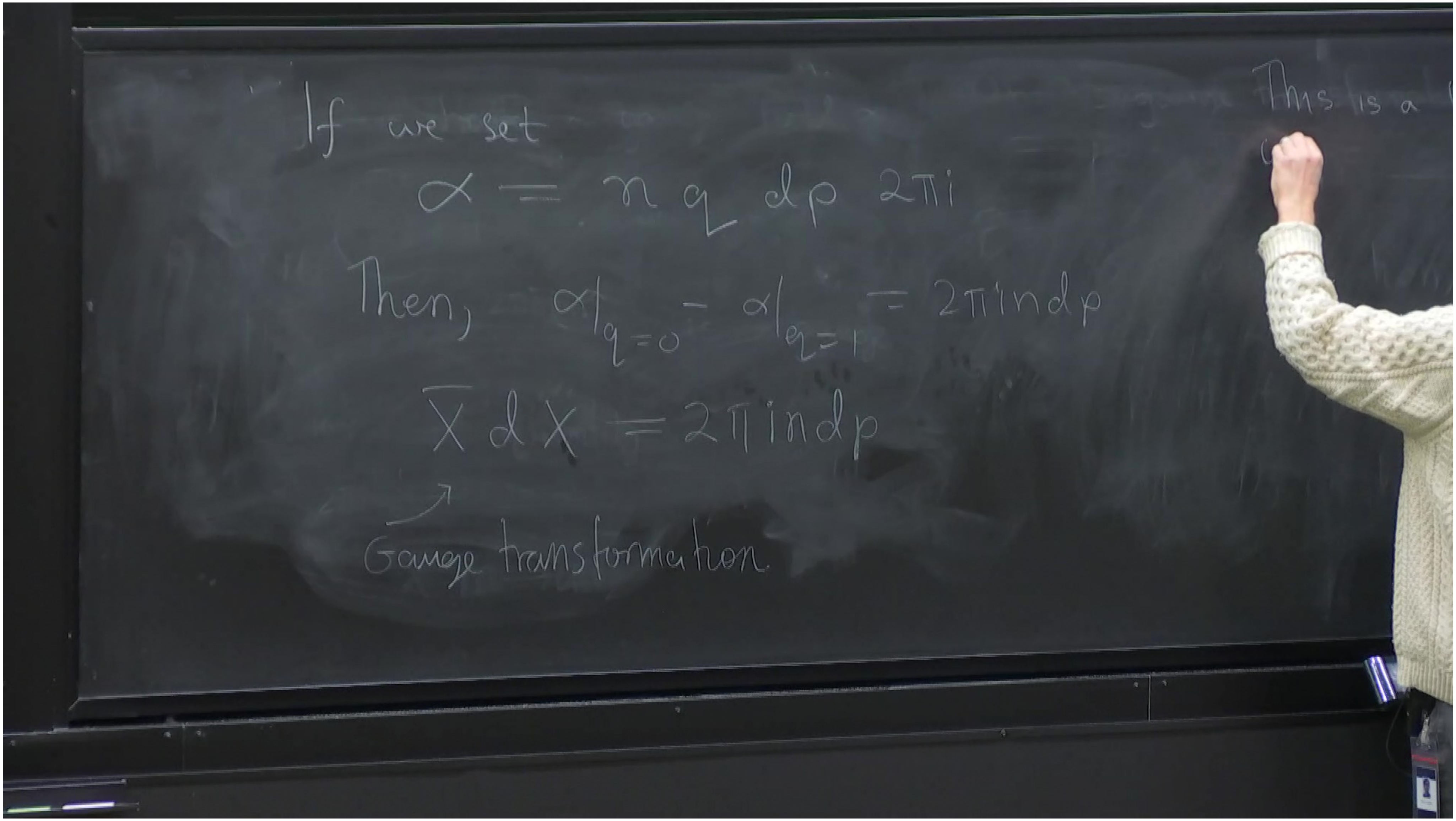
$0 \longrightarrow 1$

q

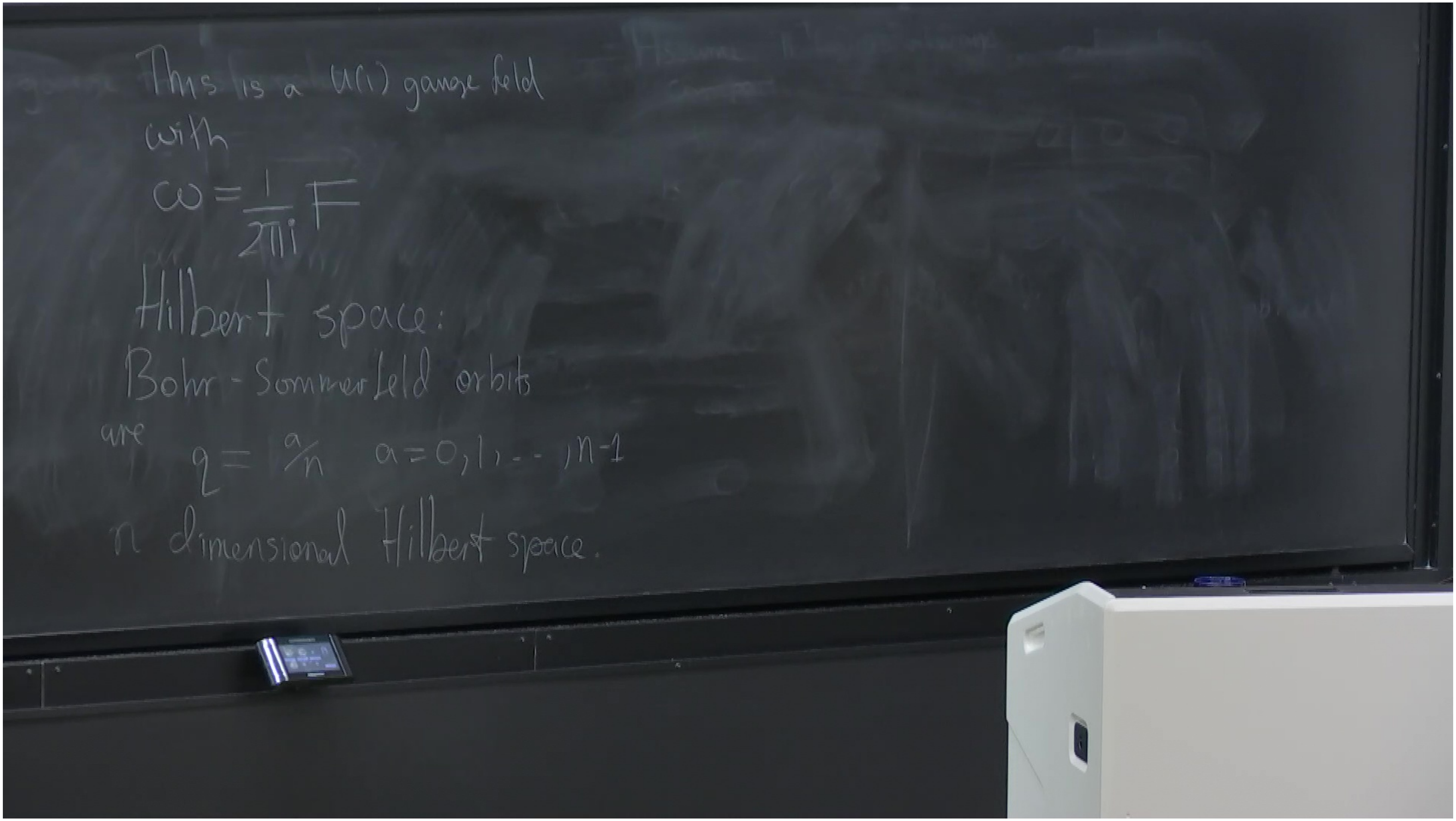
when we identify both sides we apply a gauge trans

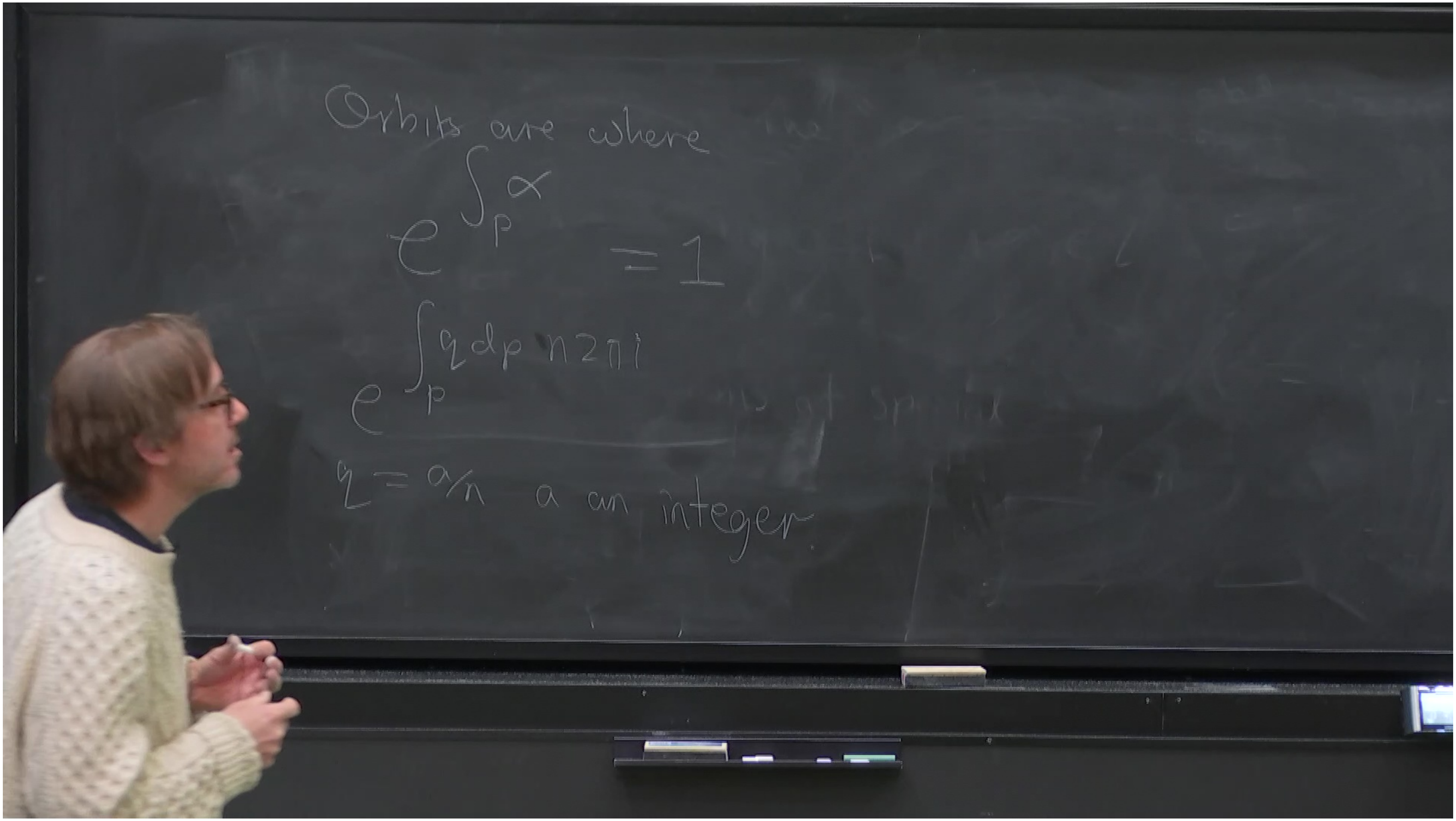
$$X = e^{2\pi i n p}$$

(GPMSS)



If we set
$\alpha = n q \, dp \, 2\pi i$
Then, $\alpha|_{q=0} - \alpha|_{q=1} = 2\pi i n \, dp$
$\bar{X} dX = 2\pi i n \, dp$
Gauge transformation
This is a


This is a U(1) gauge field
with
$\omega = \frac{1}{2\pi i} F$
Hilbert space:
Bohr - Sommerfeld orbits
are $q = a/n$ $a = 0, 1, \ldots, n-1$
n dimensional Hilbert space.


Orbits are where
e ∫p α = 1
e ∫p η dp n2πi
η = a/n a an integer

Question

We saw before that algebra of operators is given by

$$P, Q$$

satisfying $PQ = QPe^{2\pi i \hbar}$

in this case, $\hbar = 1/n$