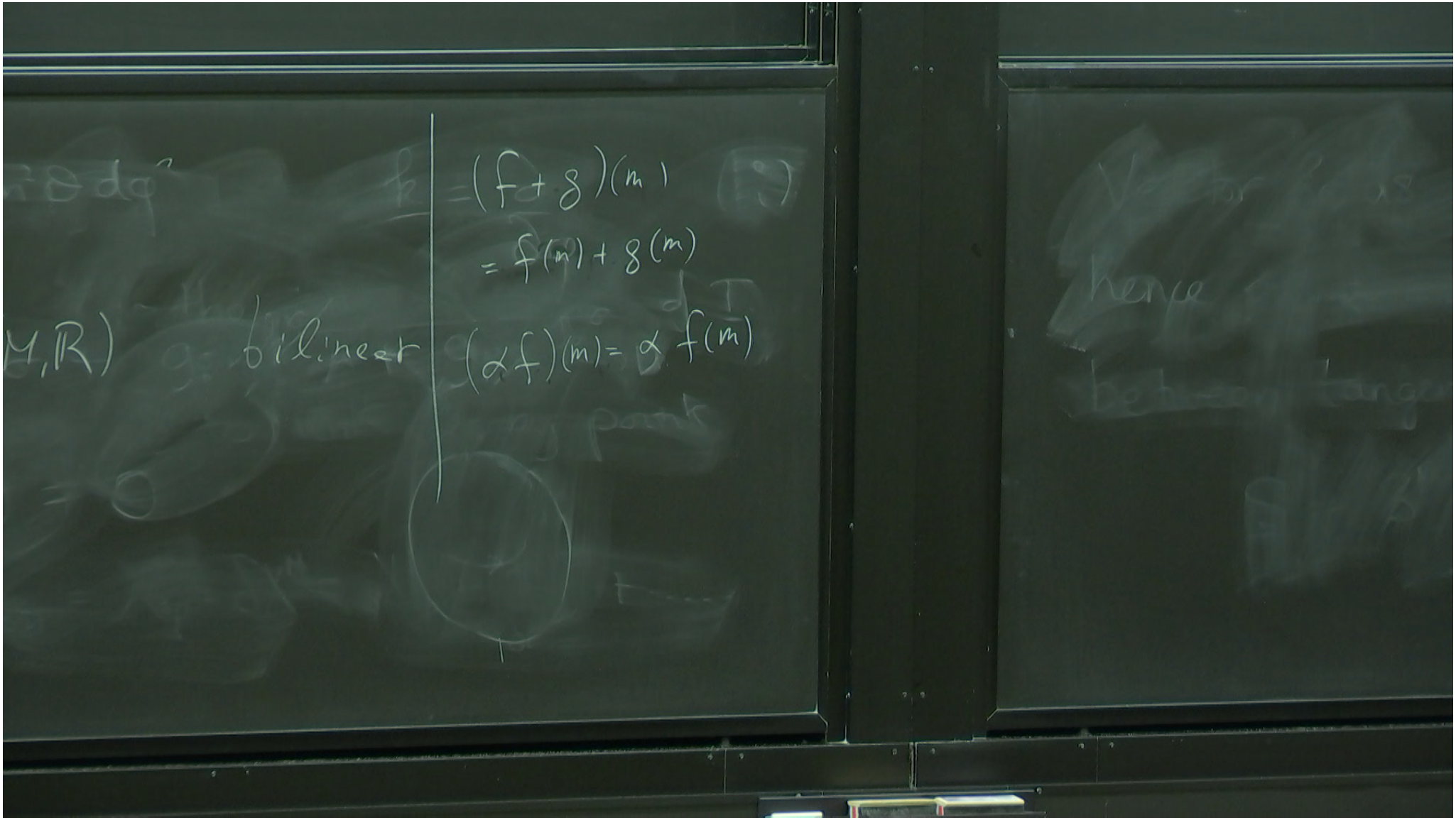
Title: Mathematical Physics Lecture - 230113

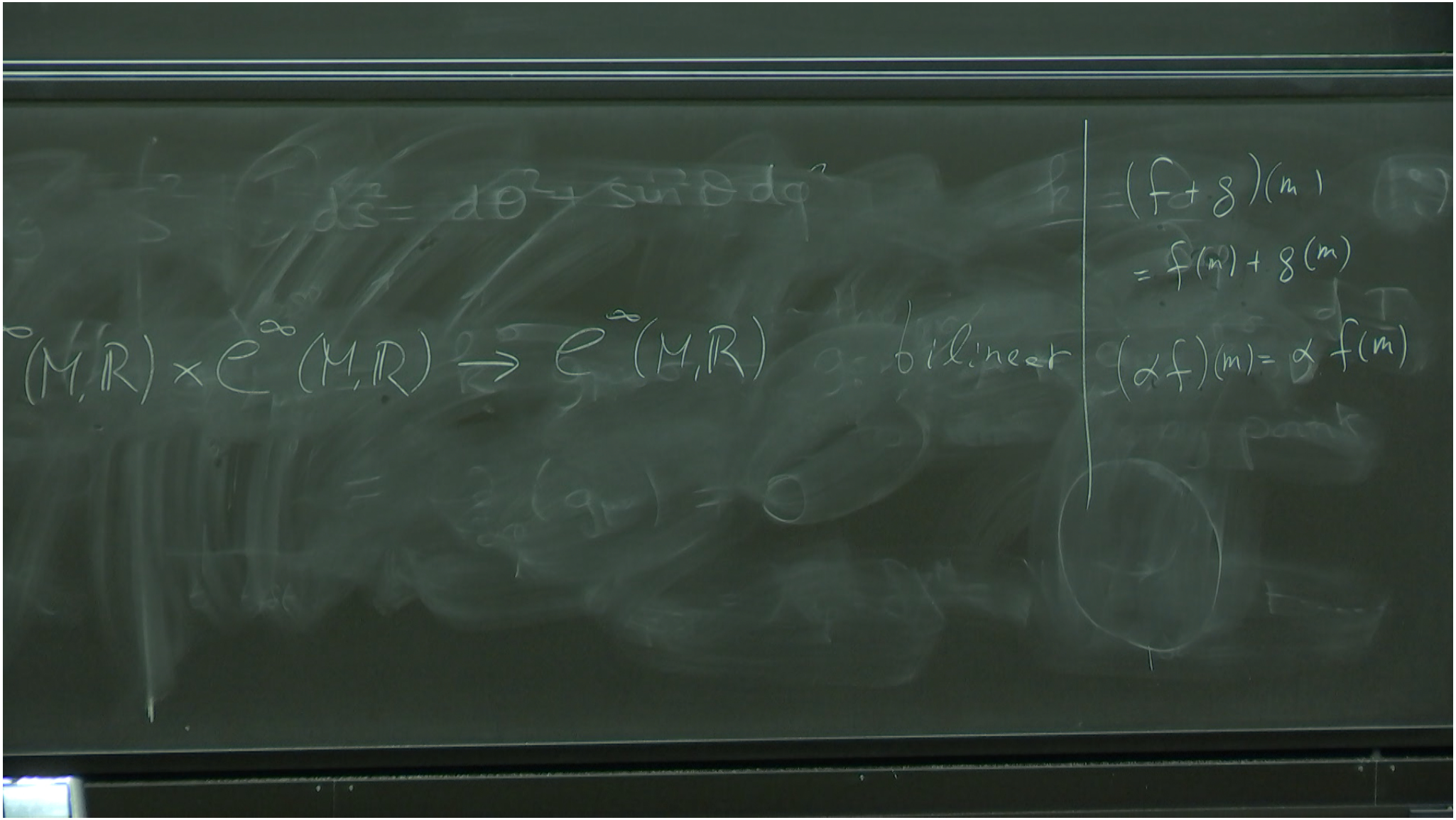
Speakers: Giuseppe Sellaroli

Collection: Mathematical Physics (2022/2023)

Date: January 13, 2023 - 2:00 PM

URL: https://pirsa.org/23010019

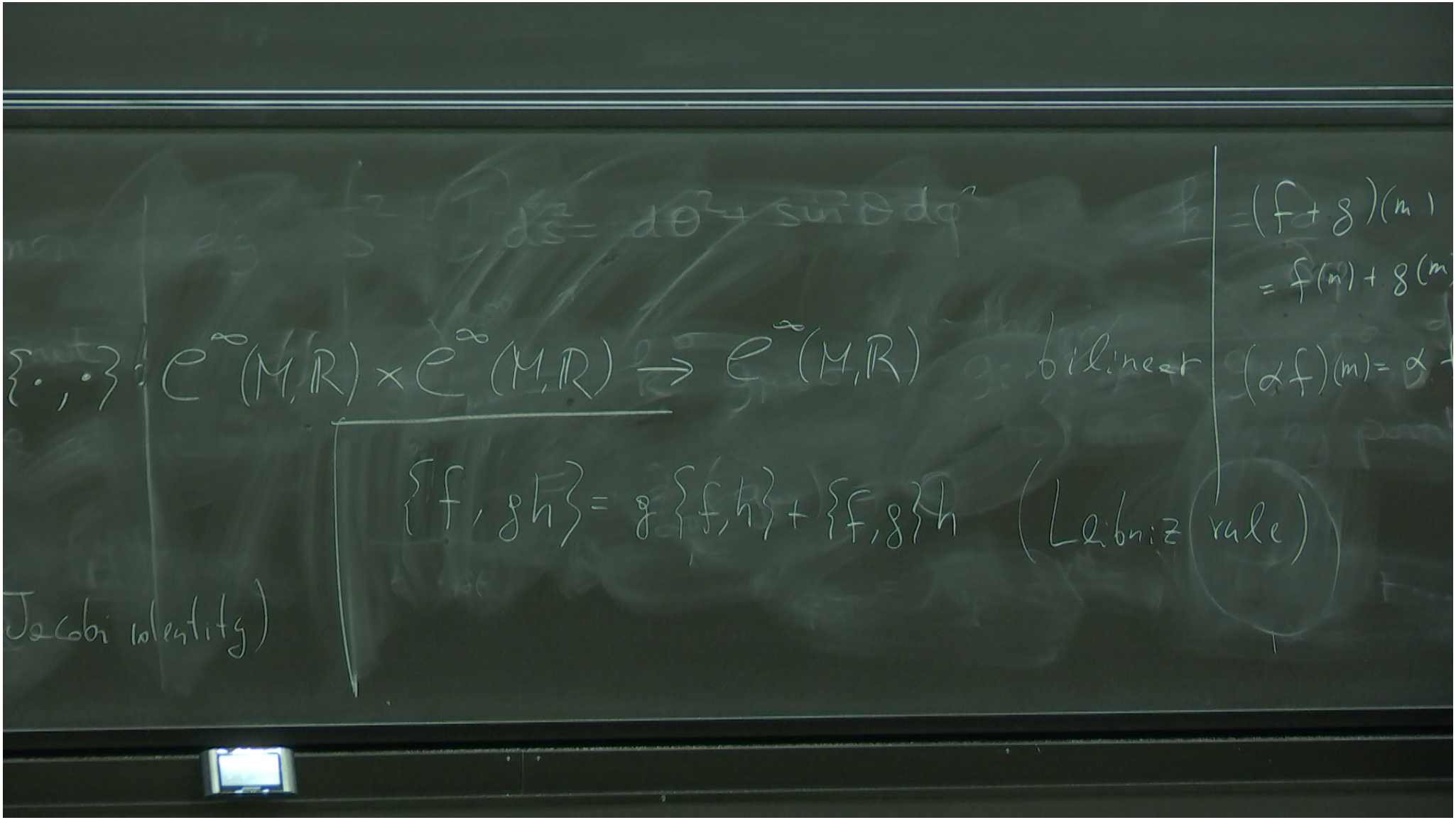

$(f+g)(m)$
$= f(m) + g(m)$
bilinear
$(\alpha f)(m) = \alpha f(m)$
hence


$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$
$C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$
bilinear
$(f+g)(m)$
$= f(m) + g(m)$
$(\alpha f)(m) = \alpha f(m)$

Symplectic manifolds

Poisson bracket on manifold M: $\{\cdot,\cdot\}: C^\infty(M,\mathbb{R}) \times C$

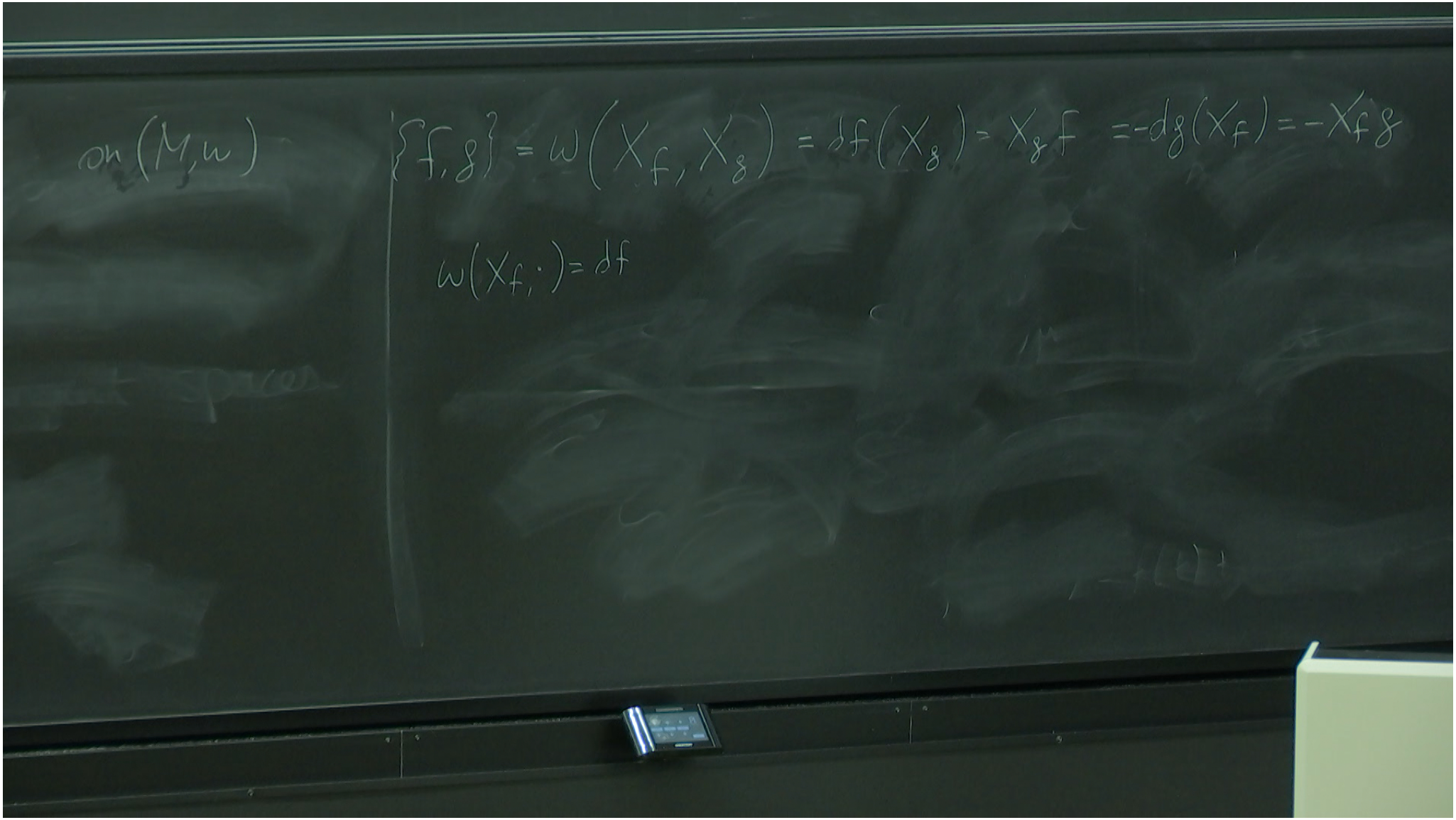
- $\{f,g\} = -\{g,f\}$
- $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$ (Jacobi identity)

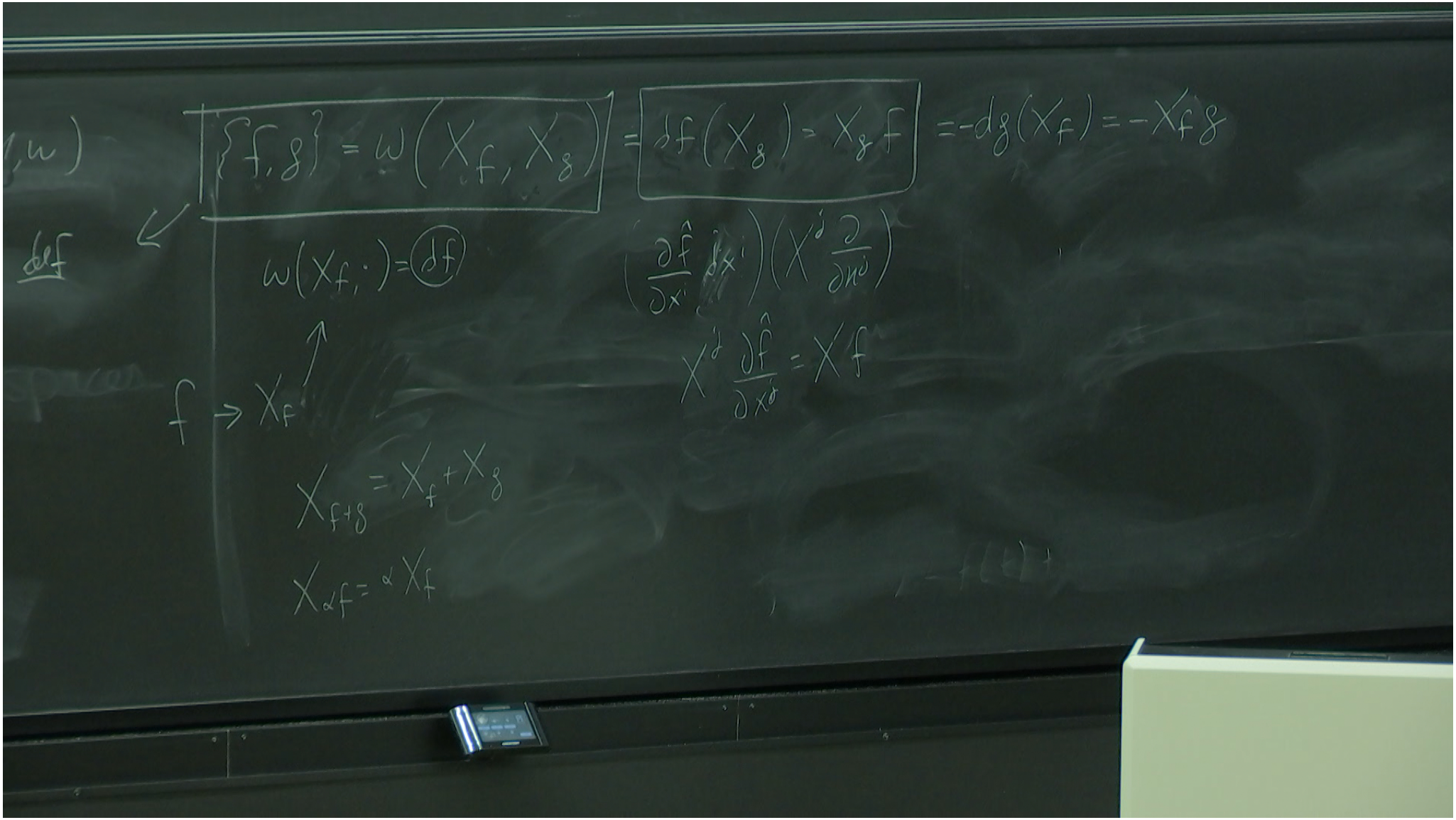

$ds^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$
$\{\cdot,\cdot\} : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$
bilinear
$(f+g)(m) = f(m) + g(m)$
$(\alpha f)(m) = \alpha$
$\{f, gh\} = g\{f,h\} + \{f,g\}h$ (Leibniz rule)
Jacobi identity)

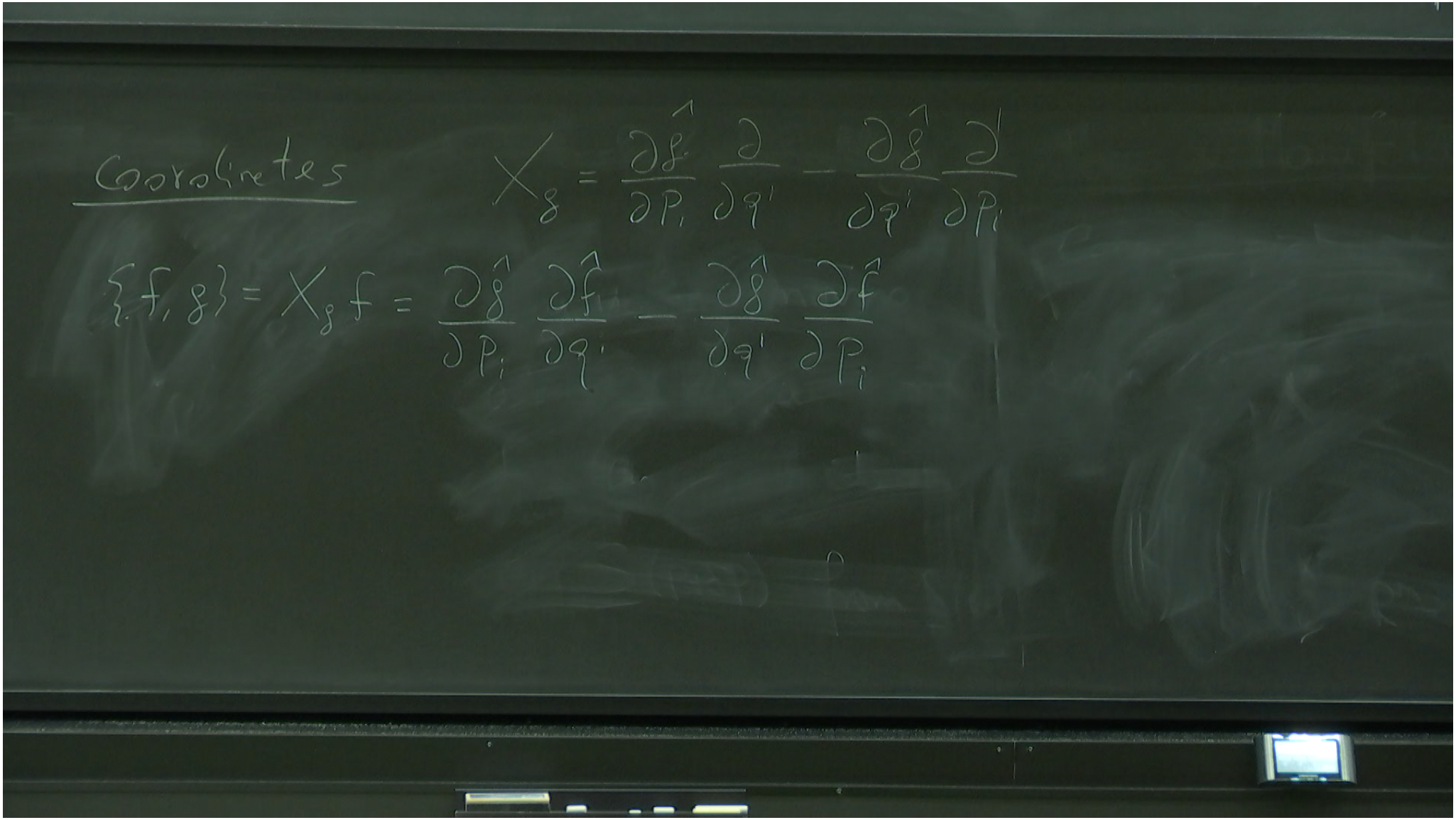
$(M, \{\cdot,\cdot\})$ Poisson manifold

(M, ω) symplectic

→ define Poisson bracket


on (M,ω)
$\{f,g\} = \omega(X_f, X_g) = df(X_g) = X_g f = -dg(X_f) = -X_f g$
$\omega(X_f, \cdot) = df$


$\{f,g\} = \omega(X_f, X_g)$
$= df(X_g) = X_g f$
$= -dg(X_f) = -X_f g$
def
$\omega(X_f, \cdot) = df$
$f \to X_f$
$X_{f+g} = X_f + X_g$
$X_{\alpha f} = \alpha X_f$
$\left(\frac{\partial \hat{f}}{\partial x^i} dx^i\right)\left(X^j \frac{\partial}{\partial x^j}\right)$
$X^j \frac{\partial \hat{f}}{\partial x^j} = X f$


Coordinates
$X_g = \frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}$
$\{f, g\} = X_g f = \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}$

or identity)

Darboux–Weinstein if $(M, \{\cdot,\cdot\})$ Poisson manifold, then et loc

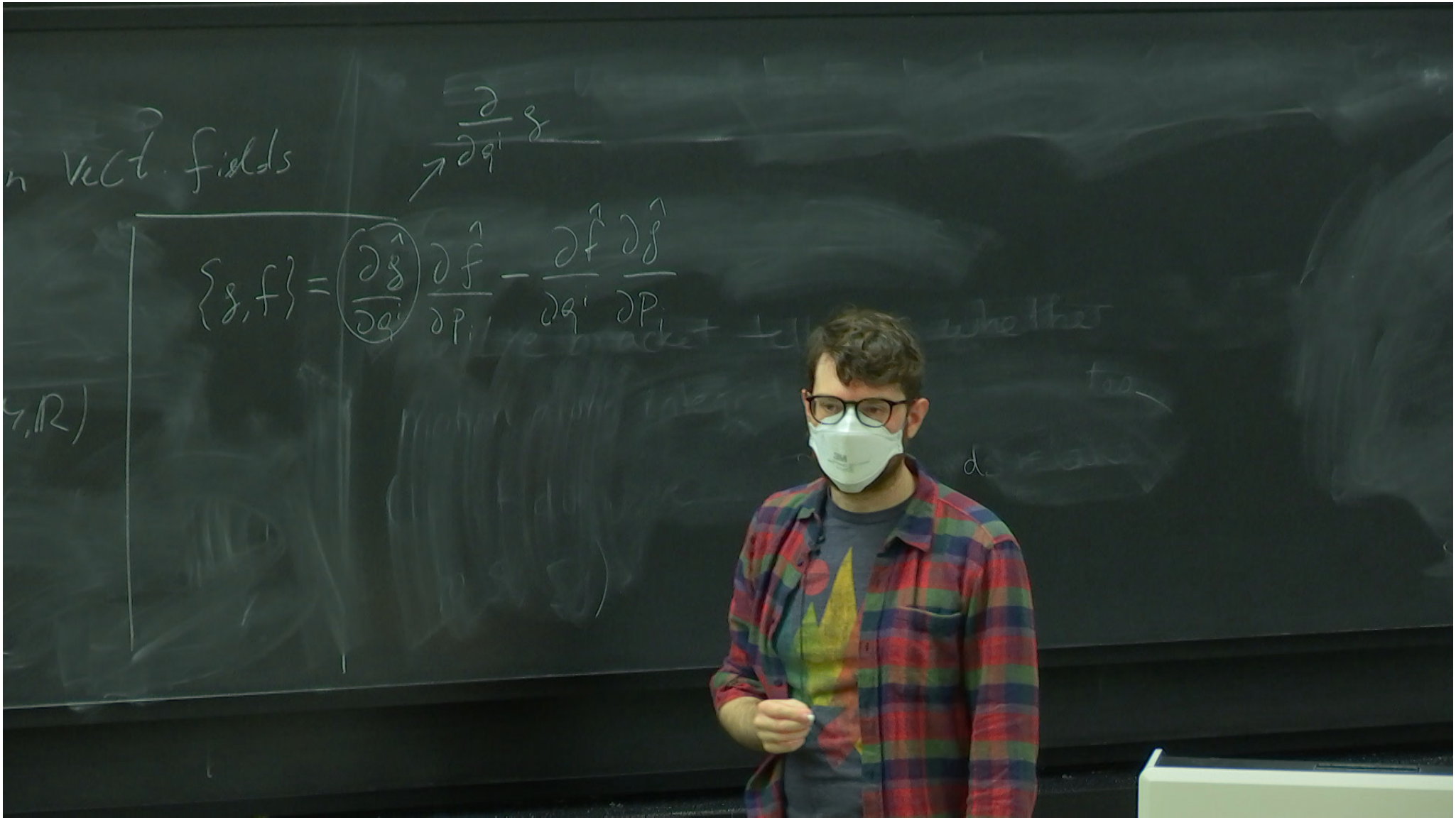
Point $m \in M$ then is a local chart $(U, (q^1, \dots, q^n, p_1, \dots, p_n, g^1, \dots, g^k))$

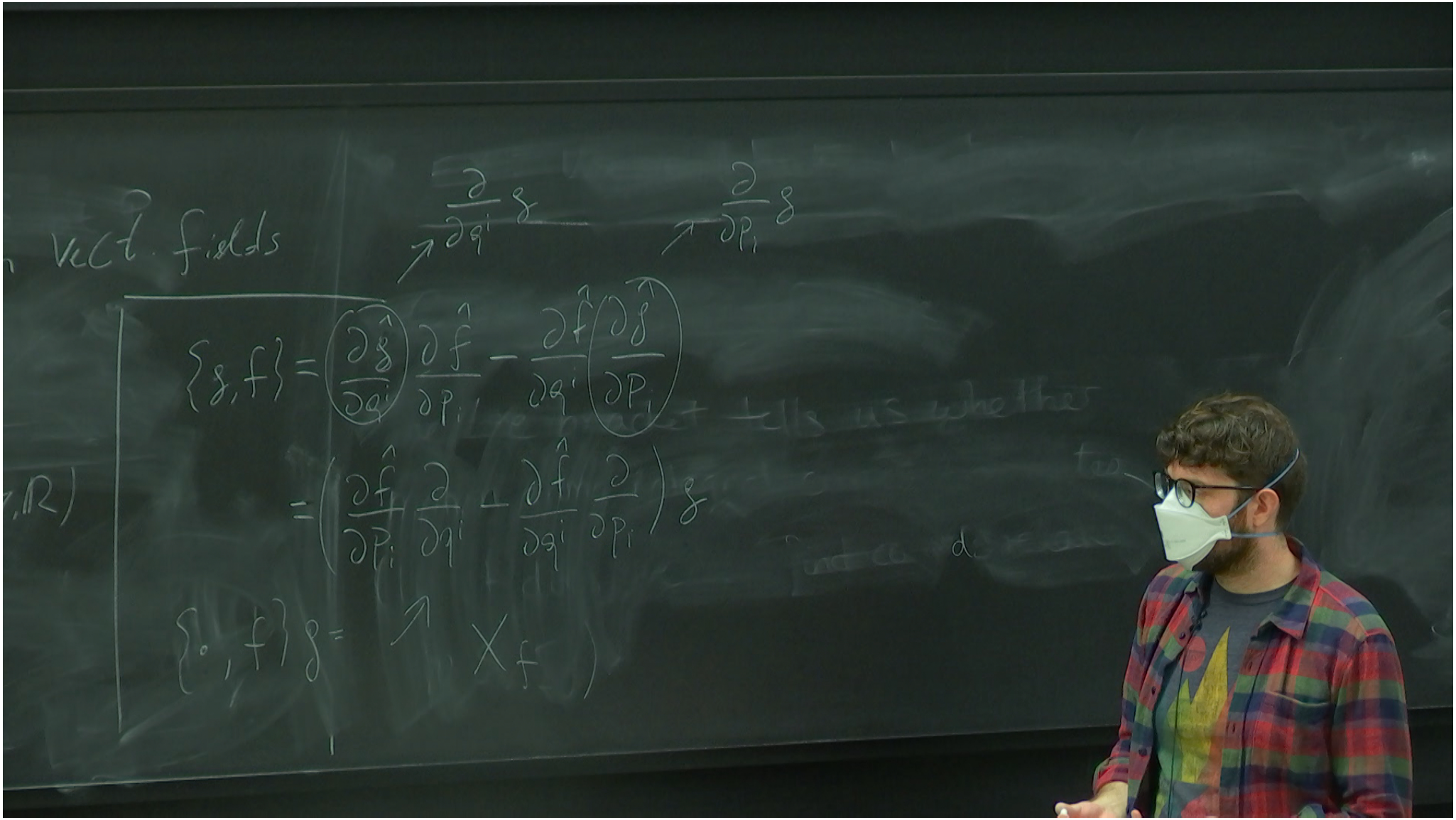
such that $\{f, g\} = \frac{\partial \hat{f}}{\partial q^i}\frac{\partial \hat{g}}{\partial p_i} - \frac{\partial \hat{g}}{\partial q^i}\frac{\partial \hat{f}}{\partial p_i}$

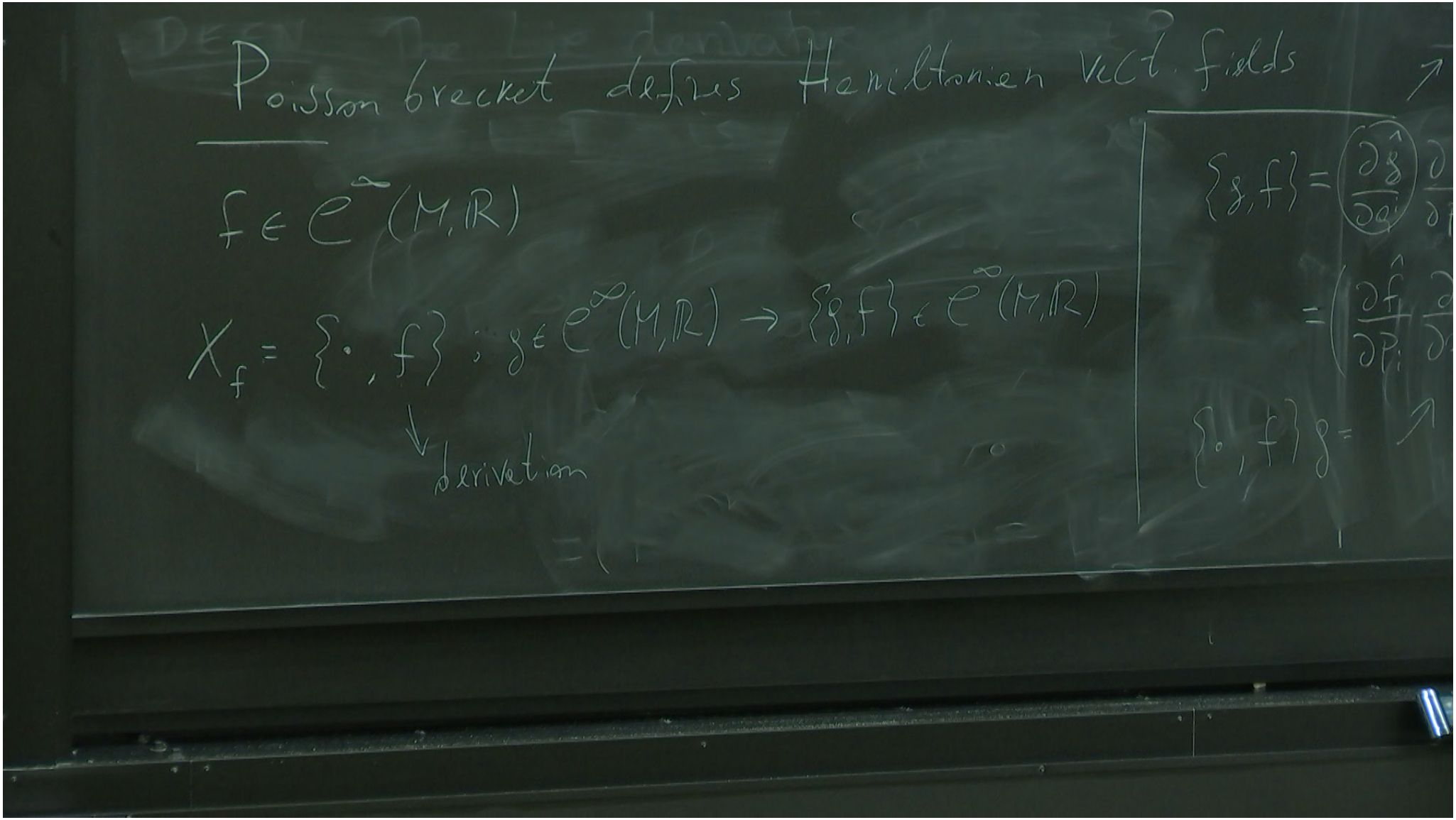
$2n+k$

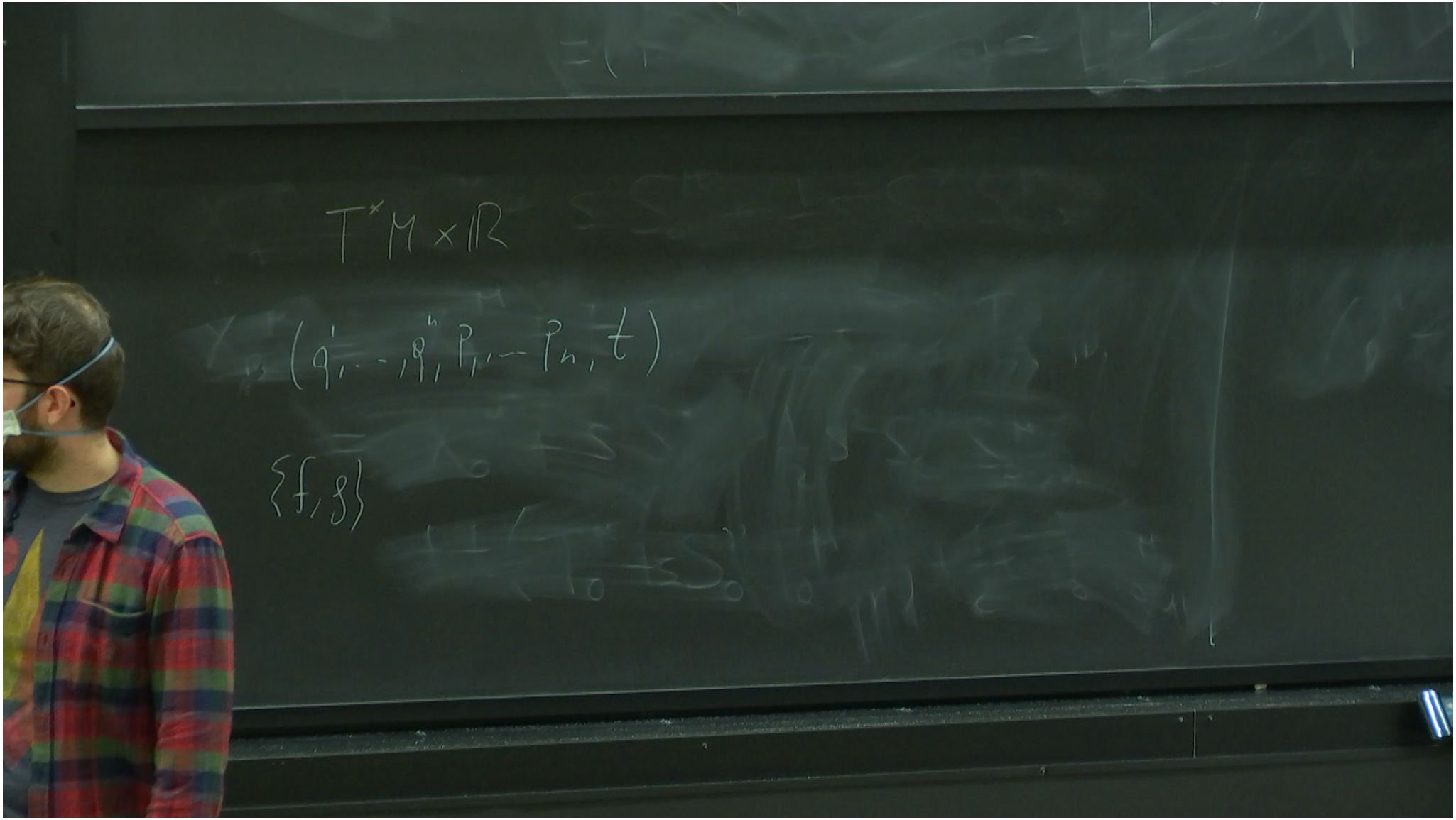
manifold

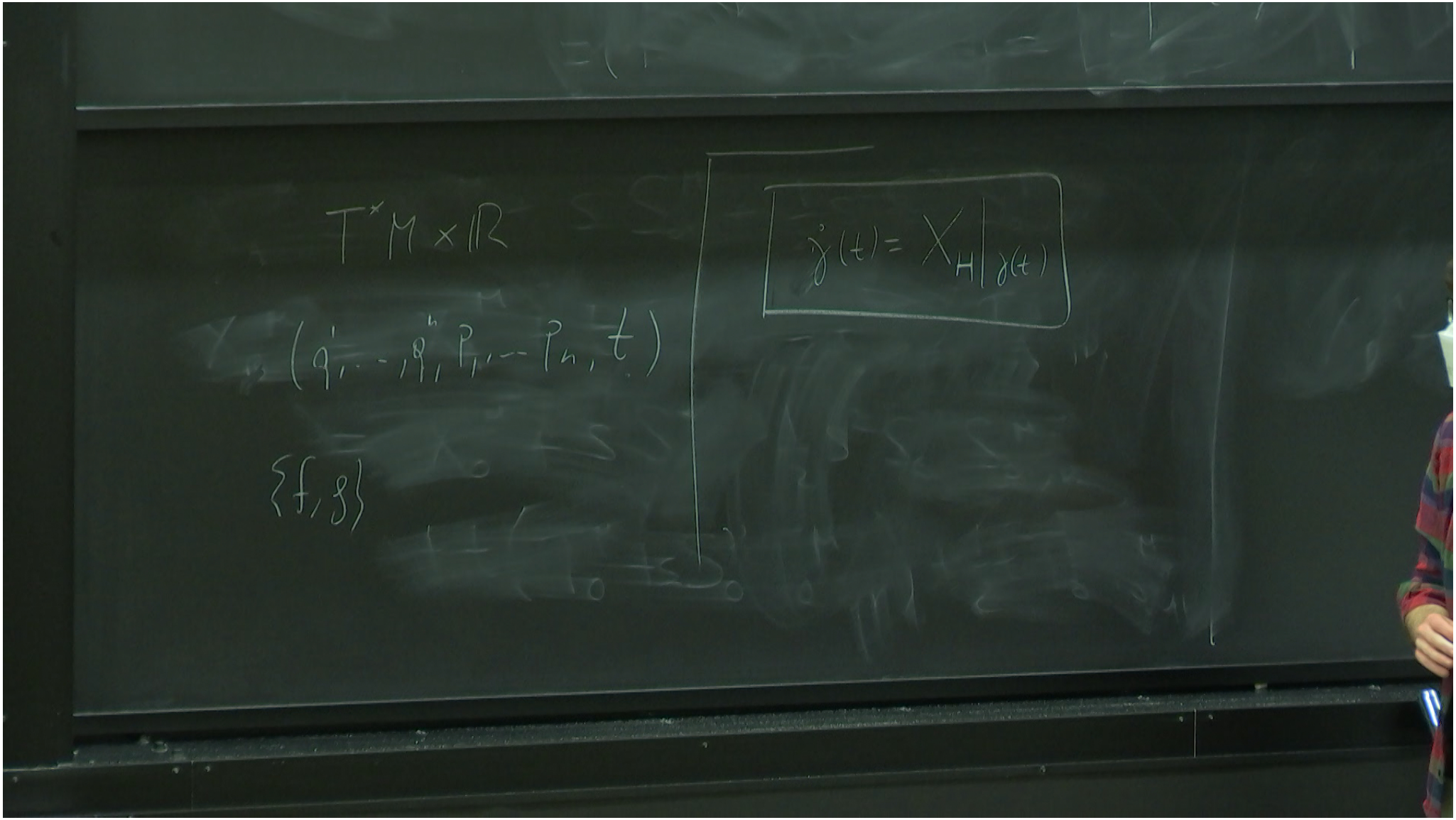
$M \times \mathbb{R}$

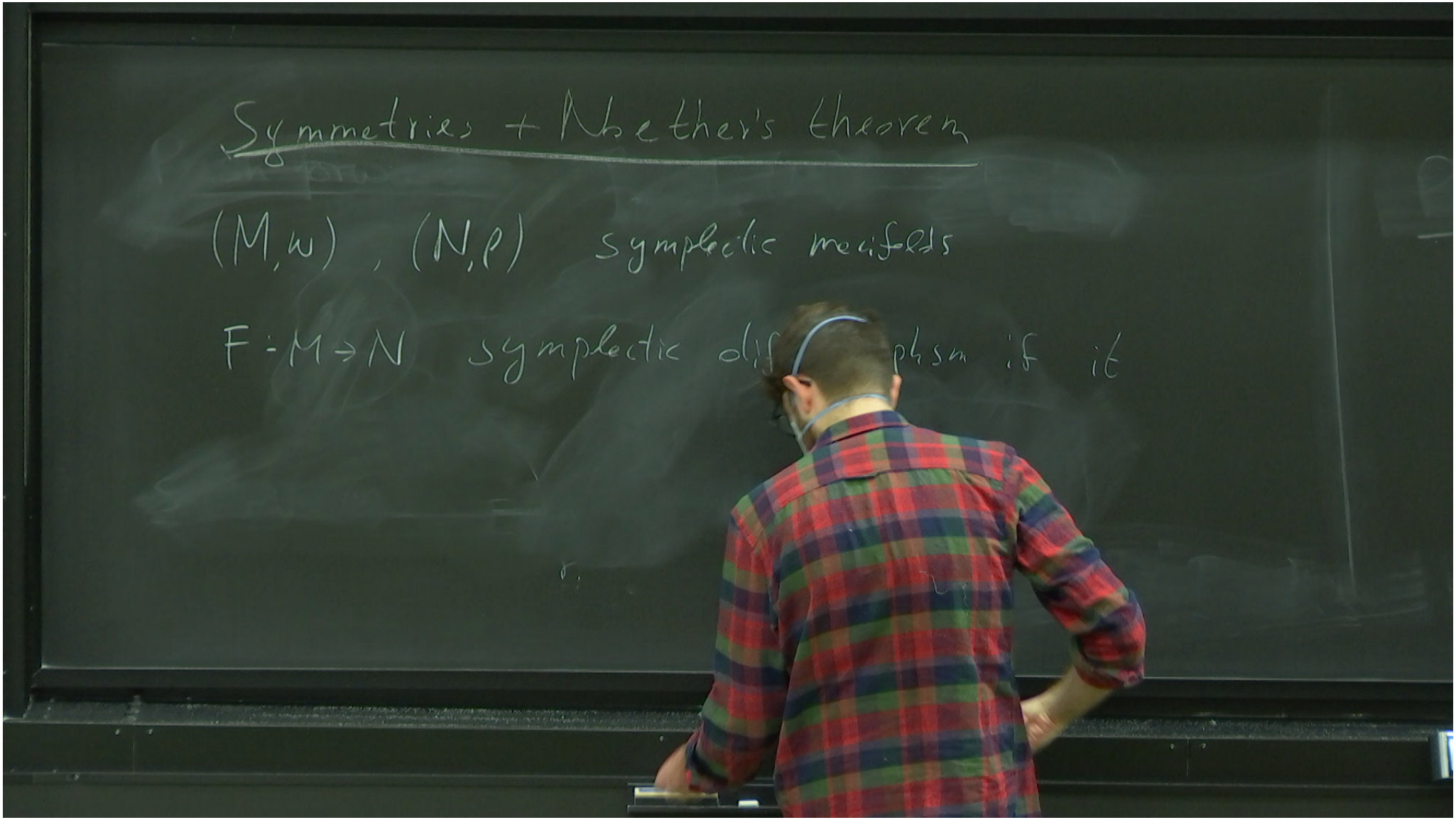

vect. fields
$\frac{\partial}{\partial q^i} g$
$\{g, f\} = \left(\frac{\partial g}{\partial q^i}\right) \frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i}$


vect. fields
$\frac{\partial}{\partial q^i} g$
$\frac{\partial}{\partial p_i} g$
$\{g, f\} = \left(\frac{\partial g}{\partial q^i}\right) \frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial q^i} \left(\frac{\partial g}{\partial p_i}\right)$
$= \left(\frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}\right) g$
$\{\cdot, f\} g = \nearrow X_f$


Poisson bracket defines Hamiltonian Vect. fields
$f \in C^\infty(M,\mathbb{R})$
$X_f = \{\cdot, f\} : g \in C^\infty(M,\mathbb{R}) \to \{g,f\} \in C^\infty(M,\mathbb{R})$
derivation
$\{g,f\} = \left(\frac{\partial g}{\partial q^i}\right)$
$= \left(\frac{\partial f}{\partial p_i}\right.$
$\{\cdot, f\} g =$


$T^*M \times \mathbb{R}$
$(q^1, \ldots, q^n, p_1, \ldots p_n, t)$
$\{f, g\}$

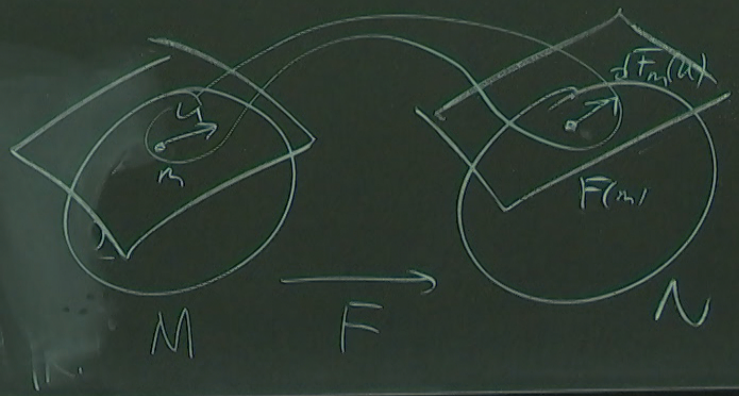

$T^*M \times \mathbb{R}$
$(q^1, \ldots, q^n, p_1, \ldots p_n, t)$
$\{f, g\}$
$\dot{\gamma}(t) = X_H|_{\gamma(t)}$


Symmetries + Noether's theorem
(M,ω), (N,ρ) symplectic manifolds
F: M → N symplectic diffeomorphism if it

$\in T_m M$

$F^* \rho = \omega$

$(F^* \rho)_m (u, v) = \rho_{F(m)} (dF_m u, dF_m v)$

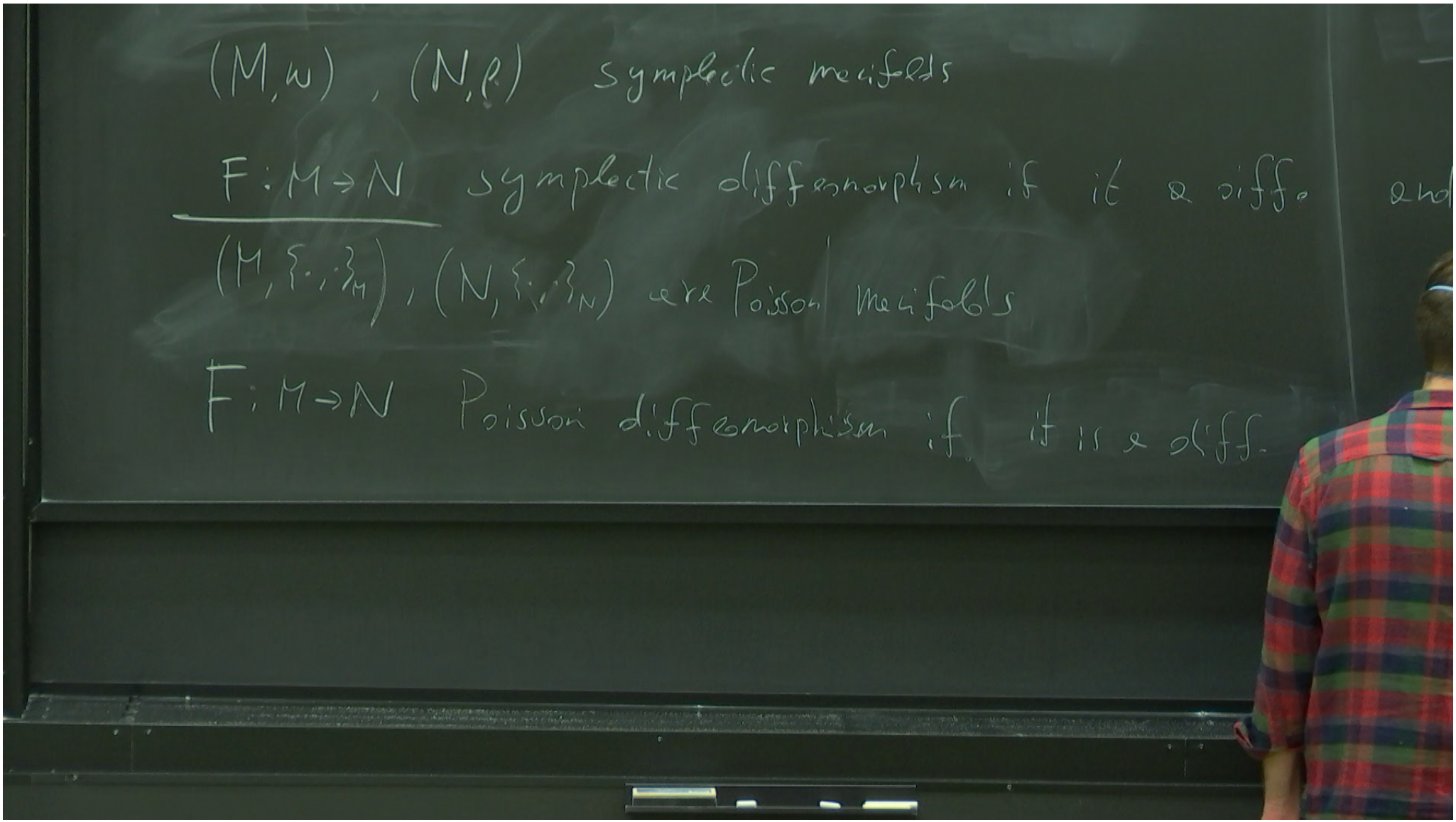


(M, ω), (N, ρ) symplectic manifolds

$F: M \to N$ symplectic diffeomorphism if it is a diff. and

$(M, \{\cdot,\cdot\}_M)$, $(N, \{\cdot,\cdot\}_N)$ are Poisson manifolds

$F: M \to N$ Poisson diffeomorphism if it is a diff.



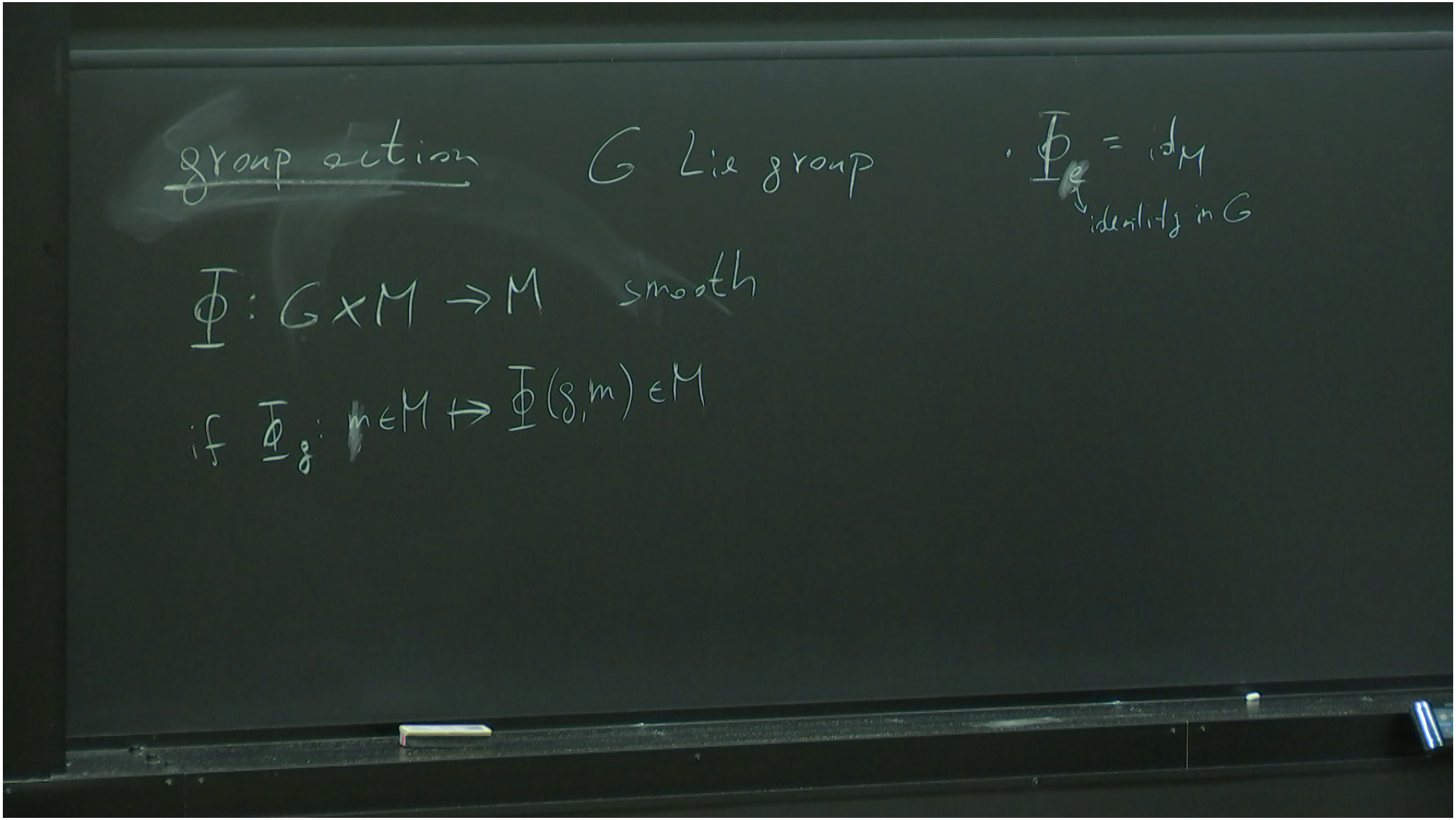
) symplectic manifolds

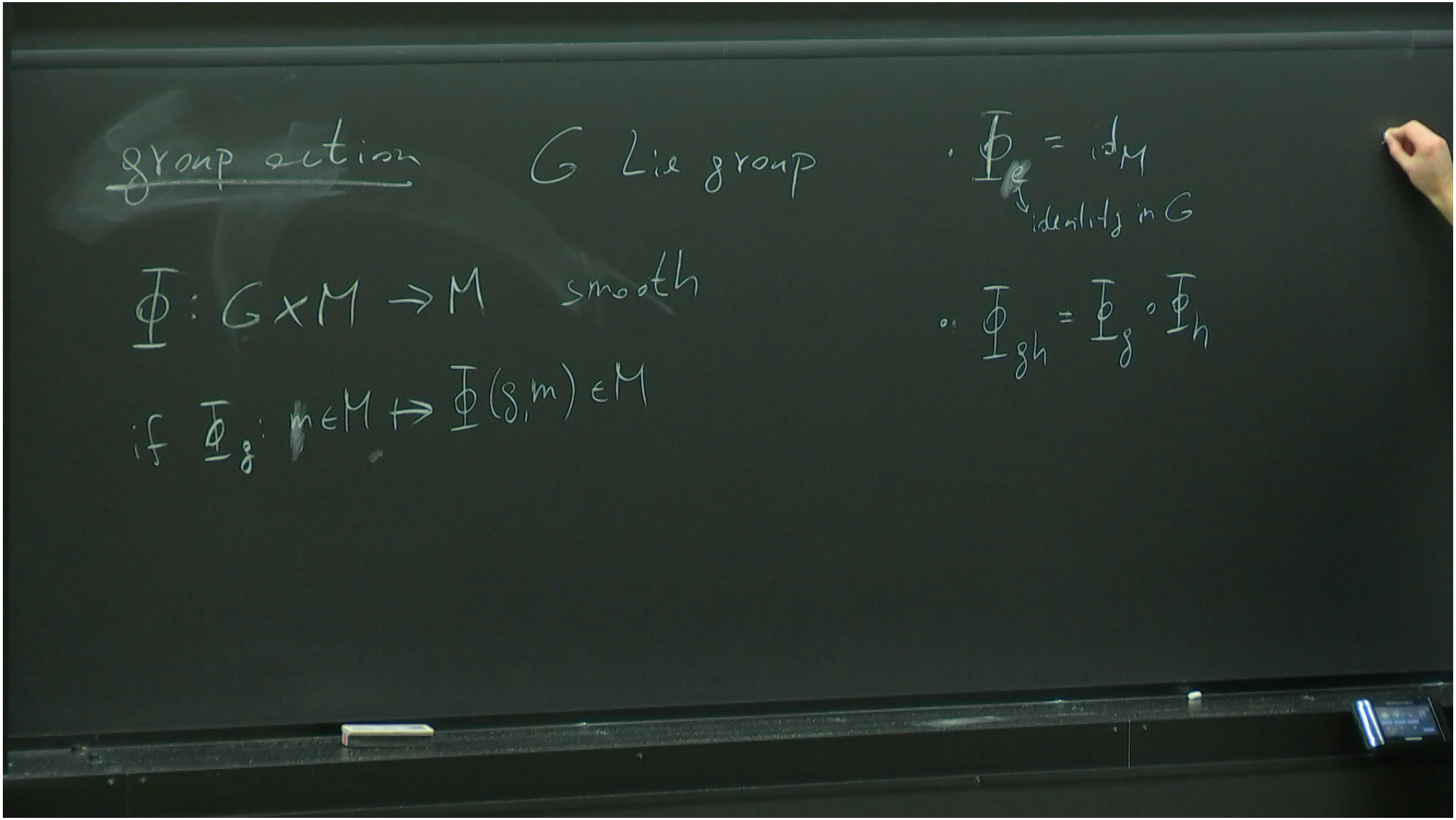
ymplectic diffeomorphism if it a diff. and $\boxed{F^*\rho = \omega}$ $(F^*\rho)_x$

$\{\cdot,\cdot\}_N$) are Poisson manifolds

$f, g \in C^\infty$

sson diffeomorphism if it is a diff. and $\{F^*f, F^*g\}_M = F^*\{f,g\}_N$


group action
G Lie group
$\Phi_e = id_M$
identity in G
$\Phi : G \times M \to M$ smooth
if $\Phi_g : m \in M \mapsto \Phi(g, m) \in M$

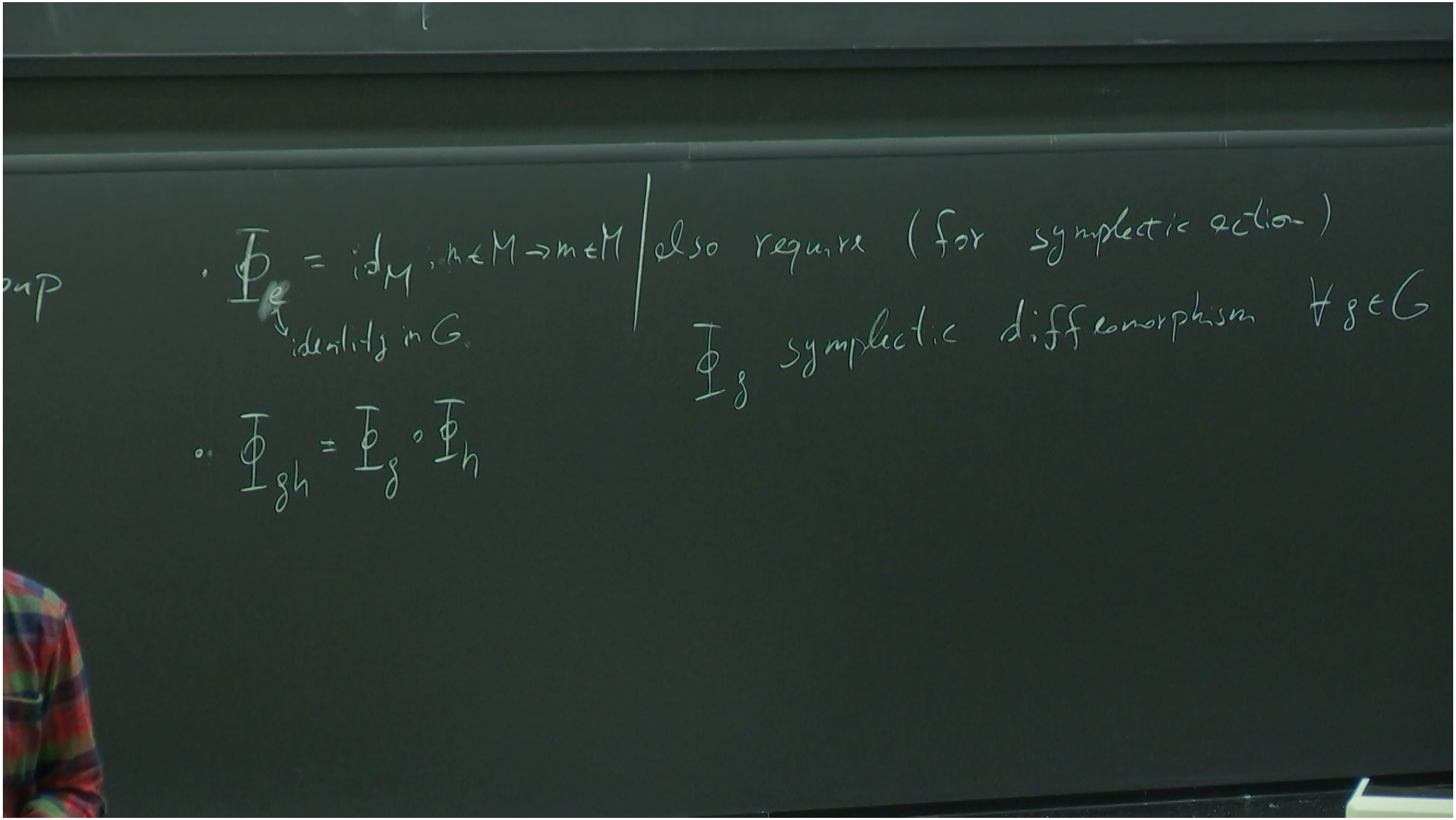

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$= id_M : m \in M \to m \in M$ | also require (for symplectic action)

identity in G.

Φ_g symplectic diffeomorphism $\forall g \in G$

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oup
$\Phi_e = id_M : m \in M \to m \in M$
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also require (for symplectic action)
Φ_g symplectic diffeomorphism $\forall g \in G$

action if $\xi \in Lie(G)$ define $\Phi^{\xi} : \mathbb{R} \times M \longrightarrow M$

$(t, m) \mapsto \Phi(e^{t\xi}, m)$

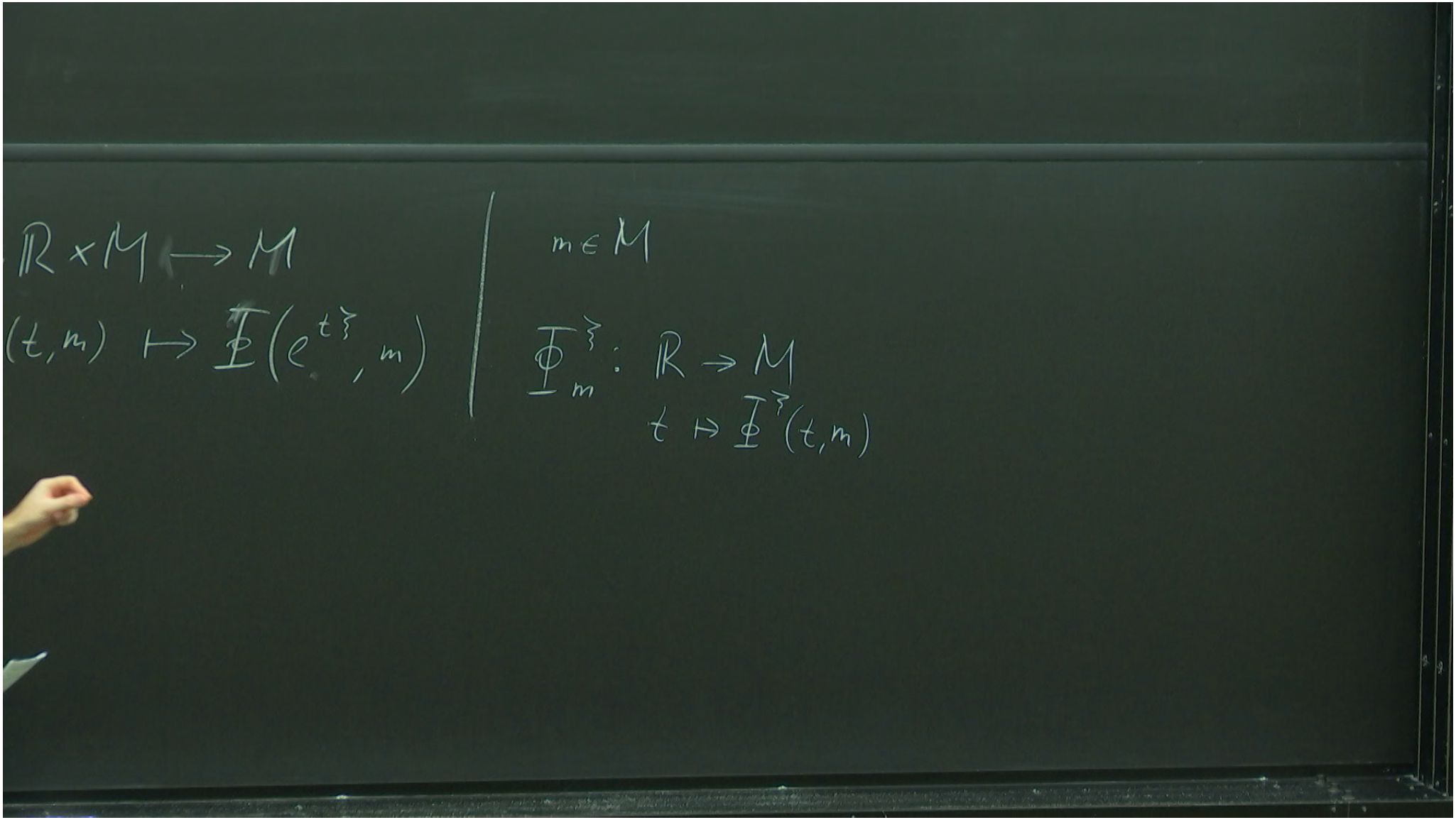
infinitesimal action

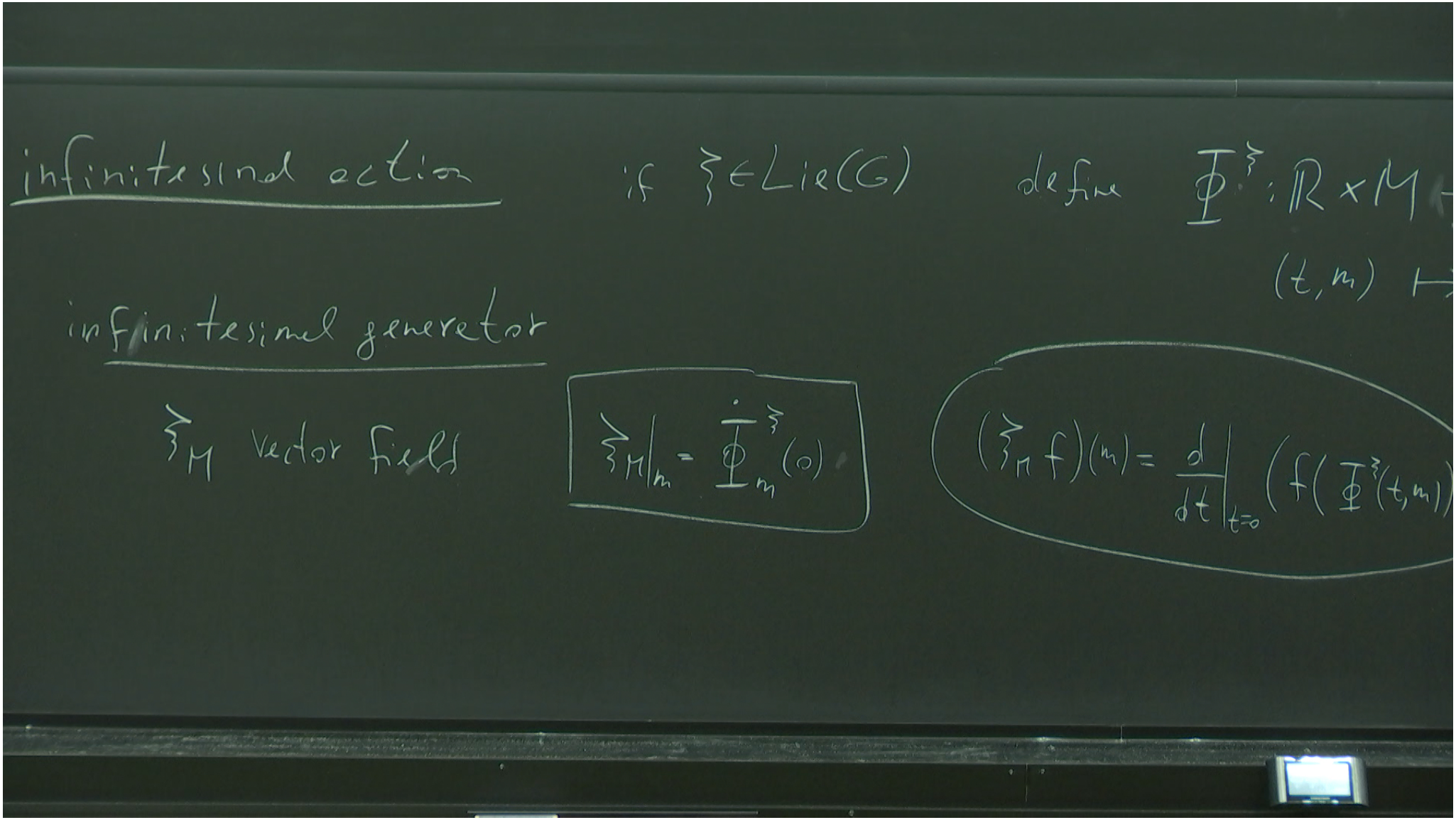
if $\xi \in Lie(G)$ define $\underline{\Phi}^{\xi} : \mathbb{R} \times M$

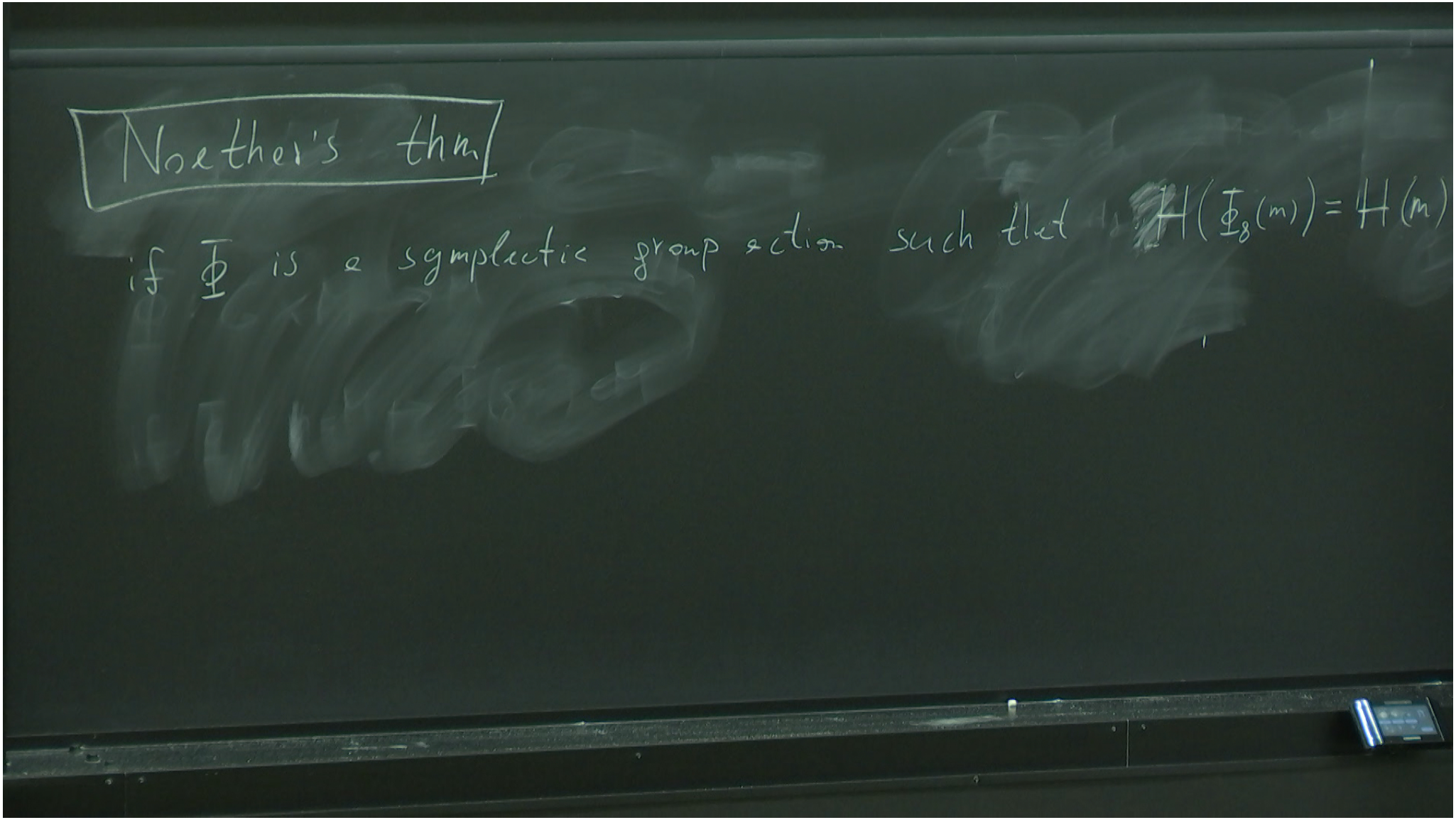
$(t, m) \mapsto$

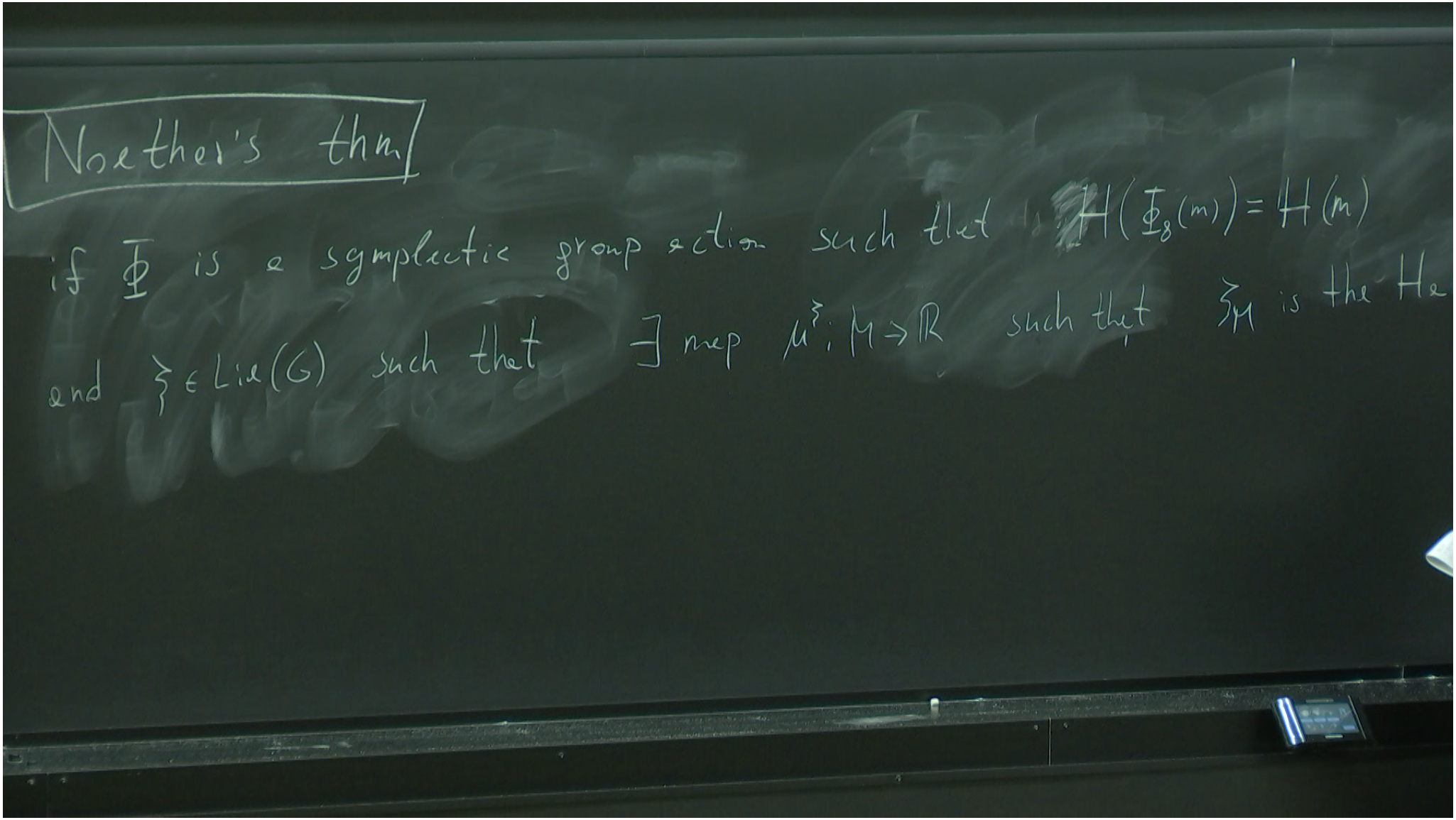
infinitesimal generator

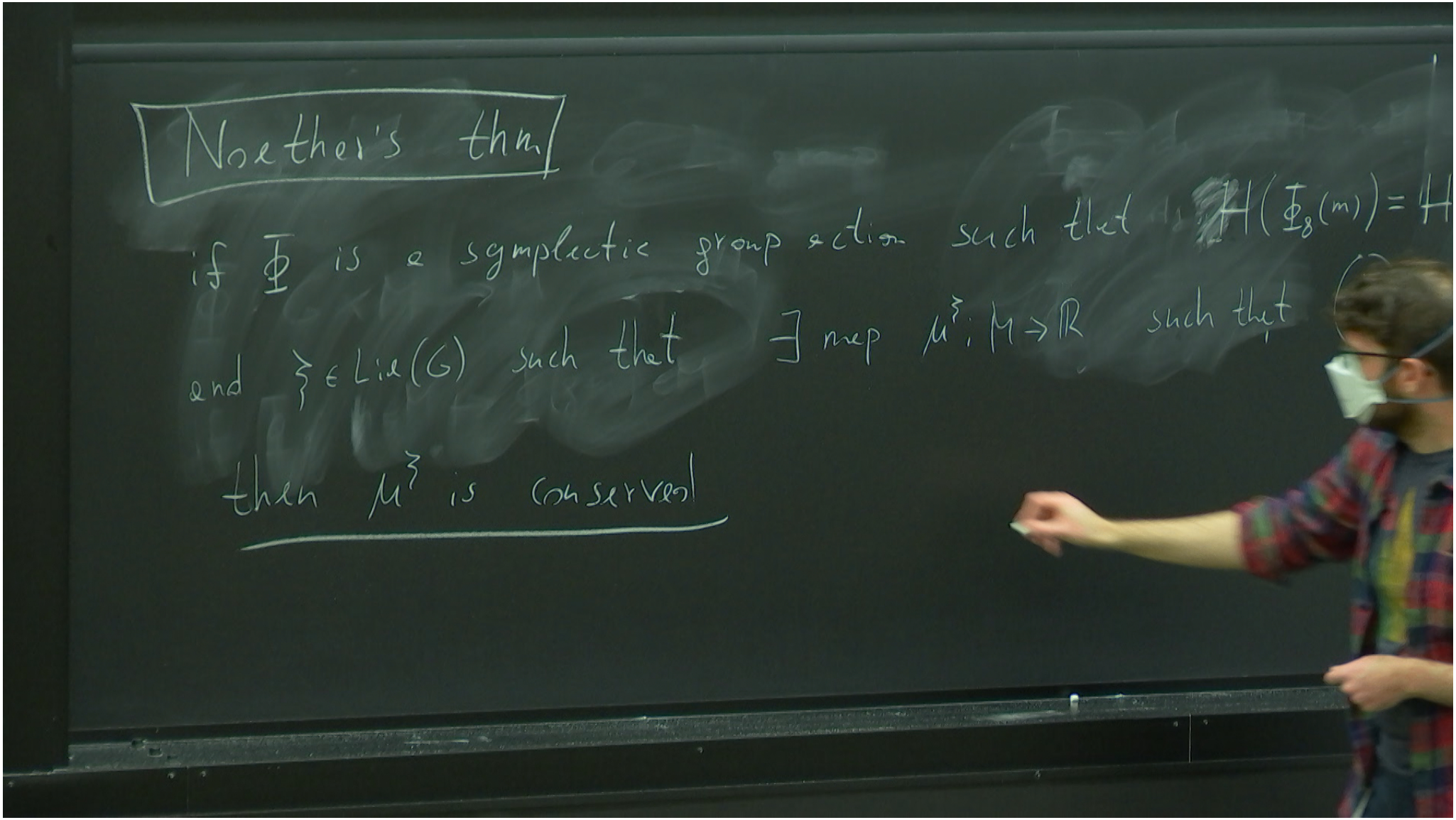
ξ_M vector field $\quad \xi_M|_m = \dot{\underline{\Phi}}^{\xi}$


$\mathbb{R} \times M \longrightarrow M$
$(t, m) \mapsto \Phi(e^{t\vec{\xi}}, m)$
$m \in M$
$\Phi_m^{\vec{\xi}} : \mathbb{R} \to M$
$t \mapsto \Phi^{\vec{\xi}}(t, m)$


infinitesimal action
if $\xi \in Lie(G)$
define $\Phi^{\xi}: \mathbb{R} \times M \to$
$(t, m) \mapsto$
infinitesimal generator
ξ_M vector field
$\xi_M|_m = \dot{\Phi}^{\xi}_m(0)$
$(\xi_M f)(m) = \frac{d}{dt}\Big|_{t=0} \left(f(\Phi^{\xi}(t, m)) \right.$


Noether's thm
if Φ is a symplectic group action such that $H(\Phi_g(m)) = H(m)$


Noether's thm
if Φ is a symplectic group action such that $H(\Phi_g(m)) = H(m)$
and $\xi \in Lie(G)$ such that $\exists$ map $\mu^\xi : M \to \mathbb{R}$ such that ξ_M is the Ha


Noether's thm.
if Φ is a symplectic group action such that $H(\Phi_g(m)) = H$
and $\xi \in Lie(G)$ such that $\exists$ map $\mu^\xi : M \to \mathbb{R}$ such that
then μ^ξ is conserved