Title: Mathematical Physics Lecture - 230131

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Collection: Mathematical Physics (2022/2023)

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Example

$M = S^1 \times S^1$

$p, q \quad p \sim p+1 \quad q \sim q+1$

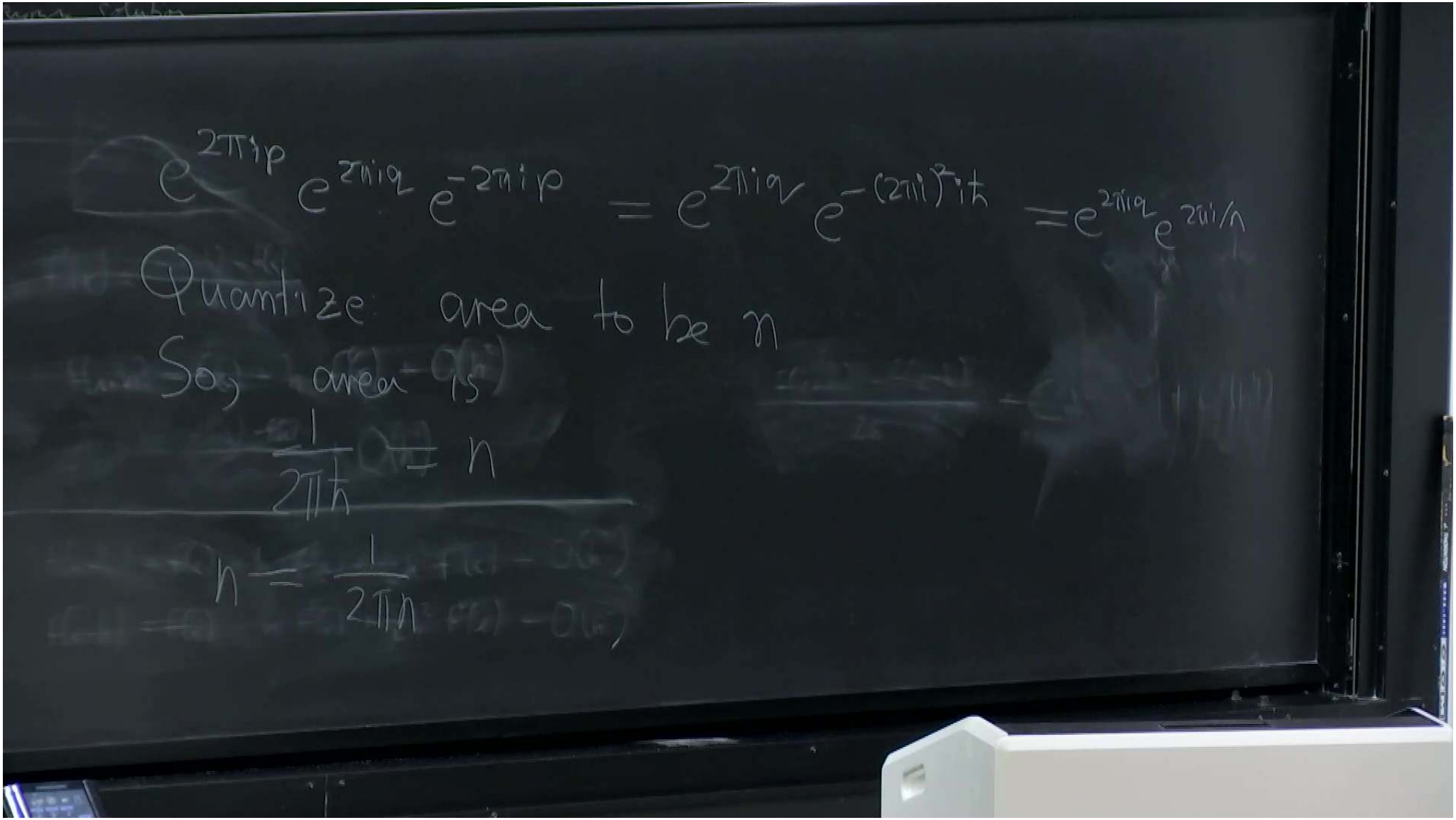
$P = e^{2\pi i p}$

$Q = e^{2\pi i q}$

$\omega = \frac{dq\,dp}{2\pi\hbar}$

Convention (corrected by factor of i)

$[P, Q] = -i\hbar = \{P, Q\}\frac{1}{2\pi i}$


$e^{2\pi ip} e^{2\pi iq} e^{-2\pi ip} = e^{2\pi iq} e^{-(2\pi i)^2 ih} = e^{2\pi iq} e^{2\pi i/n}$
Quantize area to be n
So, area is
$\frac{1}{2\pi h} = n$
$h = \frac{1}{2\pi n}$

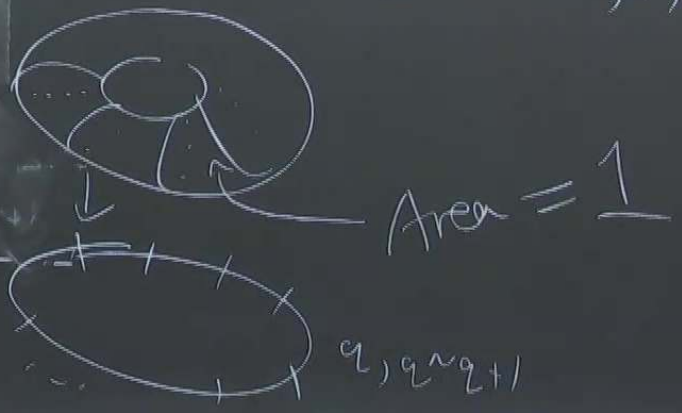
$PQ = QP e^{2\pi i/n}$

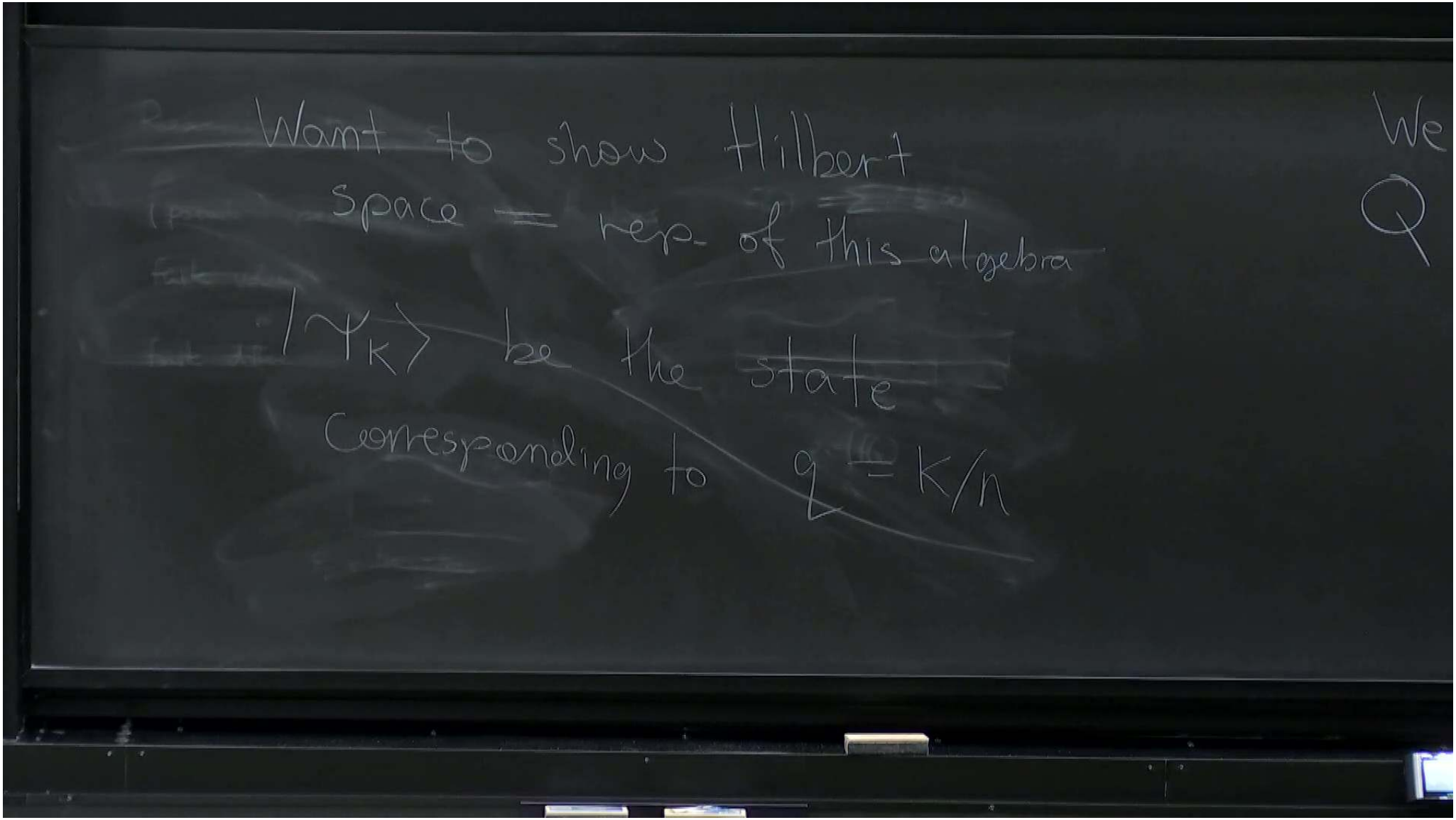
$Q = e^{2\pi i k/n}$

(i.e. $q = k/n$, $k = 0, 1, \ldots$

Last time

Bohr-Sommerfeld orbits are the circles where




Want to show Hilbert
space = rep. of this algebra
$|\Psi_K\rangle$ be the state
Corresponding to $q = K/n$
We
Q

We define

$$Q|\psi_k\rangle = e^{2\pi i k/n}|\psi_k\rangle$$

as on the BS orbit, $Q = e^{2\pi i k/n}$

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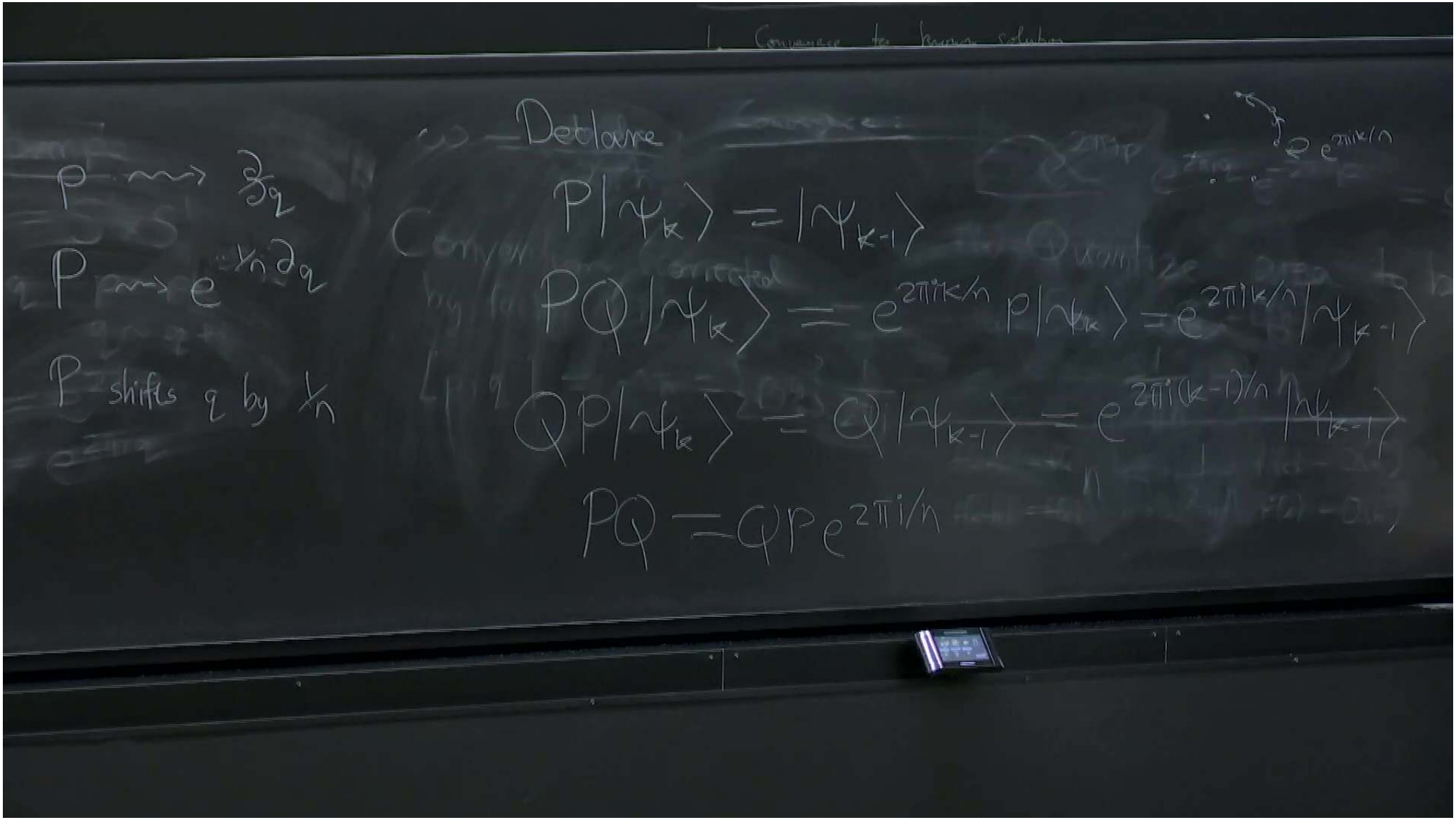
What about P?


Declare
$P|\psi_k\rangle = |\psi_{k-1}\rangle$
$PQ|\psi_k\rangle = e^{2\pi i k/n} P|\psi_k\rangle = e^{2\pi i k/n}|\psi_{k-1}\rangle$
$QP|\psi_k\rangle = Q|\psi_{k-1}\rangle = e^{2\pi i(k-1)/n}|\psi_{k-1}\rangle$
$PQ = Qe^{2\pi i/n}$
$P \rightsquigarrow \mathbb{Z}_2$
$P \rightsquigarrow e^{1/n\,\partial_q}$
P shifts q by 1/n

Harmonic Oscillator

$M = \mathbb{R}^2$

$\omega = \frac{dp\,dq}{2\pi\hbar}$

$H = p^2 + q^2$

Want to use the Bohr-Sommerfeld method to determine the spectrum.

Recall:

$p \longrightarrow -i\frac{\partial}{\partial q}\hbar$

$H \rightsquigarrow -\hbar^2\frac{\partial^2}{\partial q} + q^2$

$H = (q + \hbar\partial_q)($

Want to use the Bohr-Sommerfeld method to determine the spectrum.

Recall,

$$p \longrightarrow -i\frac{\partial}{\partial q}\hbar$$

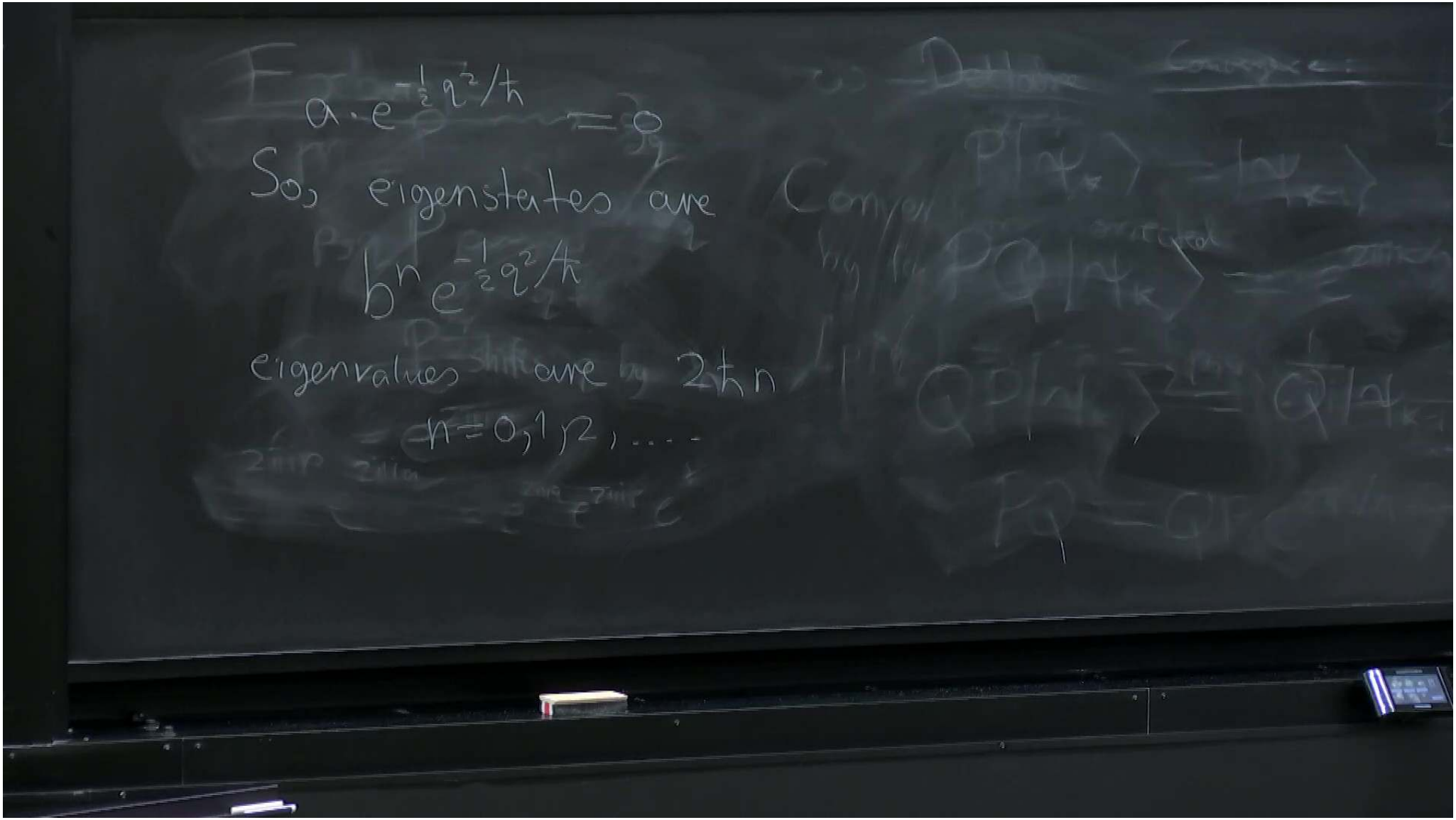
$$H \rightsquigarrow -\hbar^2\frac{\partial^2}{\partial q} + q^2$$

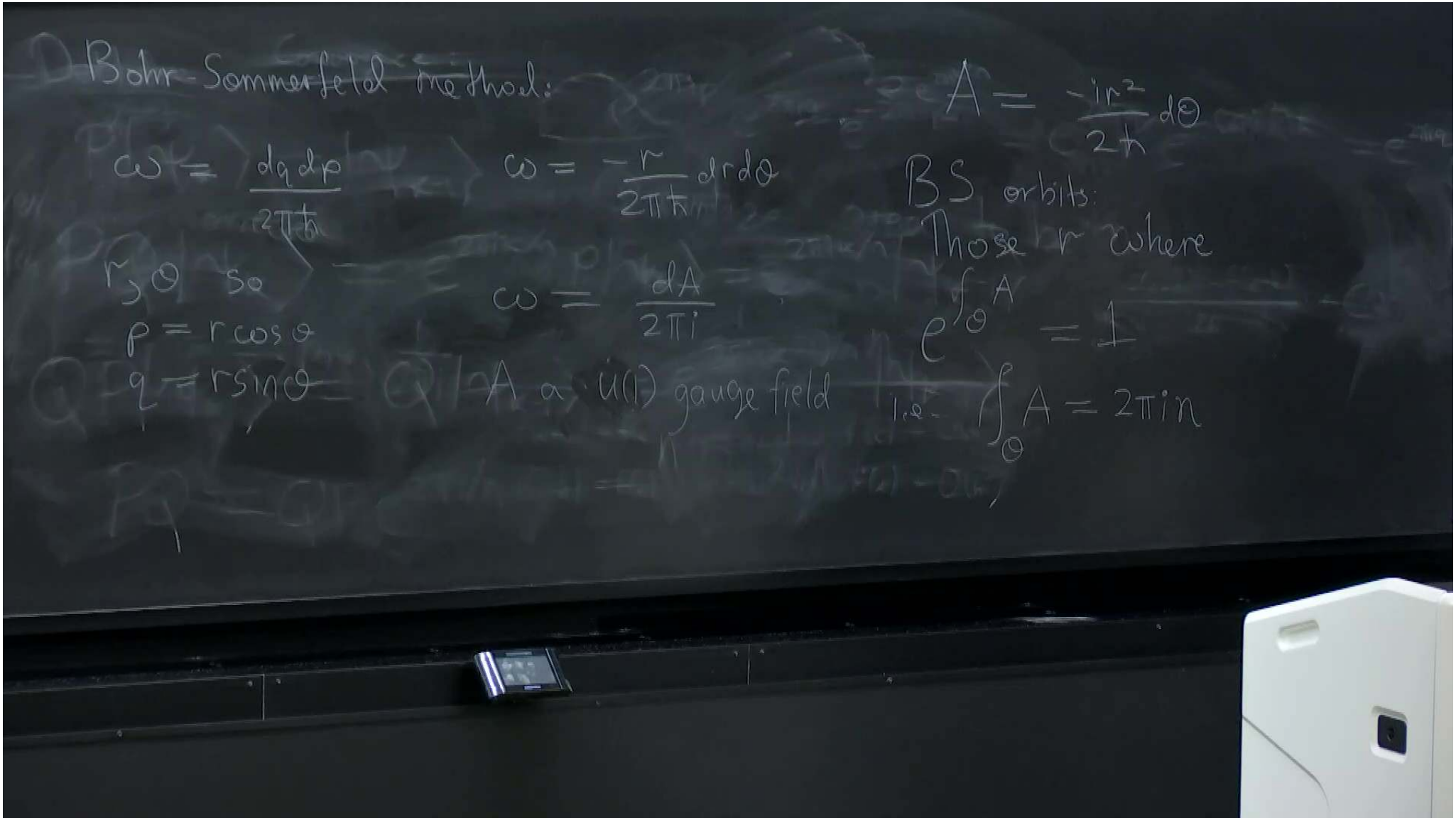
$$H = \underbrace{(q+\hbar\partial_q)}_{a}\underbrace{(q-\hbar\partial_q)}_{b} \quad (+\hbar)$$

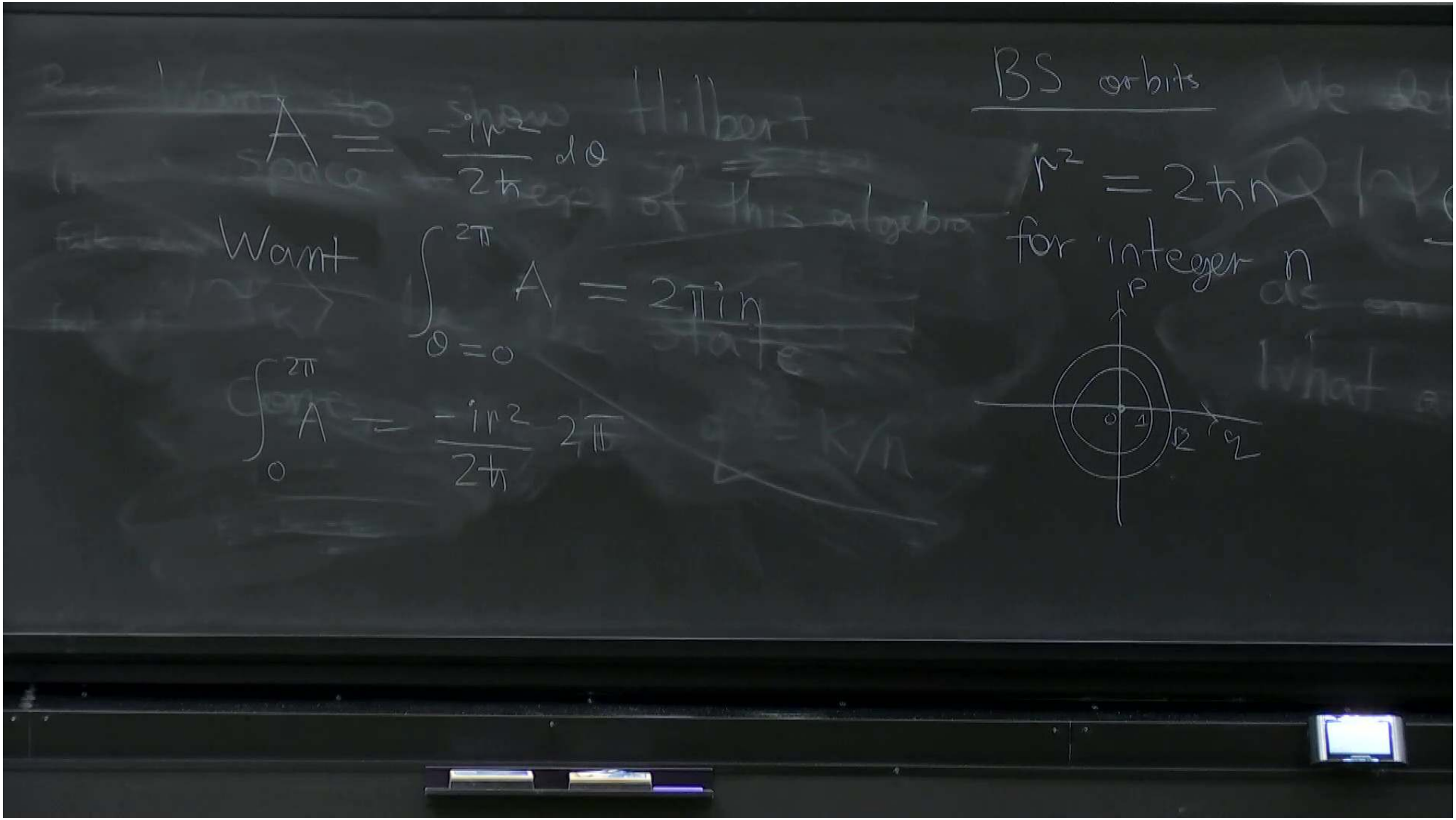
$$[a,b] = 2\hbar$$

b = Raising

a = Lowering


$a \cdot e^{-\frac{1}{2}q^2/\hbar} = 0$
So, eigenstates are
$b^n e^{-\frac{1}{2}q^2/\hbar}$
eigenvalues are by $2\hbar n$
$n = 0, 1, 2, \ldots$


Bohr-Sommerfeld method:
$\omega = \frac{dq\,dp}{2\pi\hbar}$
r, θ so
$p = r\cos\theta$
$q = r\sin\theta$
$\omega = -\frac{r}{2\pi\hbar}\,dr\,d\theta$
$\omega = \frac{dA}{2\pi i}$
A a U(1) gauge field
$A = -\frac{ir^2}{2\hbar}\,d\theta$
BS orbits:
Those r where
$e^{\int_0^A} = 1$
i.e. $\int_0 A = 2\pi i n$


BS orbits
$A = \frac{-ir^2}{2\hbar} d\theta$
Hilbert
$r^2 = 2\hbar n$
for integer n
Want $\int_{\theta=0}^{2\pi} A = 2\pi i n$
$\int_0^{2\pi} A = \frac{-ir^2}{2\hbar} 2\pi$
Re
Im
0
1
$\sqrt{2}$
$r\sqrt{2}$

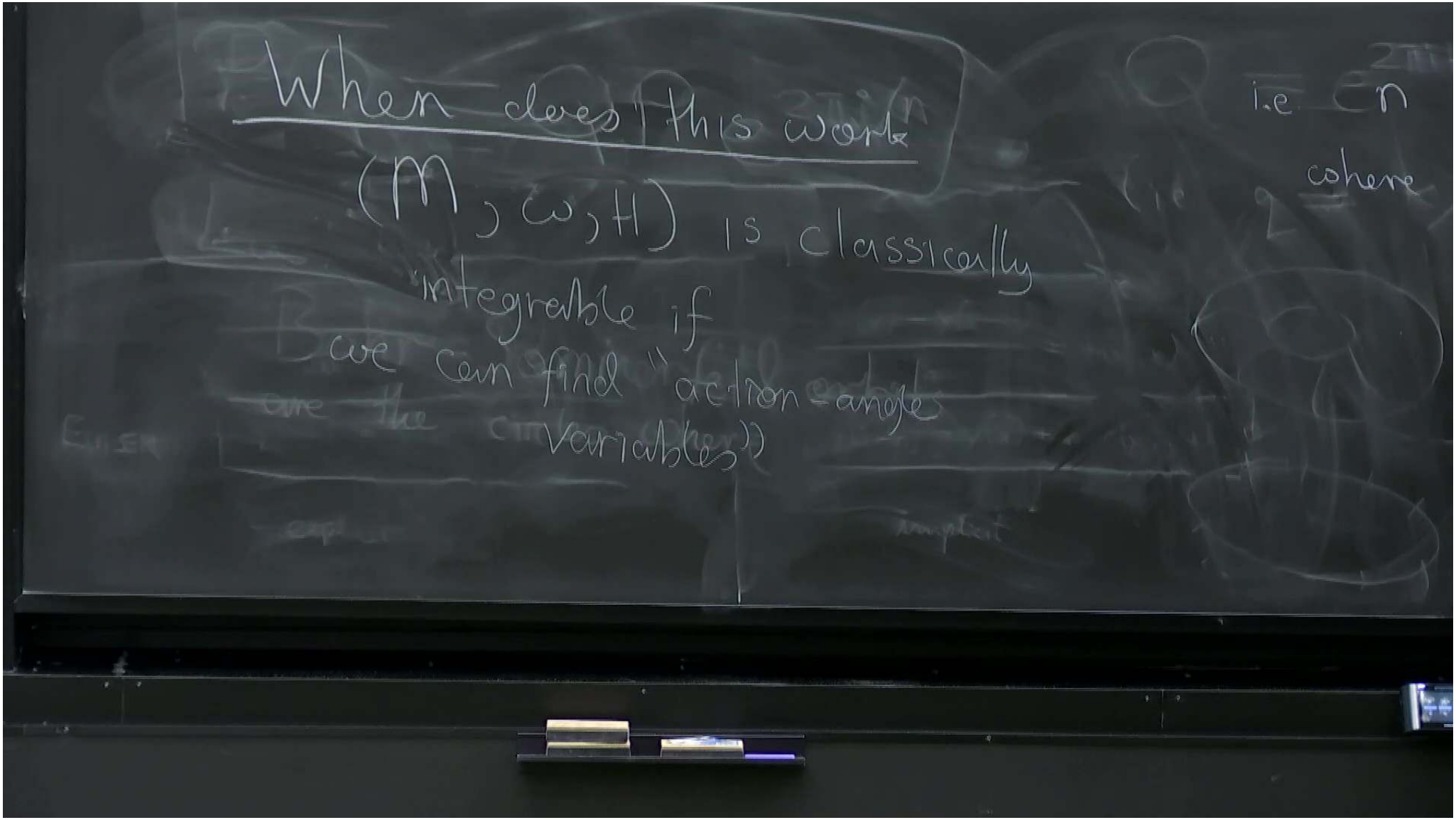
$H = r^2$

Conclude:

If ψ_n = BS orbit for $r^2 = 2\hbar n$

then $H|\psi_n\rangle = 2\hbar n|\psi_n\rangle$

Same spectrum!


When does this work
(M, ω, H) is classically integrable if we can find "action-angle variables"
i.e
coherent

i.e n functions $f_1, \ldots, f_n$ on M

cohere 1) $\{f_i, f_j\} = 0$

($\vec{f} : M \to \mathbb{R}^n$ polarizes M)

2) The fibres of $\vec{f} : M \to \mathbb{R}^n$ are tori $\underbrace{S^1 \times \cdots \times S^1}_{n \text{ copies}}$

3) $H =$ one of the f's, say f_1

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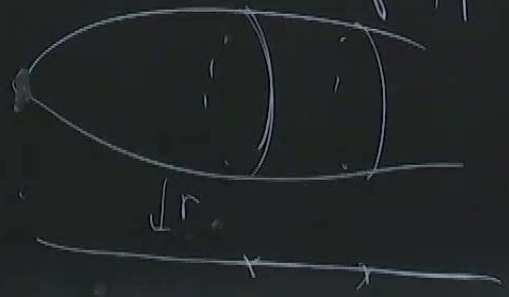
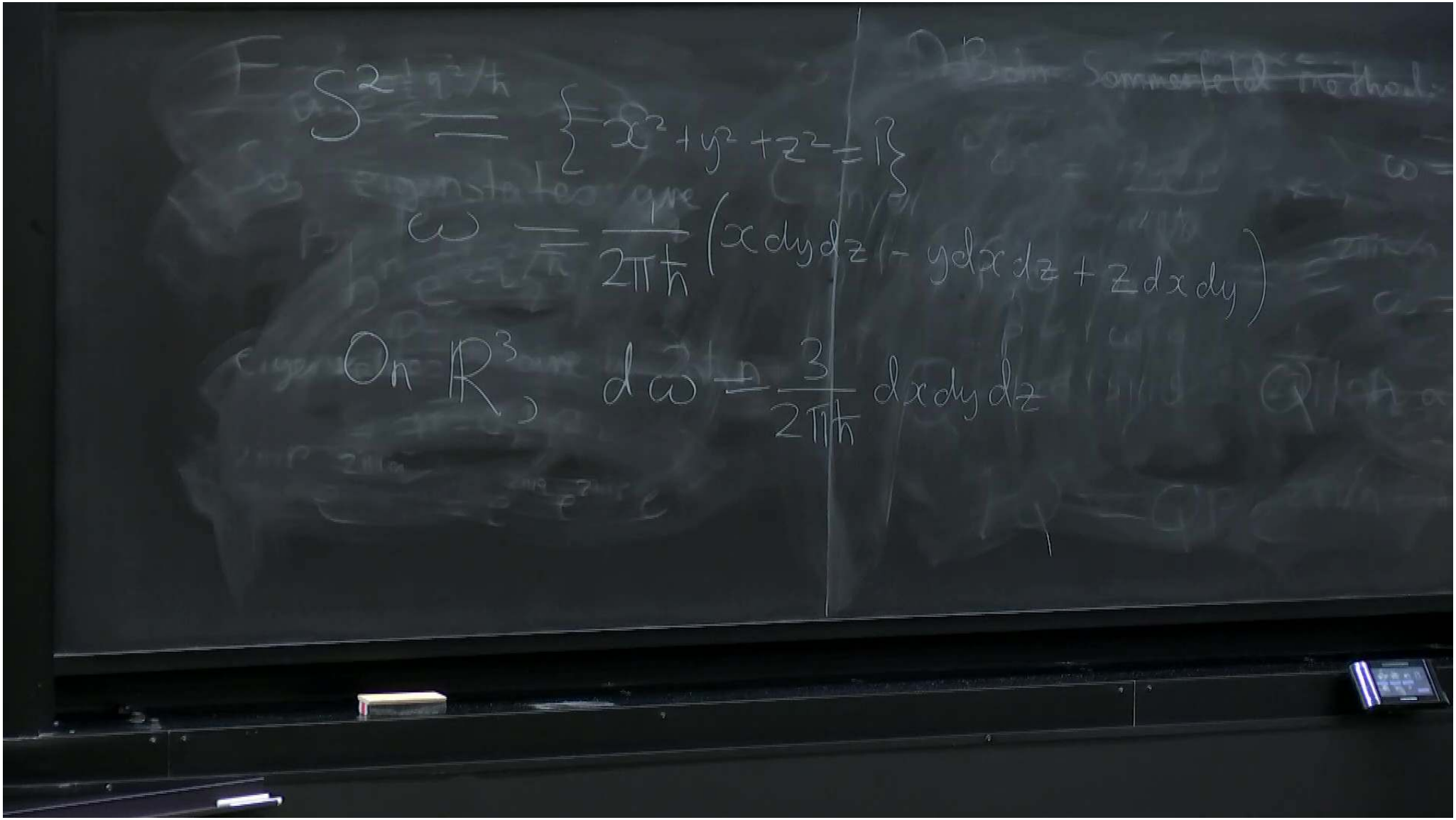
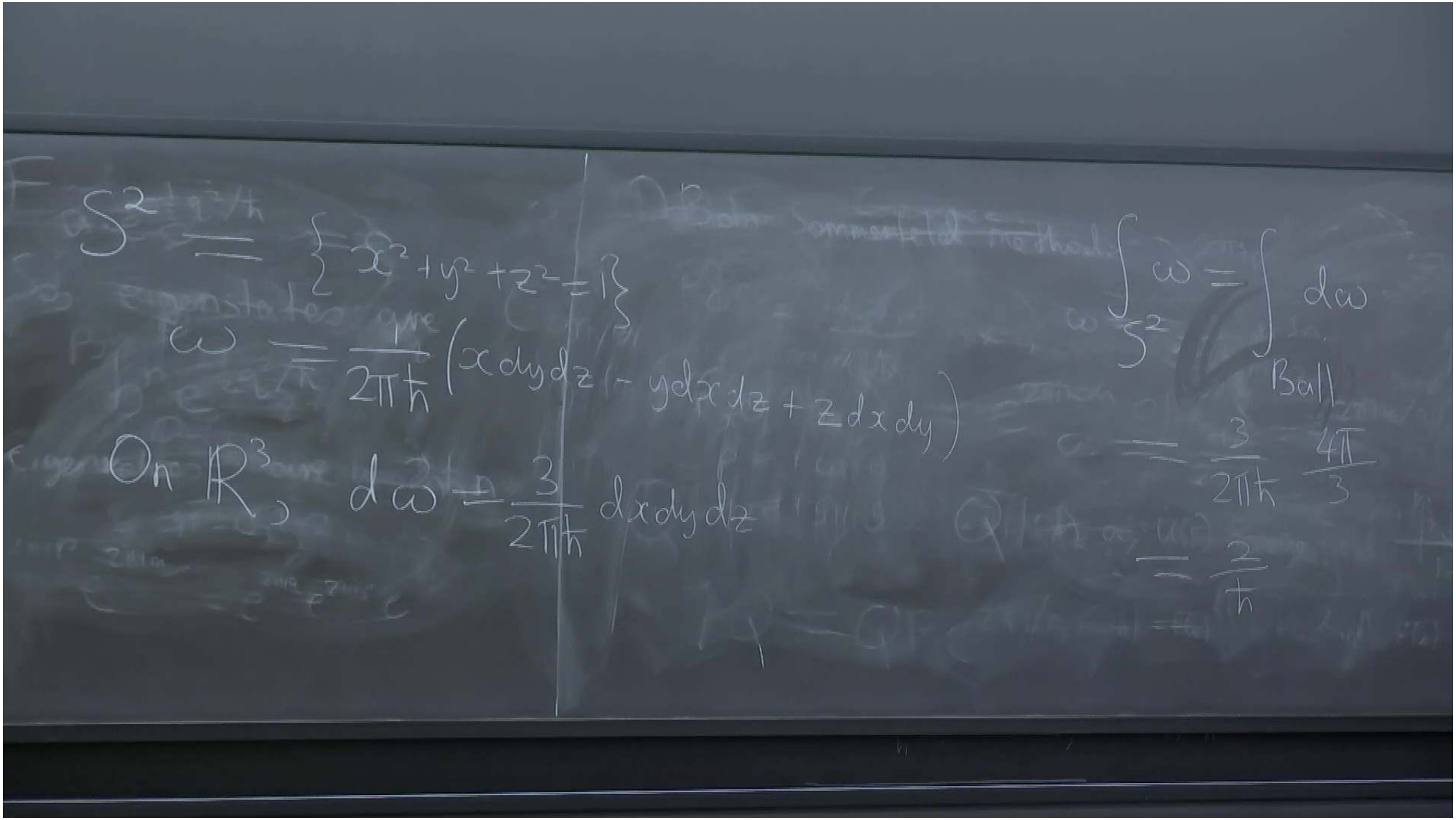
3) H = one of the f's, say f_1

In this case, can compute spectrum of H.




$S^2 = \{x^2 + y^2 + z^2 = 1\}$
$\omega = \frac{1}{2\pi\hbar}(x\,dy\,dz - y\,dx\,dz + z\,dx\,dy)$
On $\mathbb{R}^3$, $d\omega = \frac{3}{2\pi\hbar}\,dx\,dy\,dz$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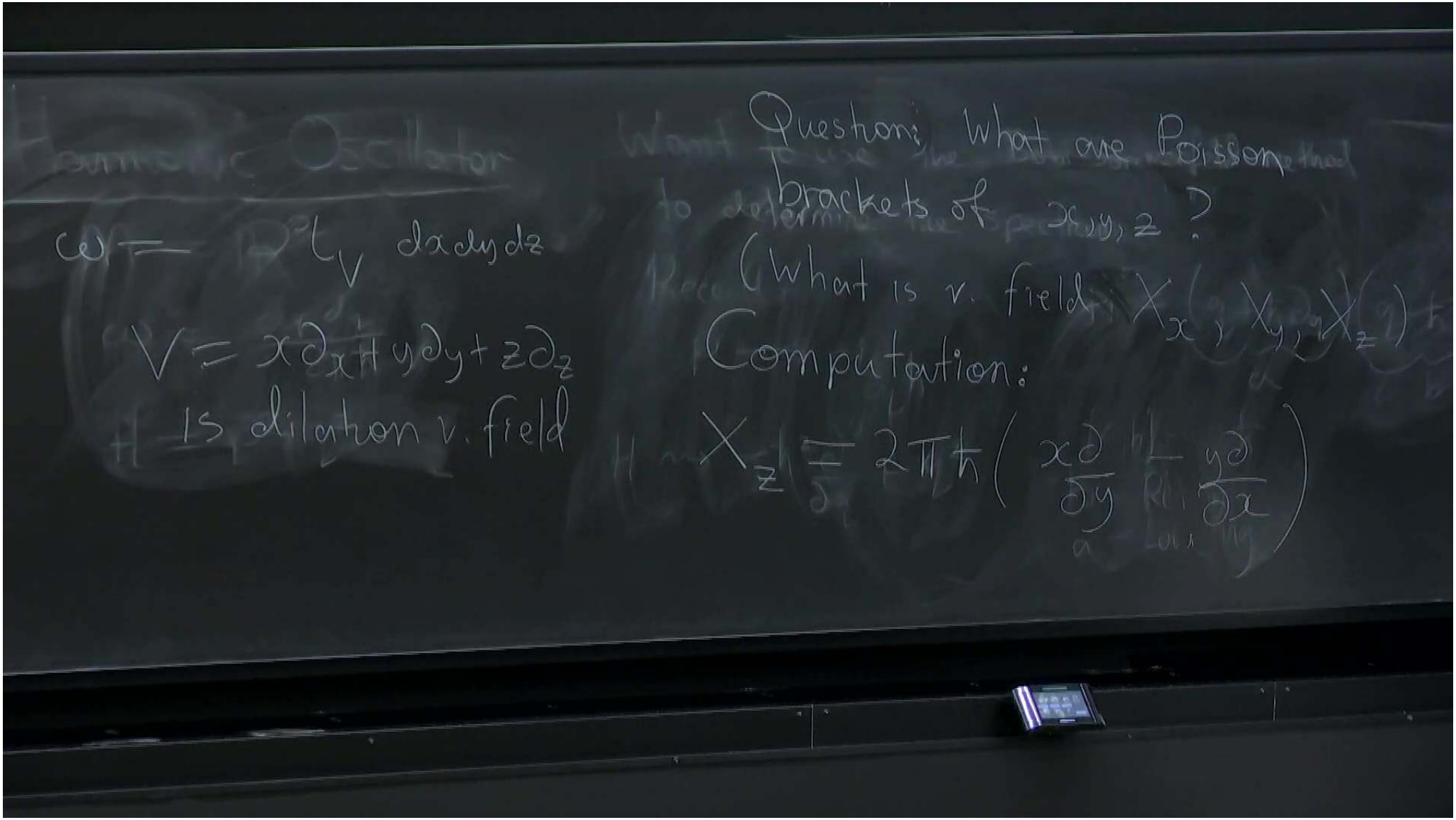
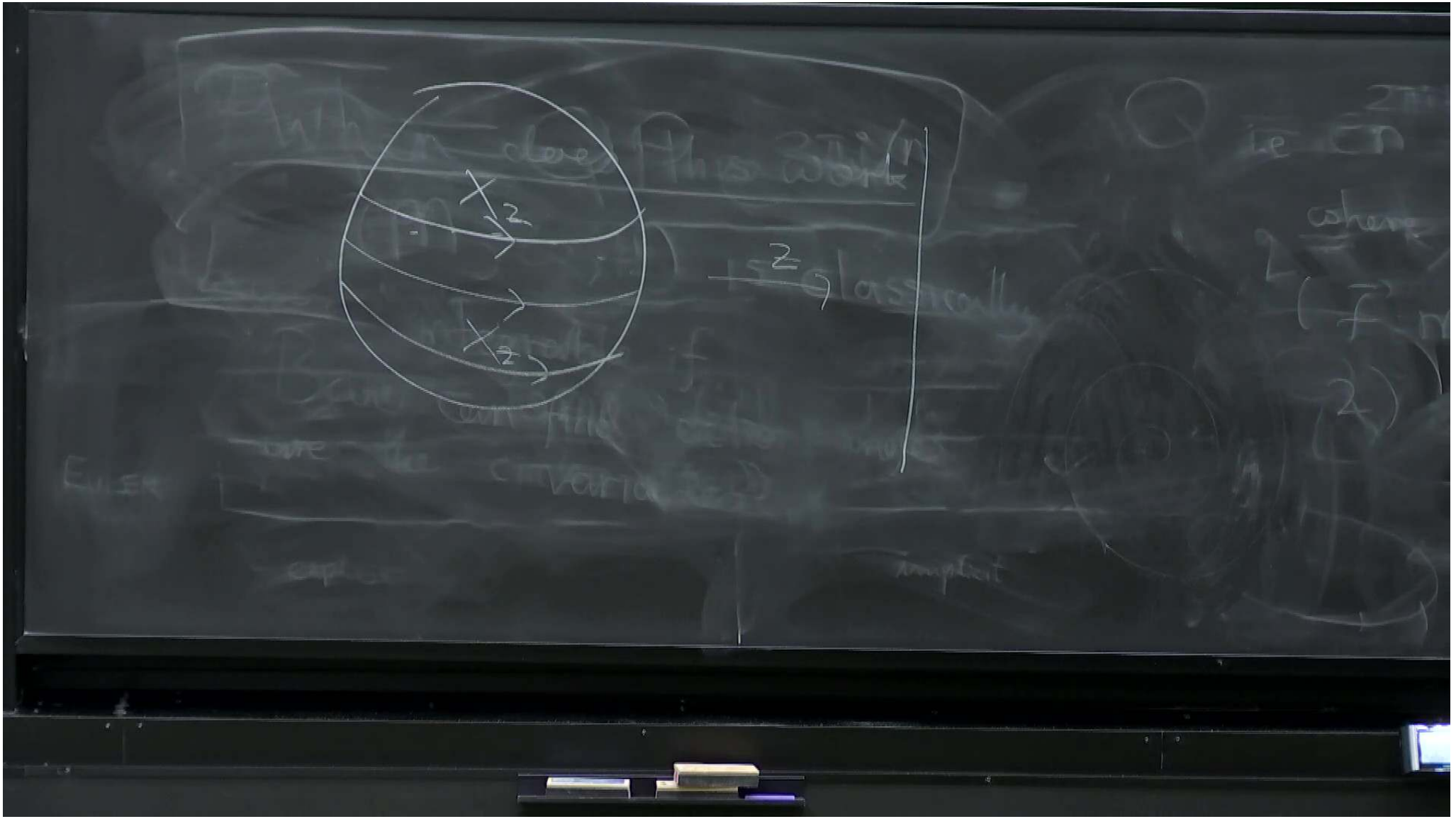
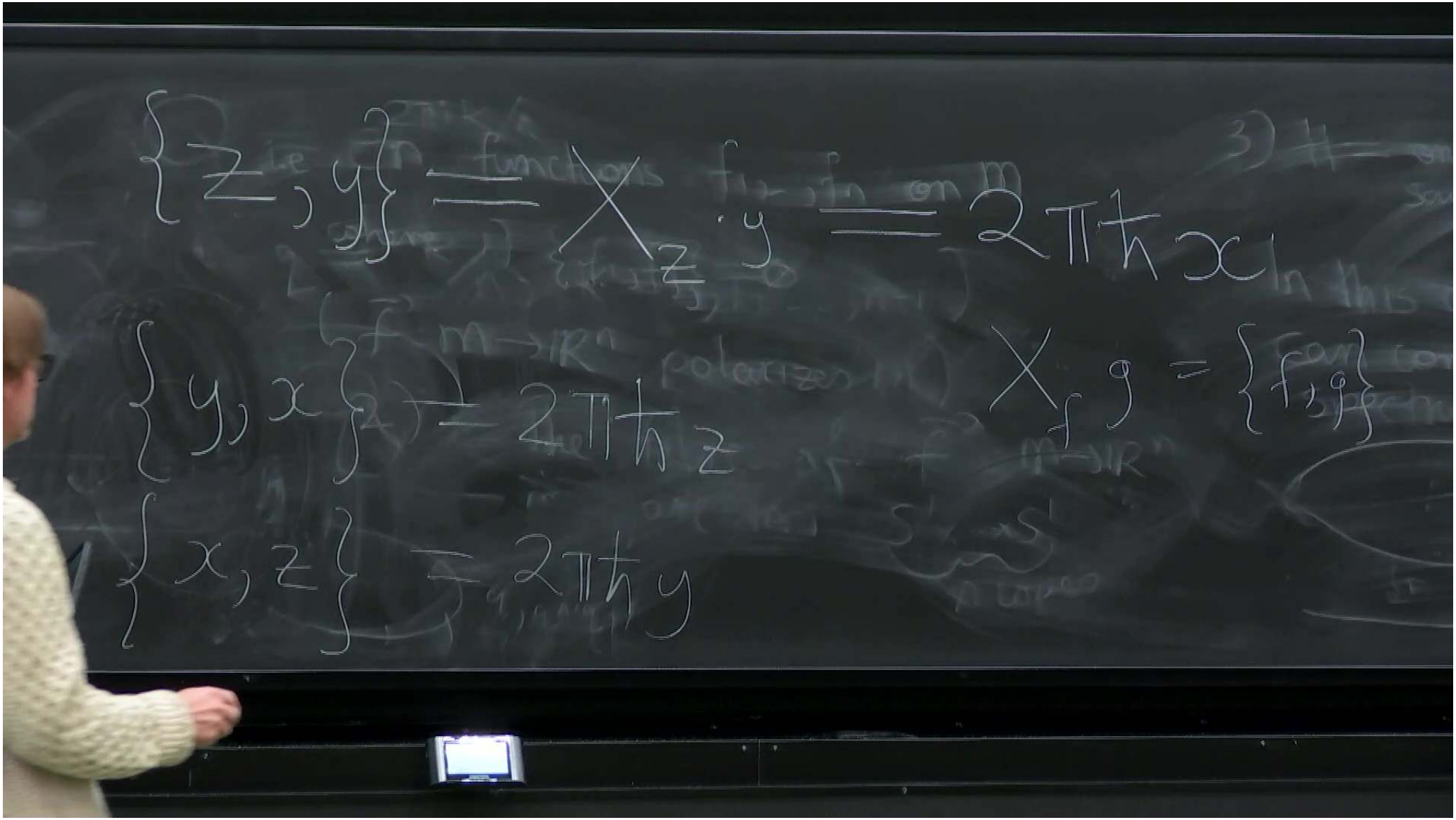
On $\mathbb{R}^3$, $d\omega = \frac{1}{2\pi\hbar}\frac{3}{2\pi\hbar}\,dx\,dy\,dz$
$\int_{S^2}\omega = \int_{\text{Ball}} d\omega$
$= \frac{3}{2\pi\hbar}\frac{4\pi}{3}$
$= \frac{2}{\hbar}$


Harmonic Oscillator
$\omega = \frac{1}{2}\sum dx_i \wedge dy_i$
$V = x\partial_x + y\partial_y + z\partial_z$
H is dilation v. field
Question: What are Poisson brackets of x, y, z?
(What is v. field X_x, X_y, X_z)
Computation:
$X_z = \frac{1}{2\pi\hbar}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$


X_2
X_2
$\mathbb{Z}_2$

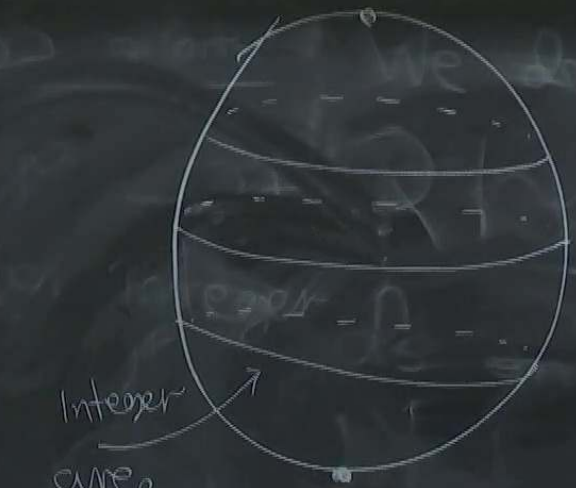

$\{z, y\} = X_z \cdot y = 2\pi\hbar x_1$
$\{y, x\} = 2\pi\hbar z$
$\{x, z\} = 2\pi\hbar y$
$X_f g = \{f, g\}$

What is Hilbert space?

Polarize using $\mathbb{Z}$

Volume $= \frac{2}{\hbar} = n$

$\hbar = \frac{2}{n}$

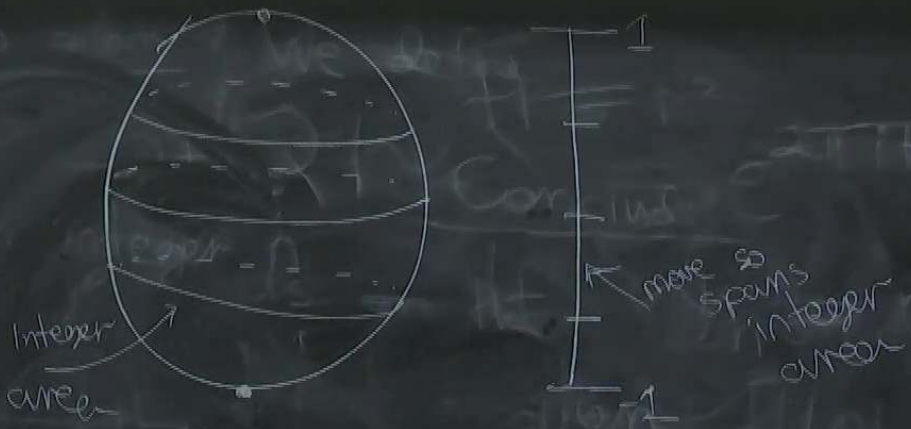


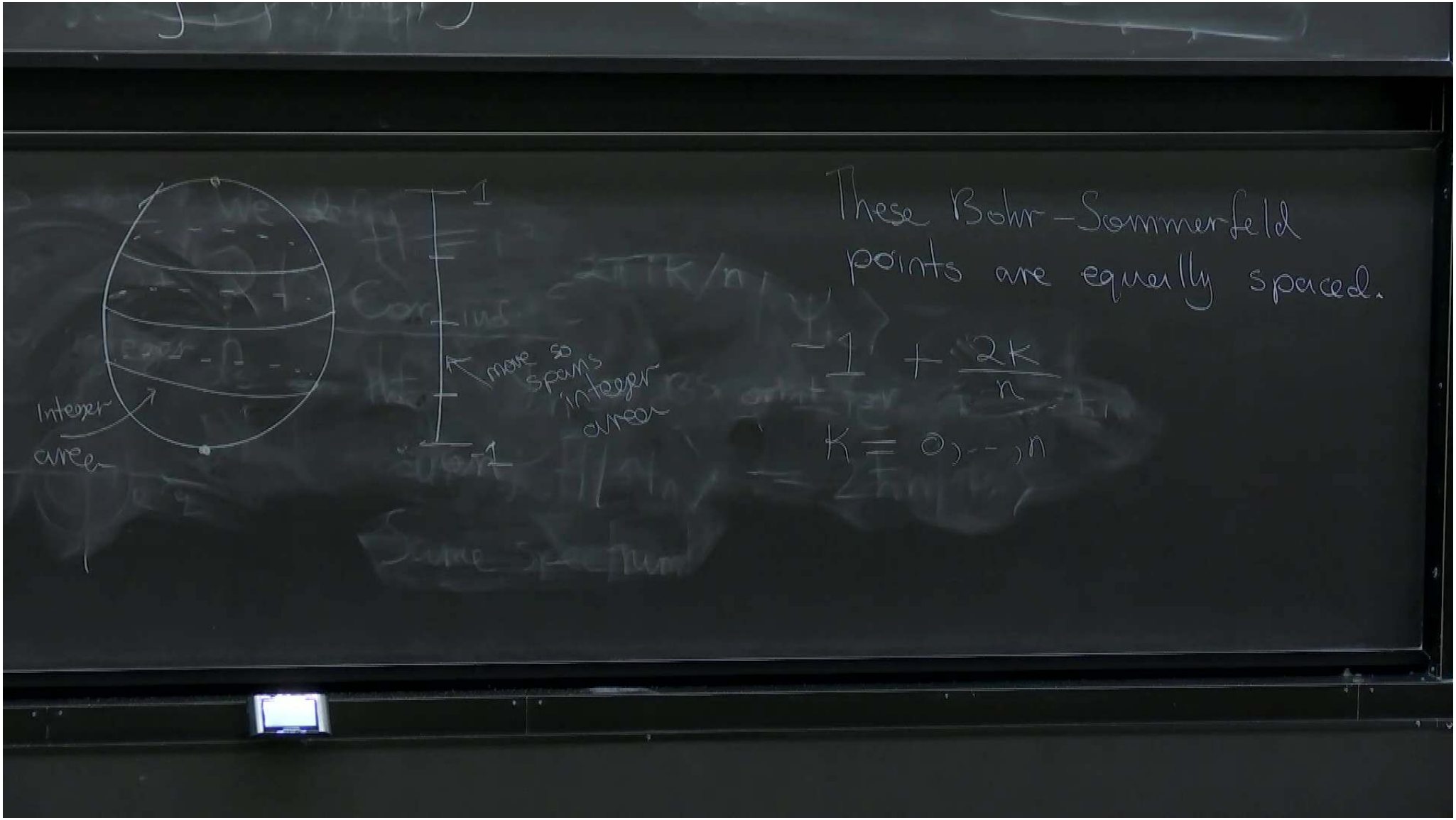
What is Hilbert space?

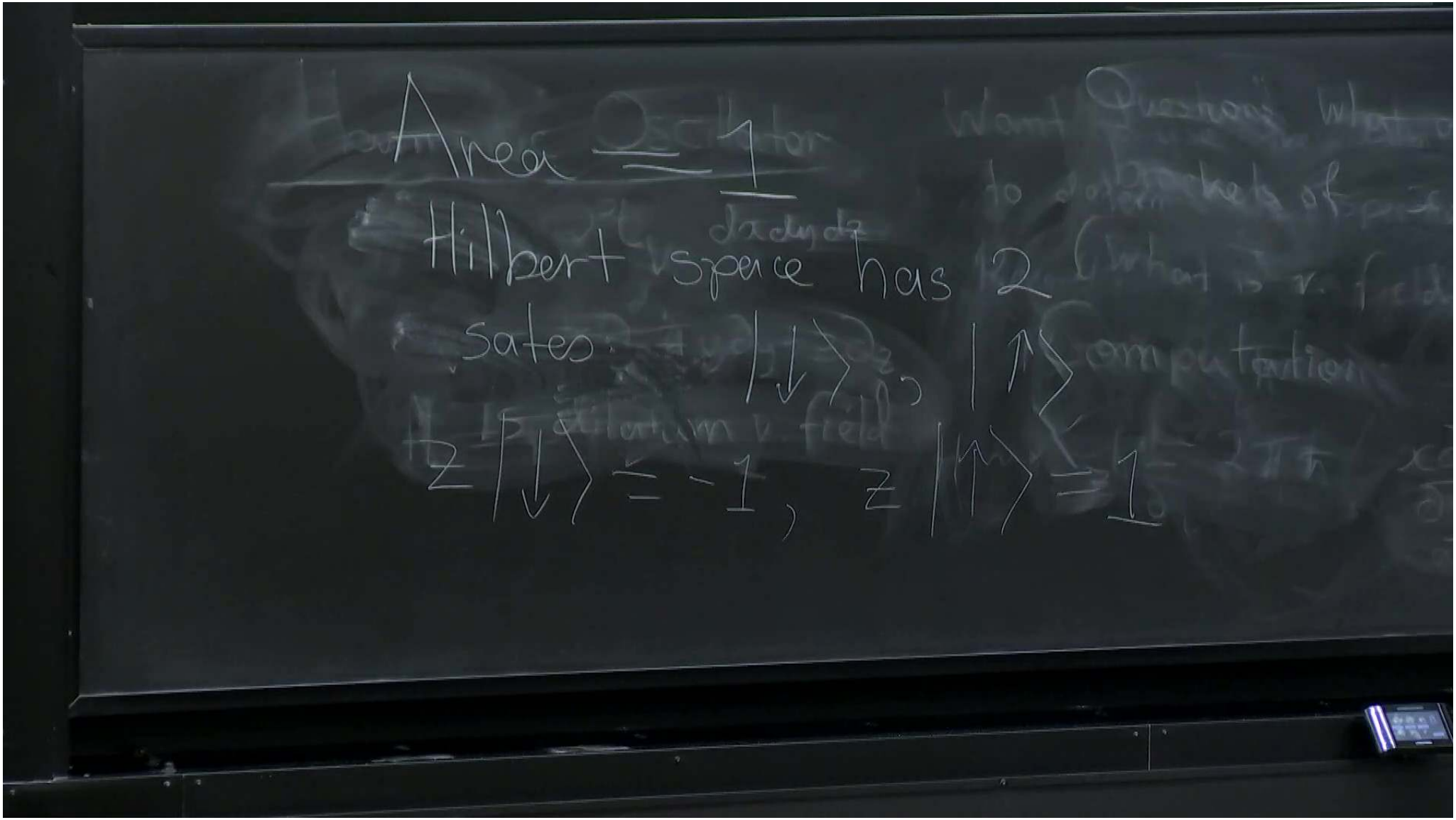
Polarize using $\vec{z}$

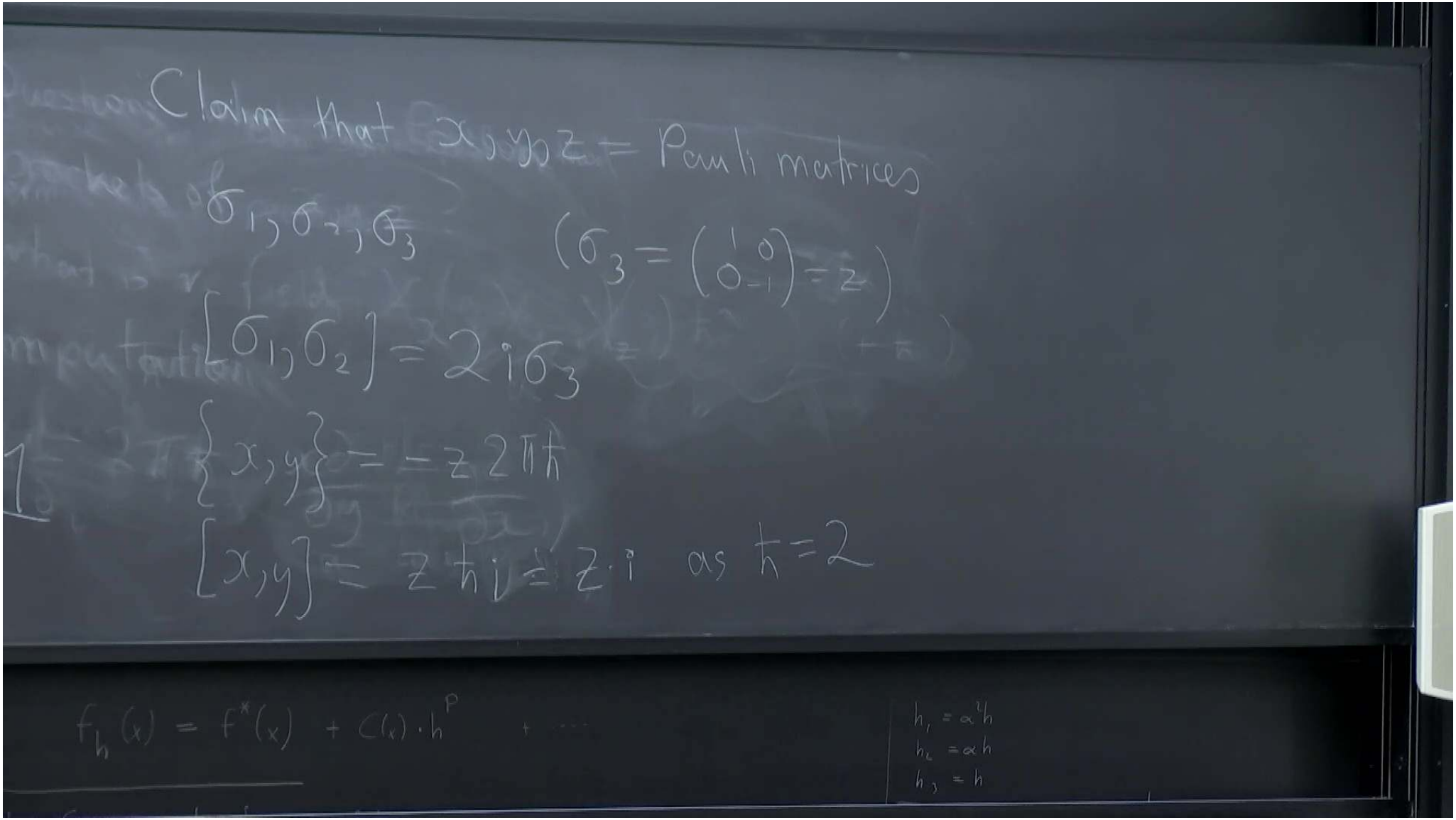
Volume $= \frac{2}{\hbar} = n$

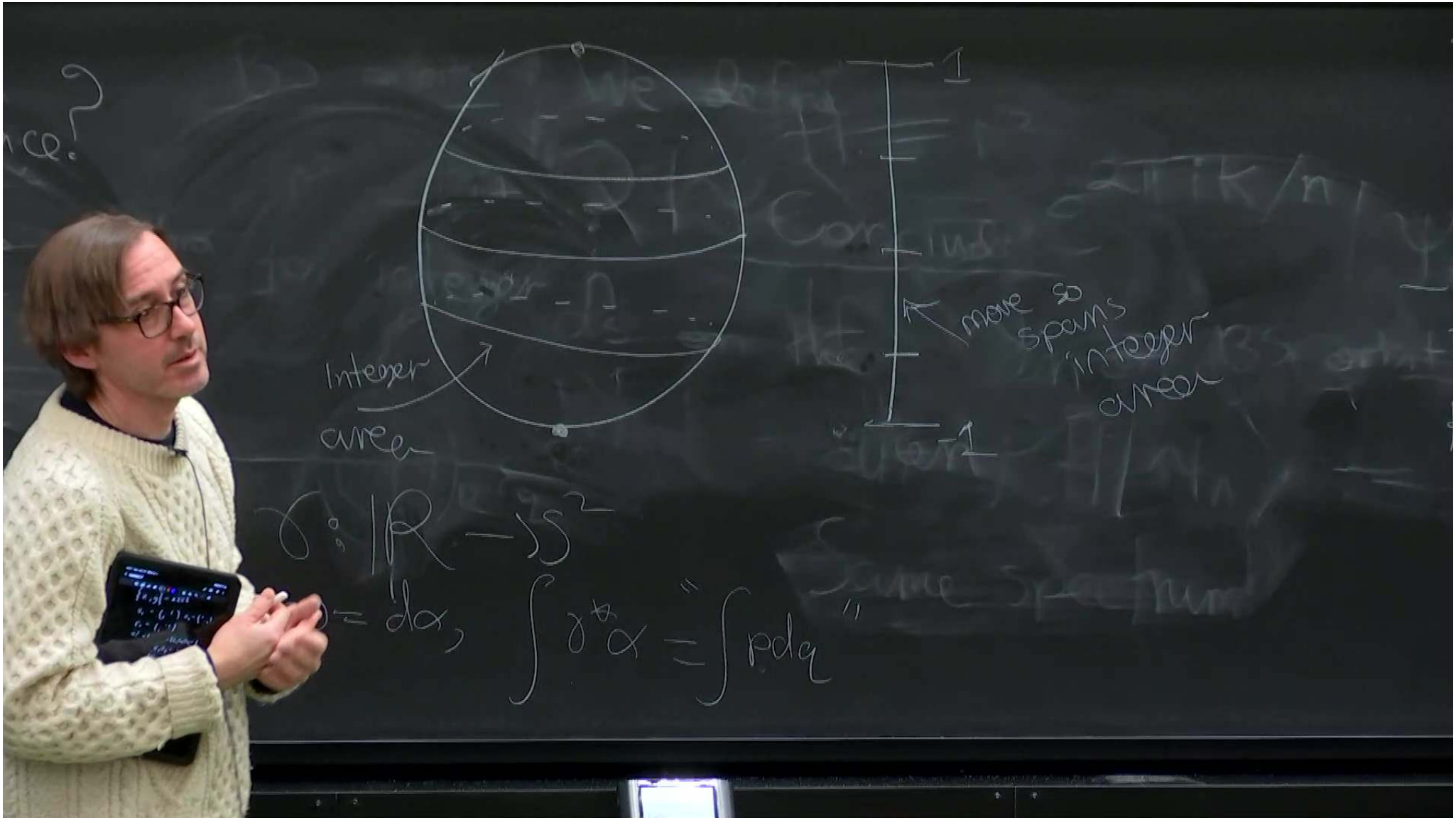
$\hbar = \frac{2}{n}$




1
-1
more so spans integer areas
Integer
area
These Bohr-Sommerfeld points are equally spaced.
$-1 + \frac{2k}{n}$
$k = 0, \ldots, n$


Area = 1
Hilbert space has 2
sates:
z
z


Claim that x, y, z = Pauli matrices
σ1, σ2, σ3
(σ3 = (1 0 / 0 −1) = z)
[σ1, σ2] = 2iσ3
{x, y} = −z 2πħ
[x, y] = z ħi = z·i as ħ = 2
f_h(x) = f*(x) + C(x)·h^p + ···
h1 = α^−1 h
h2 = α h
h3 = h


1
more so
spans
integer
area
Integer
area
1

$x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

These flip north + south poles

If $\mathrm{Vol}\, S^2 = n$?

Then, Hilbert space has $n+1$ states

nz takes integer values

$\pm n,\ 2-n,\ \ldots,\ n$

n_x, n_y still satisfy algebra of Pauli matrices

They give a representation of $su(2)$ of dim $n+1$

(these are all $su(2)$ reps)