Title: Mathematical Physics Lecture - 230131

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Collection: Mathematical Physics (2022/2023)

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Example

$$M = S^1 \times S^1$$

$p, q \quad p \sim p+1$

$\qquad\quad q \sim q+1$

$$P = e^{2\pi i p}$$

$$Q = e^{2\pi i q}$$

$$\omega = \frac{dq\, dp}{2\pi\hbar}$$

Convention (corrected by factor of i)

$$[p, q] = -i\hbar = \{p, q\} \frac{1}{2\pi i}$$

$$e^{2\pi i p} e^{2\pi i q} e^{-2\pi i p} = e^{2\pi i q} e^{-(2\pi i)^2 i\hbar} = e^{2\pi i q} e^{2\pi i/n}$$

Quantize: area to be n

So, area is

$$\frac{1}{2\pi\hbar} = n$$

$$n = \frac{1}{2\pi\hbar}$$

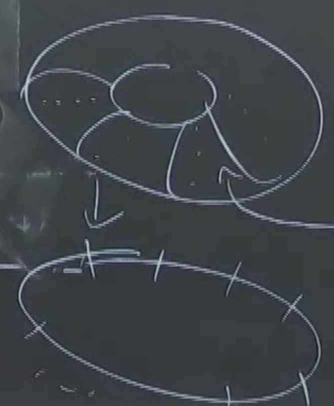
$$\boxed{PQ = QP\,e^{2\pi i/n}}$$

Last time

Bohr-Sommerfeld orbits are the circles where

$$Q = e^{2\pi i k/n}$$

(i.e. $q = k/n$, $k = 0, 1, \dots$



Area $= 1$

$q,\ q \sim q+1$

Want to show Hilbert space = rep. of this algebra

$|\psi_K\rangle$ be the state corresponding to $q = K/n$

We

Q

We define

$$Q|\psi_k\rangle = e^{2\pi i k/n}|\psi_k\rangle$$

as on the BS orbit, $Q = e^{2\pi i k/n}$

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What about P?

$$P \rightsquigarrow \frac{\partial}{\partial q}$$

$$P \rightsquigarrow e^{\frac{1}{n}\partial_q}$$

P shifts q by $\frac{1}{n}$

Declare

$$P|\psi_k\rangle = |\psi_{k-1}\rangle$$

$$PQ|\psi_k\rangle = e^{2\pi i k/n}\, P|\psi_k\rangle = e^{2\pi i k/n}|\psi_{k-1}\rangle$$

$$QP|\psi_k\rangle = Q|\psi_{k-1}\rangle = e^{2\pi i (k-1)/n}|\psi_{k-1}\rangle$$

$$PQ = QPe^{2\pi i/n}$$

Harmonic Oscillator

$$M = \mathbb{R}^2$$

$$\omega = \frac{dp\,dq}{2\pi\hbar}$$

$$H = p^2 + q^2$$

Want to use the Bohr-Sommerfeld method to determine the spectrum.

Recall,

$$p \longrightarrow -i\frac{\partial}{\partial q}\hbar$$

$$H \rightsquigarrow -\hbar^2\frac{\partial^2}{\partial q} + q^2$$

$$H = (q + \hbar\partial_q)($$

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$$H = \underbrace{(q+\hbar\partial_q)}_{a}\underbrace{(q-\hbar\partial_q)}_{b} \quad (+\hbar\ )$$

$$[a,b] = 2\hbar$$

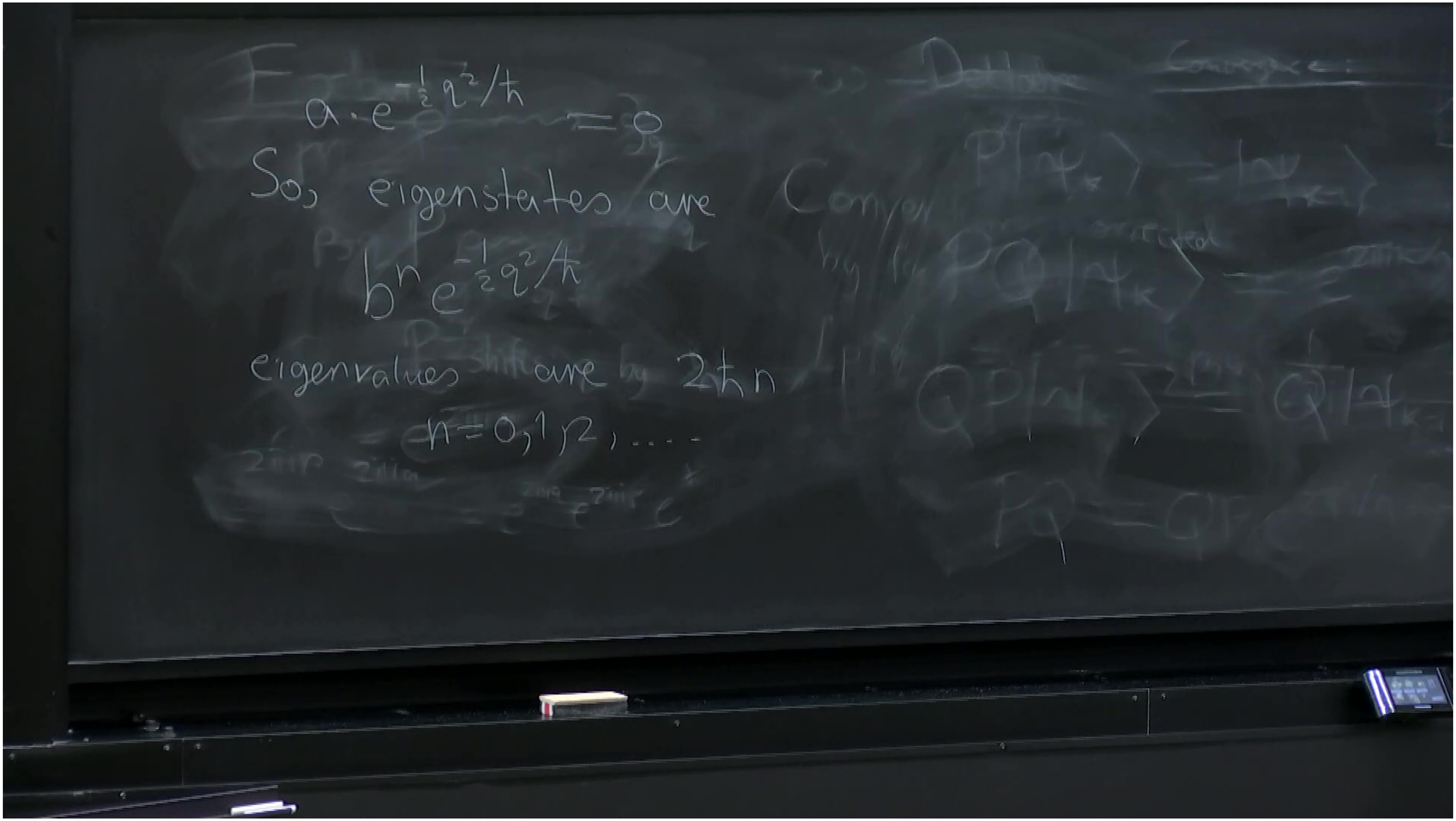
b = Raising

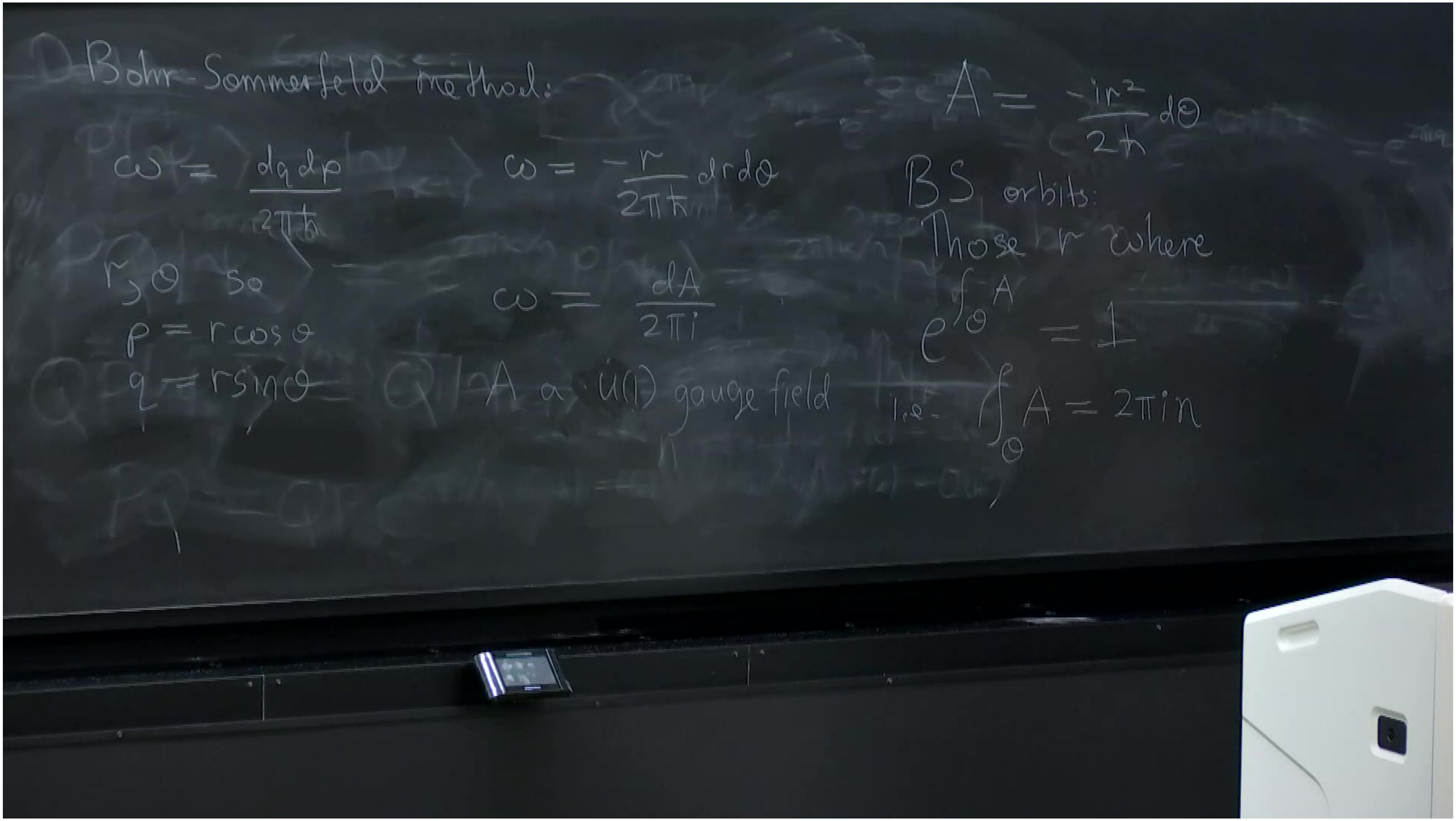
a = Lowering

$f_h(x) = f(x) + C(x)\cdot h + \cdots$

$h_1 = \alpha' h$

$h_1 = \alpha h$


$a \cdot e^{-\frac{1}{2}q^2/\hbar} = 0$
So, eigenstates are
$b^n e^{-\frac{1}{2}q^2/\hbar}$
eigenvalues are $2\hbar n$
$n = 0, 1, 2, \ldots$


Bohr-Sommerfeld method:
$\omega = \frac{dq\,dp}{2\pi\hbar}$
$\omega = -\frac{r}{2\pi\hbar}\,dr\,d\theta$
r, θ so
$p = r\cos\theta$
$q = r\sin\theta$
$\omega = \frac{dA}{2\pi i}$
A a U(1) gauge field
$A = \frac{-ir^2}{2\hbar}\,d\theta$
BS orbits:
Those r where
$e^{\int_\theta A} = 1$
i.e. $\int_\theta A = 2\pi i n$

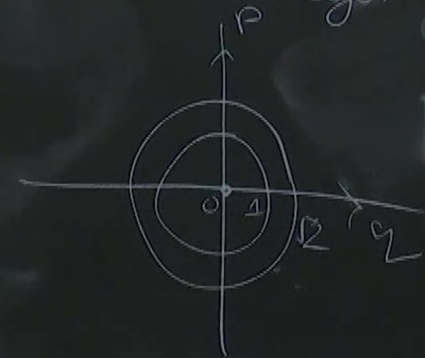
$$A = \frac{-ir^2}{2\hbar}\, d\theta$$

Want $\displaystyle\int_{\theta=0}^{2\pi} A = 2\pi i n$

$$\oint_0^{2\pi} A = \frac{-ir^2}{2\hbar}\, 2\pi$$

BS orbits

$r^2 = 2\hbar n$ for integer n



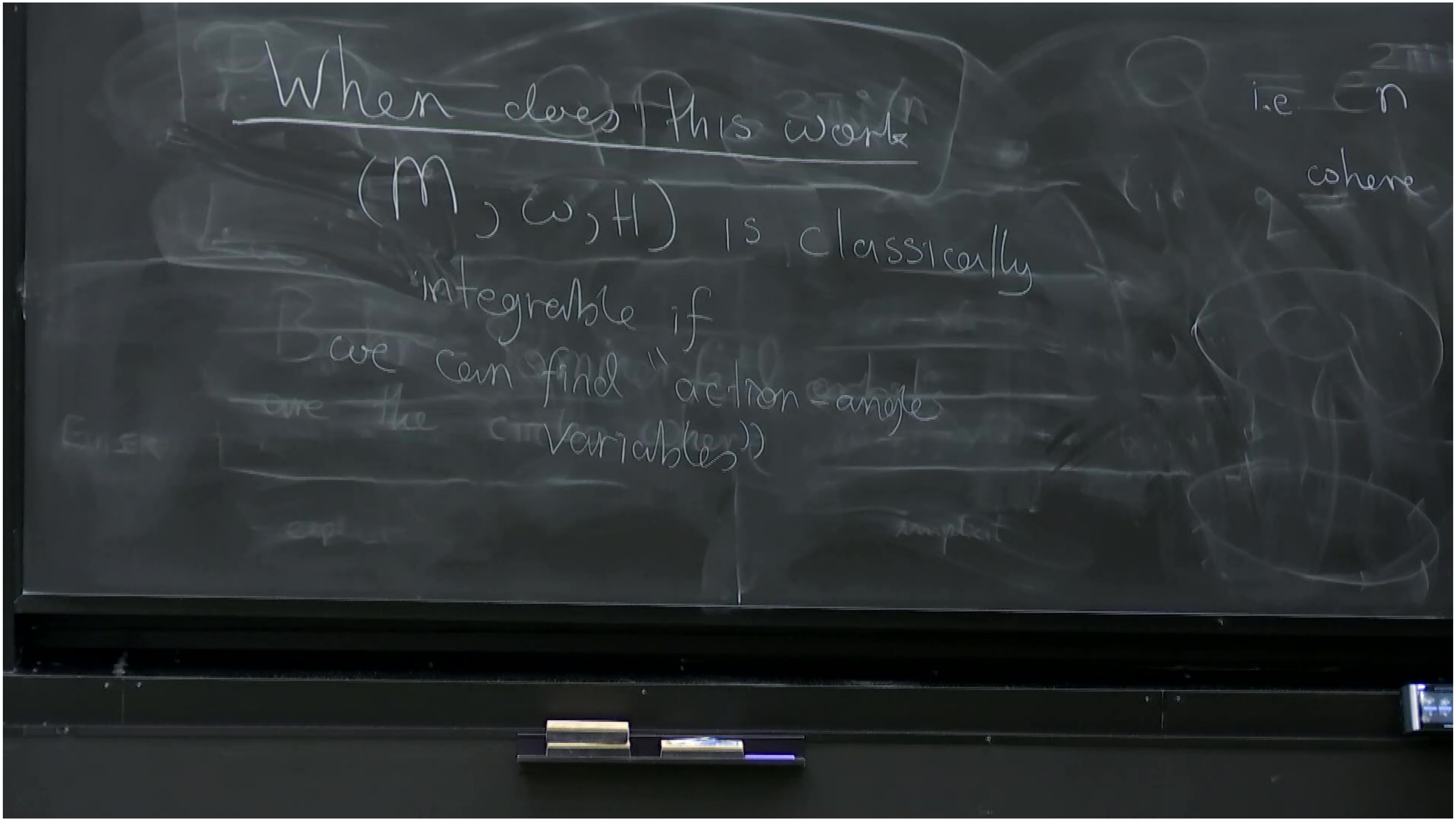
$H = r^2$

Conclude:

If ψ_n = BS orbit for $r^2 = 2\hbar n$

then $H|\psi_n\rangle = 2\hbar n|\psi_n\rangle$

Same spectrum!


When does this work
(M, ω, H) is classically
"integrable" if
we can find "action-angle
variables"

i.e. $\exists$ n functions $f_1, \dots, f_n$ on M

where 1) $\{f_i, f_j\} = 0$

$(\vec{f} : M \to \mathbb{R}^n$ polarizes $M)$

2) The fibres of $\vec{f} : M \to \mathbb{R}^n$ are tori $\underbrace{S^1 \times \dots \times S^1}_{n \text{ copies}}$

3) $H =$ one of the f's, say f_1

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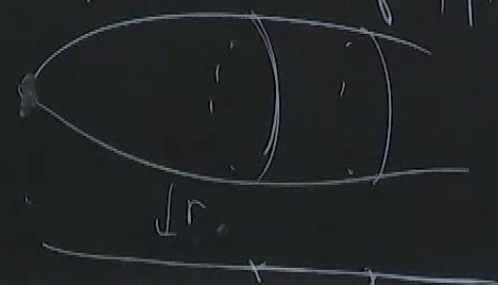
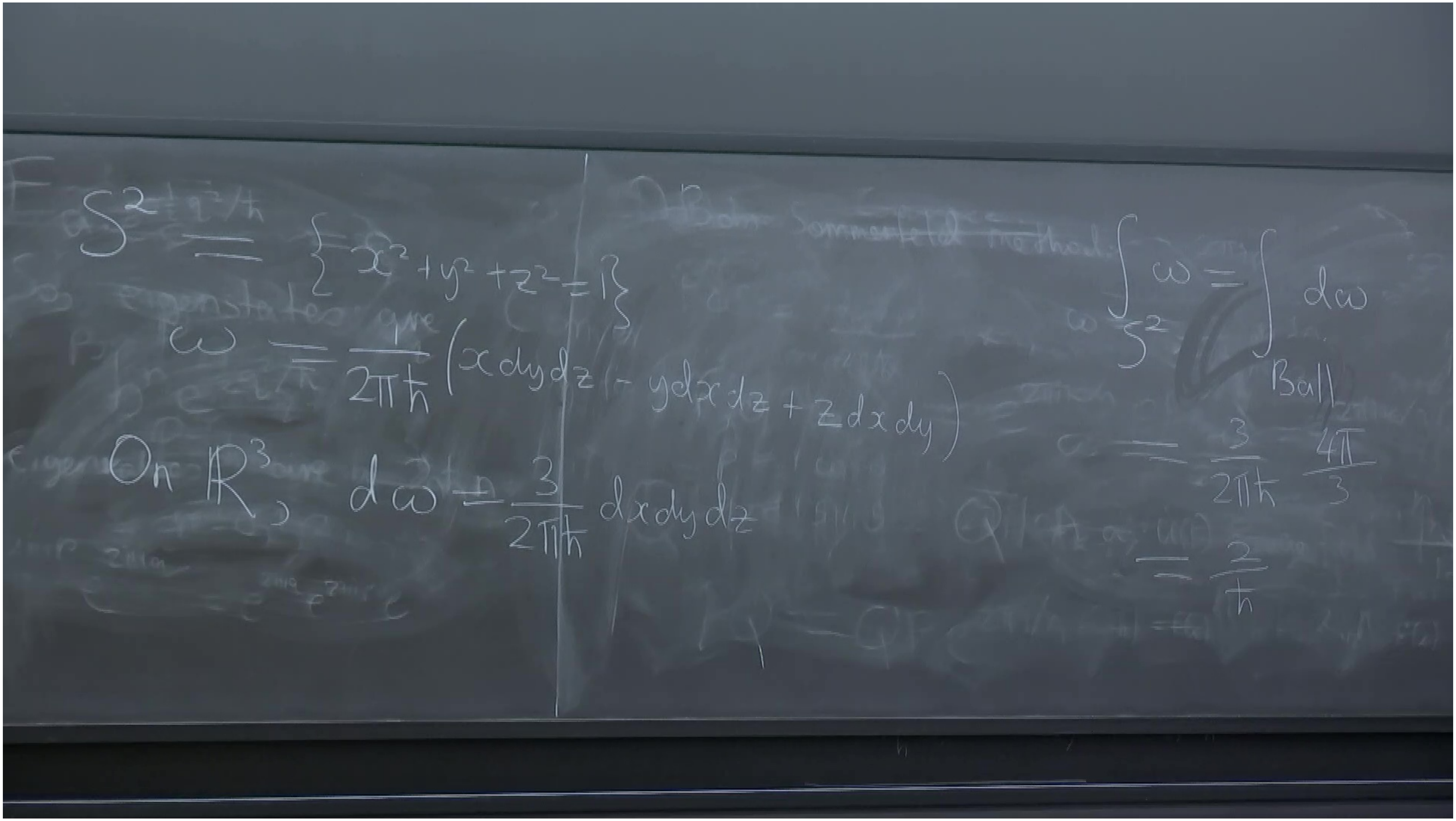
In this case, can compute spectrum of H.

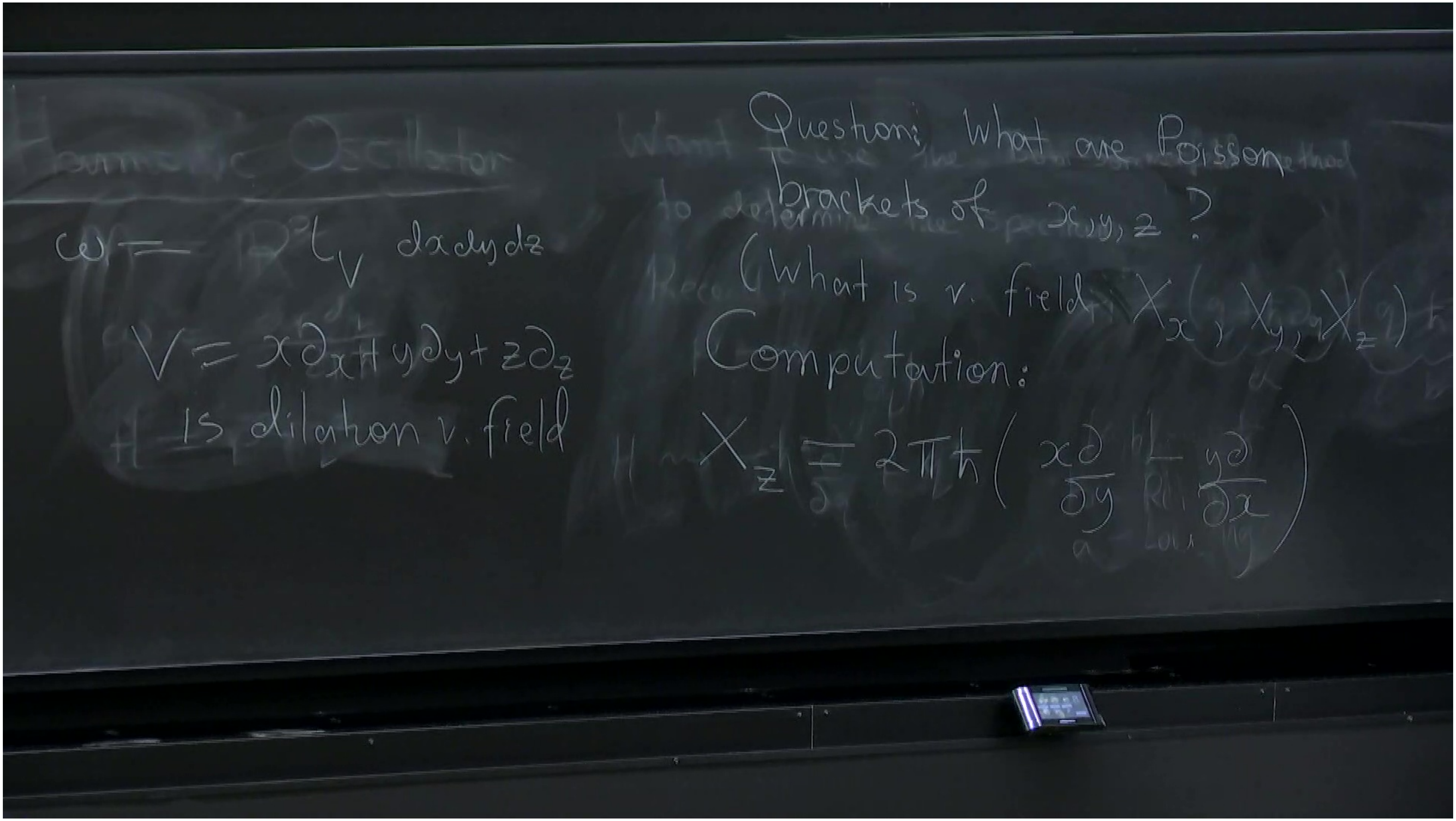


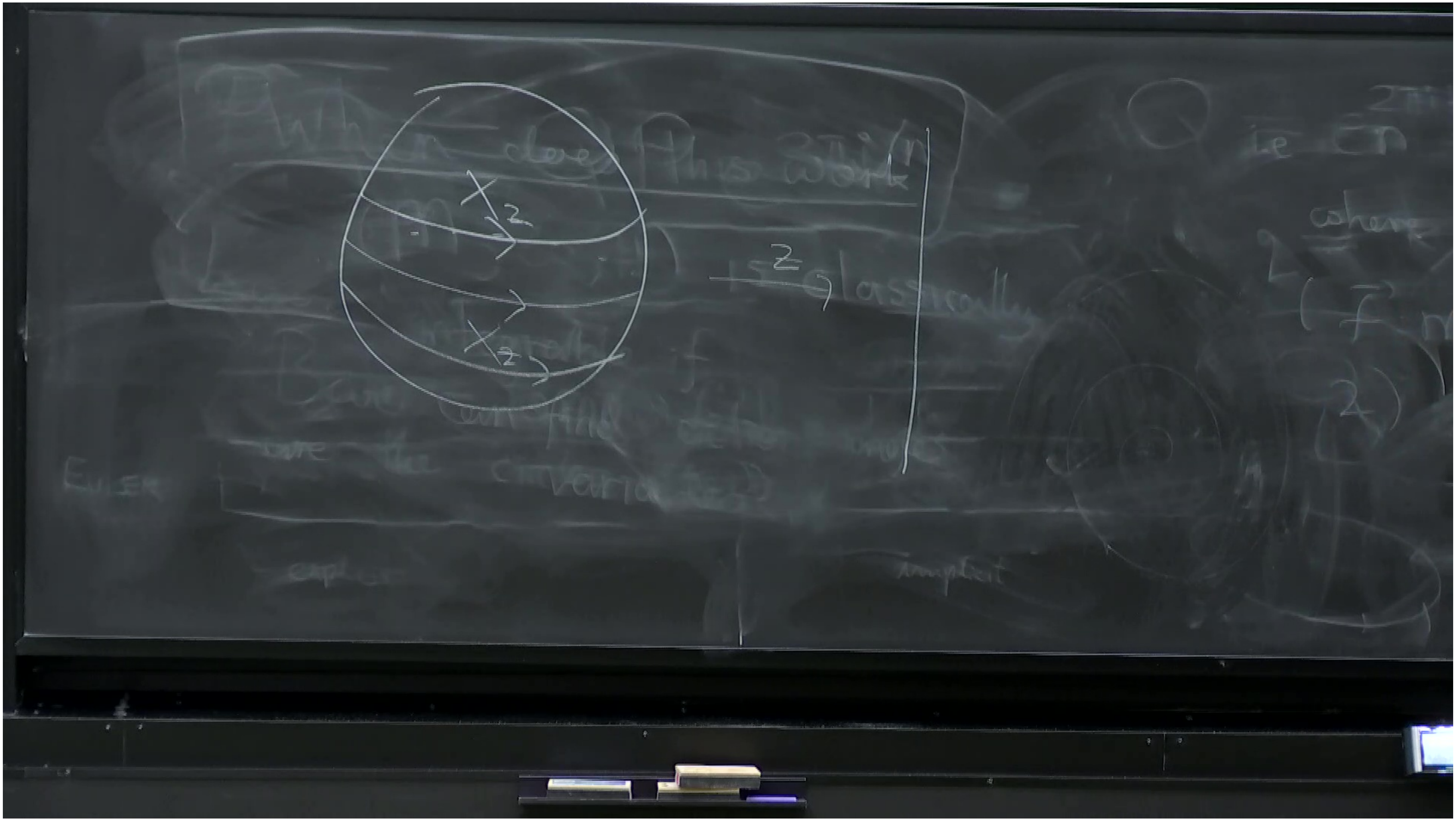
$$S^2 = \{x^2+y^2+z^2=1\}$$

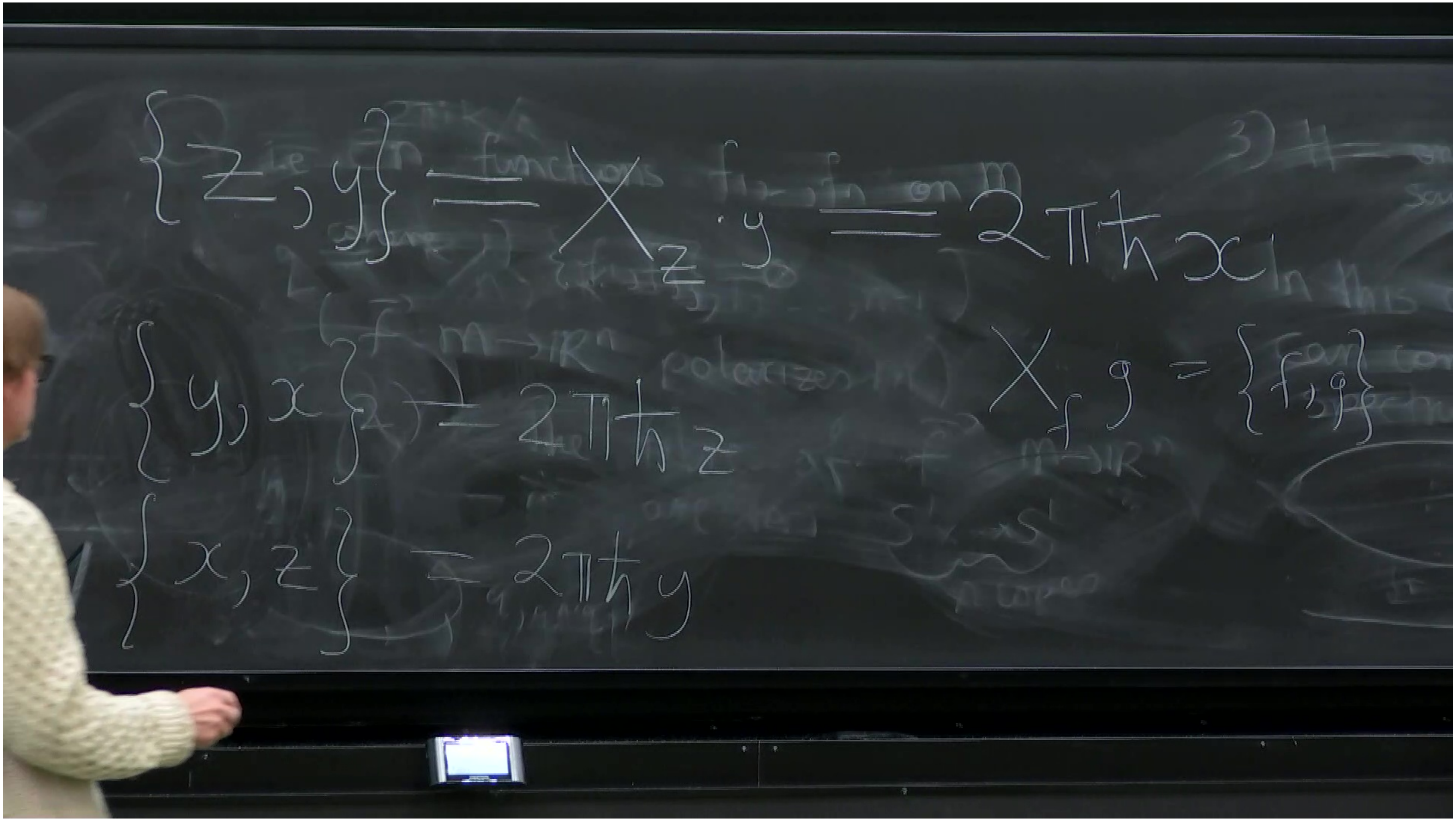
$$\omega = \frac{1}{2\pi\hbar}\left(x\,dy\,dz - y\,dx\,dz + z\,dx\,dy\right)$$

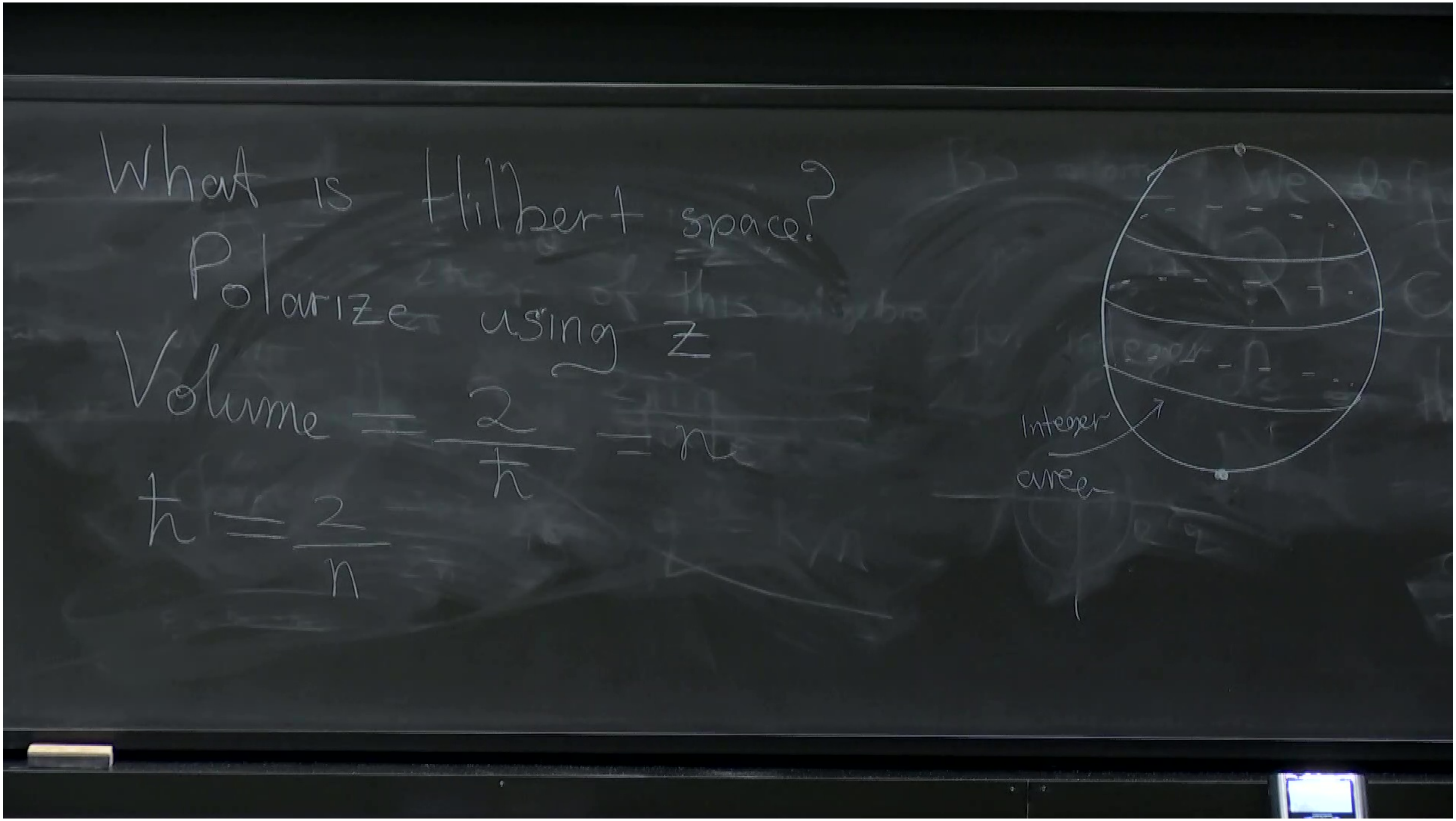
On $\mathbb{R}^3$, $d\omega = \frac{3}{2\pi\hbar}\,dx\,dy\,dz$


$S^2 = \{x^2+y^2+z^2=1\}$
$\omega = \frac{1}{2\pi\hbar}(x\,dy\,dz - y\,dx\,dz + z\,dx\,dy)$
On $\mathbb{R}^3$, $d\omega = \frac{3}{2\pi\hbar}\,dx\,dy\,dz$
Bohr Sommerfeld method
$\int_{S^2}\omega = \int_{\text{Ball}} d\omega$
$= \frac{3}{2\pi\hbar}\,\frac{4\pi}{3}$
$= \frac{2}{\hbar}$


$\omega = \iota_V\, dx\,dy\,dz$
$V = x\partial_x + y\partial_y + z\partial_z$
is dilation v. field
Question: What are Poisson brackets of x, y, z?
(What is v. field X_x, X_y, X_z)
Computation:
$X_z = 2\pi\hbar\left(x\frac{\partial}{\partial y} \quad y\frac{\partial}{\partial x} \right)$


classically
where


$\{z, y\} = X_z \cdot y = 2\pi\hbar x$
$\{y, x\} = 2\pi\hbar z$
$\{x, z\} = 2\pi\hbar y$
$X_f\, g = \{f, g\}$

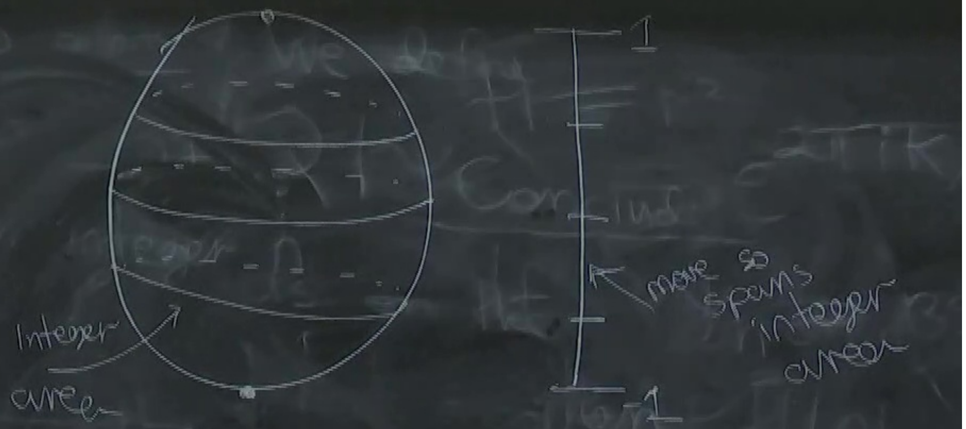

What is Hilbert space?
Polarize using z
Volume = 2/ħ = n
ħ = 2/n
Integer area

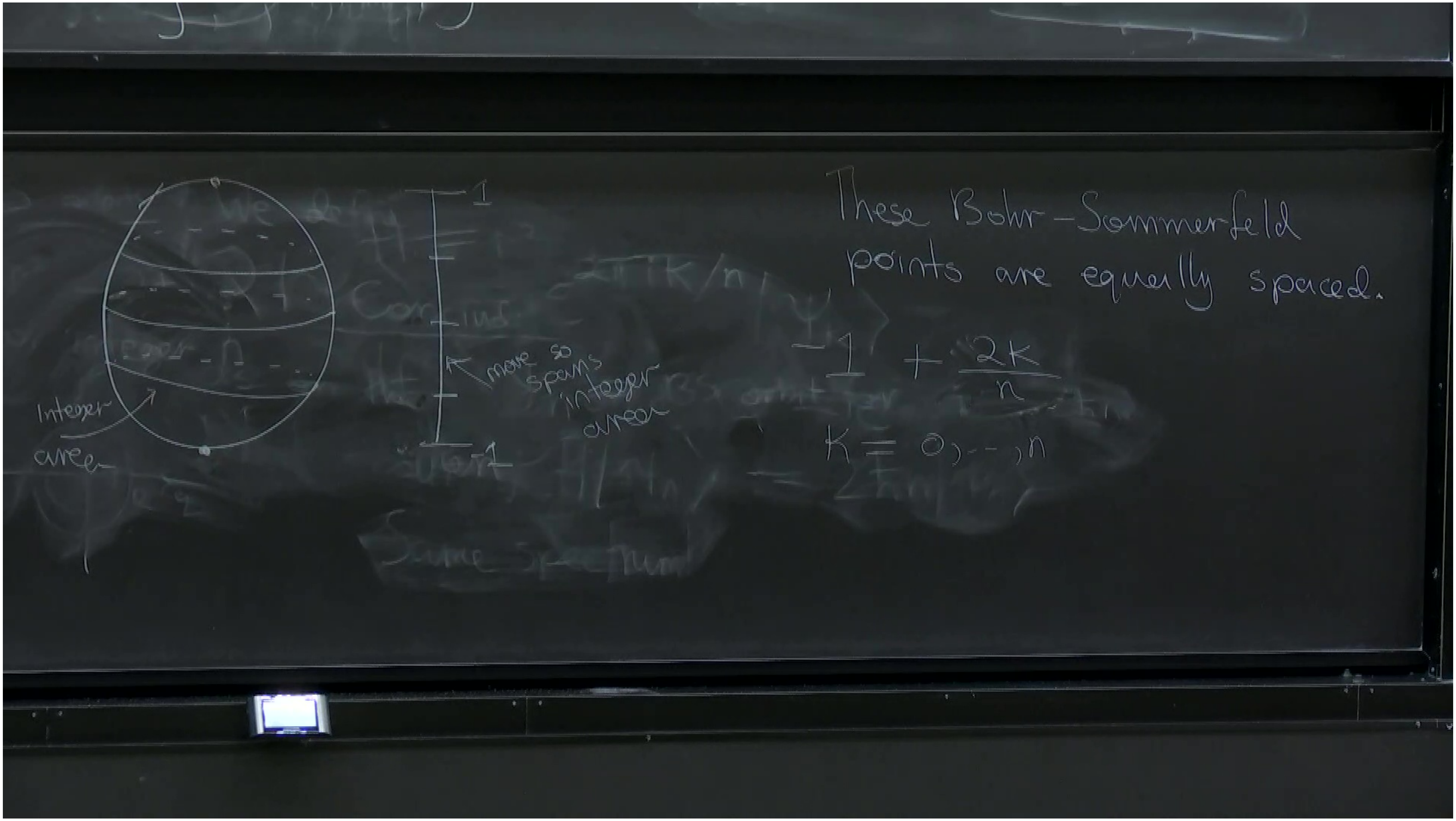
What is Hilbert space?

Polarize using z

$$\text{Volume} = \frac{2}{\hbar} = n$$

$$\hbar = \frac{2}{n}$$




These Bohr-Sommerfeld points are equally spaced.
$-1 + \frac{2k}{n}$
$k = 0, \dots, n$
1
-1
move so spans 'integer' area
Integer area

Area 1

Hilbert space has 2 sates: $|\downarrow\rangle, |\uparrow\rangle$

$z|\downarrow\rangle = -1, \quad z|\uparrow\rangle = 1$

Claim that x, y, z = Pauli matrices

$\sigma_1, \sigma_2, \sigma_3$ $\quad \left(\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = z\right)$

$$[\sigma_1, \sigma_2] = 2i\sigma_3$$

$$\{x, y\} = -z\, 2\pi\hbar$$

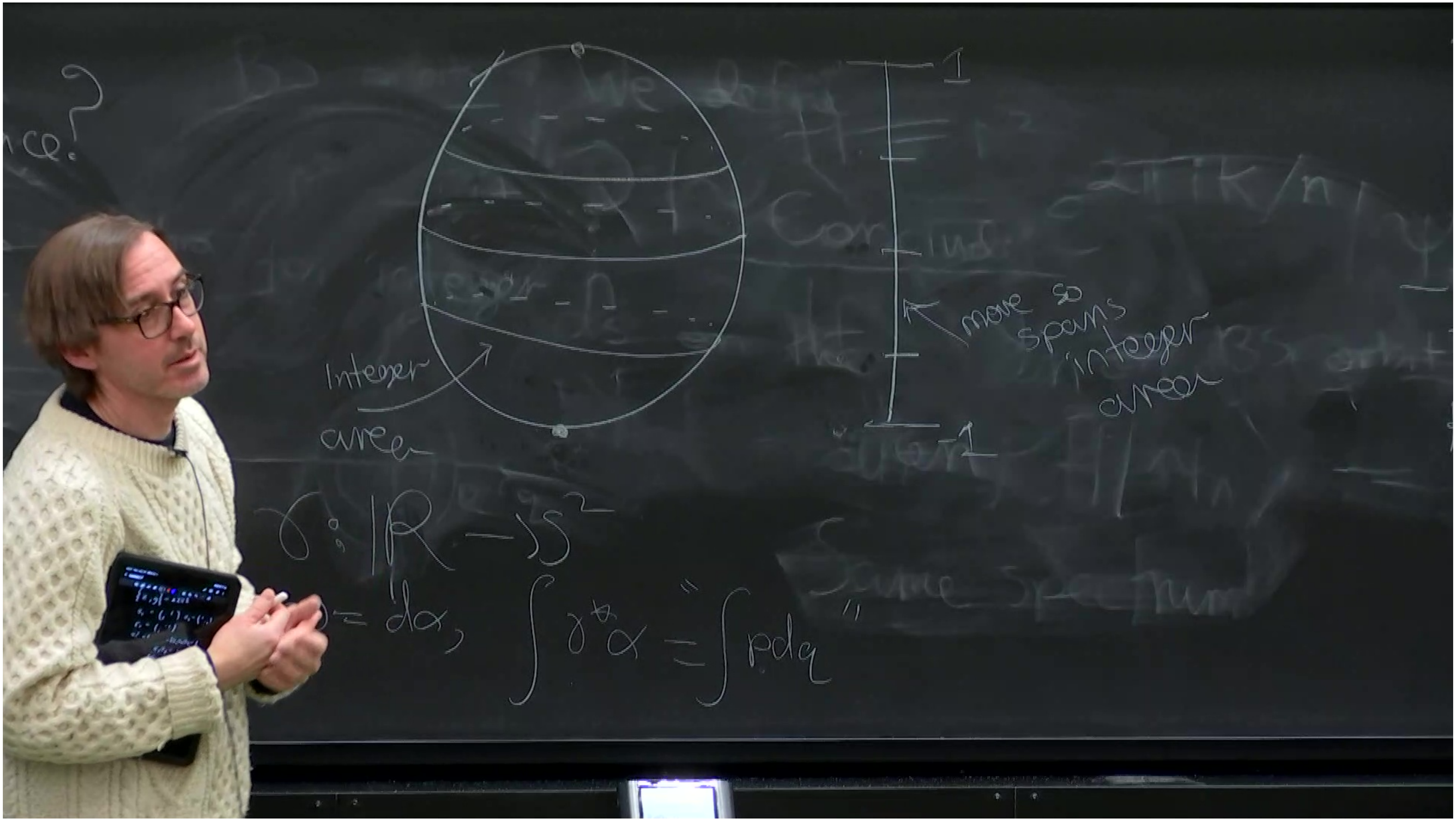
$$[x, y] = z\hbar i \doteq z \cdot i \quad \text{as } \hbar = 2$$

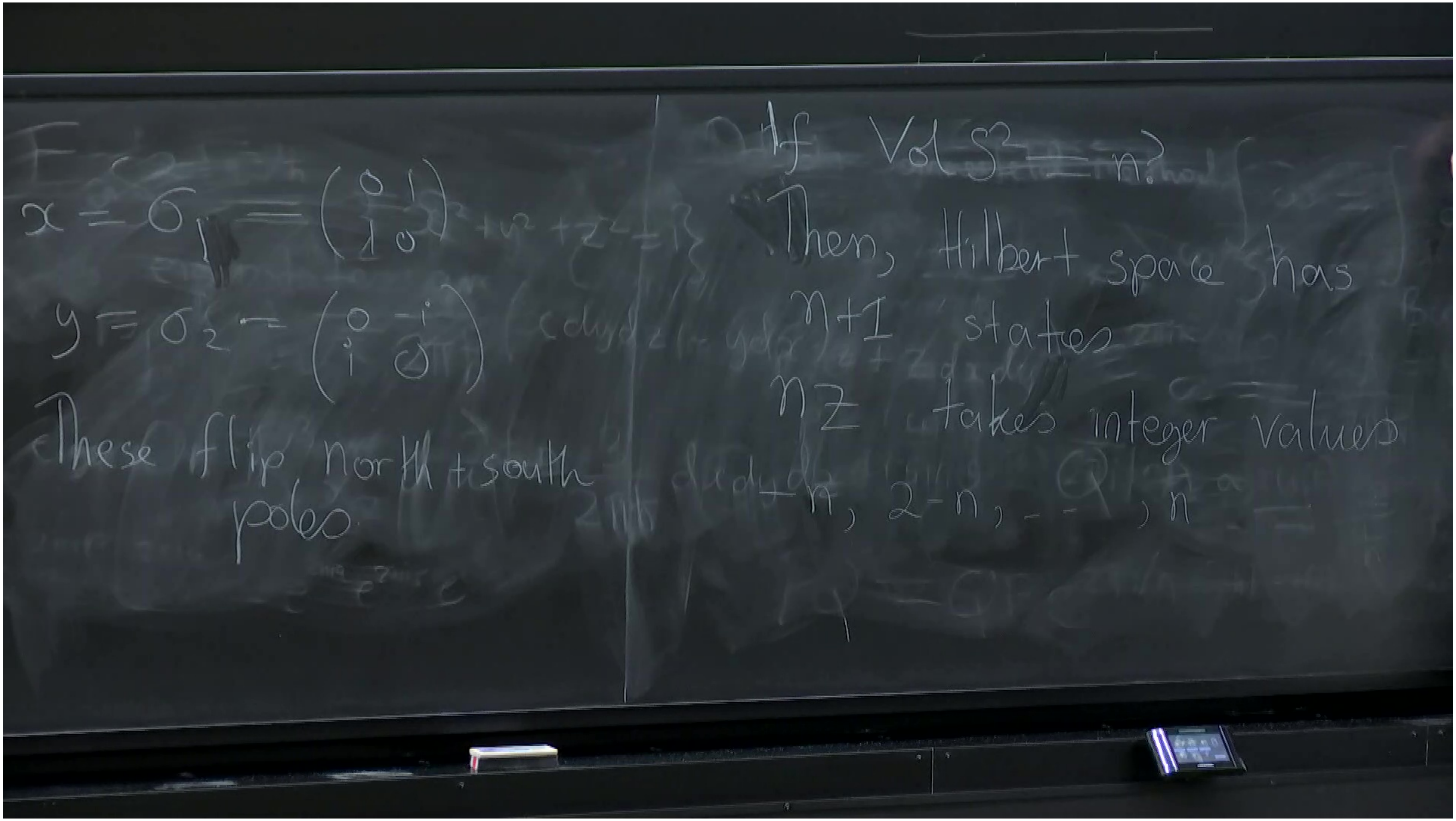
$$f_h(x) = f^*(x) + C(x) \cdot h^p + \cdots$$

$h_1 = \alpha^2 h$

$h_2 = \alpha h$

$h_3 = h$


Integer
area
1
-1
move so
spans
integer
area
Same Spectrum



x = σ₁ = (0 1 / 1 0)
y = σ₂ = (0 −i / i 0)
These flip north + south poles.
If Vol S² = n?
Then, Hilbert space has n+1 states
n_Z takes integer values
−n, 2−n, ..., n

n_x, n_y still satisfy
algebra of Pauli matrices
They give a representation
of $su(2)$ of dim^n $n+1$
(these are all $su(2)$ reps)

$\oint A = 2\pi i n$