Title: Mathematical Physics Lecture - 230126

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Collection: Mathematical Physics (2022/2023)

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Last time

Gauge theory in 1+1 dimension:

Before symplectic reduction

$$Q(\theta) = A_\theta(\theta)$$

$$P(\theta) = \partial_t A_\theta$$

$$\{P(\theta), Q(\theta')\} = \delta_{\theta-\theta'}$$

$$\mu(\theta) = \partial_\theta P(\theta)$$

is used to symplectically

dimension.

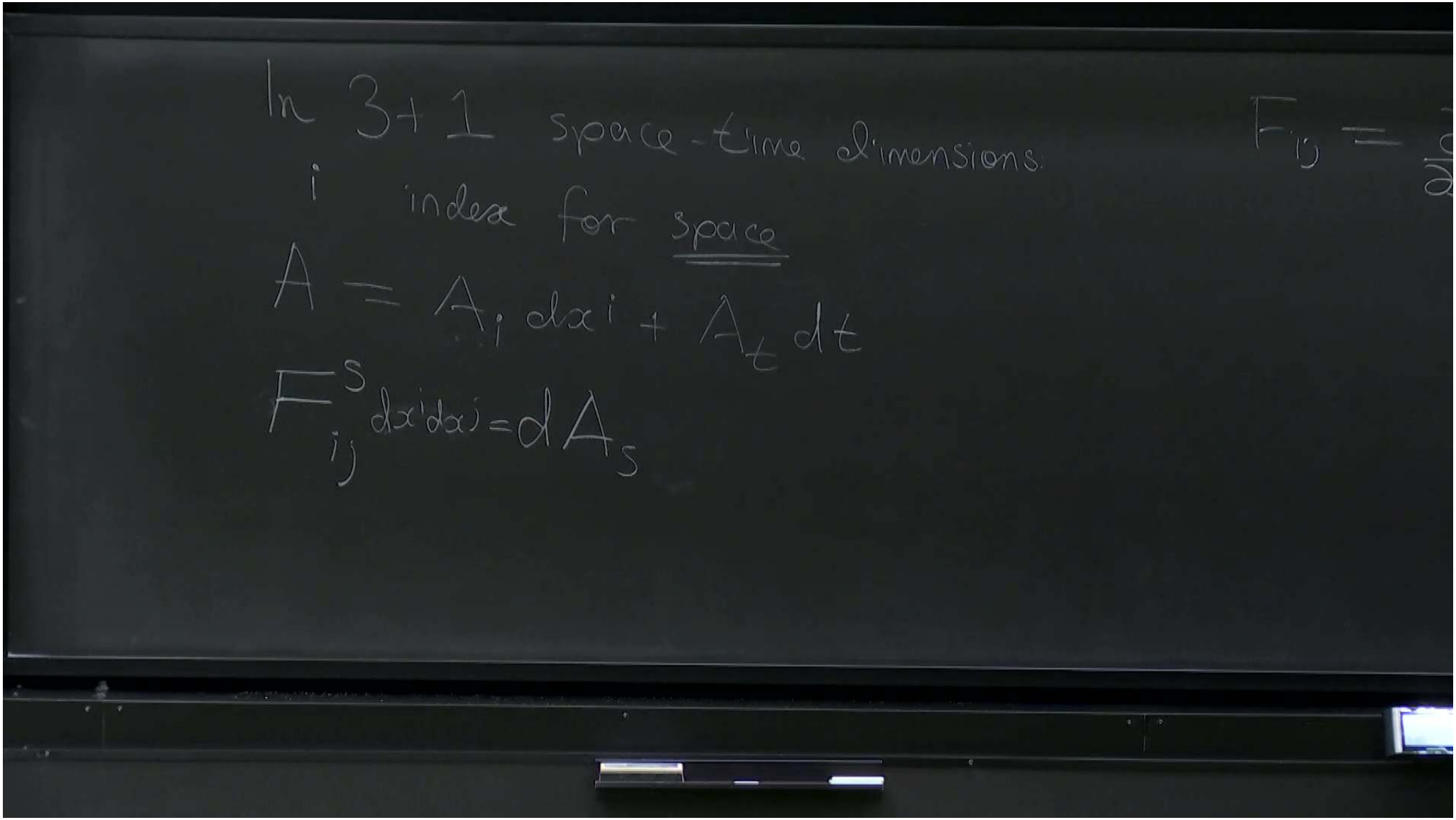
uction

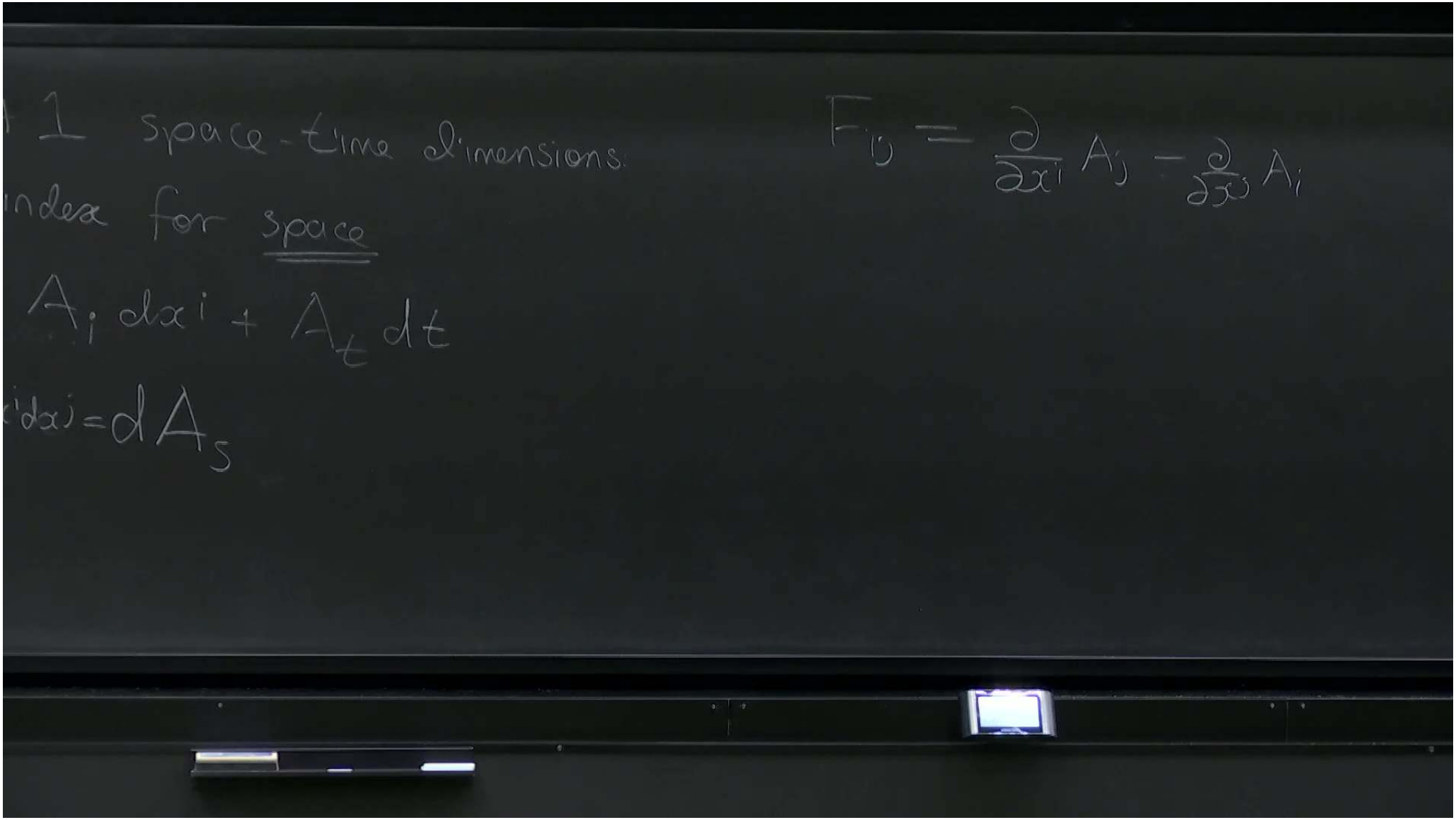
(θ)

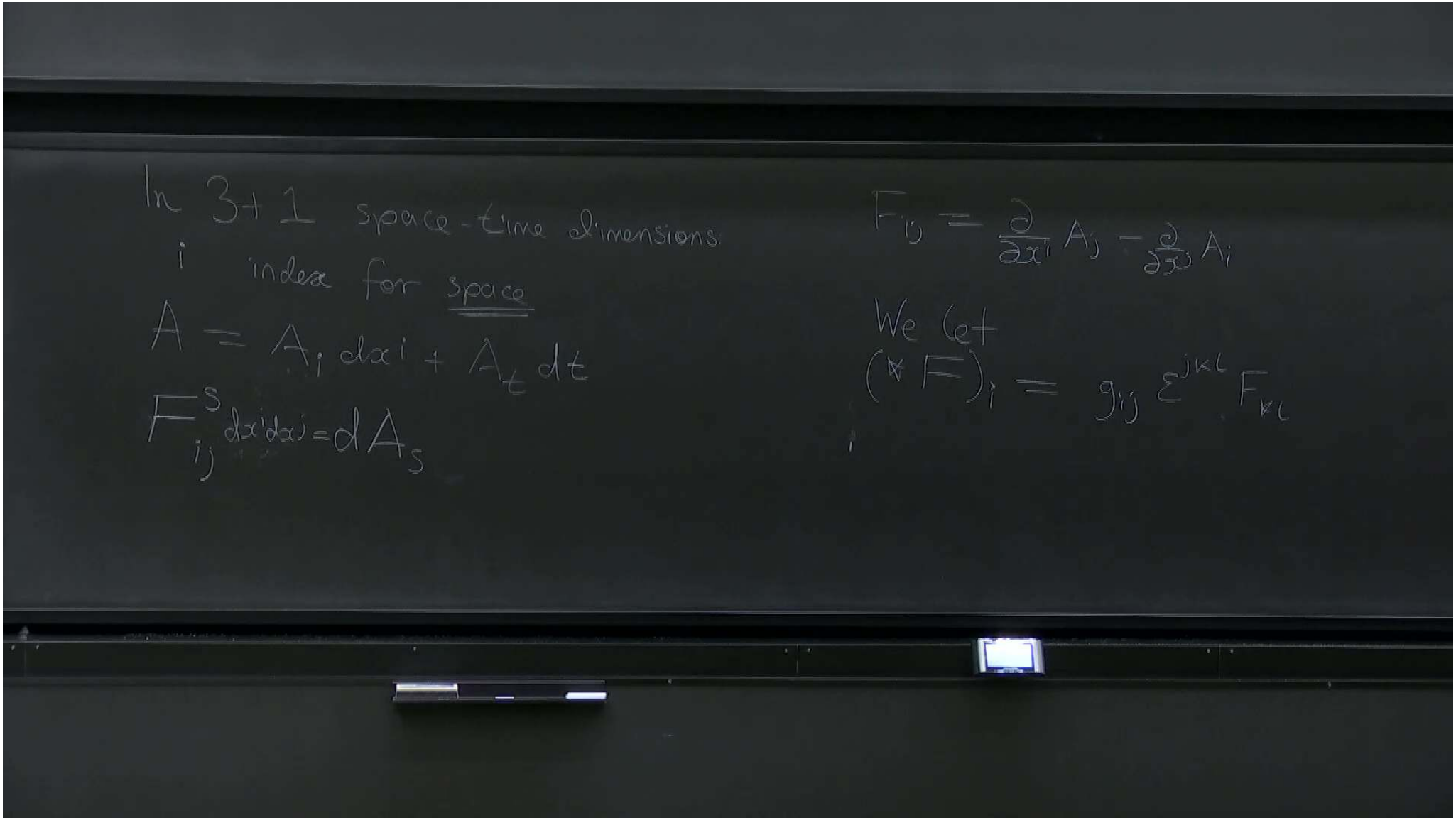
$$\{P(\theta), Q(\theta')\} = \delta_{\theta-\theta'}$$

$$\mu(\theta) = \partial_\theta P(\theta)$$

is used to symplectically reduce.


In 3+1 space-time dimensions.
i index for space
$A = A_i dx^i + A_t dt$
$F^S_{ij} dx^i dx^j = dA_S$


1 space-time dimensions
ndex for space
$A_i dx^i + A_t dt$
$i dx^i = dA_s$
$F_{ij} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i$

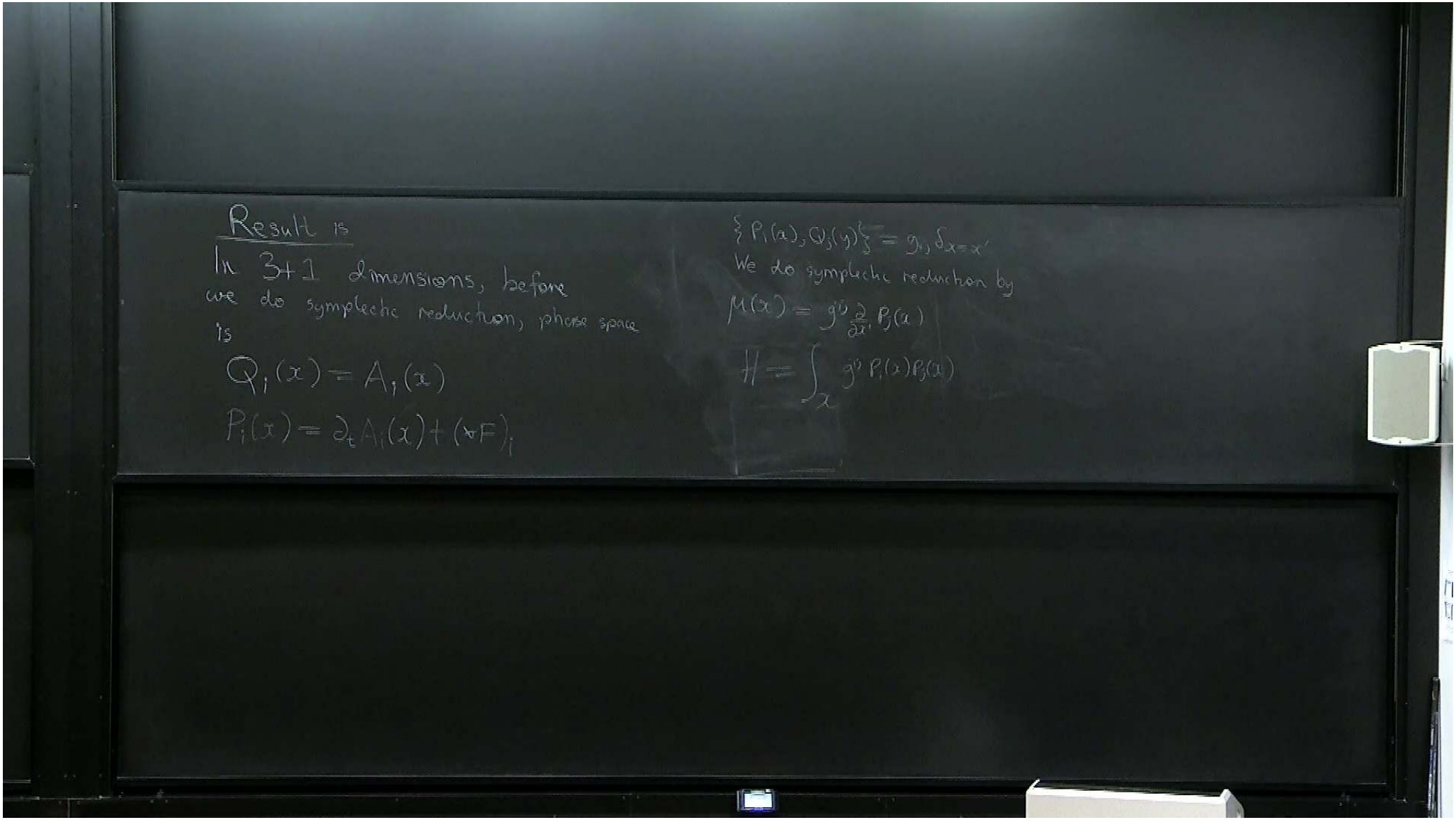

In 3+1 space-time dimensions.
i index for space
$A = A_i dx^i + A_t dt$
$F^S_{ij} dx^i dx^j = dA_S$
$F_{ij} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i$
We get
$(*F)_i = g_{ij} \varepsilon^{jkl} F_{kl}$

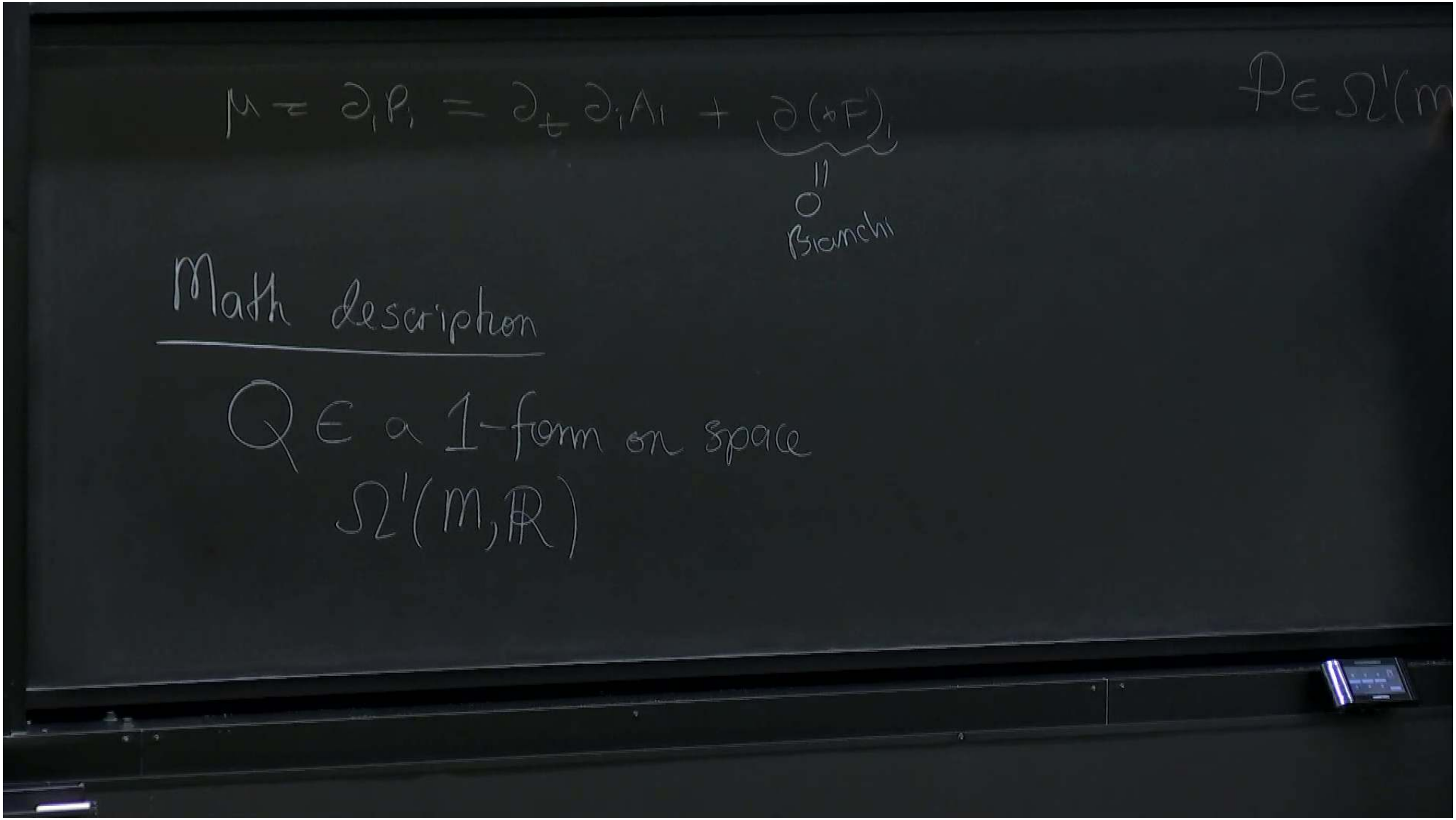
Result is

In 3+1 dimensions, before we do symplectic reduction, phase space is

$$Q_i(x) = A_i(x)$$

$$P_i(x) = \partial_t A_i(x) + (\nabla F)_i$$


Result is
In 3+1 dimensions, before we do symplectic reduction, phase space is
$Q_i(x) = A_i(x)$
$P_i(x) = \partial_t A_i(x) + (\nabla F)_i$
$\{P_i(x), Q_j(y)\} = g_{ij}\delta_{x=x'}$
We do symplectic reduction by
$\mu(x) = g^{ij}\frac{\partial}{\partial x^i}B_j(x)$
$H = \int_\Sigma g^{ij}P_i(x)B_j(x)$


$\mu = \partial_i P_i = \partial_t \partial_i A_i + \underbrace{\partial (*F)_i}_{0}$
Bianchi
$P \in \Omega^1(m$
Math description
$Q \in$ a 1-form on space
$\Omega^1(M, \mathbb{R})$

$P \in \Omega^1(m, \mathbb{R})$ or

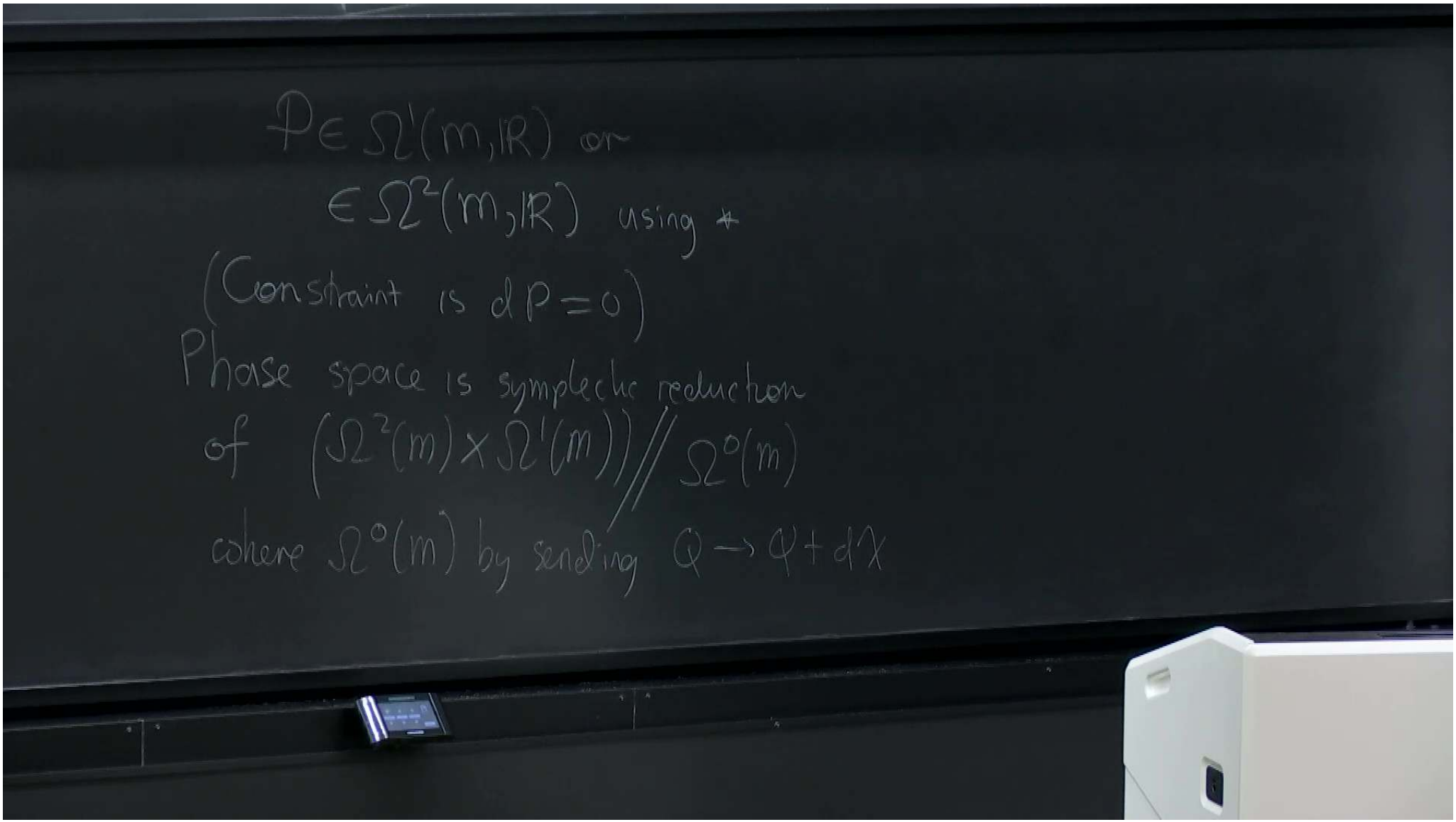
$\in \Omega^2(m, \mathbb{R})$ using $*$

(Constraint is $dP = 0$)

Phase space is symplectic reduction

of $(\Omega^2(m) \times \Omega^1(m)) /\!/ \Omega^0(m)$

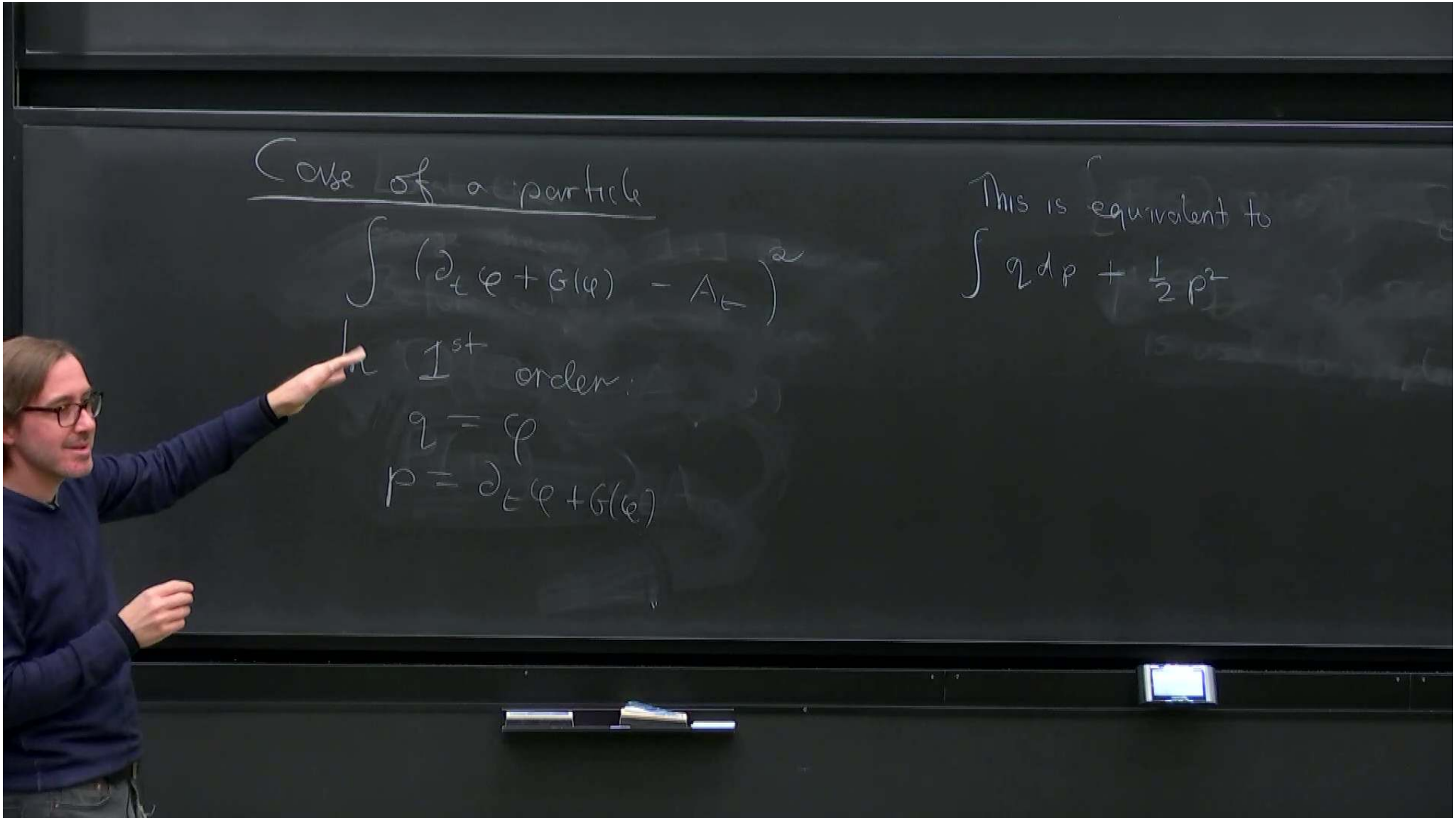
where $\Omega^0(m)$ by sending $Q \to Q + d\chi$



$$F_{full} = F_{spatial} + \partial_t A_i - \partial_i A_t$$

$$S_{YM} = \int F_{full}^2 = \int \left(\underbrace{\partial_t A_i + (*F_{spatial})_i}_{\mathcal{P}_i} - \partial_i A_t \right)^2$$

... exactly as before.


Case of a particle
$\int (\partial_t \varphi + G(\varphi) - A_t)^2$
In 1st order:
$q = \varphi$
$p = \partial_t \varphi + G(\varphi)$
This is equivalent to
$\int q\,dp + \frac{1}{2}p^2$

Deformation Quantization $\neq$ Canonical Quantization

If $M = \mathbb{R}^2$, $\omega = dq \wedge dp$

$\{p, q\} = 1$

When we quantize, p, q be

Canonical Quantization

When we quantize, p, q become non-commuting and $[p, q] = \hbar$.

Locally, in coords x^i $\omega = \omega_{ij}\, dx^i \wedge dx^j$

ω_{ij} is constant

$\omega^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$

Formula for $*$ product (Moyal):

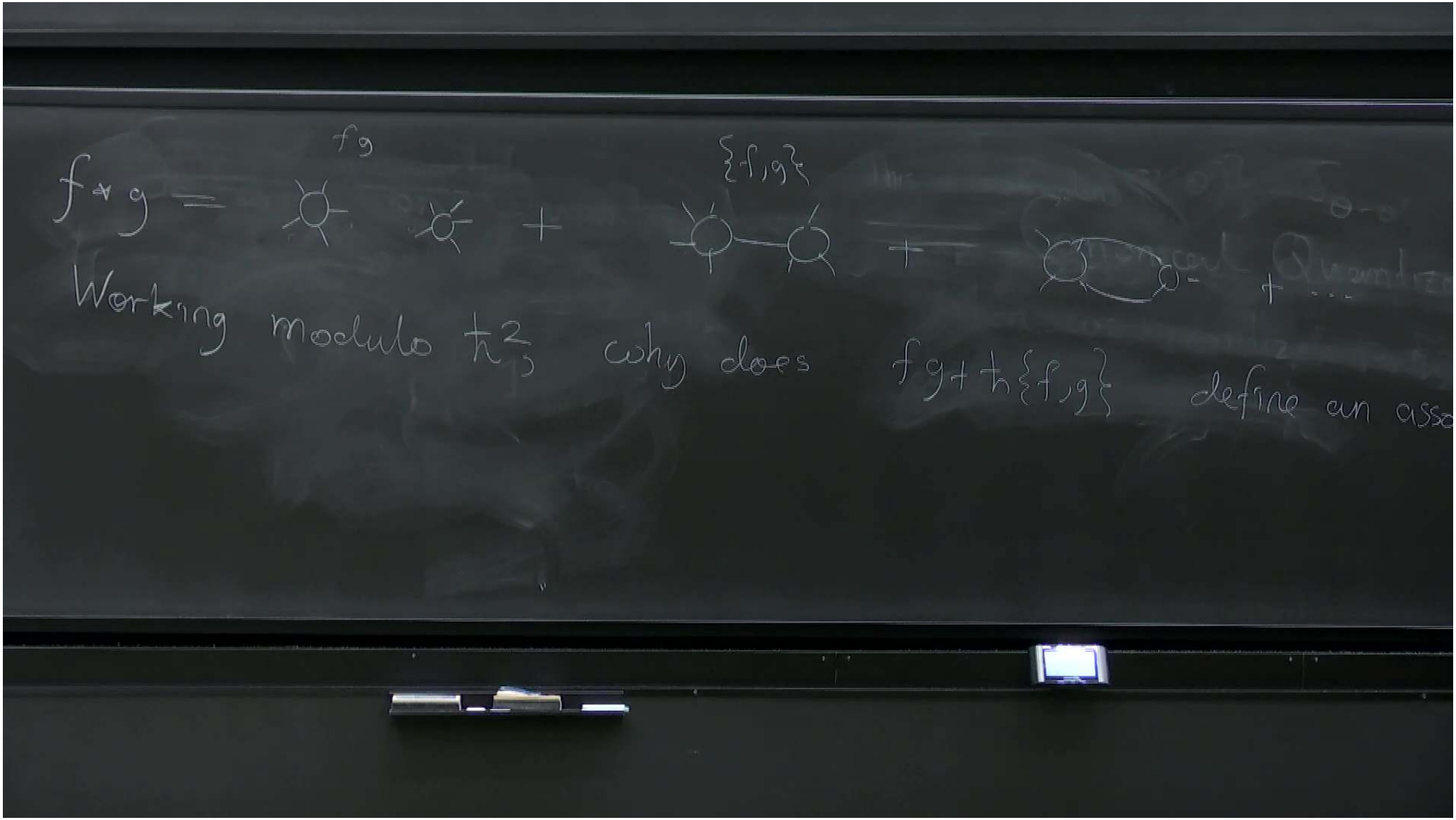
$$f \star g = fg + \hbar \omega^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} + \hbar^2 \frac{\omega^{ij}\omega^{kl}}{2} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial^2 g}{\partial x_j \partial x_l} + \cdots$$

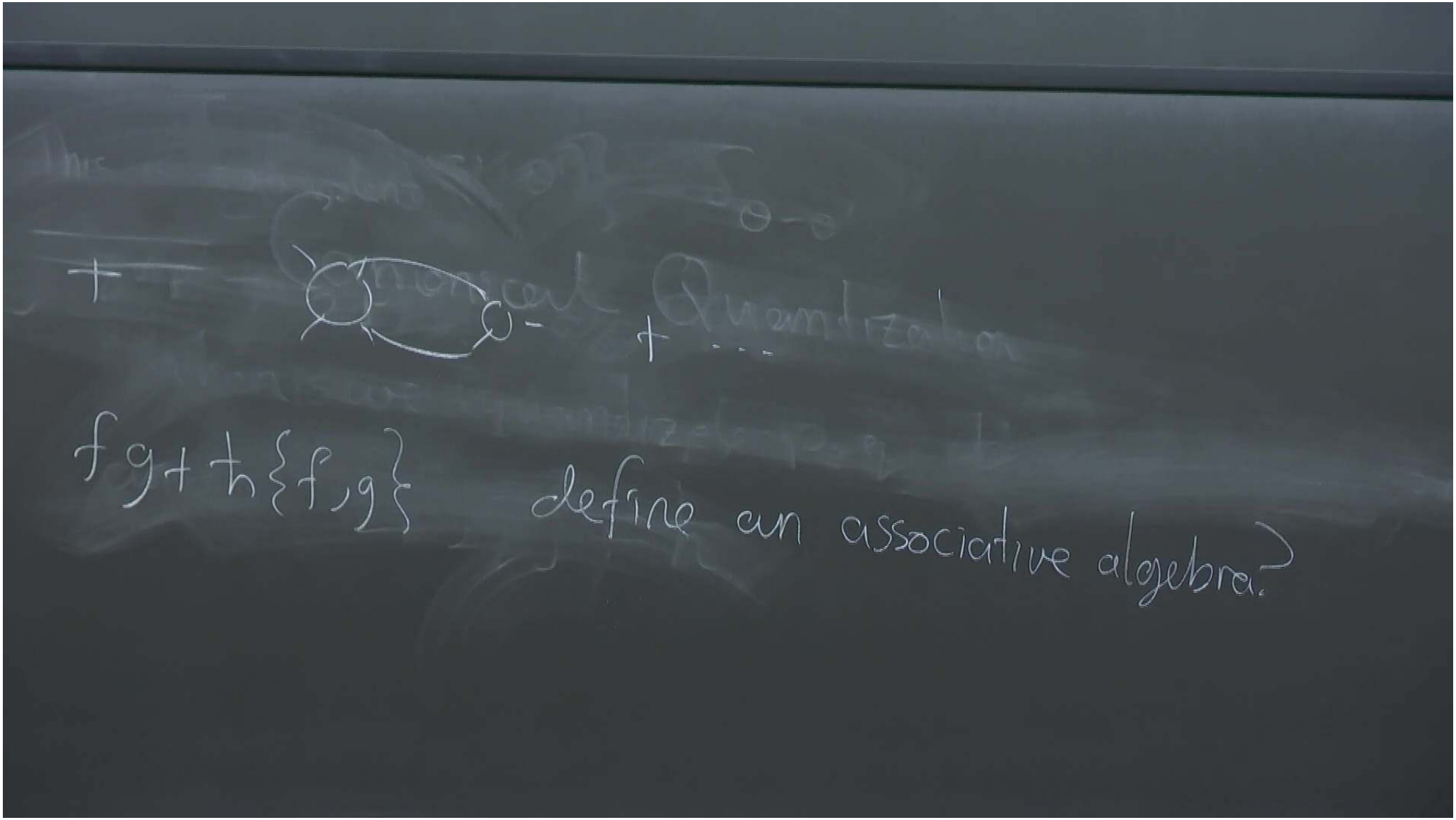
$$= e^{\hbar \omega^{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}} f \otimes g$$

hits f, hits g

If $f = x^n$

$g = x^m$


f * g = fg + {f,g} + ...
Working modulo ħ²; why does fg + ħ{f,g} define an


+ + ...
f g + h {f, g} define an associative algebra?

$\{f, -\}$ is locally $\omega^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}$

It involves differentiating once.

The Leibniz rule says

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

Similarly $$\{fg, h\} = f\{g, h\} + g\{f, h\}$$

Let's check that

$f \star (g \star h) = (f \star g) \star h$ modulo $\hbar^2$

LHS $fgh + \frac{\hbar}{i} f\{g,h\} + \frac{\hbar}{i}\{f,gh\} + O(\hbar^2)$

RHS: $fgh + \frac{\hbar}{i}\{f,g\}h + \frac{\hbar}{i}\{fg,h\} + O(\hbar^2)$

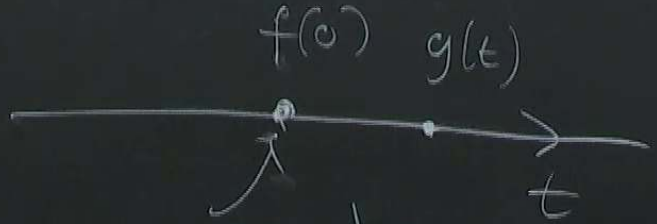
The Leibniz rule $\Rightarrow$ these are the same.

modulo $\hbar^2$

$-\hbar\{f,gh\} + O(\hbar^2)$

$+\hbar\{fg,h\} + O(\hbar^2)$

are the same.



$= C^\infty(m)$

$f \star g = (\lim_{t \to 0_+} f(0)\, g(t)$

If we have n free particles $\omega =$

$p_i, q_i \quad \omega = dq_i dp_i$

$\mu =$ quadratic expressions

$p\ q$

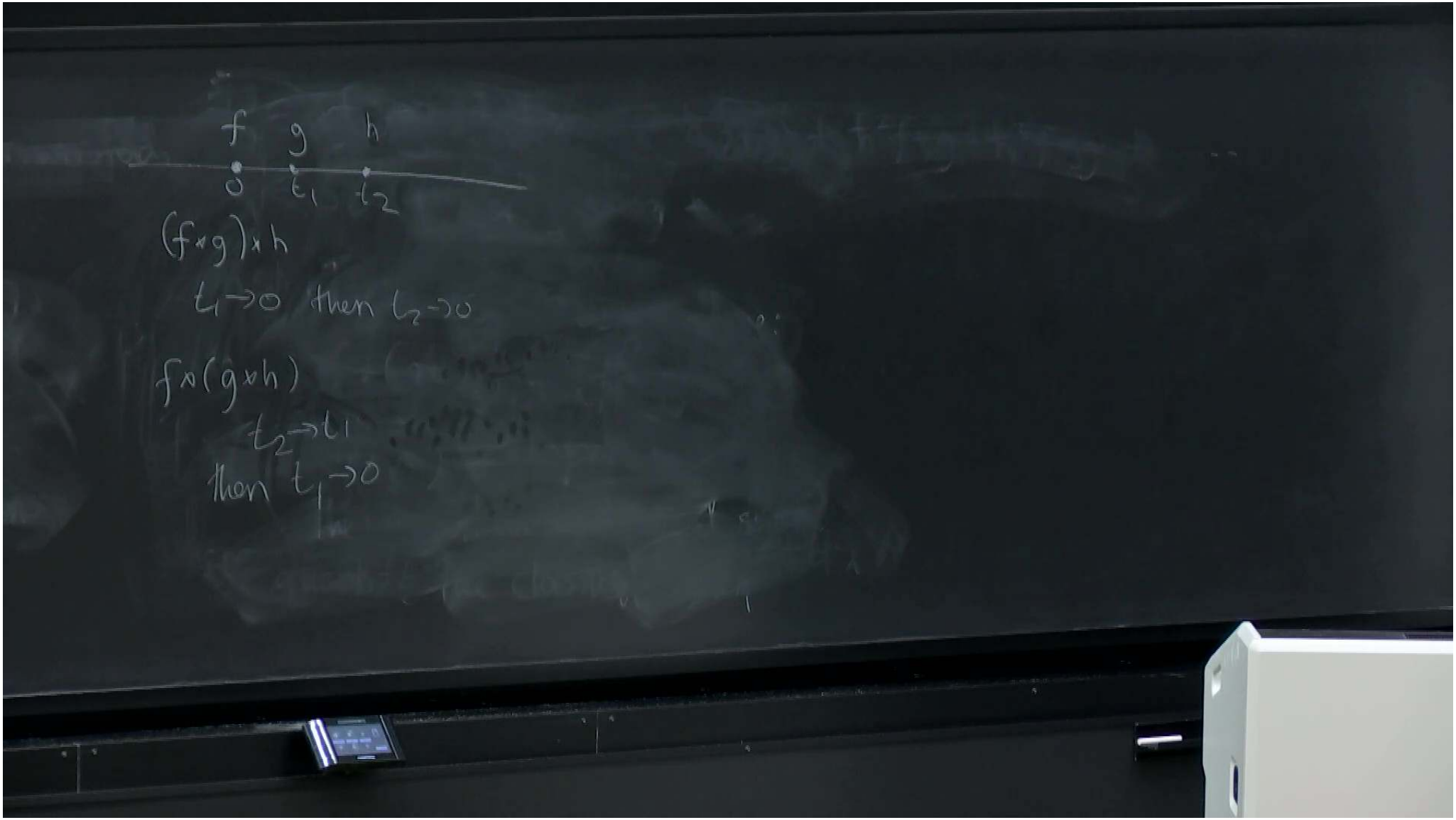
or p^2, q^2

$\{\mu, \mu\}$ satisfy some algebra (classical symmetry)

$$[p^2, q^2] = -2\hbar p q + \hbar^2 1$$

$$p = -\hbar \frac{\partial}{\partial q}$$

$$\left[\hbar^2 \frac{\partial^2}{\partial q^2}, q^2\right] = 2\hbar^2 q \frac{\partial}{\partial q} + \hbar^2 1$$


f g h
0 t_1 t_2
$(f*g)*h$
$t_1 \to 0$ then $t_2 \to 0$
$f*(g*h)$
$t_2 \to t_1$
then $t_1 \to 0$

Where does $[p,q] = \hbar$ come from?

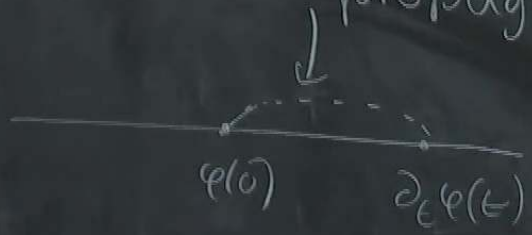
Example:

$$\int_{\mathbb{R}} \varphi \, \partial_t^2 \varphi \, dt$$

$q = \varphi(0)$,

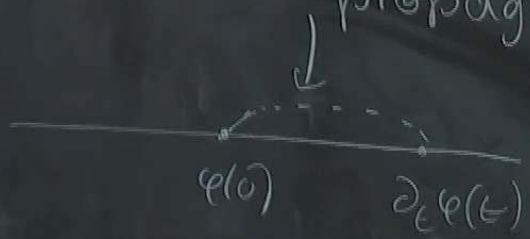
$p = \partial_t \varphi(0)$

What is $\langle \varphi(0)\, \partial_t \varphi(t) \rangle$?



$P(t)$ so that $\partial_t^2 P(t) = \delta_{t=0}$.

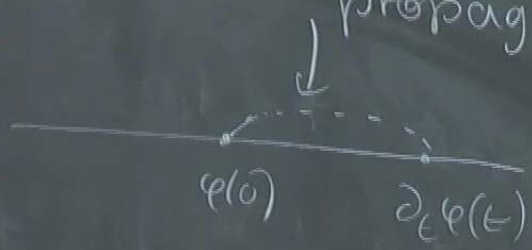
What is $\langle \varphi(0)\, \partial_t \varphi(t) \rangle$?



$P(t)$ so that $\partial_t^2 P(t) = \delta_{t=0}$

Solve EOM in presence of $\varphi(0)$

What is $\langle \varphi(0)\, \partial_t \varphi(t) \rangle$?



$P(t)$ so that $\partial_t^2 P(t) = \delta_{t=0}$

Solve EOM in presence of $\varphi(0)$

$\frac{1}{2}\int \varphi \partial_t^2 \varphi + \varphi(0) = \int \frac{1}{2}\varphi \partial_t^2 \varphi + \varphi \delta_{t=0}$ Vary φ, $\partial_t^2 \varphi = \delta_{t=0}$

$$\int_{\varphi} e^{\int_{\mathbb{R}} \frac{1}{2} \varphi \partial_t^2 \varphi} \varphi(0)\, \partial_t \varphi(\epsilon)$$

$$= \langle \varphi(0)\, \partial_t \varphi(\epsilon) \rangle$$

$$P(t) =$$

$P(t) = \frac{1}{2}|t|$

$t > 0$

$P(t) = \frac{1}{2}t$

$\frac{1}{2}$ —— $t < 0,\ P(t) = -t\frac{1}{2}$

$\partial_t P =$ —— 0

$-\frac{1}{2}$ ——

$= \frac{1}{2}\ t > 0,\ -\frac{1}{2}\ t < 0$

$$\partial_t^2 P = \delta_{t=0}$$

$$\langle \varphi(0)\ \partial_t \varphi(t) \rangle$$

$$= \partial_t P$$

$$= \begin{cases} 1/2, & t>0 \\ -1/2, & t<0 \end{cases}$$

$$\langle q \star p \rangle = \lim_{t \to 0^+} \langle \varphi(0)\ \partial_t \varphi(t) \rangle = 1/2$$

$$\langle p \star q \rangle = \lim_{t \to 0^-} \langle \varphi(0)\ \partial_t \varphi(t) \rangle = -1/2$$

So,

$$\langle q \star p - p \star q \rangle = 1$$

(or, really, $\hbar$)

$$[q, p^2] = \hbar 2p$$