Title: Mathematical Physics Lecture - 230126

Speakers: Kevin Costello

Collection: Mathematical Physics (2022/2023)

Date: January 26, 2023 - 11:00 AM

URL: https://pirsa.org/23010017

Last time

Gauge theory in 1+1 dimension:

Before symplectic reduction

$$Q(\theta) = A_\theta(\theta)$$

$$P(\theta) = \partial_t A_\theta$$

$$\{P(\theta), Q(\theta')\} = \delta_{\theta-\theta'}$$

$$\mu(\theta) = \partial_\theta P(\theta)$$

is used to symplectically

$$\{ P(\theta), Q(\theta') \} = \delta_{\theta-\theta'}$$

1 dimension:

$$\mu(\theta) = \partial_\theta P(\theta)$$

uction

is used to symplectically reduce.

(θ)

In 3+1 space-time dimensions

i index for space

$$A = A_i \, dx^i + A_t \, dt$$

$$F^s_{ij} \, dx^i dx^j = dA_s$$

$F_{ij} =$

+ 1 space-time dimensions:

index for space

$$F_{ij} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i$$

$$A_i \, dx^i + A_t \, dt$$

$${}^{i} dx^j = dA_s$$

In 3+1 space-time dimensions:

i index for space

$$A = A_i \, dx^i + A_t \, dt$$

$$F^s_{ij} \, dx^i dx^j = dA_s$$

$$F_{ij} = \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i$$

We let

$$({*}F)_i = g_{ij} \, \varepsilon^{jkl} F_{kl}$$

Result is

In 3+1 dimensions, before we do symplectic reduction, phase space is

$$Q_i(x) = A_i(x)$$

$$P_i(x) = \partial_t A_i(x) + (\star F)_i$$

Result is

In 3+1 dimensions, before we do symplectic reduction, phase space is

$$Q_i(x) = A_i(x)$$

$$P_i(x) = \partial_t A_i(x) + (\star F)_i$$

$$\{P_i(x), Q_j(y)\} = g_{ij}\,\delta_{x=x'}$$

We do symplectic reduction by

$$\mu(x) = g^{ij}\frac{\partial}{\partial x^i} P_j(x)$$

$$H = \int_x g^{ij} P_i(x) P_j(x)$$

$$\mu = \partial_i P_i = \partial_t \partial_i A_i + \underbrace{\partial(*F)_i}_{\substack{\shortparallel \\ 0 \\ \text{Bianchi}}}$$

$P \in \Omega^1(m$

Math description

$Q \in$ a 1-form on space
$\Omega^1(M, \mathbb{R})$

$P \in \Omega^1(M, \mathbb{R})$ or

$\in \Omega^2(M, \mathbb{R})$ using $*$

(Constraint is $dP = 0$)

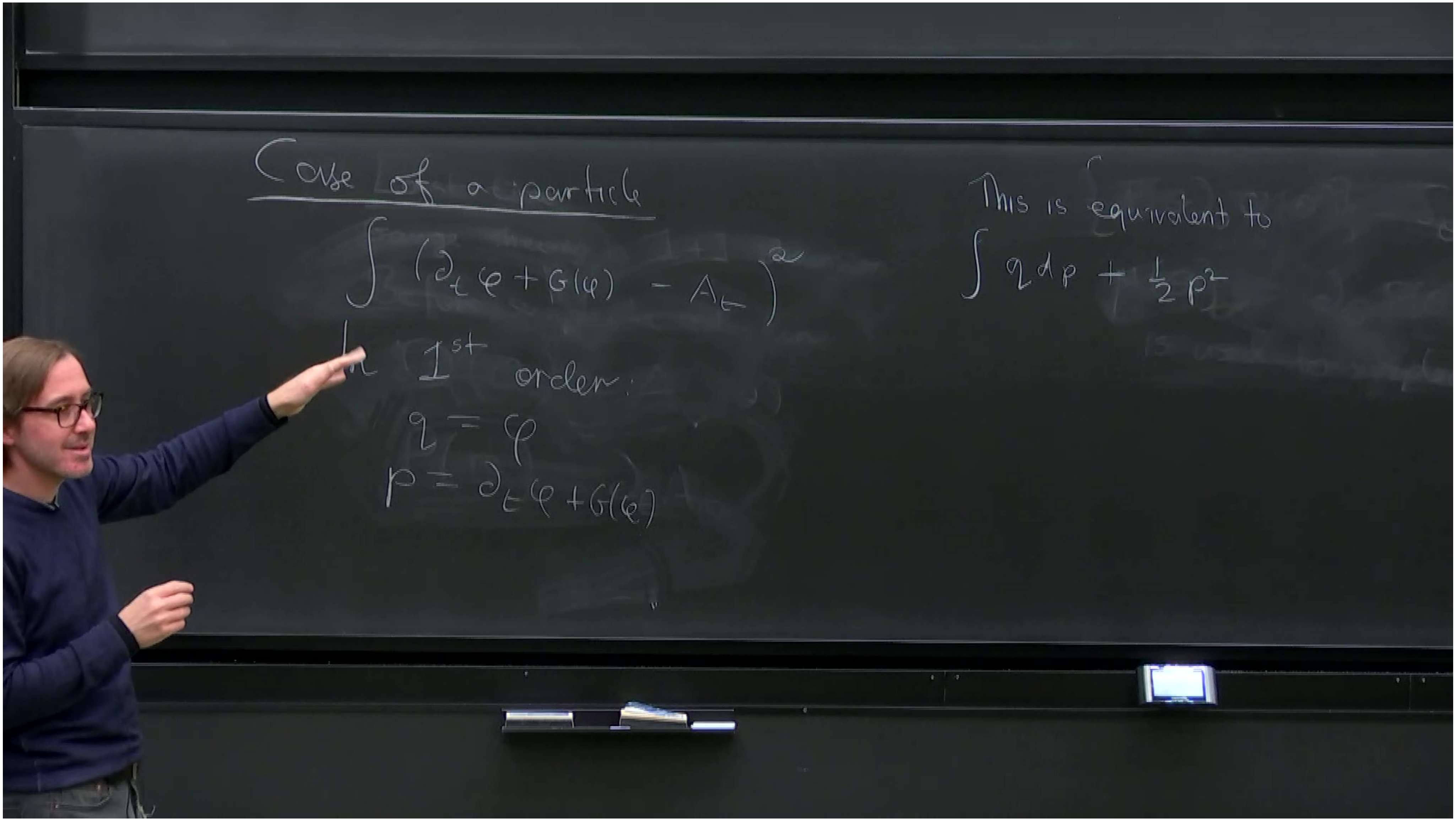
Phase space is symplectic reduction of $\left(\Omega^2(M) \times \Omega^1(M)\right) /\!/ \Omega^0(M)$

where $\Omega^0(M)$ by sending $Q \to Q + d\chi$

$$F_{full} = F_{spatial} + \partial_t A_i - \partial_i A_t$$

$$S_{YM} = \int F_{full}^2 = \int \Big(\underbrace{\partial_t A_i + (*F_{spatial})_i}_{P} + \partial_i A_t \Big)^2$$

... exactly as before.


Case of a particle
$\int (\partial_t \varphi + G(\varphi) - A_t)^2$
In 1st order:
$q = \varphi$
$p = \partial_t \varphi + G(\varphi)$
This is equivalent to
$\int q\,dp + \frac{1}{2} p^2$

Deformation Quantization $=$ Canonical Quantization

If $M = \mathbb{R}^2$, $\omega = dq \wedge dp$

$$\{p, q\} = 1$$

When we quantize, p, q be

m = Canonical Quantization

When we quantize, p, q become non-commuting

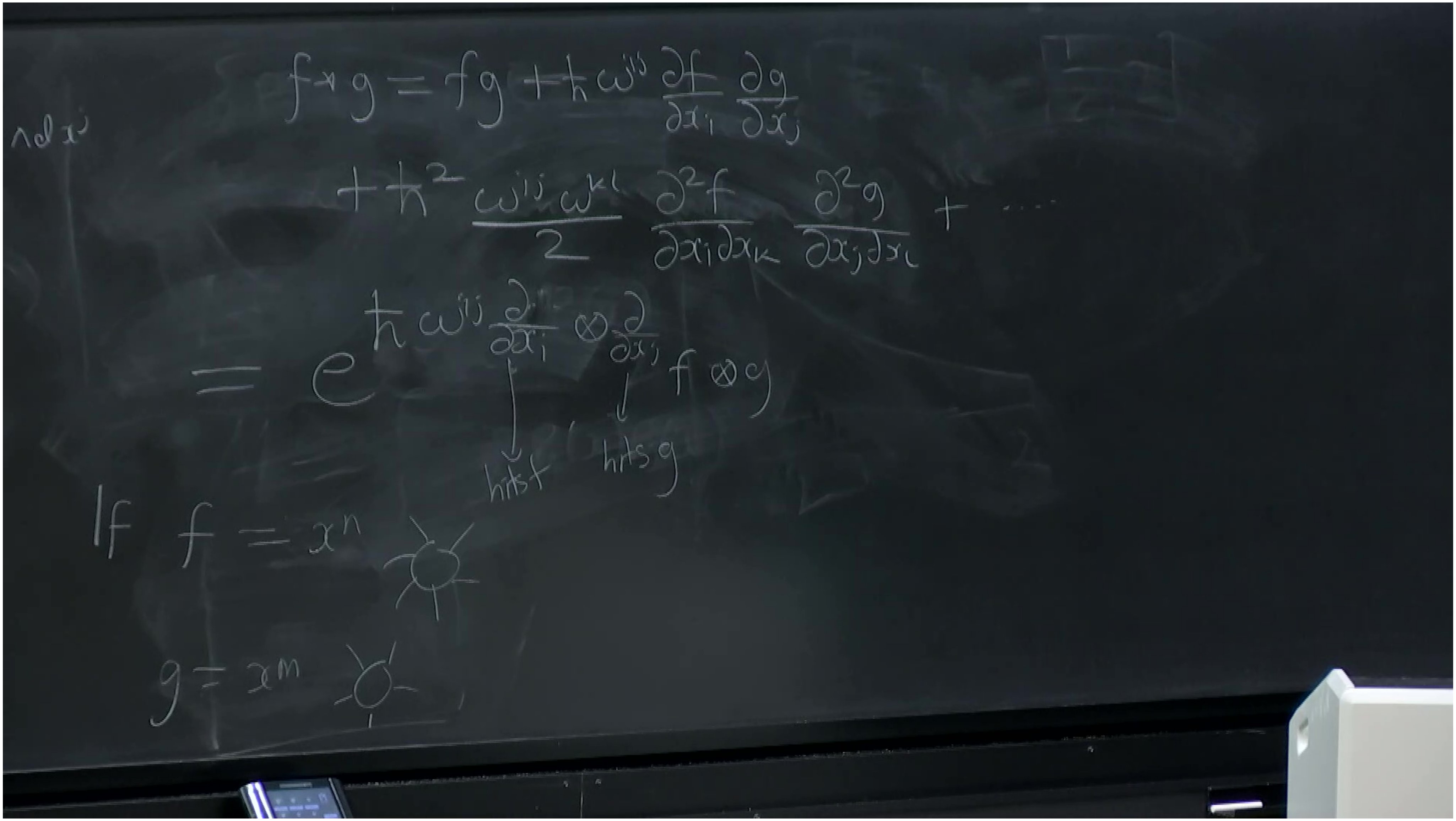
and $[p, q] = \hbar$.

Locally, in coords x^i $\quad \omega = \omega_{ij}\, dx^i \wedge dx^j$

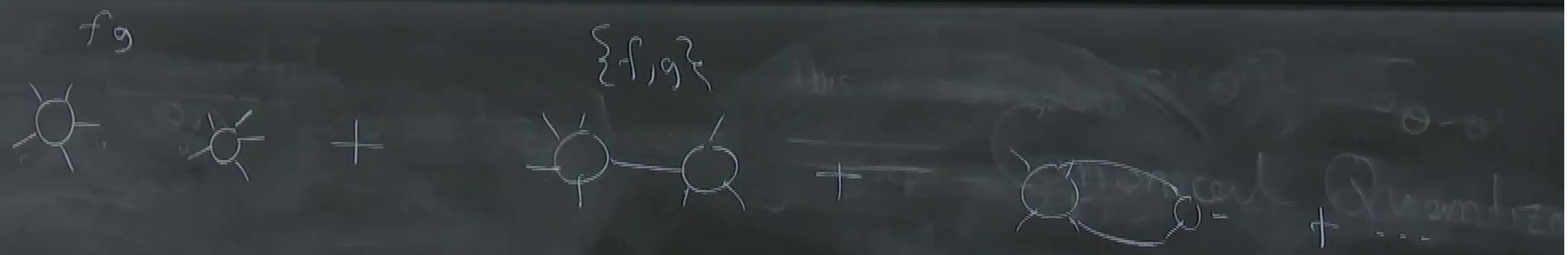
ω_{ij} is constant

$\omega^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$

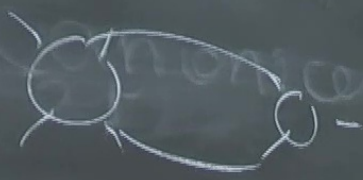
Formula for $\star$ product (Moyal):


$\wedge dx^j$
$f \star g = fg + \hbar \omega^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}$
$+ \hbar^2 \frac{\omega^{ij} \omega^{kl}}{2} \frac{\partial^2 f}{\partial x_i \partial x_k} \frac{\partial^2 g}{\partial x_j \partial x_l} + \dots$
$= e^{\hbar \omega^{ij} \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j}} f \otimes g$
hits f
hits g
If $f = x^n$
$g = x^m$

$f \star g =$



Working modulo $\hbar^2$, why does $fg + \hbar\{f,g\}$ define an asso

$+$  $+ \cdots$

$fg + \hbar\{f,g\}$ define an associative algebra?

$\{f, -\}$ is locally $\omega^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}$

it involves differentiating once.

The Leibniz rule says

$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

Similarly $\{fg, h\} = f\{g, h\} + g\{f, h\}$

Let's check that

$f \star (g \star h) = (f \star g) \star h$ modulo $\hbar^2$

LHS $fgh + \hbar f\{g,h\} + \hbar\{f,gh\} + O(\hbar^2)$

RHS: $fgh + \hbar\{f,g\}h + \hbar\{fg,h\} + O(\hbar^2)$

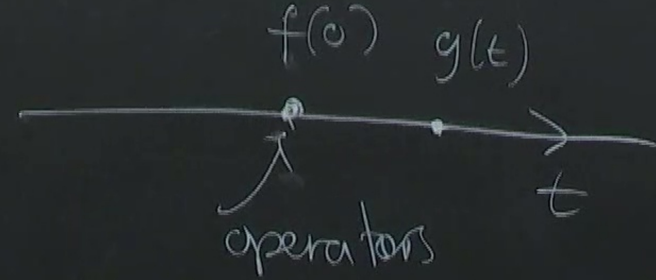
The Leibniz rule $\Rightarrow$ these are the same.

modulo $\hbar^2$

$- \hbar\{f, gh\} + O(\hbar^2)$

$h + \hbar\{fg, h\} + O(\hbar^2)$

e are the same.



$= C^\infty(M)$

$$f * g = \lim_{t \to 0_+} f(0)\, g(t)$$

If we have n free particles $\omega =$

$p_i, q_i \quad \omega = dq_i\, dp_i$

$\mu =$ quadratic expressions

$p\, q$

or p^2, q^2

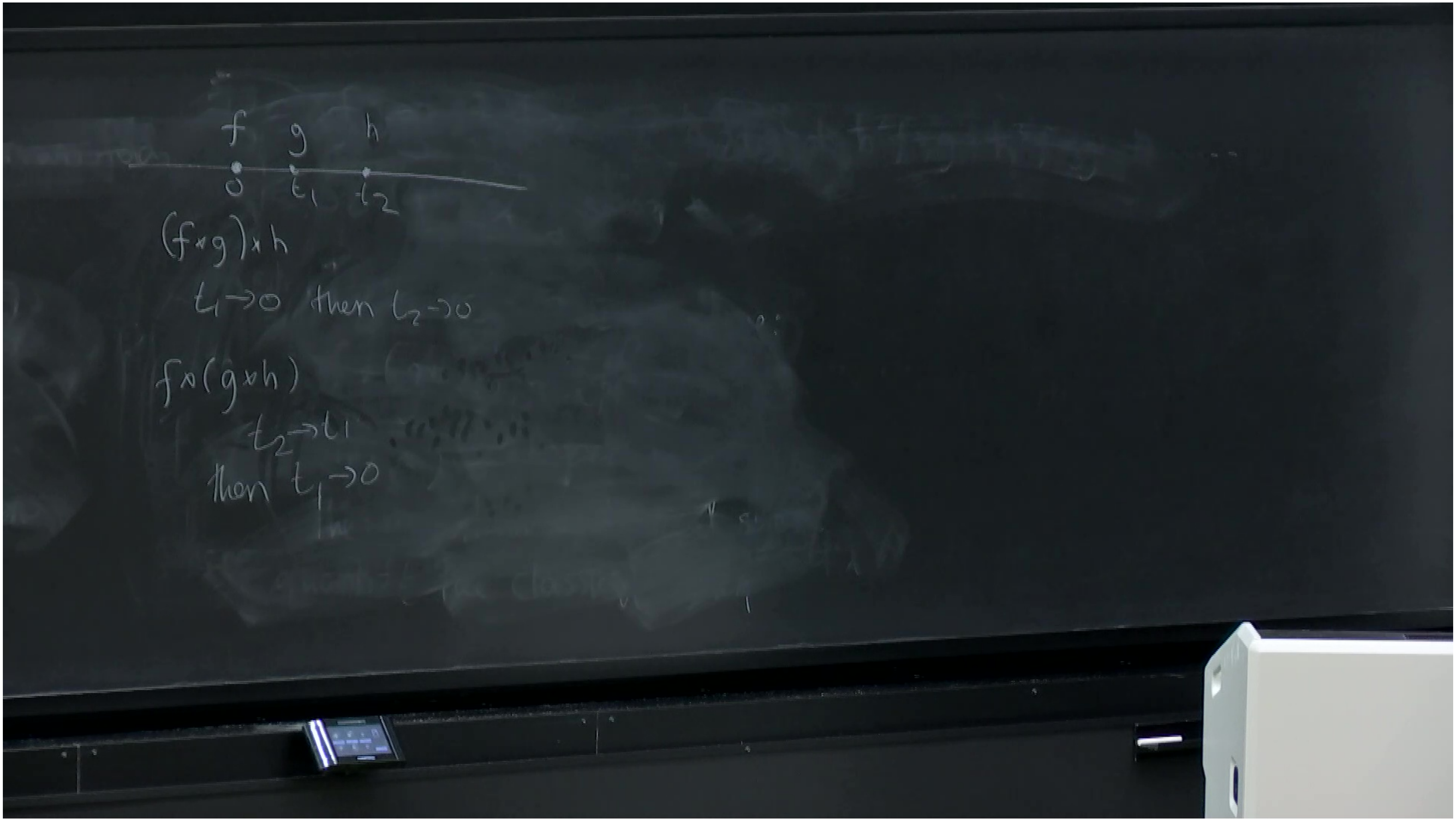
$\{\mu, \mu\}$ satisfy some algebra (classical symmetry)

$$[p^2, q^2] = \hbar 2pq + \hbar^2 1$$

$$p = \hbar \frac{\partial}{\partial q}$$

$$\left[\hbar^2 \frac{\partial^2}{\partial q^2}, q^2\right] = 2\hbar^2 q \frac{\partial}{\partial q} + \hbar^2 1$$

$2\hbar q p$


f g h
0 t1 t2
(f*g)*h
t1→0 then t2→0
f*(g*h)
t2→t1
then t1→0

Where does $[p, q] = \hbar$ come from?

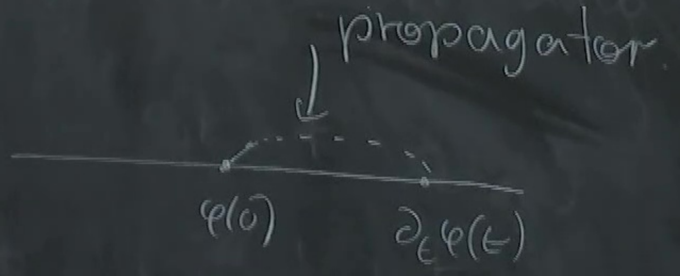
Example:

$$\int_{\mathbb{R}} \varphi\, \partial_t^2 \varphi \, dt$$

$q = \varphi(0)$,

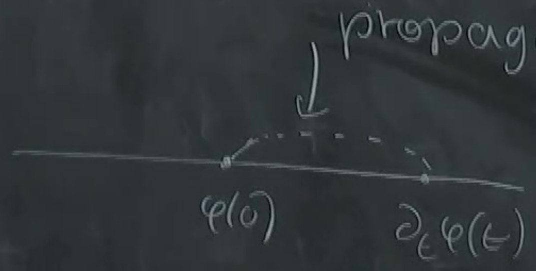
$p = \partial_t \varphi(0)$

What is $\langle \varphi(0)\, \partial_t \varphi(t) \rangle$?



$P(t)$ so that $\partial_t^2 P(t) = \delta_{t=0}$

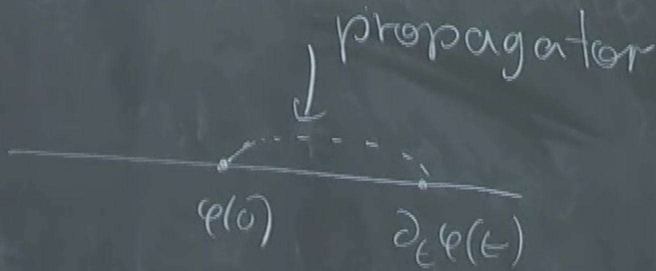
What is $\langle \varphi(0)\, \partial_t \varphi(t) \rangle$?



$P(t)$ so that $\partial_t^2 P(t) = \delta_{t=0}$

Solve EOM in presence of $\varphi(0)$

What is $\langle \varphi(0)\, \partial_t \varphi(t) \rangle$?



$P(t)$ so that $\partial_t^2 P(t) = \delta_{t=0}$

Solve EOM in presence of $\varphi(0)$

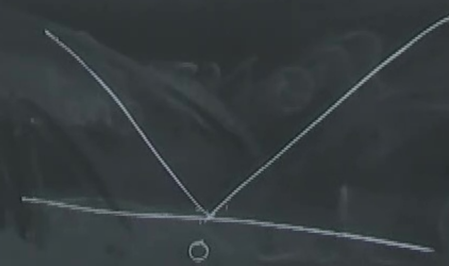
$\frac{1}{2}\int \varphi\, \partial_t^2 \varphi + \varphi(0) = \int \frac{1}{2} \varphi\, \partial_t^2 \varphi + \varphi\, \delta_{t=0}$ Vary φ, $\partial_t^2 \varphi = \delta_{t=0}$

$$\int_{\varphi} e^{\int_{\mathbb{R}} \frac{1}{2} \varphi \partial_t^2 \varphi} \varphi(0) \, \partial_t \varphi(t)$$

$$= \langle \varphi(0) \, \partial_t \varphi(t) \rangle$$

$P(t) =$

$P(t) = \frac{1}{2}|t|$

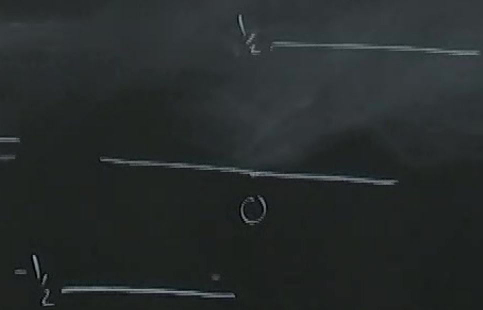


$t>0$

$P(t) = \frac{1}{2}t$

$t<0,\ P(t) = -t\frac{1}{2}$

$\partial_t P =$



$= \frac{1}{2}\ t>0,\ -\frac{1}{2}\ t<0$

$$\partial_t^2 P = \delta_{t=0}$$

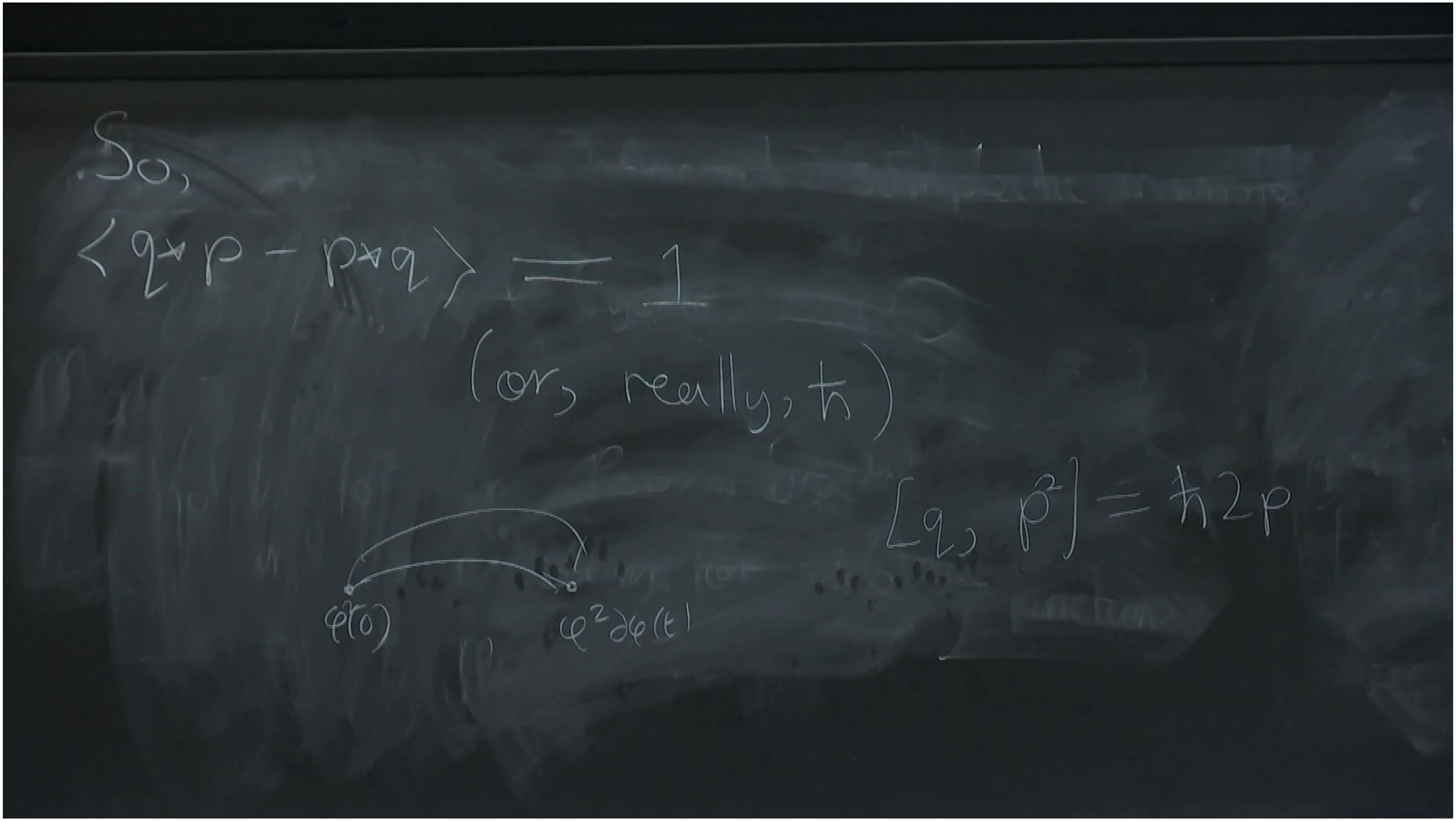
$$\langle \varphi(0)\ \partial_t \varphi(t) \rangle$$

$$= \partial_t P$$

$$= \begin{cases} 1/2, & t>0 \\ -1/2, & t<0 \end{cases}$$

$$\langle q * p \rangle = \lim_{t \to 0^+} \langle \varphi(0)\ \partial_t \varphi(t) \rangle = 1/2$$

$$\langle p * q \rangle = \lim_{t \to 0^-} \langle \varphi(0)\ \partial_t \varphi(t) \rangle = -1/2$$


So,
$\langle q \star p - p \star q \rangle = 1$
(or, really, $\hbar$)
$[q, p^2] = \hbar 2 p$
$\varphi^2(0)$
$\varphi^2 \partial \varphi(t)$