Title: Mathematical Physics Lecture - 230124

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Collection: Mathematical Physics (2022/2023)

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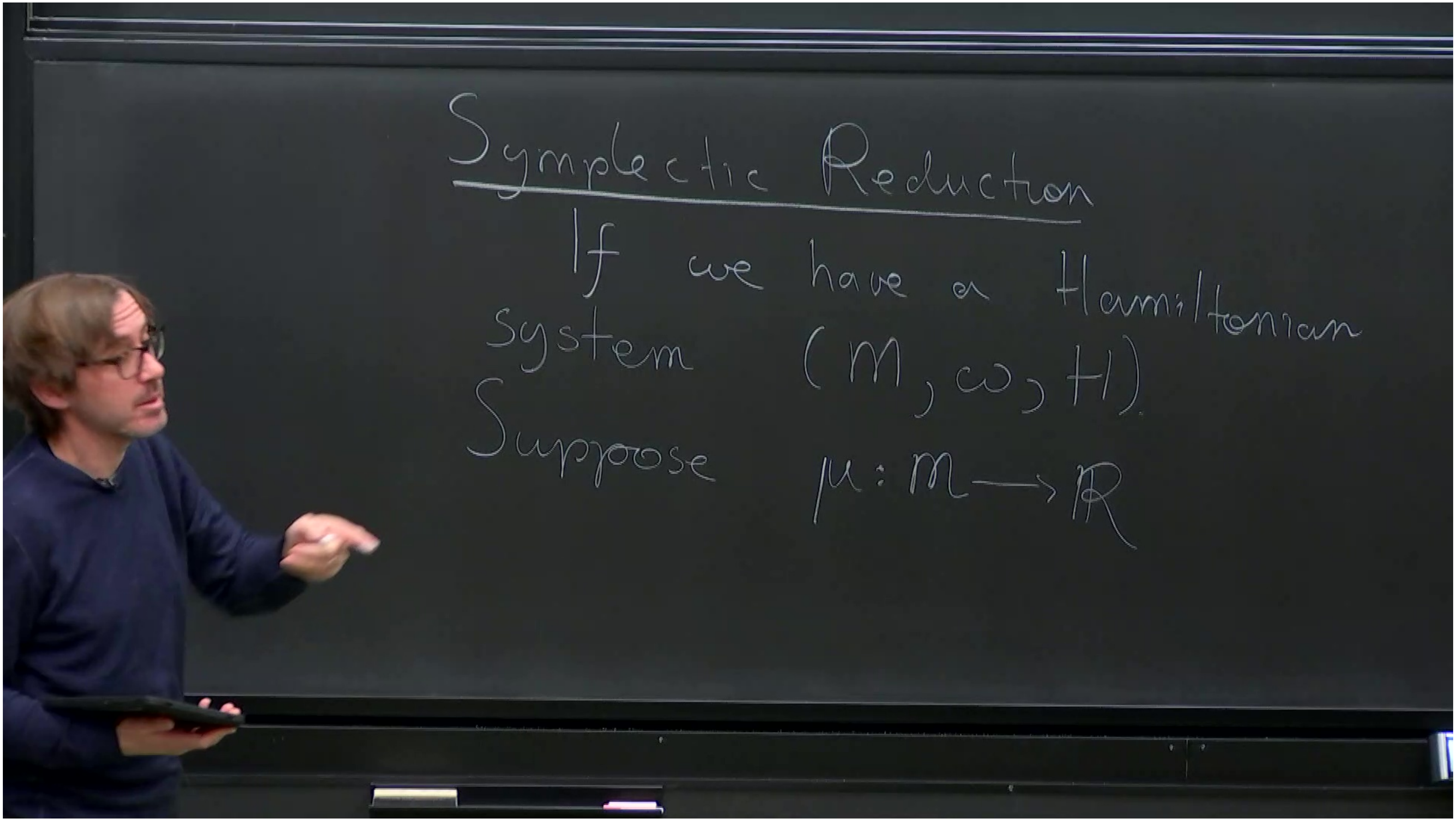
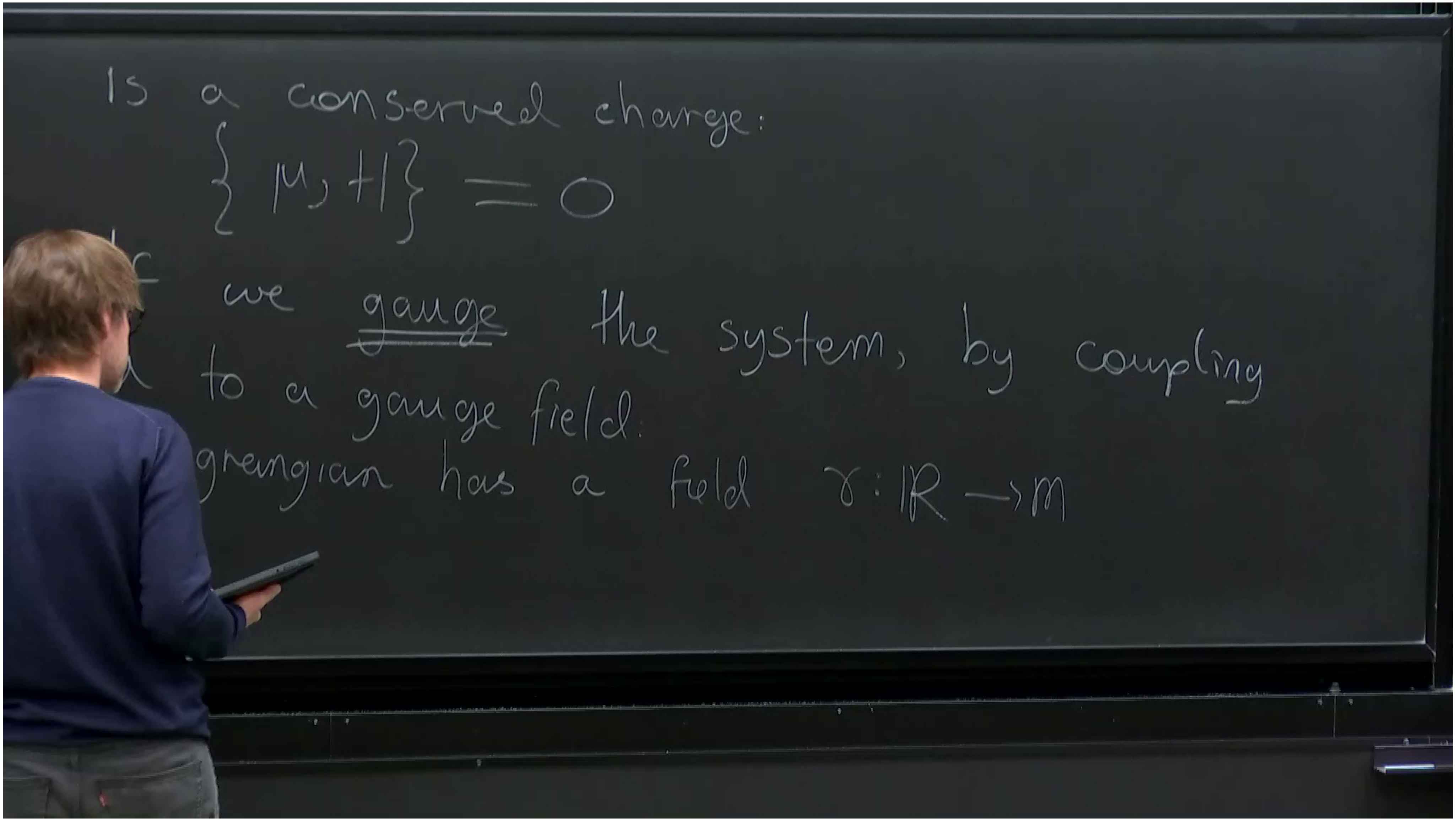

Symplectic Reduction
If we have a Hamiltonian
system (M, ω, H).
Suppose $\mu : M \longrightarrow \mathbb{R}$

is a conserved charge:

$$\{\mu, H\} = 0$$

If we gauge the system, by coupling μ to a gauge field:


Is a conserved charge:
$\{\mu, H\} = 0$
we gauge the system, by coupling
to a gauge field:
grangian has a field $\gamma : \mathbb{R} \to M$

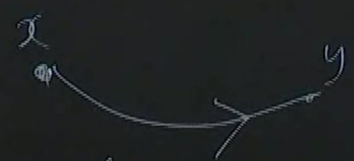
n we gauge, replace M by a new phase space

$/\!/\mathbb{R}$

we gauge, replace M by a new phase space

$$M_R = \{ x \in M, \mu(x) = 0 \} / \sim$$

where $x \sim y$ if



they are related by a flow of v. field X_μ.

$\omega = d\alpha$

$$S(\gamma) \int_{\mathbb{R}} \gamma^* \alpha + H(\gamma(t))dt + \mu(\gamma(t)) A_t$$

gauge field

Whe

m //

When we gauge, replace M by a new phase space

$$M//\mathbb{R} = \{x \in M, \mu(x) = 0\}/\sim$$

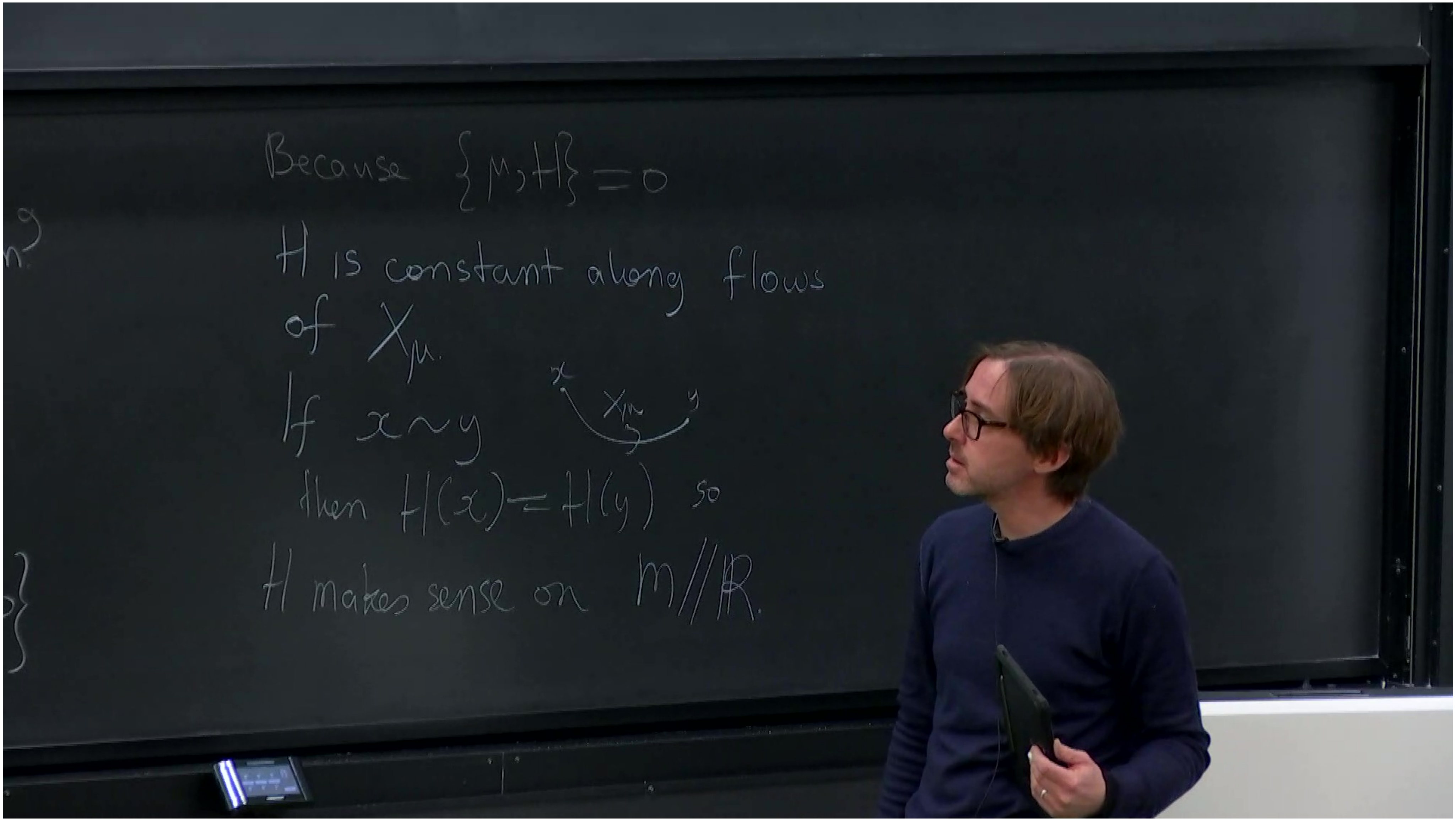
where $x \sim y$ if

they are related by a flow of v. field X_μ.

Why is this still a Hamiltonian system?

- Want a new ω on $M /\!/ \mathbb{R}$
- Want a new H.

  H is a function on set $\{x \in M, \mu(x) = 0\}$


Because $\{\mu, H\} = 0$
H is constant along flows
of X_μ.
If $x \sim y$
x X_μ y
then $H(x) = H(y)$ so
H makes sense on $M/\!/\mathbb{R}$.

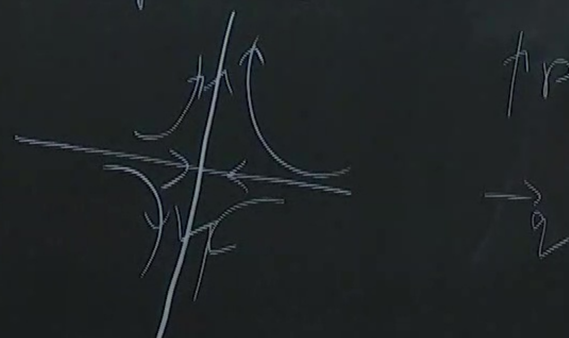
<u>Caution</u> This works well when $d\mu$ is never zero

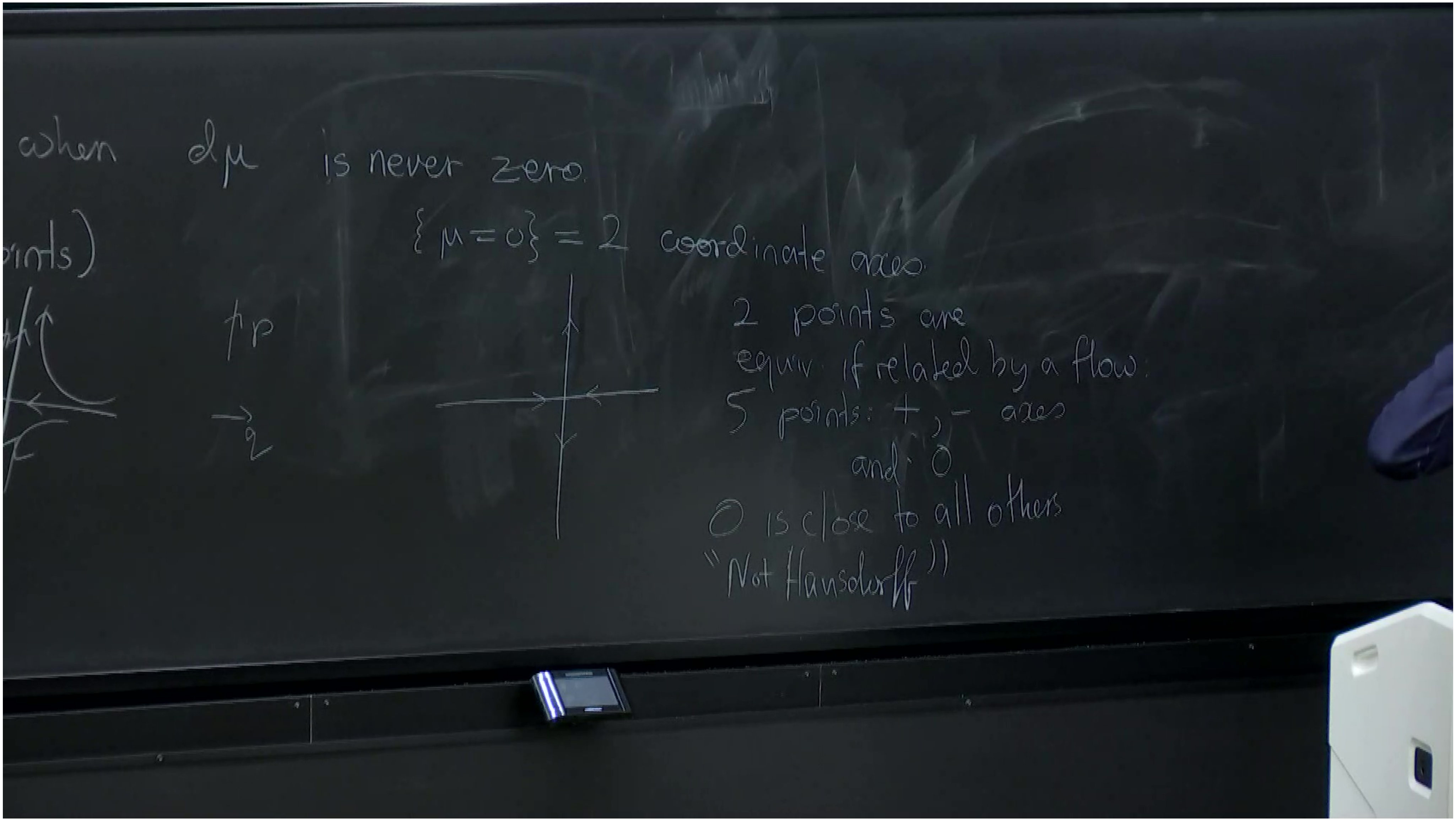
($\iff X_\mu$ has no fixed points)

$M = \mathbb{R}^2$

$\mu = pq$

$X = p\,\partial_p - q\,\partial_q$




when dμ is never zero.
{μ = 0} = 2 coordinate axes
oints)
p
q
2 points are
equiv if related by a flow.
5 points: +, − axes
and 0
0 is close to all others
"Not Hausdorff"

If we have Lagrangian

$$\int \partial_t \varphi \, \partial_t \varphi$$

where φ is periodic,

$\varphi \sim \varphi + 1$

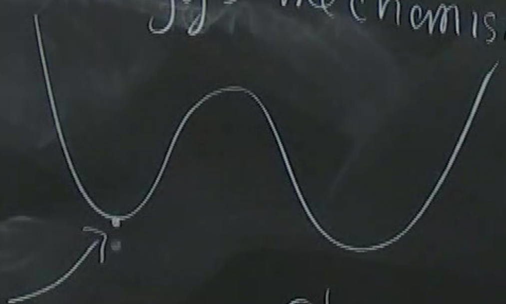
and we gauge the $U(1)$ symmetry

$\varphi \sim$

$\varphi \sim \varphi + c, \quad c \in \mathbb{R}/\mathbb{Z}$

$U(1)$ symmetry.

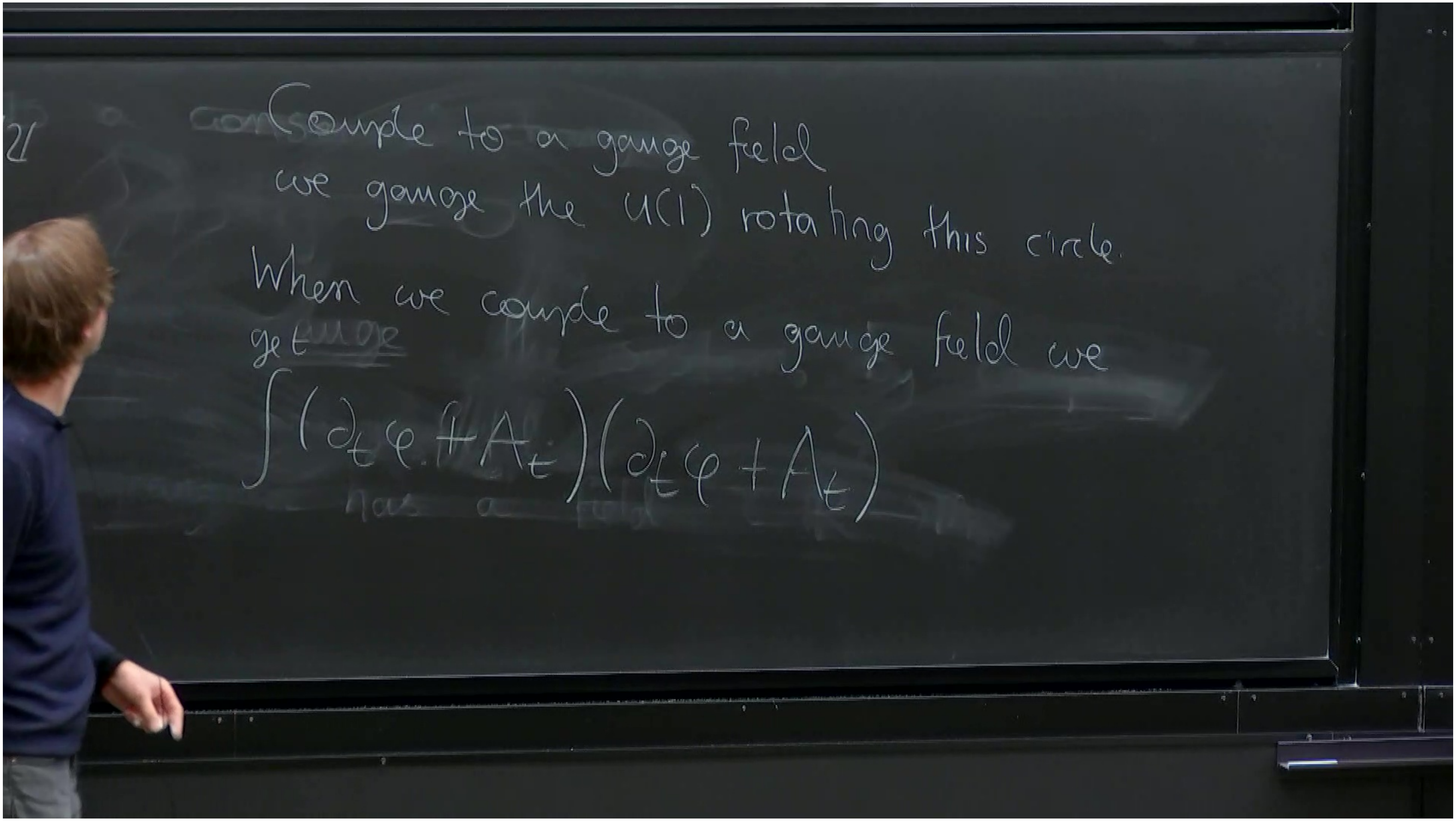
Higgs mechanism:



minima $= S^1$

Couple to a gauge field

we gauge the $U(1)$ rota


Couple to a gauge field
we gauge the U(1) rotating this circle.
When we couple to a gauge field we
get
$\int (\partial_t \varphi + A_t)(\partial_t \varphi + A_t)$

Phase space before reduction

$q \in S^1$

$q \sim q+1$

and p

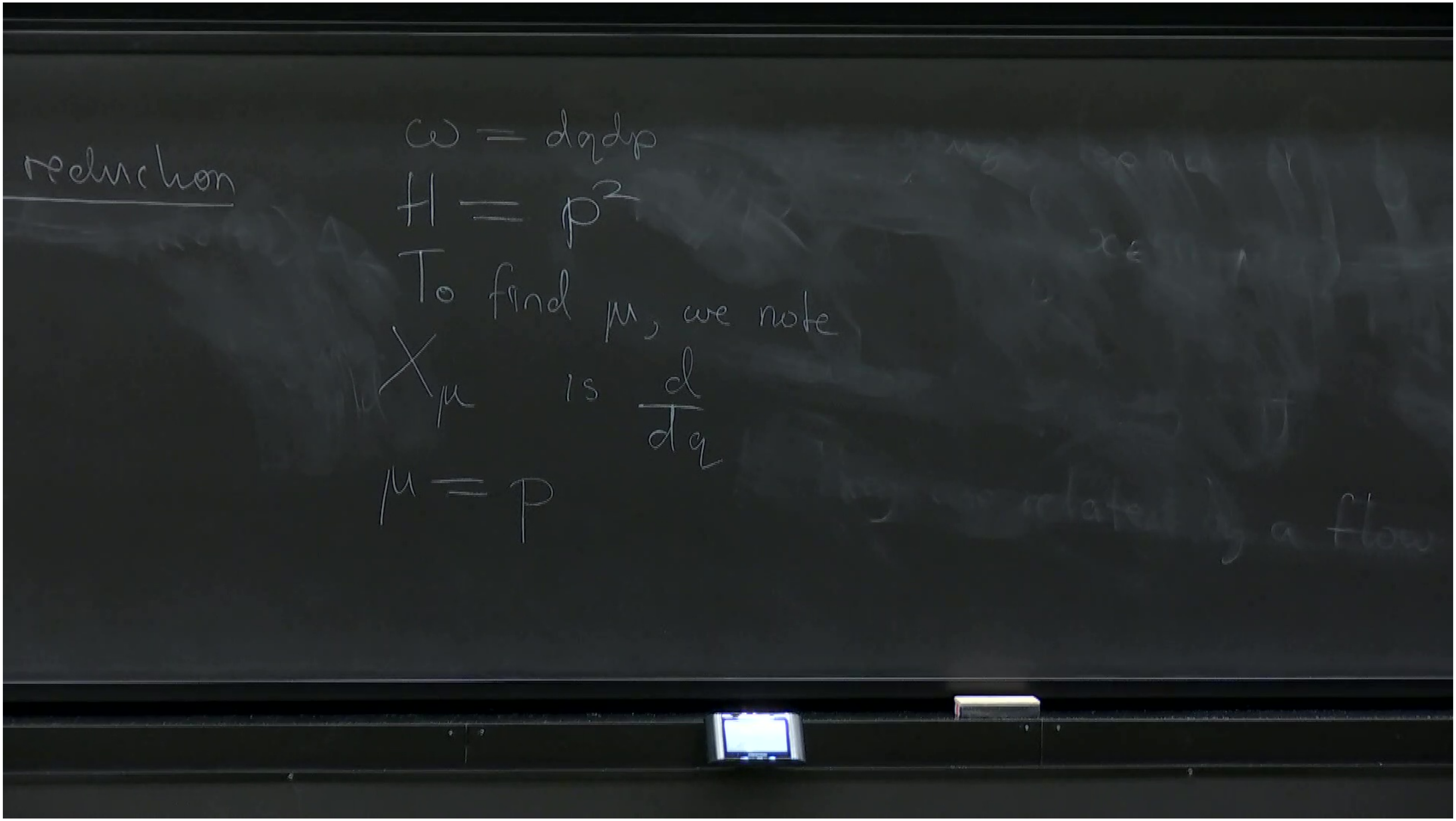
$q = \varphi(t=0)$

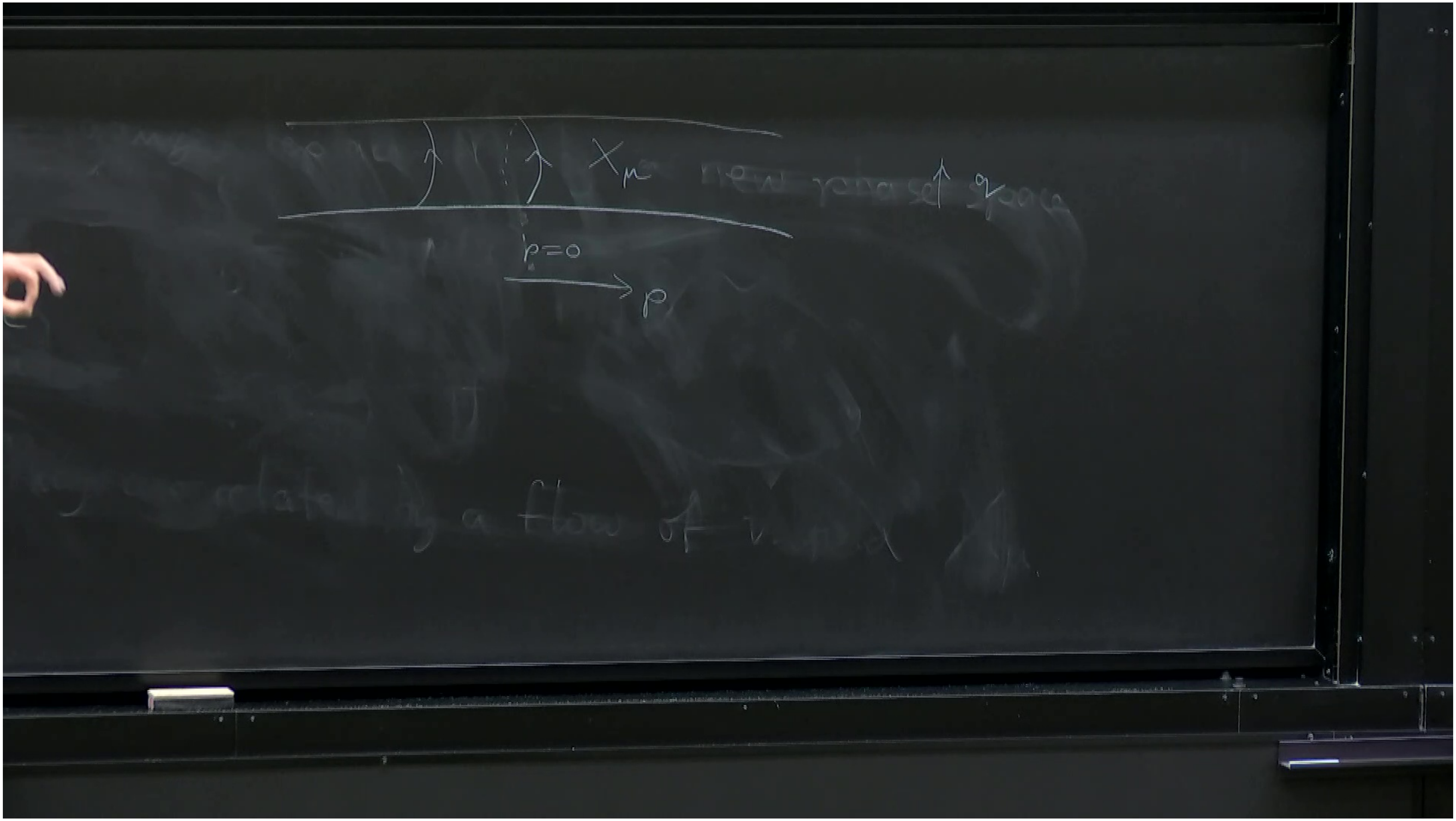
$p = D_t \varphi$

$\omega = dq\,dp$

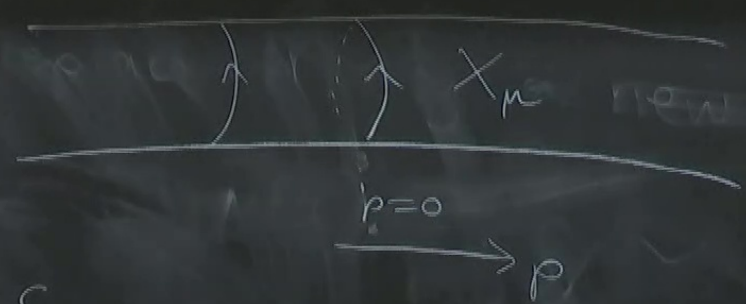
$H = p^2$

To find μ,


reduction
ω = dqdp
H = p²
To find μ, we note
X_μ is d/dq
μ = p



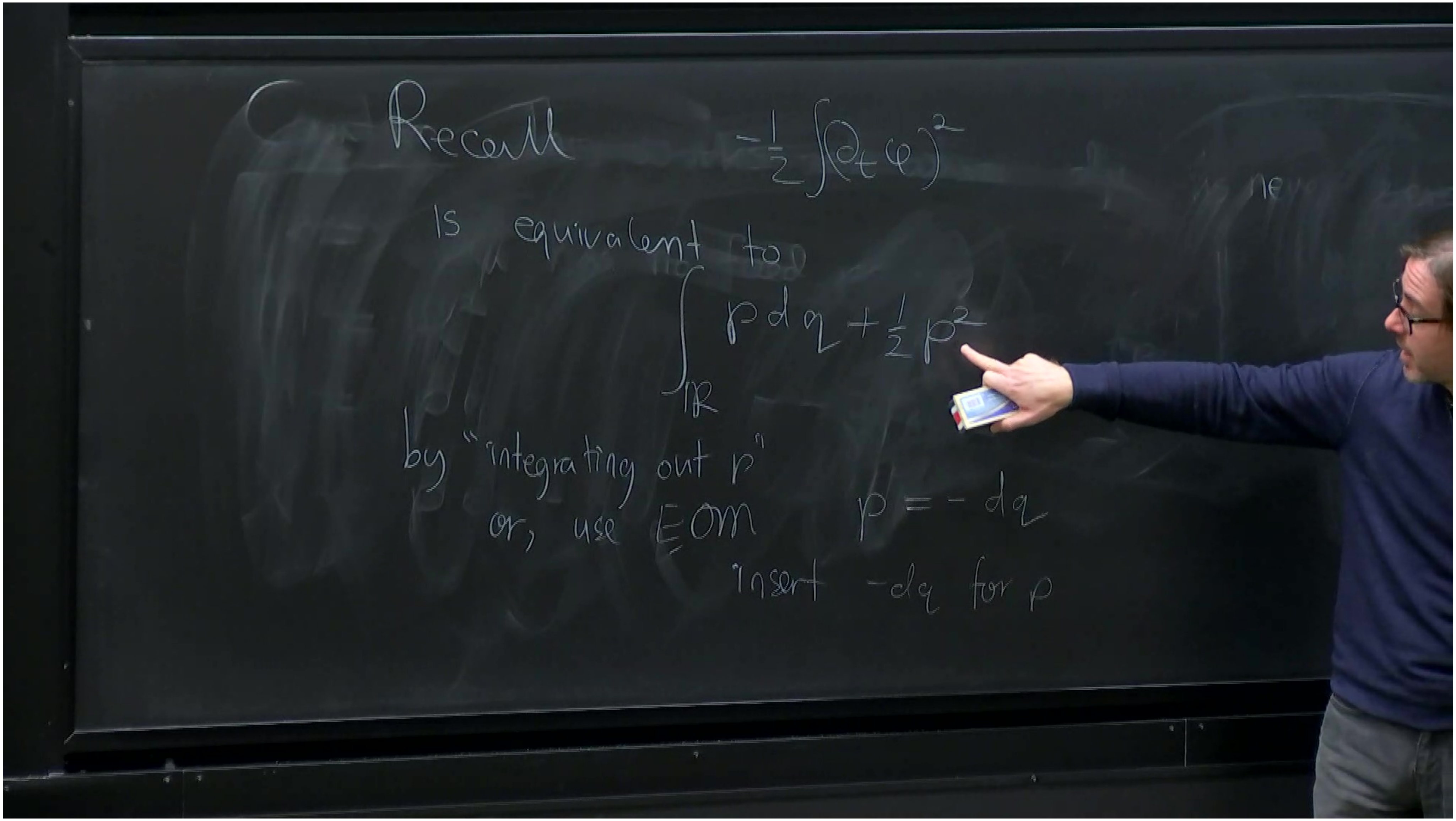
X_μ
new phase space
$\rho=0$
ρ
a flow of

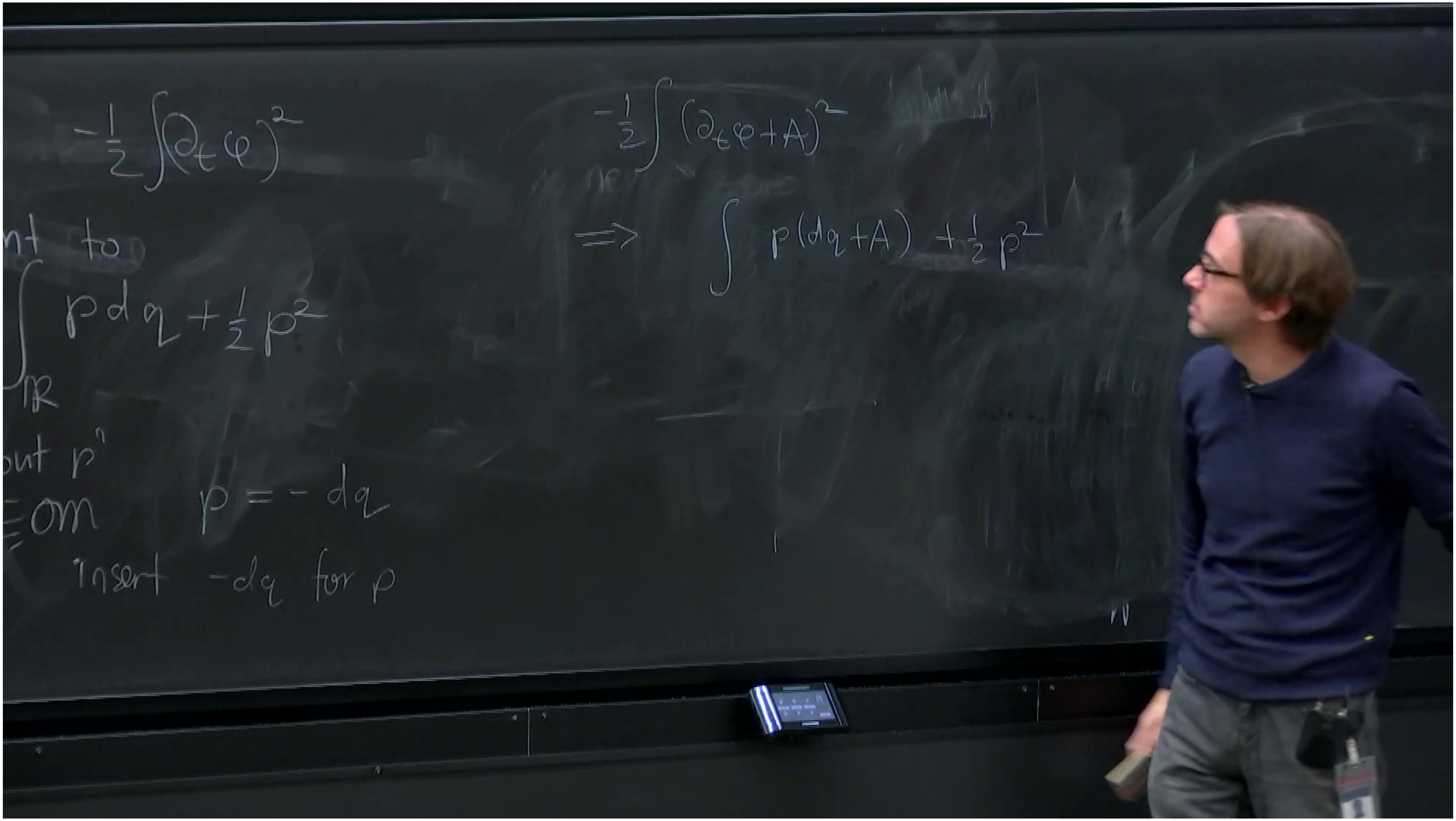


Symplectic reduction:

$M /\!/ S^1 =$ a point.

$\{p=0\} = S^1$, and any 2 points are related by rotation.


Recall $-\frac{1}{2}\int(\partial_t\varphi)^2$
is equivalent to
$\int_{\mathbb{R}} p\,dq + \frac{1}{2}p^2$
by "integrating out p"
or, use EOM $p = -dq$
insert $-dq$ for p



$-\frac{1}{2}\int(\partial_t\varphi)^2$
nt to
$\int_{\mathbb{R}} p\,dq + \frac{1}{2}p^2$
out p
on $p = -dq$
insert $-dq$ for p
$-\frac{1}{2}\int(\partial_t\varphi + A)^2$
$\Longrightarrow \int p(dq + A) + \frac{1}{2}p^2$

$$\int p(dq+A) + p^2 + cA$$

this is gauge invariant

$$\delta A \to A + d\chi$$

This sets $p + c = 0$.

$+A) + p^2 + cA$

gauge invariant

$\rightarrow A + d\chi$

ets $p + c = 0.$

General (QFT) mechanism

μ = the thing we find when place $A = \delta_{t=0}$ in Lagrangian.

Consider YM in $1+1$ space-time dimensions

with coordinates θ, t $\qquad \theta \sim \theta + 2\pi$

gauge group $U(1)$

Field is $A \in \Omega^1(S^1 \times \mathbb{R})$

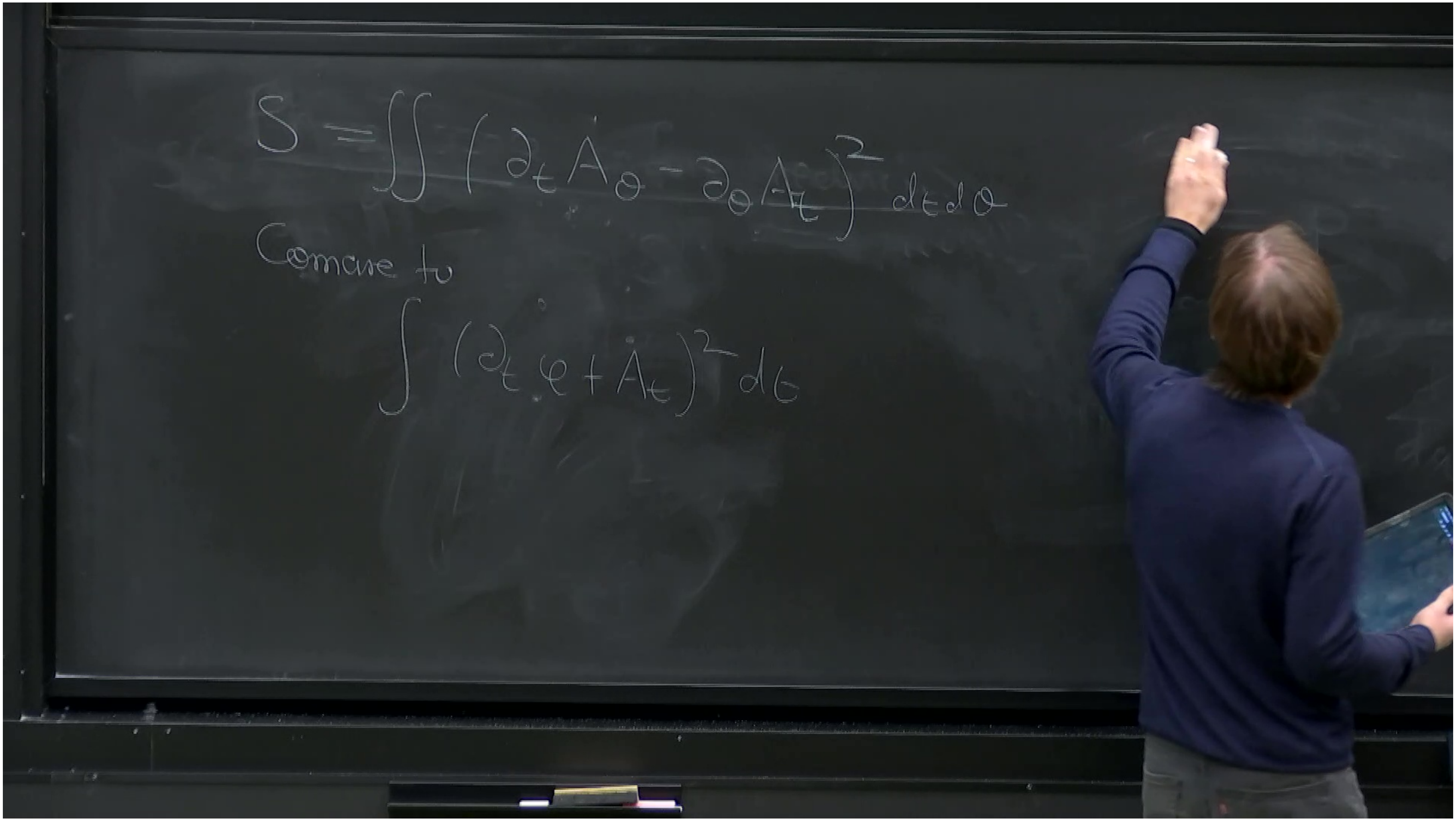
$$A = A_\theta \, d\theta + A_t \, dt$$

$$F = \partial_t A_\theta - \partial_\theta A_t$$

$$\left(F \, dt \, d\theta = dA \right)$$

Lagrangian is

$$\int F \, F \, dt \, d\theta$$


$S = \iint (\partial_t \dot{A}_\theta - \partial_\theta A_t)^2 \, dt \, d\theta$
Comare to
$\int (\partial_t \varphi + \dot{A}_t)^2 \, dt$

$d t \, d\theta$

If we expand in Fourier modes

$$A_\theta = \sum A_\theta^{n,+} \cos n\theta + A_\theta^{n,-} \sin n\theta$$

$$S = \iint (\partial_t A_\theta - \partial_\theta A_t)^2 \, dt \, d\theta$$

Comare to

$$\int (\partial_t \varphi + A_t)^2 \, dt$$

$P(\theta) = \partial_t A_\theta$

$q(\theta) = A_\theta$

$\int dp(\theta) \, dq(\theta)$

$H = \int p(\theta)^2$

$\mu(\theta) = \partial_\theta p(\theta)$

If we expo

$A_\theta =$

$A_t =$

$S =$

If we expand in Fourier modes

$$A_\theta = \sum A_\theta^{n,+} \cos n\theta + A_\theta^{n,-} \sin n\theta$$

$$A_t = \sum \quad '' \qquad\qquad ''$$

$$) = \partial_\theta \rho(\theta)$$

$$S = \sum_n \int_t \left(\partial_t A_\theta^{n,+} - n A_t^{n,-}\right)^2 + \sum_n \int_t \left(\partial_t A_\theta^{n,-} + n A_t^{n,+}\right)^2$$

...and in Fourier modes

$$\sum A_\theta^{n,+} \cos n\theta + A_\theta^{n,-} \sin n\theta$$

$$\sum \quad '' \qquad ''$$

$$\sum_n \int_t \left(\partial_t A_\theta^{n,+} - n A_t^{n,-}\right)^2$$

$$+ \sum_n \int_t \left(\partial_t A_\theta^{n,-} + n A_t^{n,+}\right)^2$$

For $n \neq 0$

we have a particle

$p^{n,+}, q^{n,+}$

$H = (p^{n,+})^2$

$\mu^{n,+} = n p^{n,+}$

Symplectic reduction:

$p^{n,+} = 0$, divide by $U(1)$

action $\Rightarrow$ a point.

0 Fourier mode is all that remains:

$$\int (\partial_t A^0_\theta)^2 \quad \text{with } \mu = 0.$$

Phase space is finite-dimensional:

$\mathbb{R}^2$, coords p, q

$H = p^2$.

...pand in Fourier modes.

$$\sum A_\theta^{n,+} \cos n\theta + A_\theta^{n,-} \sin n\theta$$

$$\sum \quad '' \qquad\qquad ''$$

$$\sum_n \int_t \left(\partial_t A_\theta^{n,+} - n A_t^{n,-}\right)^2$$

$$+ \sum_n \int_t \left(\partial_t A_\theta^{n,-} + n A_t^{n,+}\right)^2$$

For $n \neq 0$

we have a particle

$p^{n,+}, q^{n,+}$ $\qquad p^{n,-}$ $q^{n,-}$

$H = (p^{n,+})^2$ $\qquad (p^{n,-})^2$

$\mu^{n,+} = n p^{n,+}$ $\qquad \mu^{n,-} = -n p^{n,-}$

Symplectic reduction:

$p^{n,+} = 0$, divide by $U(1)$

action $\Rightarrow$ a point.

imensional:

μ

Hamiltonian system

as 2 systems, decoupled

M_1 M_2 phase spaces

Put them together:

$$M_1 \times M_2 = \{(x_1, x_2) \mid x_1 \in M_1,\ x_2 \in M_2\}$$

For a fixed Fourier mode

$$\int_t \left(\partial_t A_\theta^{n,+} - n A_t^{n,-}\right)^2$$

compare w.

$$\int_t \left(\partial_t \varphi + A_t\right)^2$$

$$p^{n,+} = \partial_t A_\theta^{n,+}$$

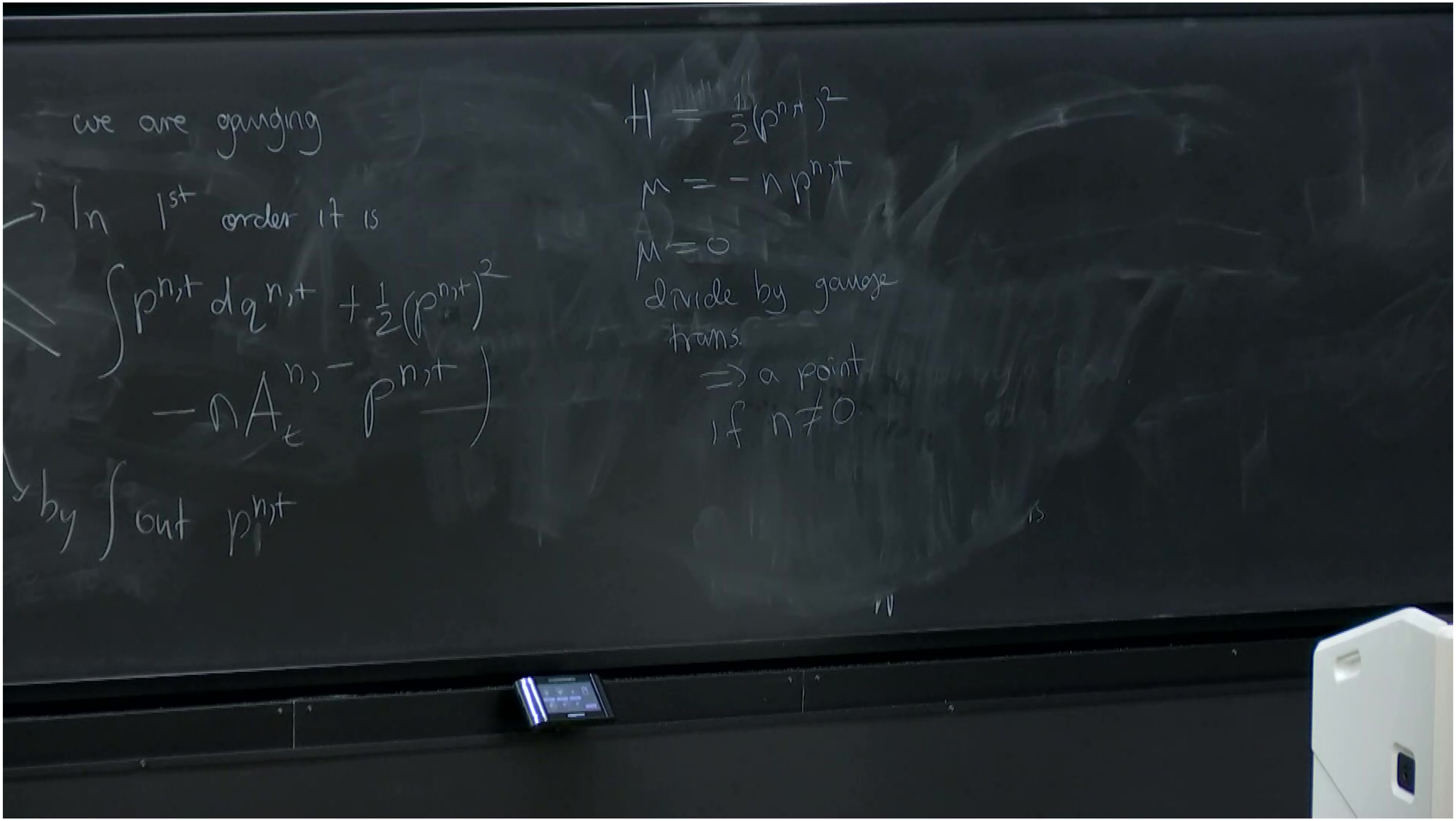
$$q^{n,+} = A_\theta^{n,+}$$

we are gauging

In 1^{st} order it is

$$\int p^{n,+}\, dq^{n,+} + \frac{1}{2}\left(p^{n,+}\right)^2 - n A_t^{n,-}\, p^{n,+}$$

by $\int$ out $p^{n,+}$


we are gauging
In 1st order it is
$\int p^{n,+} dq^{n,+} + \frac{1}{2}(p^{n,+})^2 - nA_t^{n,-} p^{n,+}$
by $\int$ out $p^{n,+}$
$H = \frac{1}{2}(p^{n,+})^2$
$\mu = -n p^{n,+}$
$\mu = 0$
divide by gauge trans.
$\Rightarrow$ a point
if $n \neq 0$

Redoing without Fourier

$$S = \iint (\partial_t A_\theta - \partial_\theta A_t)^2 \, d\theta \, dt$$

This looks like $\iint (\partial_t A_\theta)^2 \, d\theta \, dt$

with a covariantized ∂_t

space before gauging:

$= A_\theta(\theta)$

$) = \partial_t A_\theta(\theta)$

$)\, dq(\theta) \int$

$\theta)^2.$

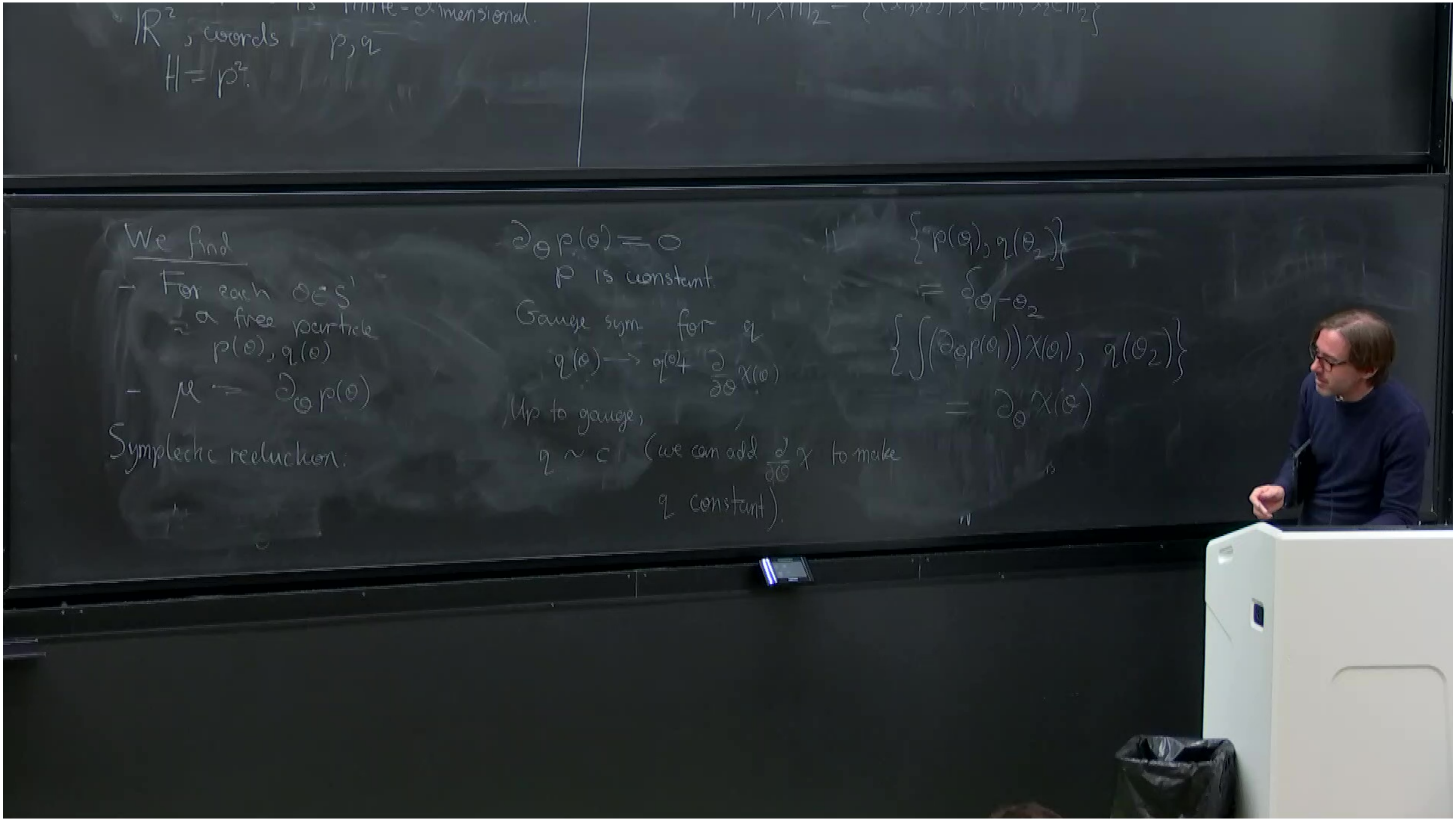
First order action

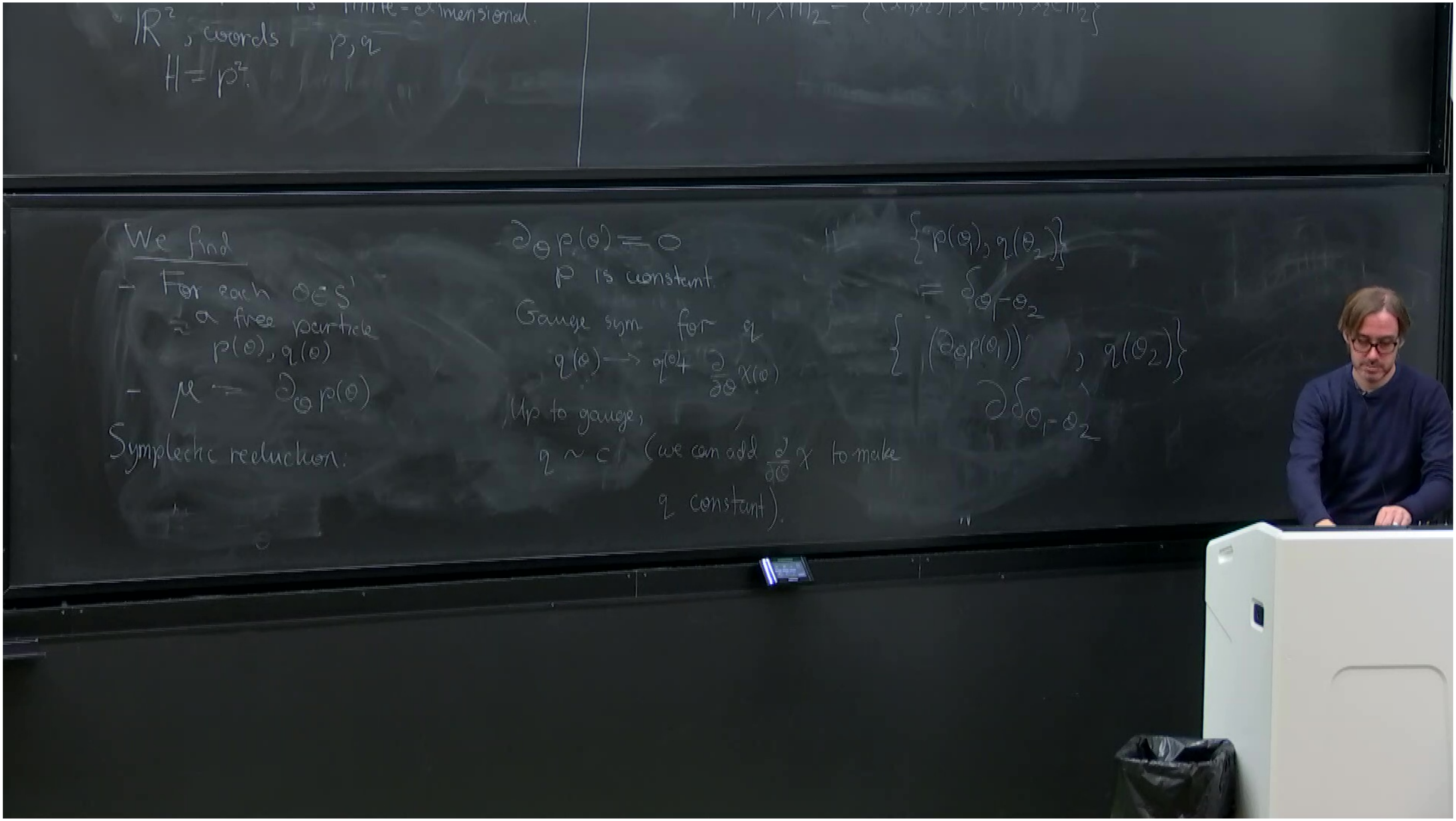
$$\iint p(\theta)\,\partial_t q(\theta) + \tfrac{1}{2} p(\theta)^2$$

First order gauged version:

$$\iint p(\theta)\left[\partial_t q(\theta) - \partial_\theta A_t\right] + \tfrac{1}{2} p(\theta)^2$$

$$= \iint p(\theta)\,\partial_t q(\theta) + \left(\partial_\theta p(\theta)\right) A_t + \tfrac{1}{2} p(\theta)^2$$


finite-dimensional.
$\mathbb{R}^2$, words p, q
$H = p^2$.
We find
– For each $\theta \in S^1$
$\exists$ a free particle
$p(\theta), q(\theta)$
– $\mu \to \partial_\theta p(\theta)$
Symplectic reduction:
$\partial_\theta p(\theta) = 0$
p is constant
Gauge sym for q
$q(\theta) \to q(\theta) + \frac{\partial}{\partial\theta}\chi(\theta)$
Up to gauge,
$q \sim c$ (we can add $\frac{\partial}{\partial\theta}\chi$ to make
q constant).
$\{p(\theta_1), q(\theta_2)\}$
$= \delta_{\theta_1-\theta_2}$
$\{\int(\partial_\theta p(\theta_1))\chi(\theta_1), q(\theta_2)\}$
$= \partial_\theta \chi(\theta)$


We find
- For each θ∈S¹
a free particle
p(θ), q(θ)
- μ = ∂_θ p(θ)
Symplectic reduction:
∂_θ p(θ) = 0
p is constant
Gauge sym for q
q(θ) → q(θ) + ∂/∂θ χ(θ)
Up to gauge,
q ~ c (we can add ∂/∂θ χ to make
q constant).
{p(θ₁), q(θ₂)}
= δ_{θ₁-θ₂}
{(∂_θ p(θ₁)), q(θ₂)}
∂δ_{θ₁-θ₂}
ℝ², coords p, q
H = p²