Title: Mathematical Physics Lecture - 230119

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Collection: Mathematical Physics (2022/2023)

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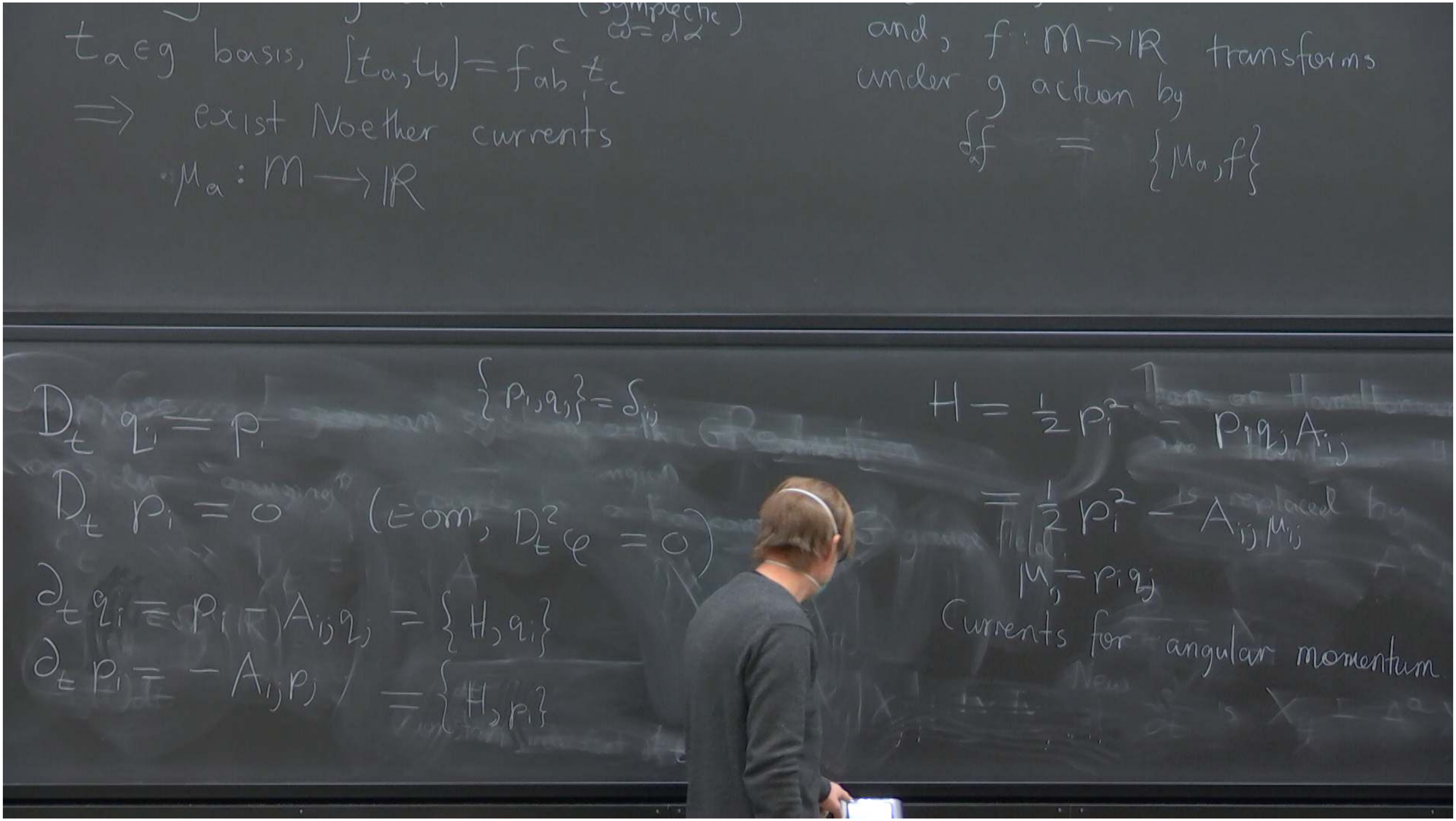

$\omega = d\alpha$
$t_a \in g$ basis, $[t_a, t_b] = f_{ab}{}^c t_c$
$\Rightarrow$ exist Noether currents
$\mu_a : M \to \mathbb{R}$
and, $f : M \to \mathbb{R}$ transforms
under g action by
$\delta_a f = \{\mu_a, f\}$
$D_t q_i = p_i$
$D_t p_i = 0$ (EOM, $D_t^2 \varphi = 0$)
$\partial_t q_i = p_i - A_{ij} q_j = \{H, q_i\}$
$\partial_t p_i = -A_{ij} p_j = \{H, p_i\}$
$\{p_i, q_j\} = \delta_{ij}$
$H = \frac{1}{2} p_i^2 - p_i q_j A_{ij}$
$= \frac{1}{2} p_i^2 - A_{ij} \mu_{ij}$
$\mu_{ij} = p_i q_j$
Currents for angular momentum

1^{st} order Lagrangian

$\gamma : \mathbb{R} \longrightarrow M$

$$\int_{\mathbb{R}} \gamma^* \alpha + H(\gamma(t))\,dt$$

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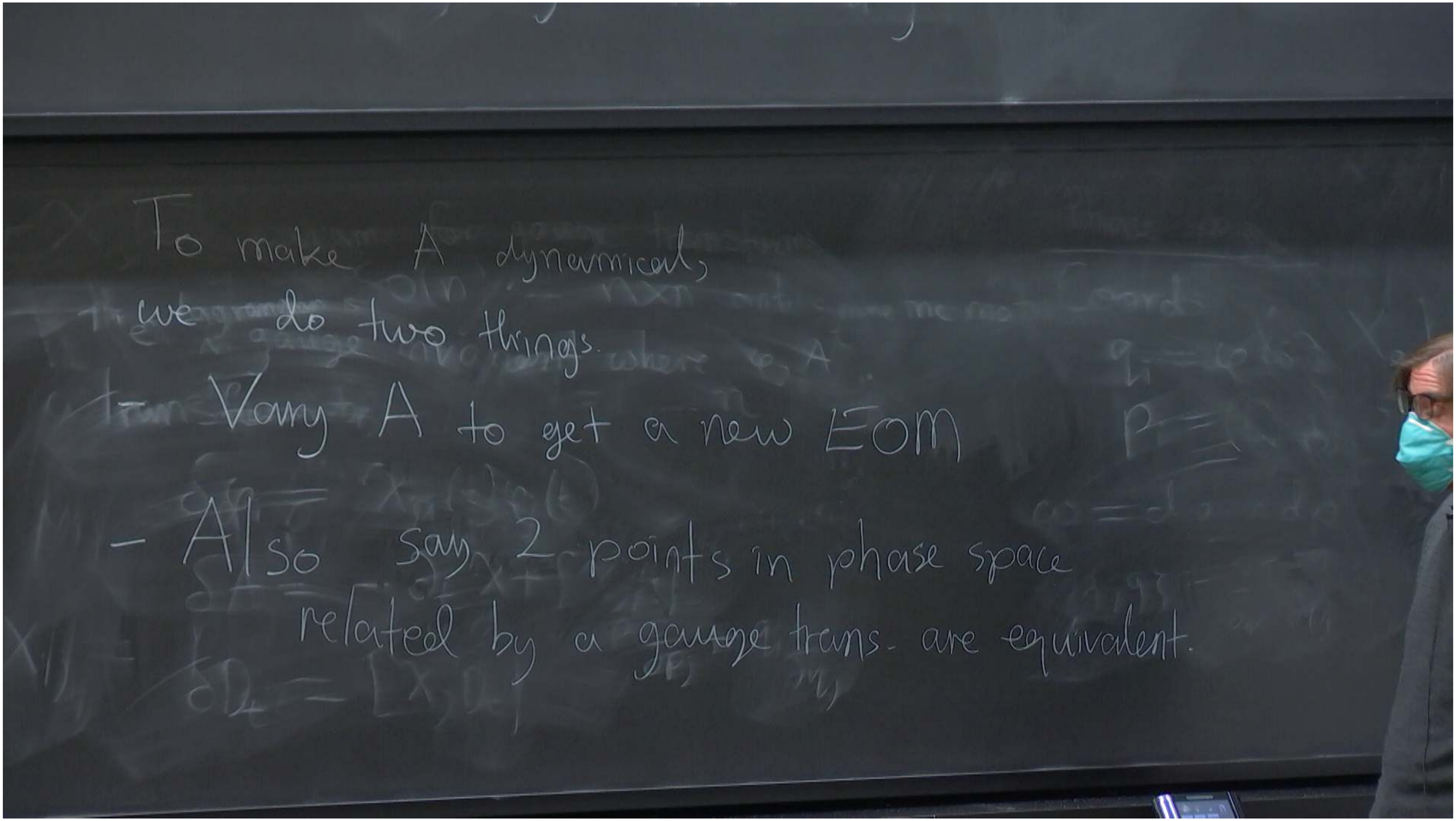
2^{nd} order $\left(\varphi \partial_t^2 \varphi \ldots\right)$

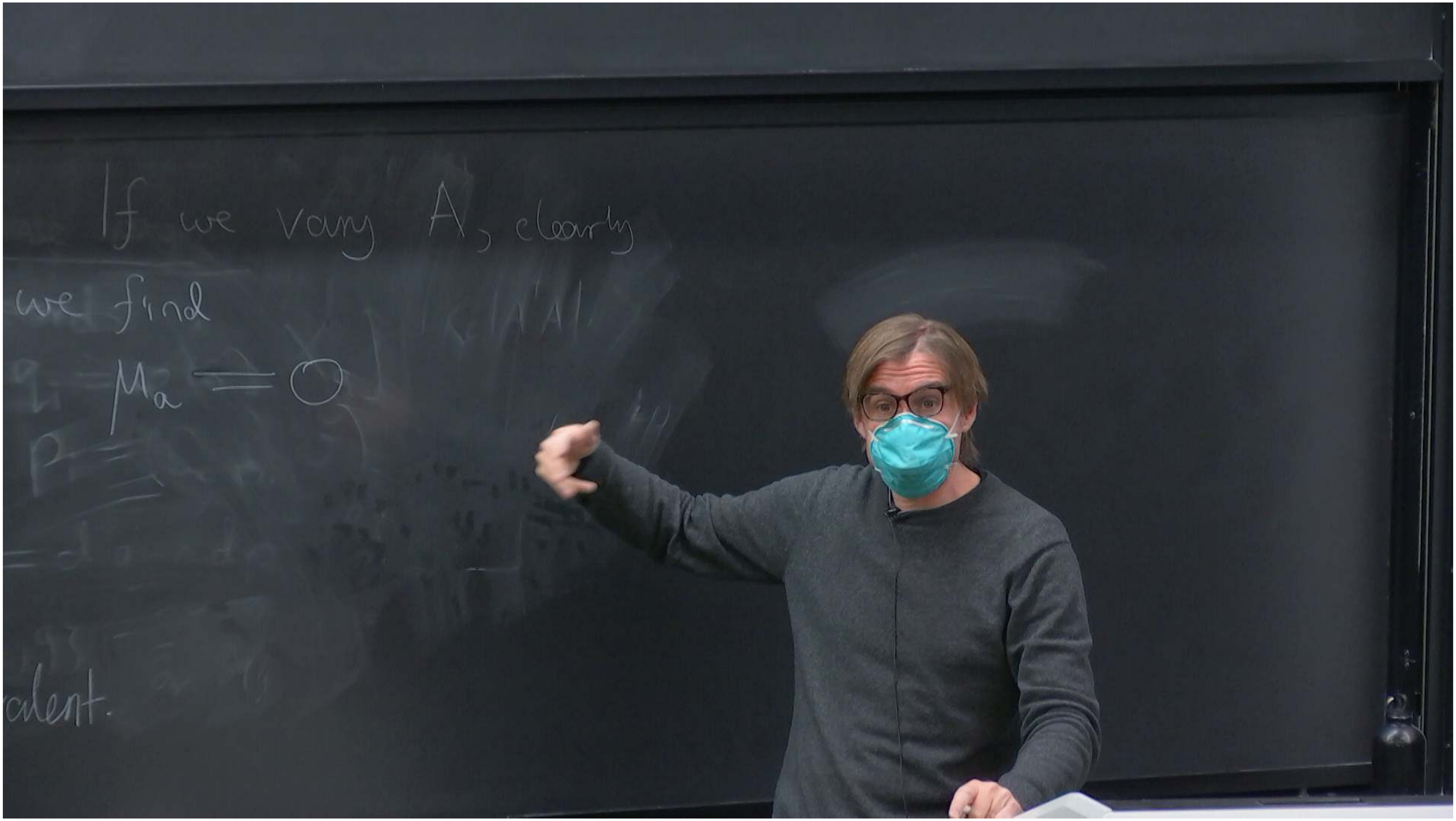
When we couple to a gauge field
we change $\underline{1}^{st}$ order Lagrangian to

$$\int_{\mathbb{R}} \gamma^* \alpha + H(\gamma(t))dt + A^a \mu_a(\gamma(t))$$

where

μ_a are Noether currents, and A^a are 1 forms on $\mathbb{R}$


To make A dynamical,
we do two things.
- Vary A to get a new EOM
- Also say 2 points in phase space
related by a gauge trans. are equivalent.


If we vary A, clearly
we find
$M_a = 0$
valent.

1st order Lagrangian

$\gamma : \mathbb{R} \longrightarrow M$

$$\int_{\mathbb{R}} \gamma^* \alpha + H(\gamma(t))\,dt$$

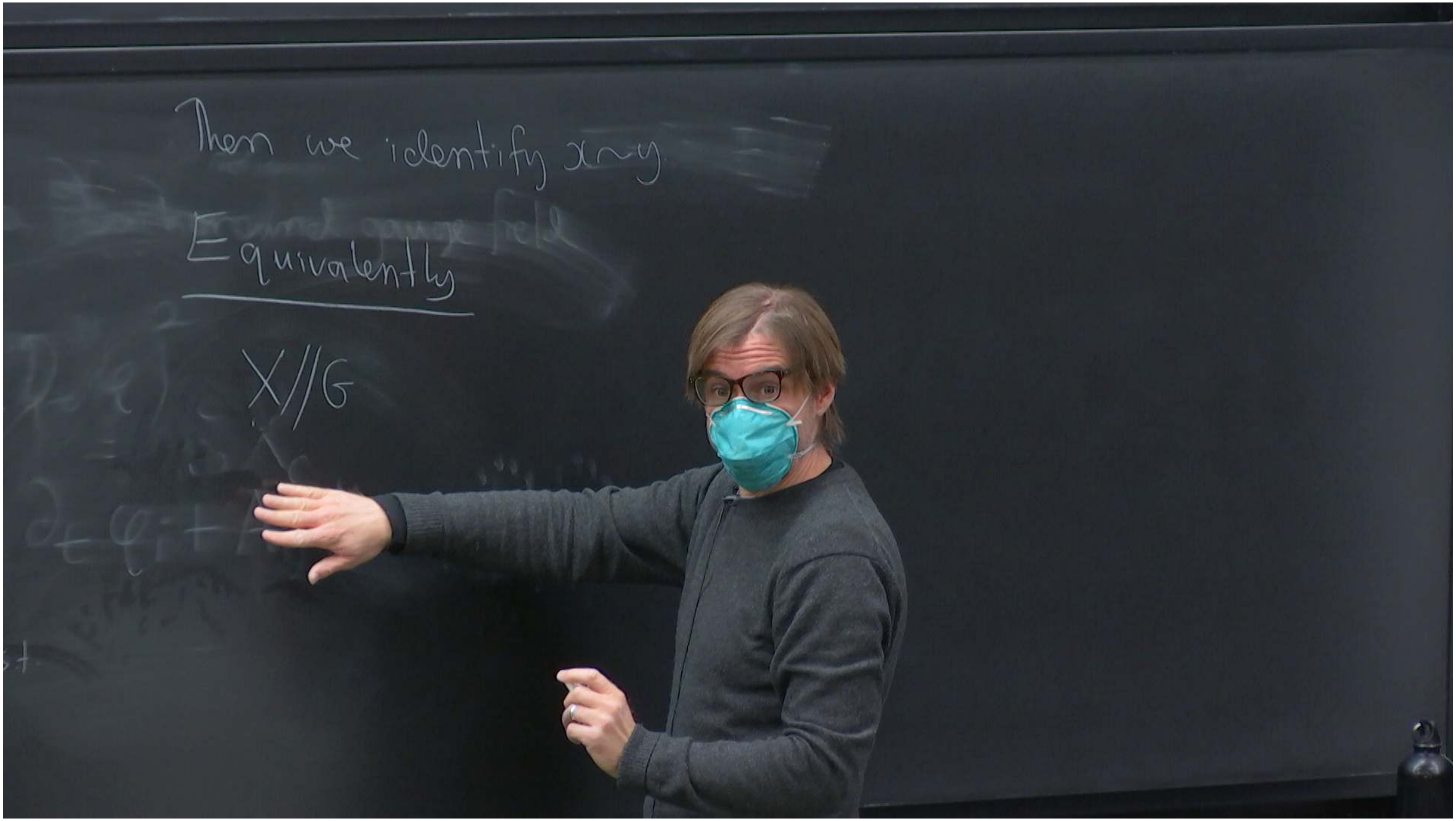
$$((\partial_t + A)\varphi)^T (\partial_t + A)\varphi)$$

Old phase space M

$\rightsquigarrow$ Reduced phase space $M/\!/G$

1) Look at set $\mu_a^{-1}(0) = \{x \in M, \mu_a(x) = 0\}$

2) We make an equivalence relation: if x, y are s.t.
$\mu_a(x) = 0$ $\mu_a(y) = 0$ and $x = gy$, some $g \in G$,


Then we identify x~y
Equivalently
X//G

Then we identify $x \sim y$.

Equivalently

$$X /\!/ G = \left\{ \begin{array}{l} \text{set of} \\ G\text{-orbits in} \\ \mu_a^{-1}(0) \end{array} \right\}$$

Then we identify $x \sim y$

Equivalently

$$M /\!/ G = \left\{ \begin{array}{l} \text{set of} \\ G\text{-orbits in} \\ \mu_a^{-1}(0) \end{array} \right\}$$

$\ldots_a(x) = 0 \}$

x, y are s.t.

$G,$

Theorem

In good cases, $M /\!/ G$ is again a symplectic manifold.

... trans. are equivalent.

Old phase space M

$\rightsquigarrow$ Reduced phase space $M/\!/G$ (of free fields)

1) Look at set $\mu_a^{-1}(0) = \{x \in M, \mu_a(x) = 0\}$

2) We make an equivalence relation: if x, y are s.t.
$\mu_a(x) = 0$, $\mu_a(y) = 0$ and $x = gy$, some $g \in G$,

Then we identify $x \sim y$

Equivalently

$$M/\!/G = \{\text{set of } G\text{-orbits in } \mu_a^{-1}(0)\}$$

$\mu_a : M \longrightarrow \mathbb{R}$

Suppose Lagrangian system with G symmetry.

Consider "gauging": couple to a background G gauge field.

$$\partial_t \rightsquigarrow D_t = \partial_t + A$$

$A_{ij} \in \Omega^1(\mathbb{R})$

$A_{ij}(t)dt$

the infinitesimal symmetry associated to $t_a \in \mathfrak{g}$

E.g. $\mathfrak{g} = so(n) = n \times n$ anti-symmetric matrices

$\omega = dq_i \wedge dp_i \qquad i = 1 \dots n$

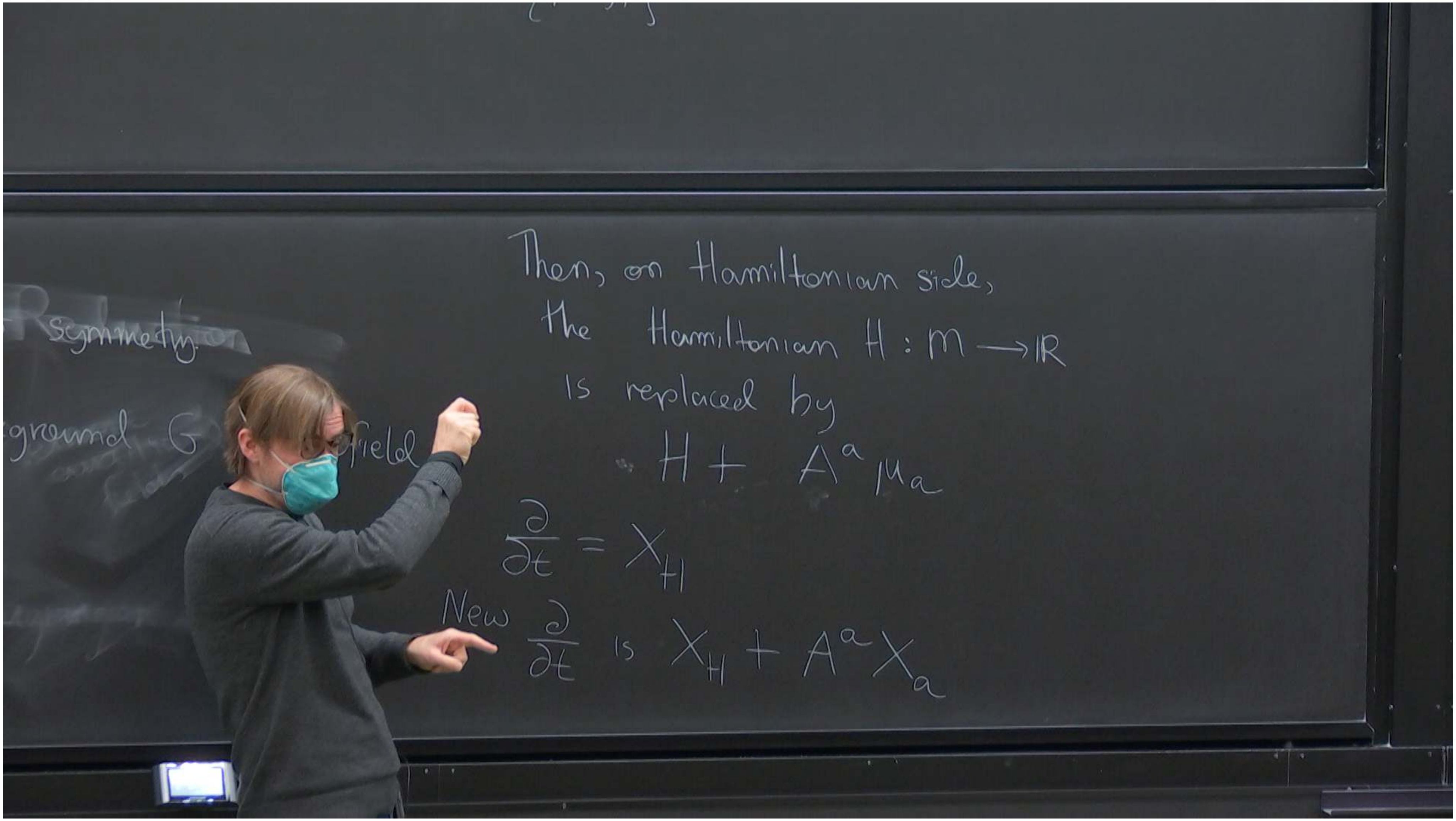
$\mu_{ij} = q_i p_j - q_j p_i$

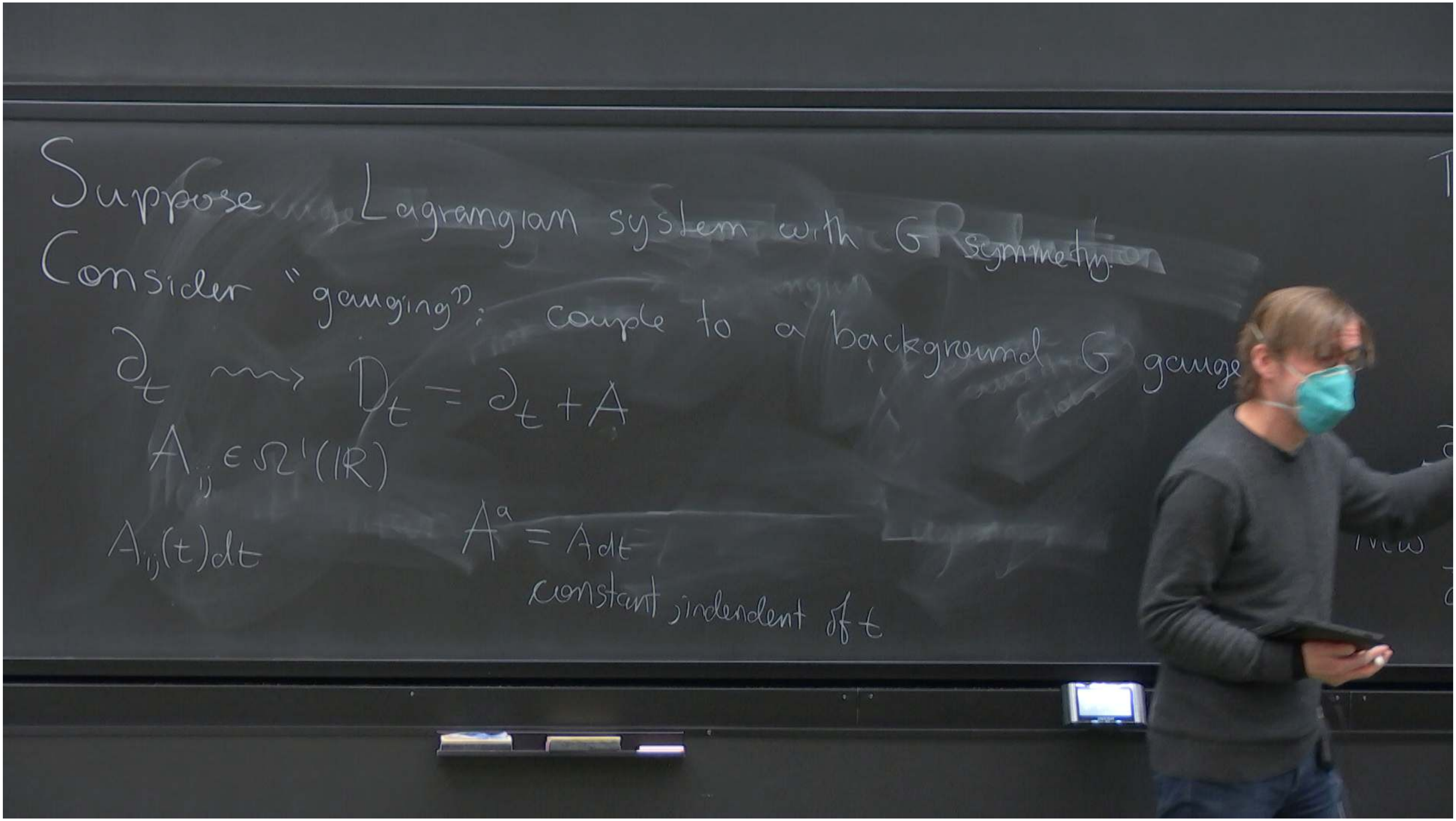
$X_{ij} = p_j \frac{\partial}{\partial p_i} - q_j \frac{\partial}{\partial q_i} - p_i \frac{\partial}{\partial p_j} + q_i \frac{\partial}{\partial q_j}$

Fact

$[X_f, X_g] = X_{\{f,g\}}$

$\{f,g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \dots$


Then, on Hamiltonian side,
the Hamiltonian $H : M \longrightarrow \mathbb{R}$
is replaced by
$H + A^a \mu_a$
$\frac{\partial}{\partial t} = X_H$
New $\frac{\partial}{\partial t}$ is $X_H + A^a X_a$


Suppose Lagrangian system with G symmetry
Consider "gauging": couple to a background G gauge
$\partial_t \rightsquigarrow D_t = \partial_t + A$
$A_{ij} \in \Omega^1(\mathbb{R})$
$A_{ij}(t)dt$
$A^a = A\,dt$
constant, indendent of t

$$\ldots{}_j = p_j \frac{\partial}{\partial p_i} - q_j \frac{\partial}{\partial q_i} - p_i \frac{\partial}{\partial p_j} + q_i \frac{\partial}{\partial q_j}$$

$$\{f, g\} = \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \ldots$$

Example

φ_i, n 1d free fields

$SO(n)$ action.

$$S(\varphi) = \int (\partial_t \varphi_i)^2$$

If A_{ij} is a background gauge field

$$S_A(\varphi) = \int (D_t \varphi)^2$$

$$(D_t \varphi)_i = \partial_t \varphi_i + A_{ij} \varphi_j$$

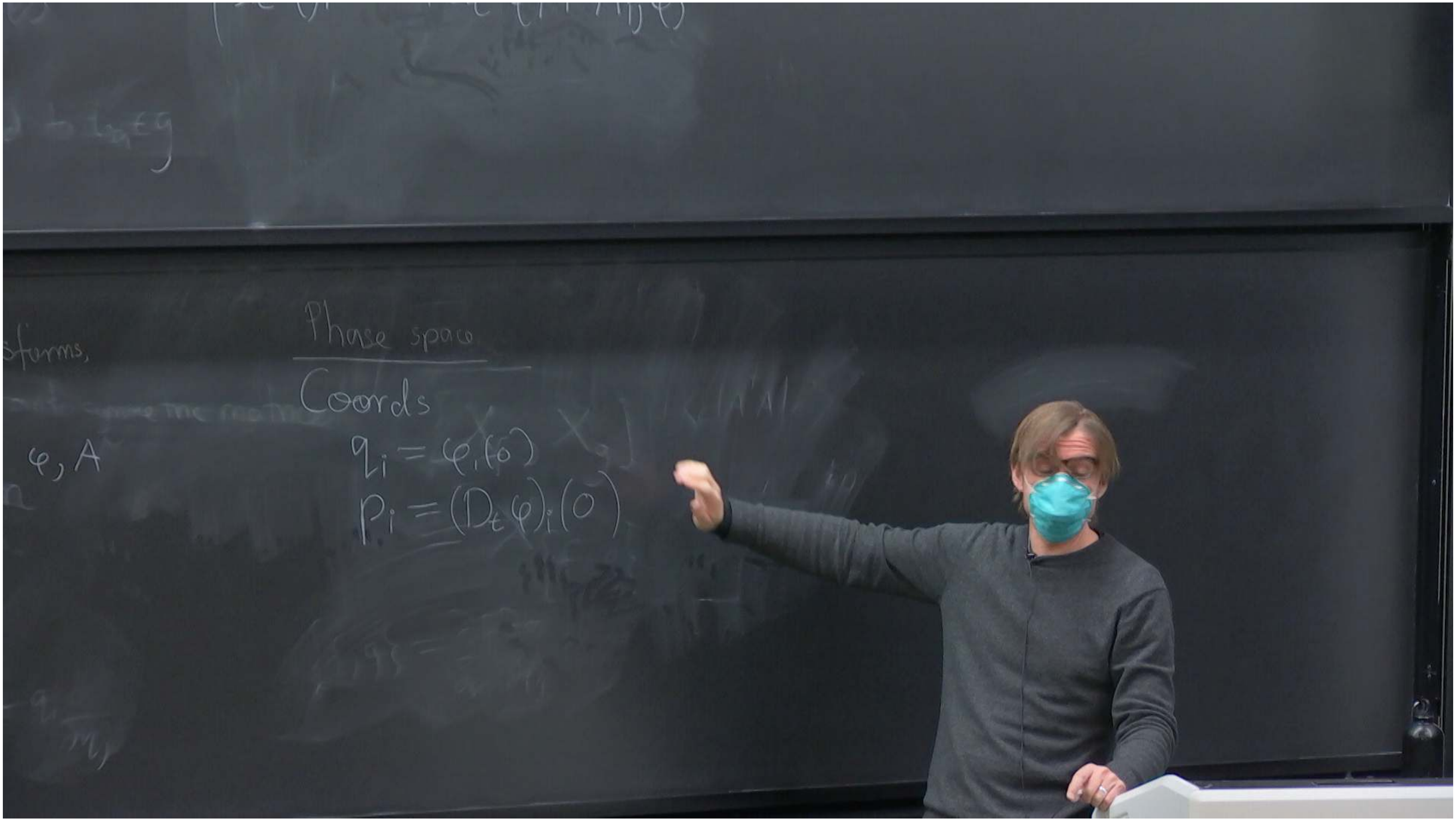
$\chi_{ij}(t)$ param. for gauge transforms.

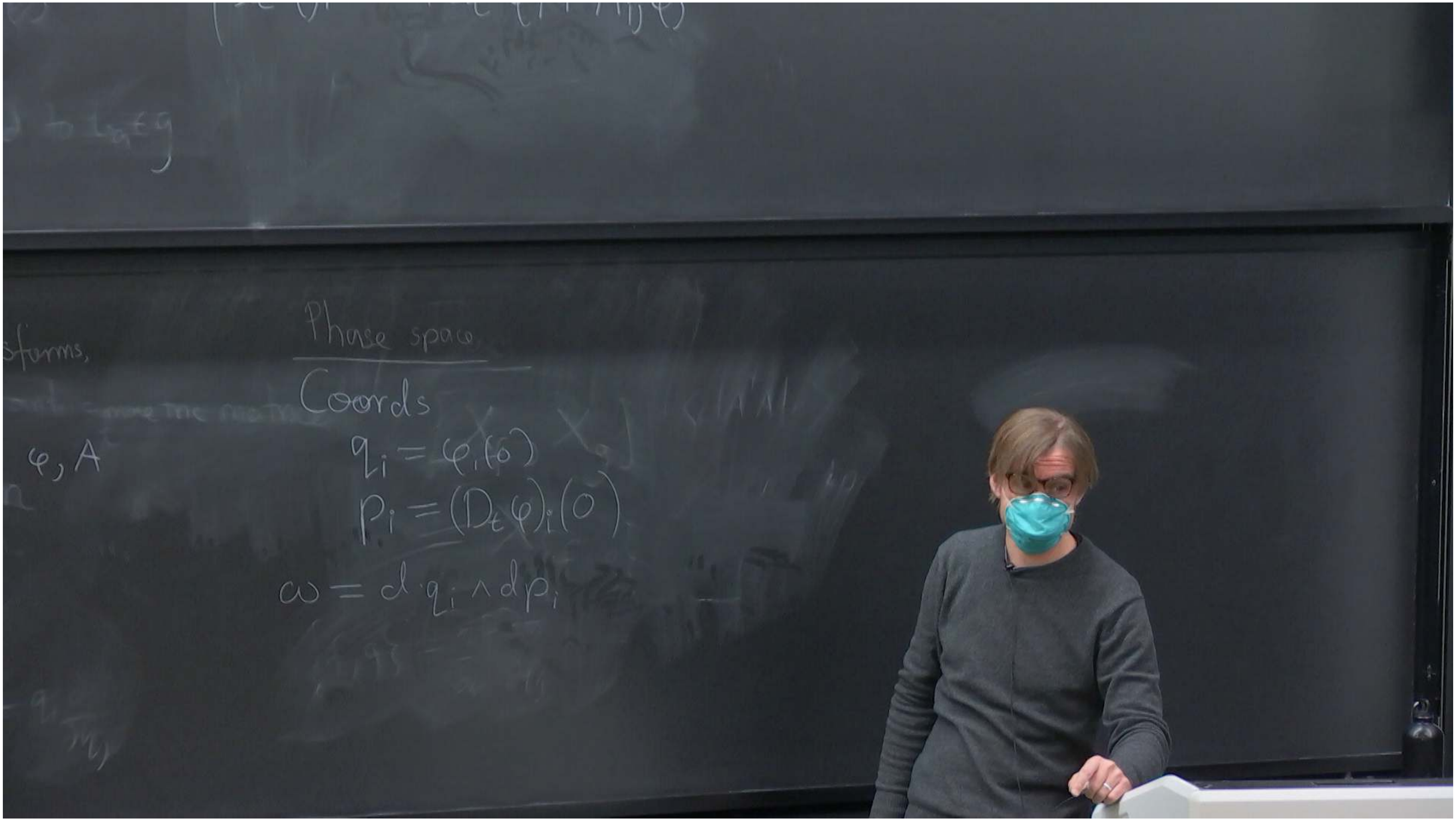
this is gauge invariant where φ, A transform by:

$$\delta \varphi_i = \chi_{ji}(t)\varphi_j(t)$$

$$\delta A = -\partial_t \chi + [\chi, A]$$

$$\delta D_t = [\chi, D_t]$$


sforms,
φ, A
Phase space
Coords
$q_i = \varphi_i(0)$
$p_i = (D_t\varphi)_i(0)$


sforms,
φ, A
Phase space
Coords
q_i = φ_i(0)
p_i = (D_t φ)_i(0)
ω = dq_i ∧ dp_i

$\chi_{ij}(t)$ param. for gauge transforms,

the Lagrangian is gauge invariant where φ, A transform by:

$$\delta \varphi_i = \chi_{ji}(t)\varphi_j(t)$$

$$\delta A = -\partial_t \chi + [\chi, A]$$

$$\delta D_t = [\chi, D_t]$$

Phase space

Coords

$$q_i = \varphi_i(0)$$

$$p_i = (D_t \varphi)_i(0)$$

$$\omega = dq_i \wedge dp_i$$

$\Longrightarrow$ exist Noether currents

$\mu_a : M \longrightarrow \mathbb{R}$

$\delta_a f = \{\mu_a, f\}$

a background G gauge field.

$$[[X_f, X_g], X_H] = [[X_f, X_H], X_g] + [X_f, [X_g, X_H]]$$

Then

the Ha

is repl

H

$\frac{\partial}{\partial t} = X_H$

New $\frac{\partial}{\partial t}$ is X

$\Longrightarrow$ exist Noether currents

$\mu_a : M \longrightarrow \mathbb{R}$

$\partial_a f$

$H = \frac{1}{2}$

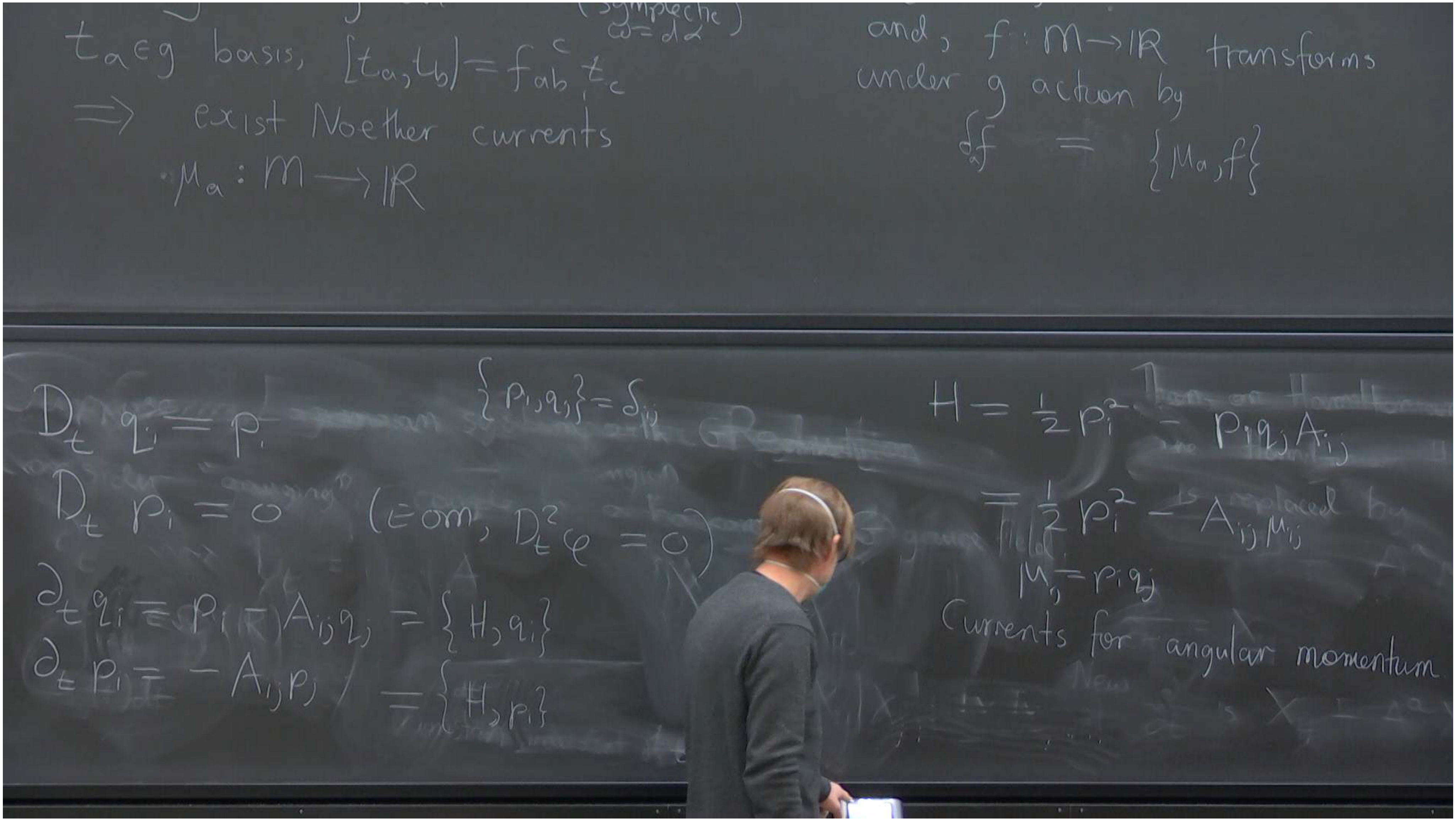
$D_t q_i = p_i$

$D_t p_i = 0$ $\quad (\text{EOM}, \; D_t^2 \varphi = 0)$

$\partial_t q_i = p_i - A_{ij} q_j = \{H, q_i\}$

$\partial_t p_i = - A_{ij} p_j = \{H, p_i\}$

$[X_f, X_g] \quad X_{\{f,g\}} \quad [X_f, [X_g$


(symplectic)
ω = dα
t_a ∈ g basis, [t_a, t_b] = f_ab^c t_c
⟹ exist Noether currents
μ_a : M → ℝ
and, f : M → ℝ transforms
under g action by
δf = {μ_a, f}
D_t q_i = p_i
{p_i, q_j} = δ_ij
D_t p_i = 0
(EOM, D_t^2 φ = 0)
∂_t q_i = p_i − A_ij q_j = {H, q_i}
∂_t p_i = − A_ij p_j = {H, p_i}
H = ½ p_i^2 − p_i q_j A_ij
= ½ p_i^2 − A_ij μ_ij
μ_ij = p_i q_j
Currents for angular momentum