Title: Mathematical Physics Lecture - 230117

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Collection: Mathematical Physics (2022/2023)

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Today

- More on Hamiltonian mechanics
- Hamiltonian $\longleftrightarrow$ Lagrangian

A Hamiltonian classical mechanical system is the following:

- A symplectic manifold M
- A function H on M, the Hamiltonian

From this we can build

X_H by $X_H \lrcorner \omega = dH$

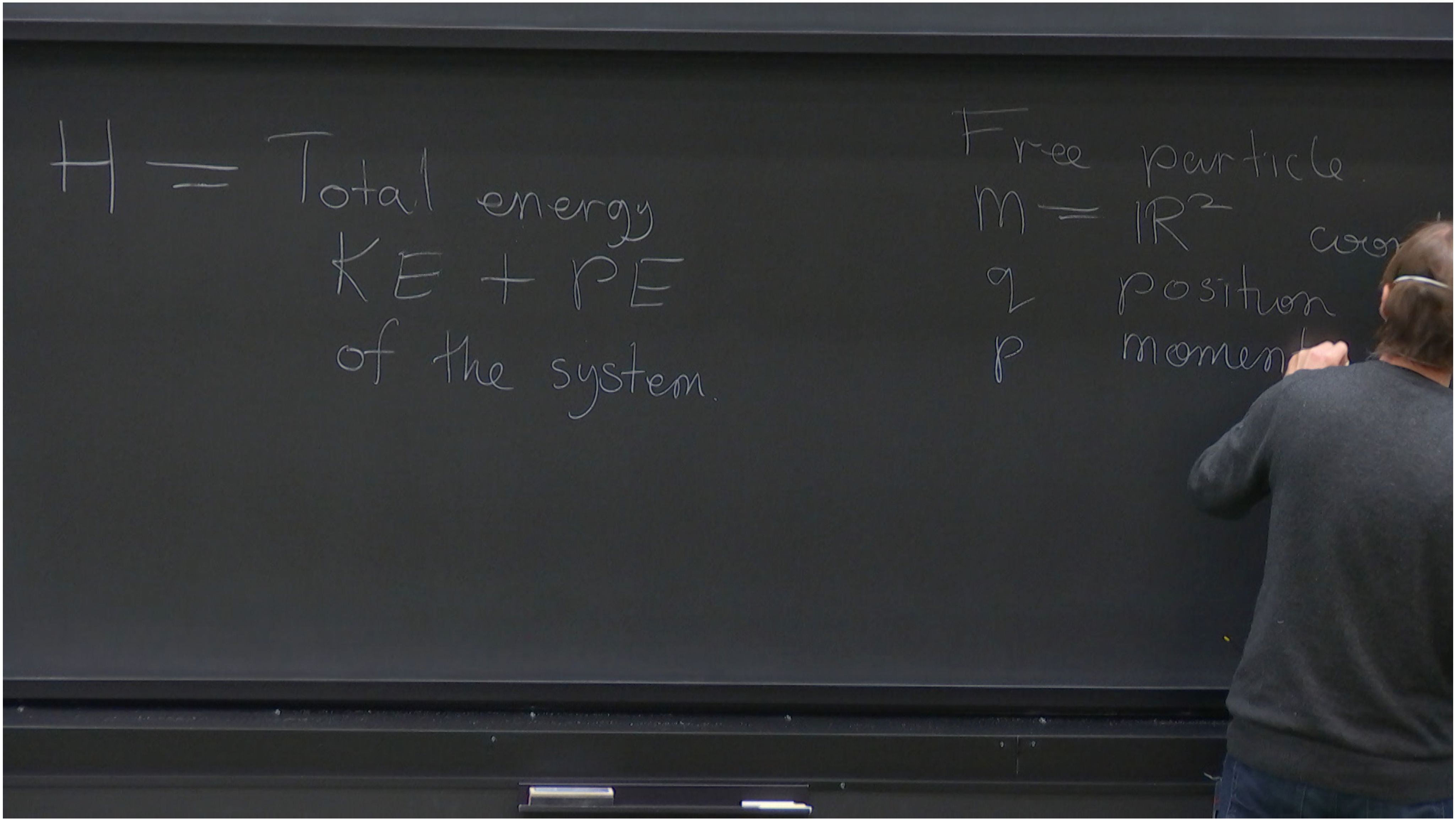
In coords, if $\omega = \sum dq^i \wedge dp_i$

$$X_H = \sum -\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}$$

$$X = \sum f_i \frac{\partial}{\partial x_i}$$

$$\omega = \sum \omega^{ij}\, dx_i \wedge dx_j$$

$$X \vee \omega = f_i\, \omega^{ij}\, dx_j - f_j\, \omega^{ij}\, dx_i$$


H = Total energy
KE + PE
of the system.
Free particle.
M = ℝ²
q position
p

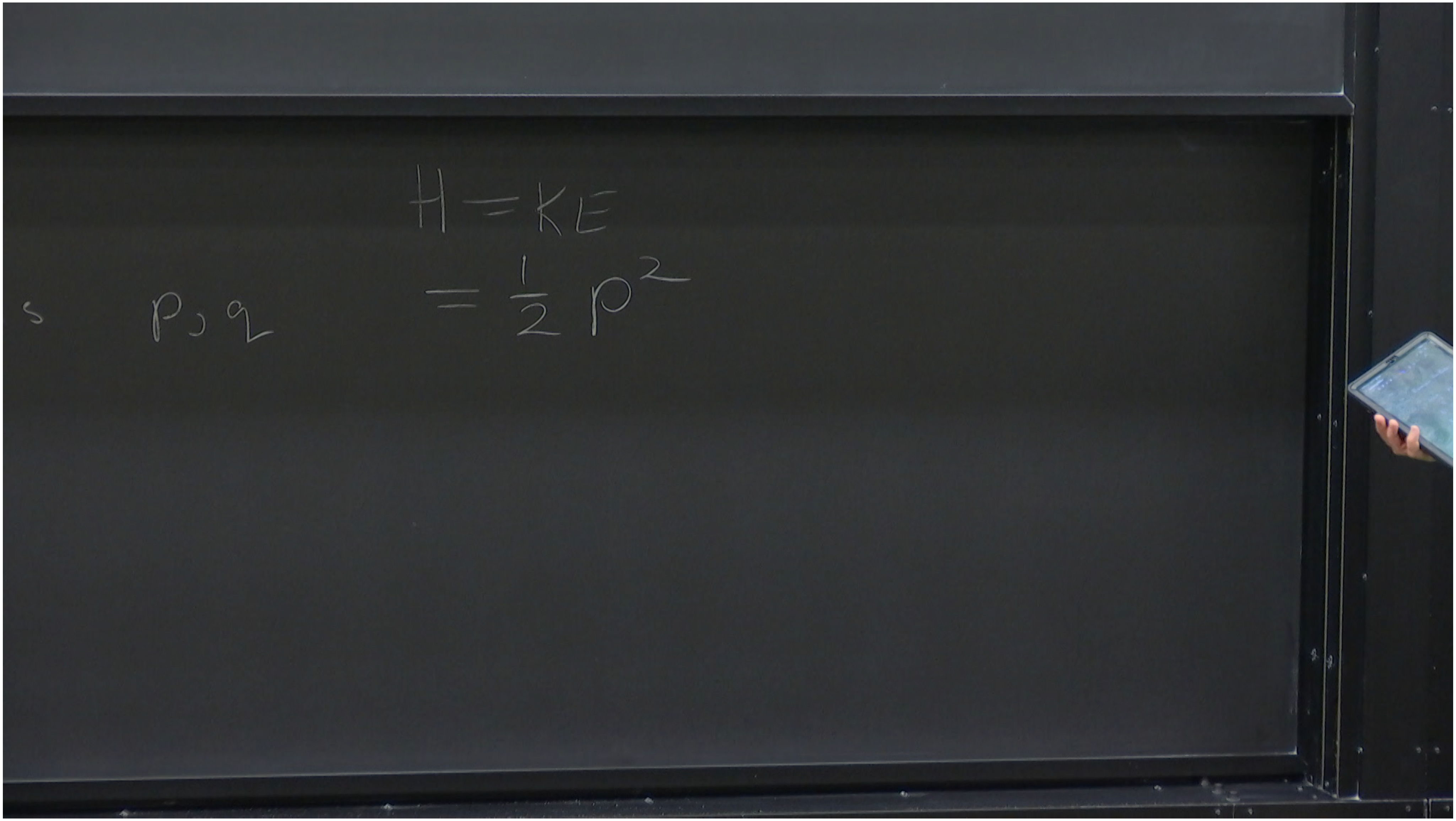
Free particle.

$M = \mathbb{R}^2$ coords p, q

q position

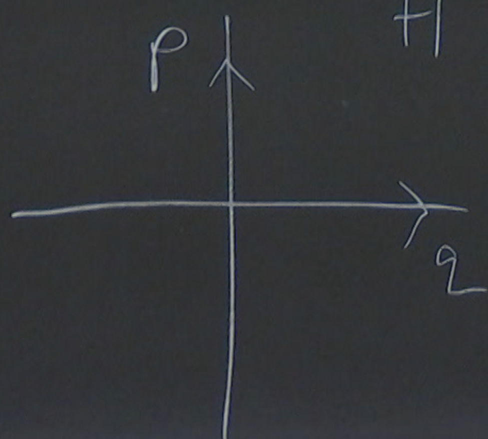
p momentum

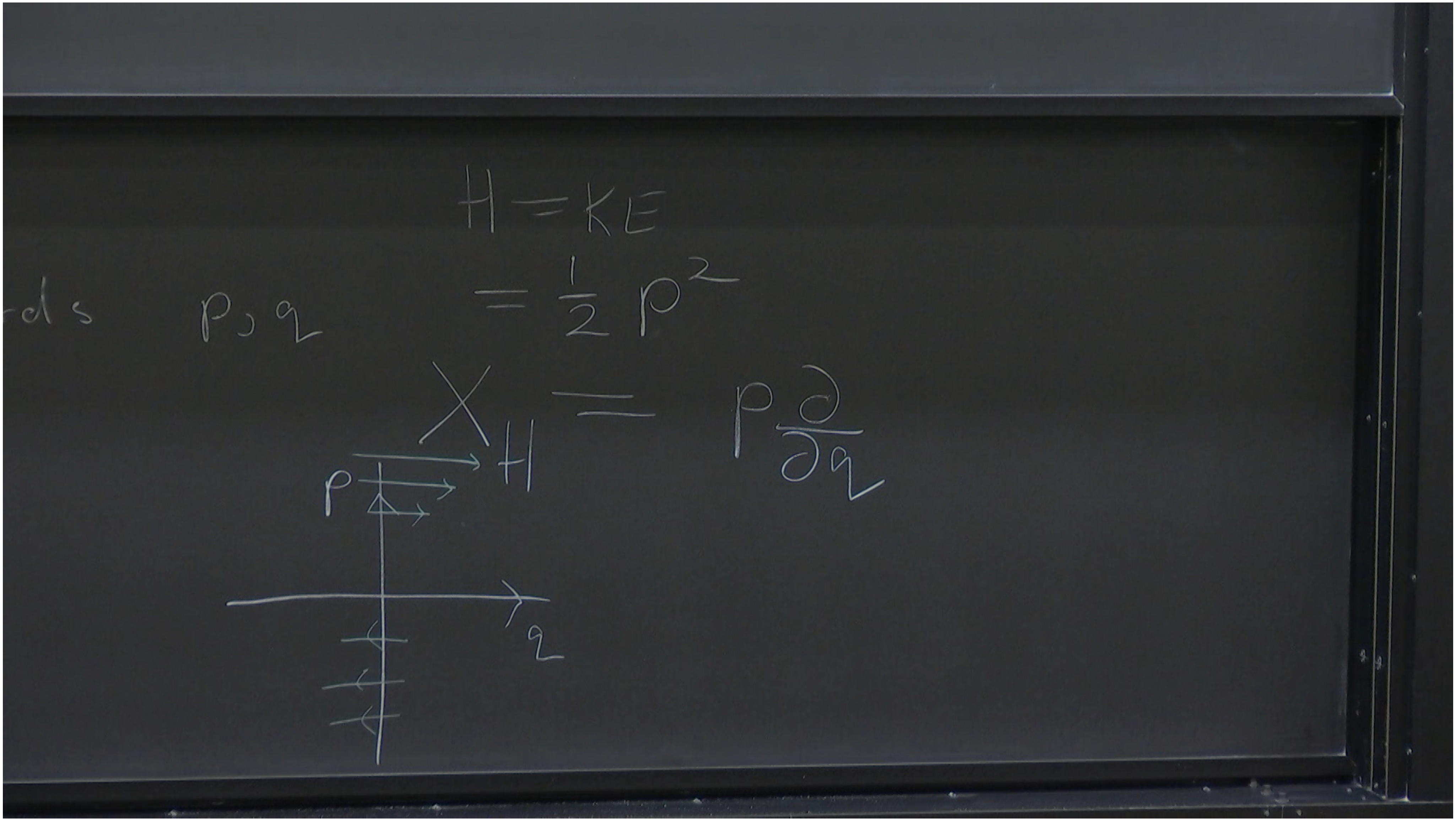
$\omega = dq \wedge dp$


s
p, q
H = KE
= 1/2 p^2

rds p, q

$$H = KE$$
$$= \frac{1}{2} p^2$$
$$X_H = p \frac{\partial}{\partial q}$$




rds p, q
H = KE
= 1/2 p^2
X_H = p ∂/∂q
p
q

Example:

Harmonic Oscillator.

$M = \mathbb{R}^2 \quad \omega = dq \wedge dp$

$$H = \frac{1}{2}(p^2 + q^2)$$

$$KE = \frac{1}{2}p^2$$

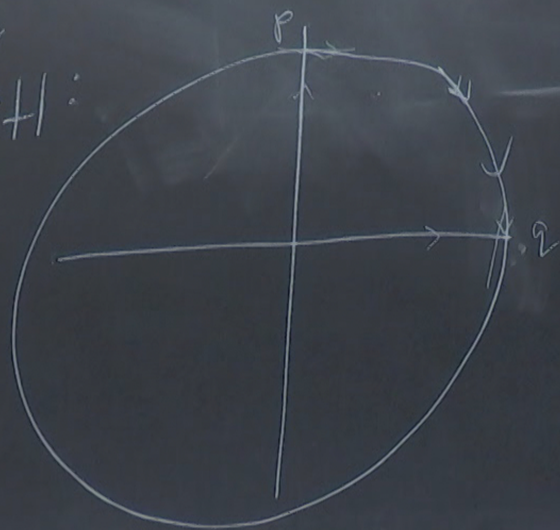
$$PE = \frac{1}{2}q^2$$

$\xrightarrow[q]{}$

$$H = \frac{1}{2}(p^2 + q^2)$$

$$X_H = p\,\partial_q - q\,\partial_p$$

Draw X_H:



Concretely,

$$p(t) = \cos t$$

$$q(t) = \sin t$$

$$(\dot{p}, \dot{q}) = (-q, p)$$

So it describes motion of a particle.

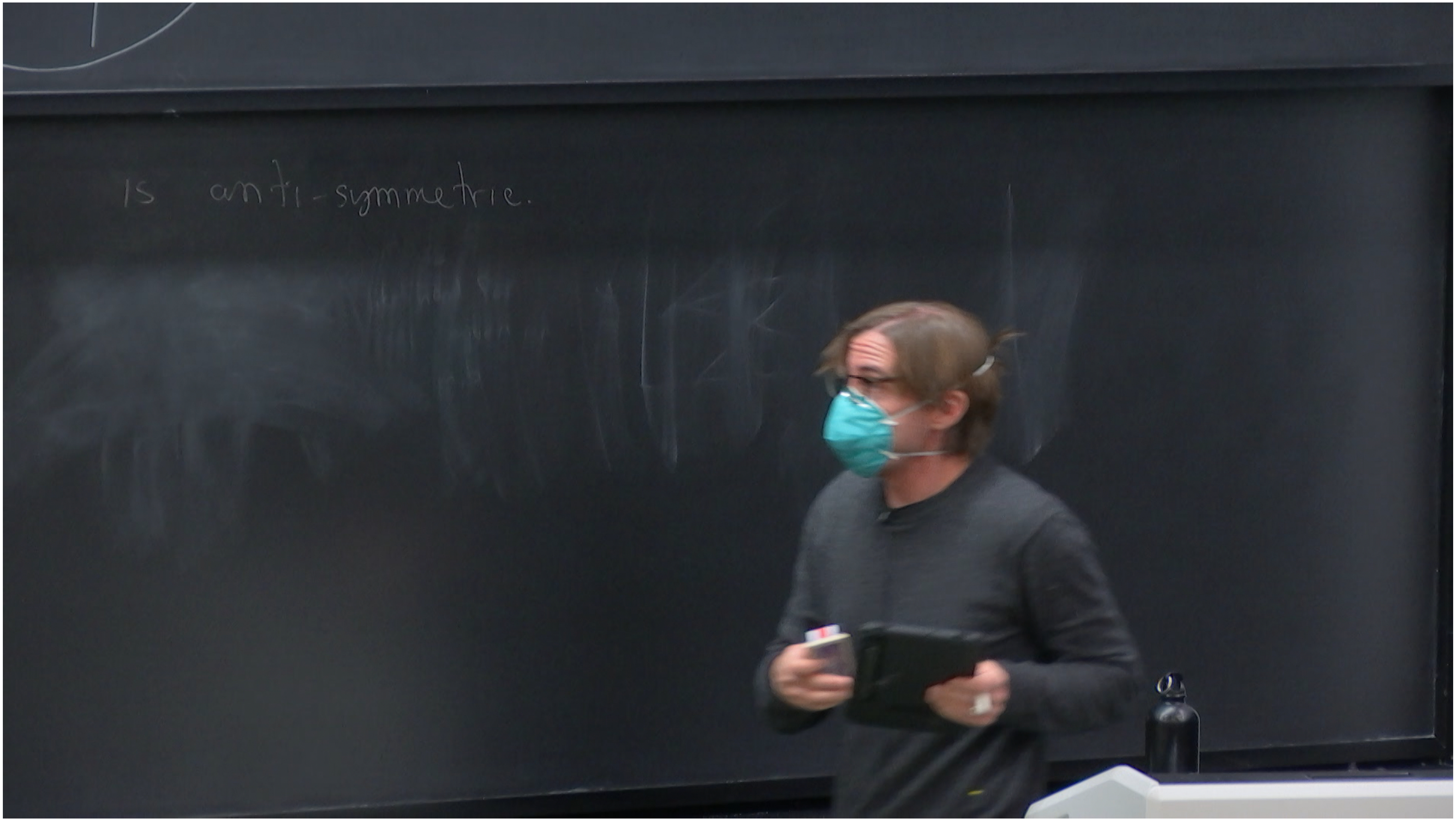
is anti-sy

H is constant along the flows of X_H

$$X_H \cdot H = 0$$

As, $X_H \cdot H = -\dfrac{\partial H}{\partial q_i}\dfrac{\partial H}{\partial p_i} + \dfrac{\partial H}{\partial p_i}\dfrac{\partial H}{\partial q_i}$

Or, $X_H \cdot f = \{H, f\}$, $\{H, H\} = 0$ as $\{\ ,\ \}$


is anti-symmetric.

Lagrangian

Suppose we have a field

$\varphi(t)$, (or many fields....)

$\mathcal{L}(\varphi, \dot{\varphi}, \ldots)$

$$S(\varphi) = \int_{\mathbb{R}} \mathcal{L}(\varphi, \dot{\varphi} \ldots)$$

From t

From this we want to build a classical mechanical system is the following

(M, ω, H)

Symplectic manifold M

function H on M the Hamiltonian

From this we want to build
(M, ω, H)

Def^n The phase space of a Lagrangian is the space of sol^{n}s to the EL equations

$\varphi = q + pt$

Example: Free particle

$$\mathcal{L}(\varphi) = \frac{1}{2}\varphi \partial_t^2 \varphi$$

$$S(\varphi) = \frac{1}{2}\int \varphi \partial_t^2 \varphi$$

EL eq^{ns}, are $\partial_t^2 \varphi = 0$.

$$\varphi = q + pt$$

$$q = \varphi(0)$$

and $p = \partial_t \varphi(0)$

are coordinates on the phase space

Where does ω come from?

H is

$X_H \cdot H$

As, X

Or, $X_H \cdot$

$\varphi = q + pt$

$q = \varphi(0)$

and $p = \partial_t \varphi(0)$

are coordinates on the phase space

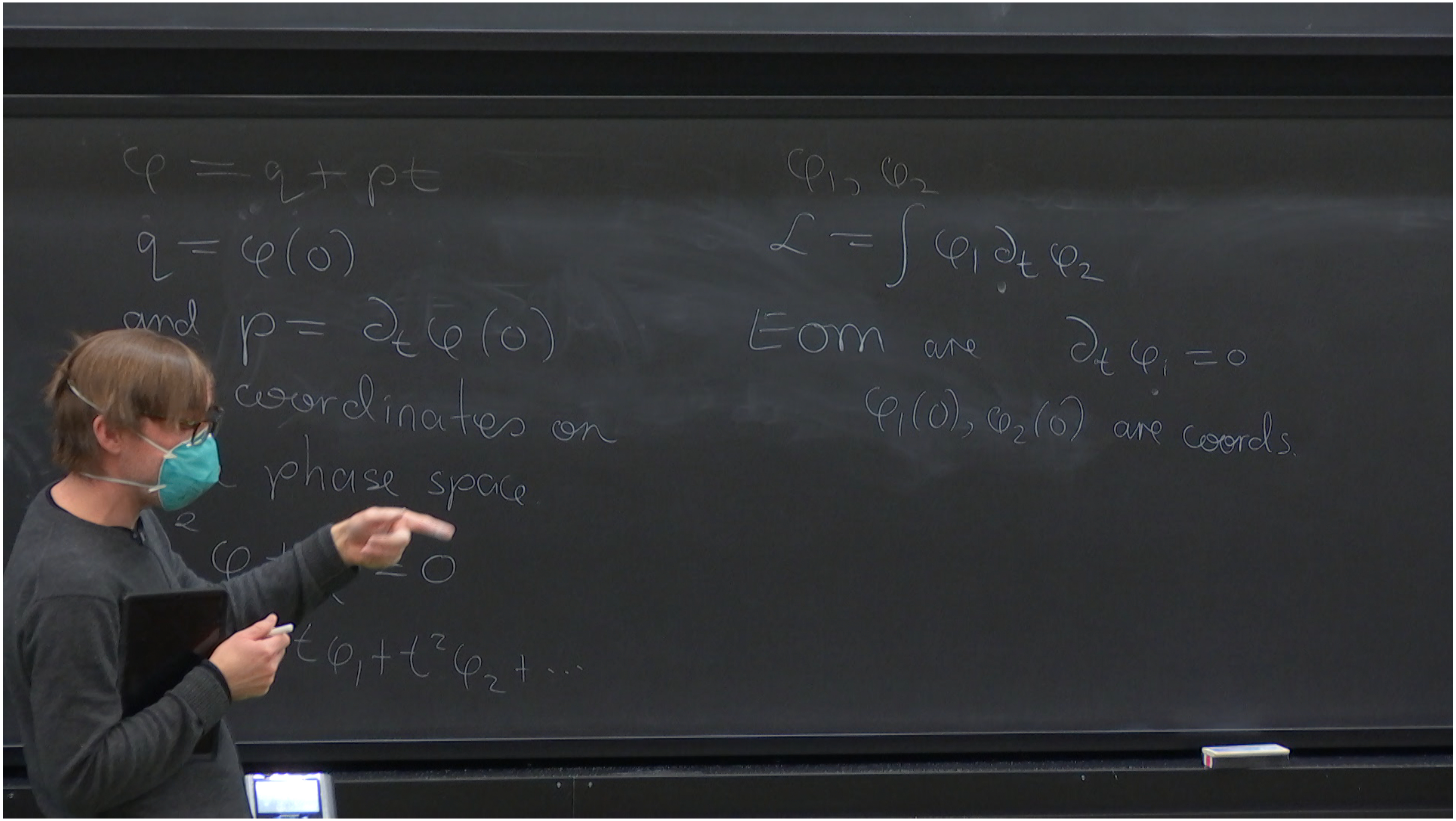
$\partial_t^2 \varphi + \varphi = 0$

$\varphi = \varphi_0 + t\varphi_1 + t^2 \varphi_2 + \cdots$

φ_1, φ_2

$\mathcal{L} = \int \varphi_1 \partial_t \varphi_2$

EOM are $\partial_t \varphi_i = 0$



φ = q + pt
q = φ(0)
and p = ∂_t φ(0)
coordinates on
phase space
tφ_1 + t^2 φ_2 + ...
φ_1, φ_2
L = ∫ φ_1 ∂_t φ_2
Eom are ∂_t φ_i = 0
φ_1(0), φ_2(0) are coords.

$\varphi = q + pt$

$q = \varphi(0)$

and $p = \partial_t \varphi(0)$

are coordinates on the phase space

$\partial_t^2 \varphi + \varphi = 0$

$\varphi = \varphi_0 + t\varphi_1 + t^2 \varphi_2 + \dots$

φ_1, φ_2

$\mathcal{L} = \int \varphi_1 (\partial_t + A) \varphi_2$

EOM are $\partial_t \varphi_i = 0$

$\varphi_1(0), \varphi_2(0)$ are coords.

$\mathcal{M} = \{ \text{sol'ns to EL equation} \}$

What is ω?

We will find a $\underline{1}$-form

α on $\mathcal{M}$, the variational $\underline{1}$-form

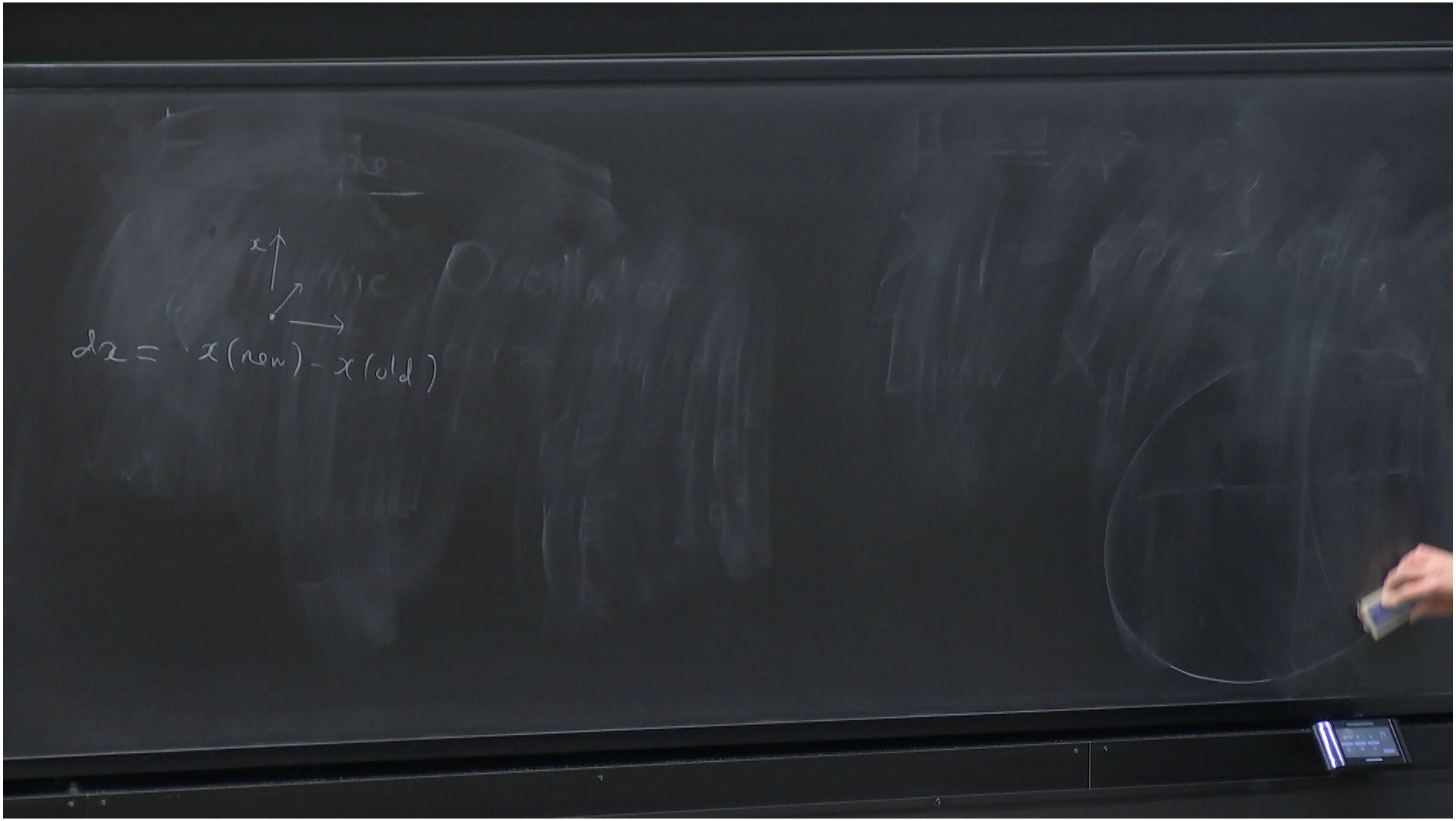
$\omega = d\alpha$

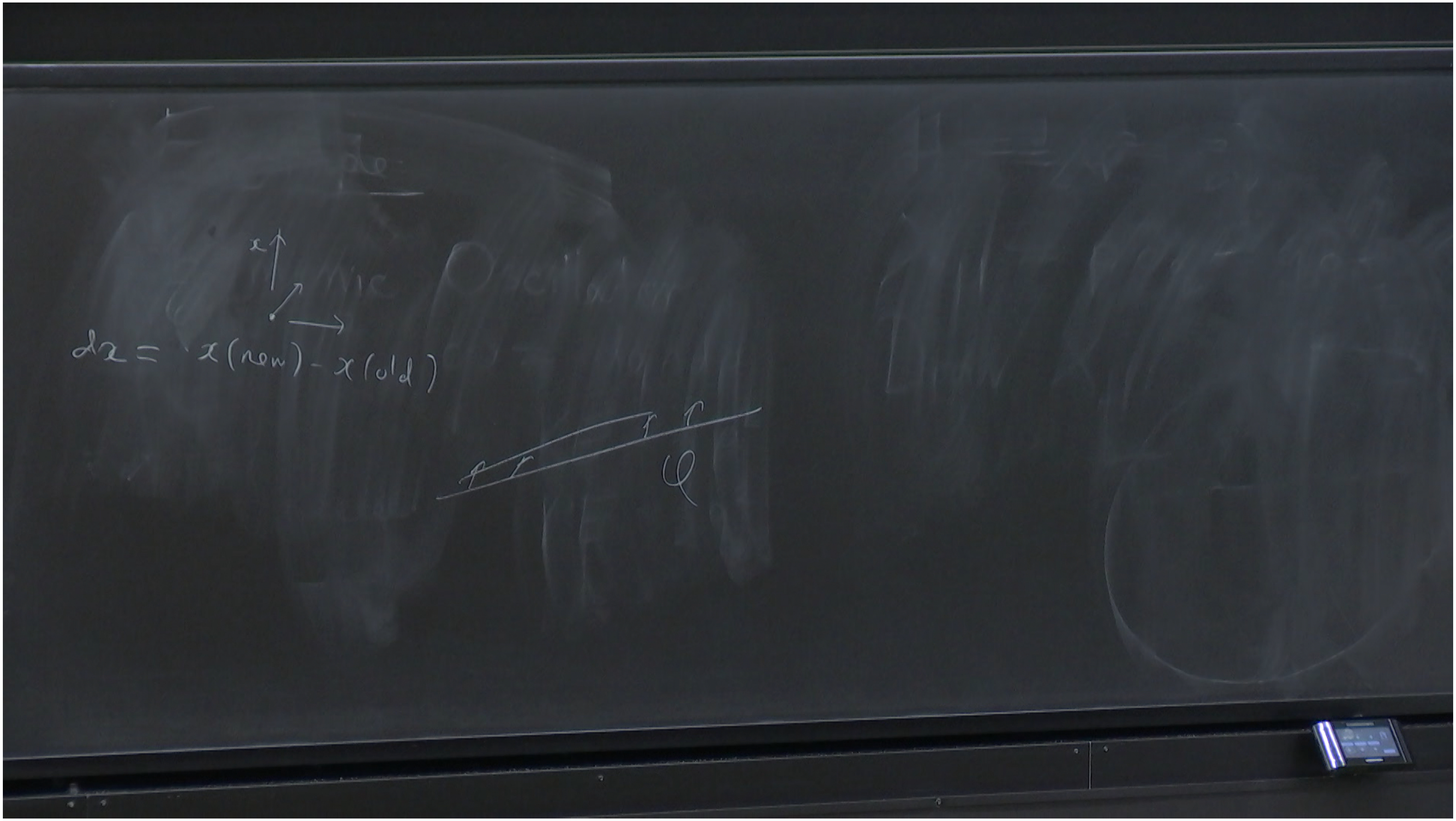
al 1-form

A point in $\mathcal{M}$ is
a field φ satisfies EL.

$T_\varphi \mathcal{M}$ is a first order variation
$\varphi + \delta\varphi$

A 1-form α, assigns to $\varphi, \delta\varphi$ a number
$\alpha(\varphi, \delta\varphi)$


x
dx = x(new) - x(old)

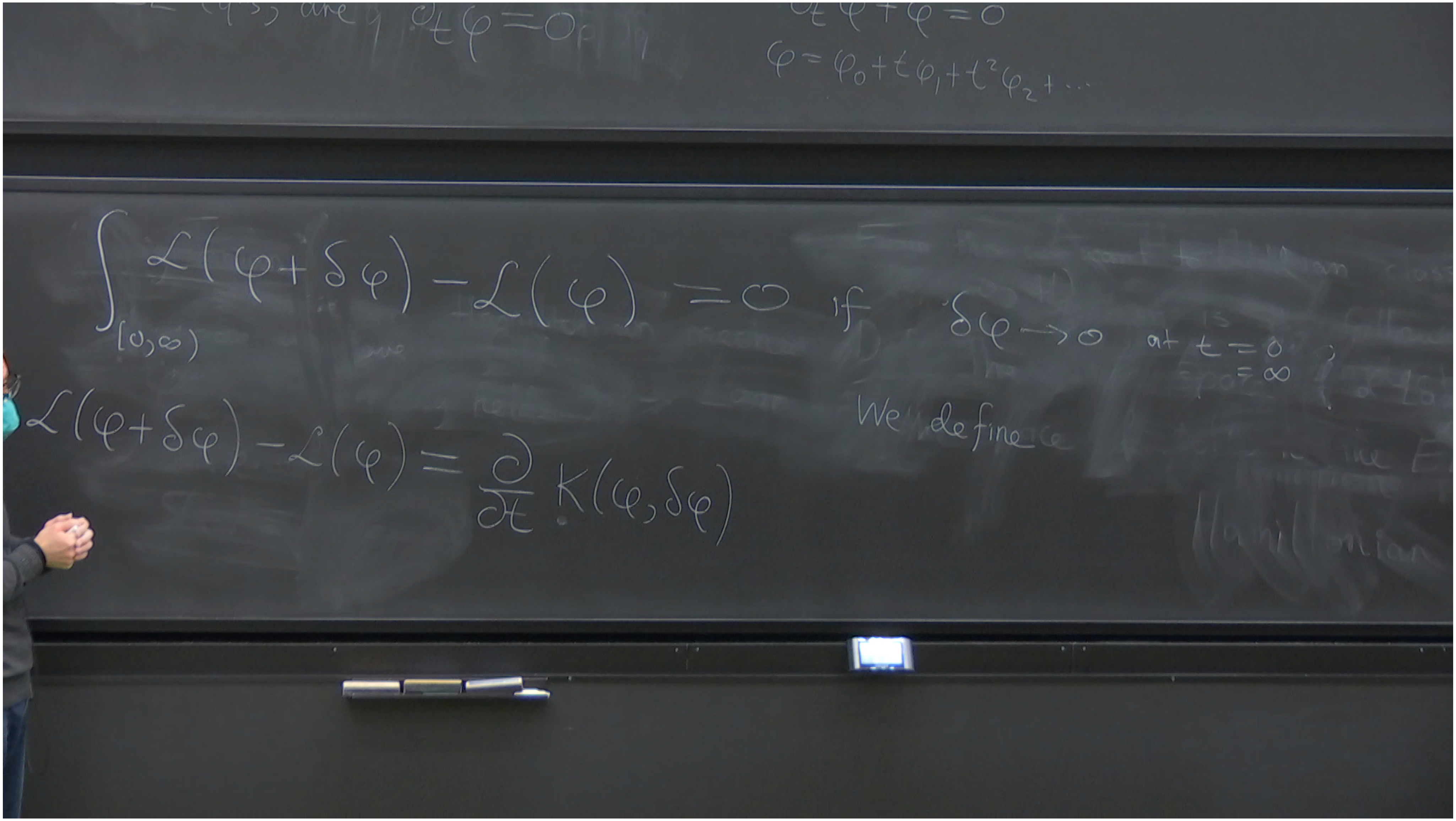

dx = x(new) - x(old)

EL eqns:

$\varphi + \delta\varphi, \quad \delta\varphi(t) \longrightarrow 0 \quad \begin{matrix} t \ll 0 \\ t \gg 0 \end{matrix}$

then $\dfrac{\delta S}{\delta \varphi} = 0$

$\Longleftrightarrow$ φ satisfies EL


$\partial_t \varphi = 0$
$\partial_t \varphi + \varphi = 0$
$\varphi = \varphi_0 + t\varphi_1 + t^2\varphi_2 + \cdots$
$\int_{[0,\infty)} \mathcal{L}(\varphi + \delta\varphi) - \mathcal{L}(\varphi) = 0$ if $\delta\varphi \to 0$ at $t = 0$, $= \infty$
We define
$\mathcal{L}(\varphi + \delta\varphi) - \mathcal{L}(\varphi) = \frac{\partial}{\partial t} K(\varphi, \delta\varphi)$
Hamiltonian

$\delta\varphi \to 0$ at $t = 0$,
space $= \infty$

We define $\alpha(\varphi, \delta\varphi) = K(\varphi, \delta\varphi)\big|_{t=0}$

Some expression in $\varphi, \partial_t \varphi, \ldots$
$\delta\varphi, \partial_t \delta\varphi, \ldots$

Example: Free particle

$q = \varphi$

$$\mathcal{L}(\varphi) = \frac{1}{2}\varphi\,\partial_t^2\varphi$$

$$S(\varphi) = \frac{1}{2}\int \varphi\,\partial_t^2\varphi$$

EL eqns, are $\partial_t^2\varphi = 0$

$q = \varphi(0)$

$p = \partial_t \varphi(0)$

Variational 1 form:

$$\mathcal{L}(\varphi + \delta\varphi) - \mathcal{L}(\varphi) = \frac{1}{2}\left(\varphi\, \partial_t^2 \delta\varphi + \delta\varphi\, \partial_t^2 \varphi\right)$$

$$\partial_t^2 \varphi = 0$$

$q = \varphi(0)$

$p = \partial_t \varphi(0)$

Variational 1 form:

$$\mathcal{L}(\varphi + \delta\varphi) - \mathcal{L}(\varphi) = \frac{1}{2}\left(\varphi \partial_t^2 \delta\varphi + \cancel{\delta\varphi \partial_t^2 \varphi}\right)$$

$$\partial_t^2 \varphi = 0$$

$$\mathcal{L}(\varphi + \delta\varphi) - \mathcal{L}(\varphi) = \frac{1}{2}\varphi \partial_t^2 \delta\varphi$$

$$\frac{1}{2}\varphi\,\partial_t^2\delta\varphi = \frac{1}{2}\partial_t\left(\varphi\,\partial_t\delta\varphi - \partial_t\varphi\,\delta\varphi\right)$$

Variational $\underline{1}$ form is

$$\alpha = \frac{1}{2}\left(\varphi(0)\,\partial_t\delta\varphi(0) - \partial_t\varphi(0)\,\delta\varphi(0)\right)$$

$$= \frac{1}{2}\left(q\,dp - p\,dq\right)$$

$\omega = d$

$= d$

$$\frac{1}{2}\varphi\,\partial_t^3\delta\varphi = \frac{1}{2}\partial_t\left(\varphi\,\partial_t\delta\varphi - \partial_t\varphi\,\delta\varphi\right)$$

$$\omega = d\alpha = dq \wedge dp$$

Variational 1 form is

$$\alpha = \frac{1}{2}\left(\varphi(0)\,\partial_t\delta\varphi(0) - \partial_t\varphi(0)\,\delta\varphi(0)\right)$$

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$$\frac{1}{2}\varphi\,\partial_t^2\delta\varphi = \frac{1}{2}\partial_t\left(\varphi\,\partial_t\delta\varphi - \partial_t\varphi\,\delta\varphi\right)$$

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$$\varphi = q + pt \qquad \delta\varphi = \delta q + t\,\delta p$$

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Variational 1 form is

$$\alpha = \frac{1}{2}\left(\varphi(0)\,\partial_t\delta\varphi(0) - \partial_t\varphi(0)\,\delta\varphi(0)\right)$$

$$= \frac{1}{2}\left(q\,dp - p\,dq\right)$$

δL

$\delta\varphi$

$\partial_t\varphi$

$$\varphi = q + pt \qquad \delta\varphi = \delta q + t\,\delta p$$

Where does Hamiltonian come from?

If φ satisfies the EOM

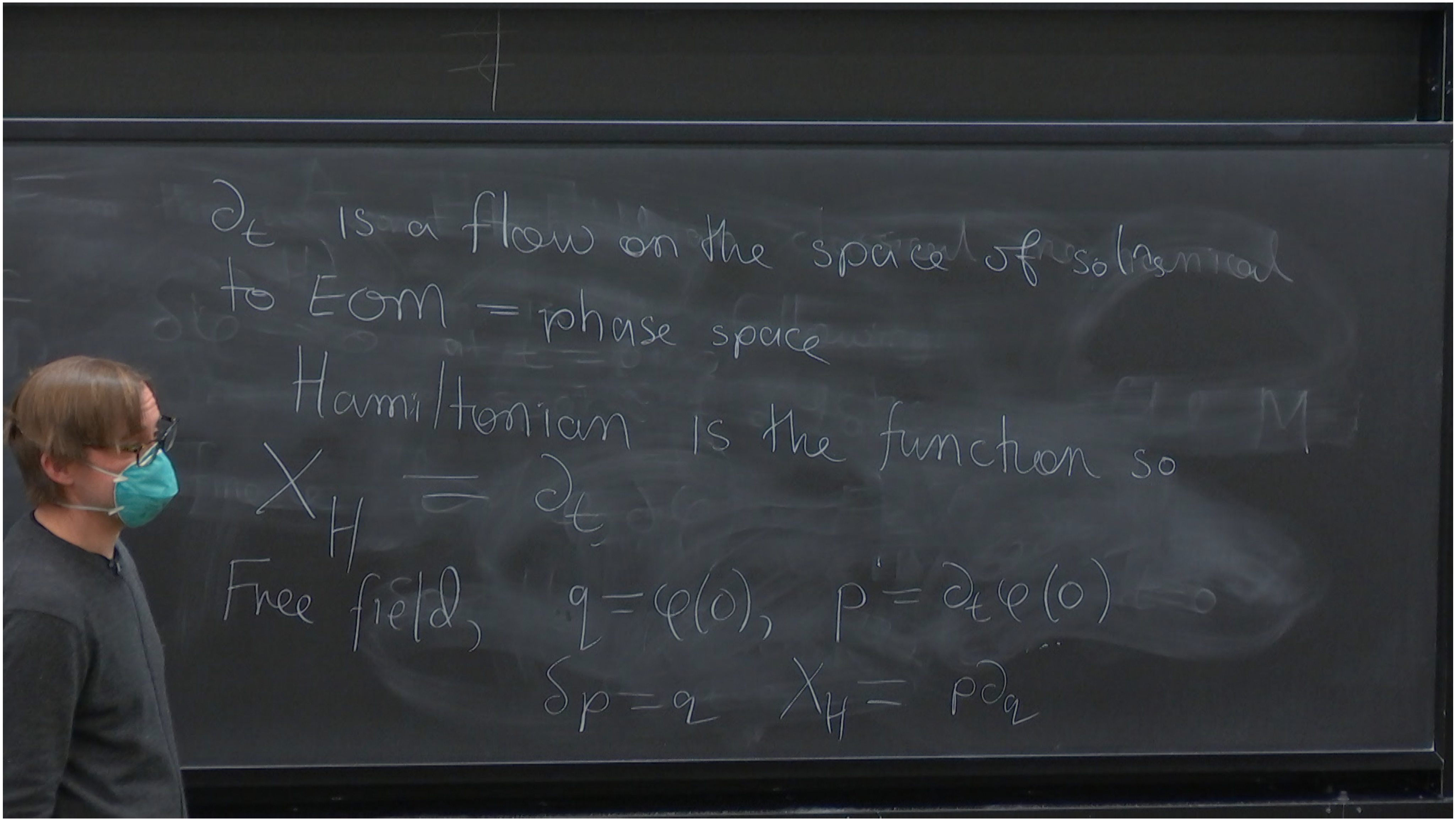
then so does $\varphi + \varepsilon \partial_t \varphi$

(mod ε^2)

∂_t is a flow on the space of solns to EOM = phase space

Hamiltonian is the function so

$$X_H = \partial_t$$


∂_t is a flow on the space of solutions
to EOM = phase space
Hamiltonian is the function so
$X_H = \partial_t$
Free field, $q = \varphi(0)$, $p = \partial_t \varphi(0)$
$\delta p = q$ $X_H = p\partial_q$

More complicated:

$$\mathcal{L}(\varphi) = \frac{1}{2}\varphi\partial_t^2\varphi + \frac{1}{2}\dot{\varphi}^2$$

$$p = \partial_t\varphi(0)$$

$$q = \varphi(0)$$

$$\partial_t^2\varphi = -\varphi$$

$$p = \partial_t q$$

$$q = -\partial_t p$$

More complicated:

$$\mathcal{L}(\varphi) = \frac{1}{2}\varphi\,\partial_t^2\varphi + \frac{1}{2}\varphi^2$$

$$p = \partial_t \varphi(0)$$

$$q = \varphi(0)$$

$$\partial_t^2 \varphi = -\varphi$$

$$p = \partial_t q$$

$$q = -\partial_t p$$

$$X_H = p\,\partial_q - q\,\partial_p$$

$$H = \frac{1}{2}(p^2 + q^2)$$

Harmonic oscillator

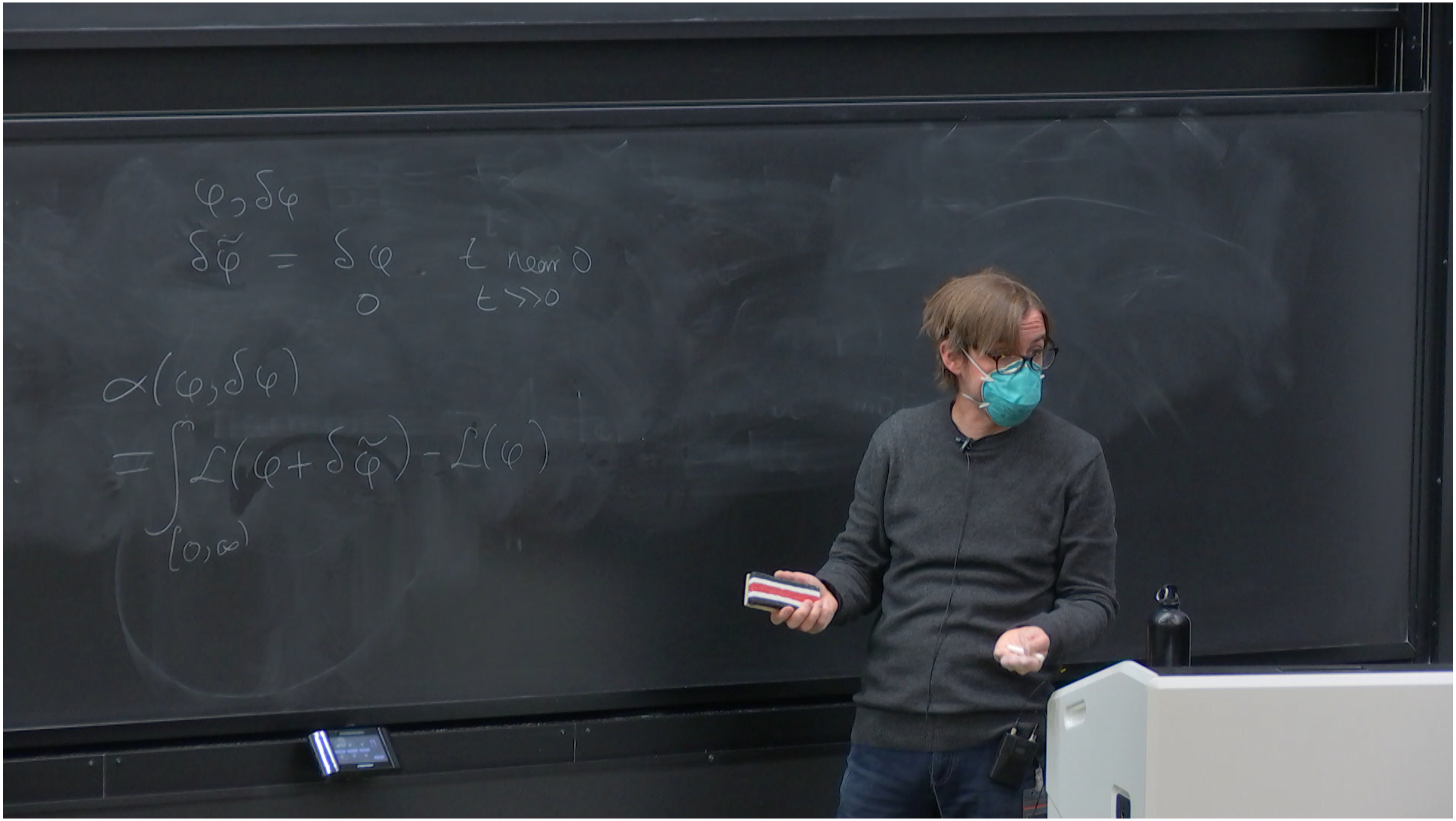
∂_t is a flow on the space of solns to EOM = phase space

Hamiltonian is the function so

$$X_H = \partial_t$$

Free field, $q = \varphi(0)$, $p = \partial_t \varphi(0)$

$\partial_t F = X F$ $\quad p = \partial_t q \quad X = p \partial_q$


$\varphi, \delta\varphi$
$\delta\tilde{\varphi} = \delta\varphi$. t near 0
0 $t \gg 0$
$\alpha(\varphi, \delta\varphi)$
$= \int_{[0,\infty)} \mathcal{L}(\varphi + \delta\tilde{\varphi}) - \mathcal{L}(\varphi)$

$$\int \varphi \partial_t^2 \varphi + m^2 \varphi^2$$

There's a v. field X which sends

$p \longrightarrow$ time derivative

$q \longrightarrow$ " "

$$q = \varphi(0)$$

$$p = \partial_t \varphi(0)$$

$$X = p\,\partial_q - m^2 q\,\partial_p$$

$$Xq = p$$

$$Xp = -m^2 q$$