Title: Mathematical Physics Lecture - 230112

Speakers: Giuseppe Sellaroli

Collection: Mathematical Physics (2022/2023)

Date: January 12, 2023 - 11:00 AM

URL: https://pirsa.org/23010013

Today: symplectic manifolds → framework of Hamiltonian dynamics

X vector field on M | $\gamma : I \subseteq_{\text{open}} \mathbb{R} \to M$ integral curve of X

the tangent vector to γ at point $\gamma(t)$ is $X_{\gamma(t)}$

folds → framework of Hamiltonian dynamics

$\gamma : I \subseteq_{\text{open}} \mathbb{R} \to M$ integral curve of X

the tangent vector to γ at point $\gamma(t)$ is $X_{\gamma(t)}$

$\dot{\gamma}(t)$ tangent vector at $\gamma(t)$
$\in T_{\gamma(t)} M$

$\dot{\gamma}(t) f = \frac{d}{dt}(f \circ \gamma)(t)$

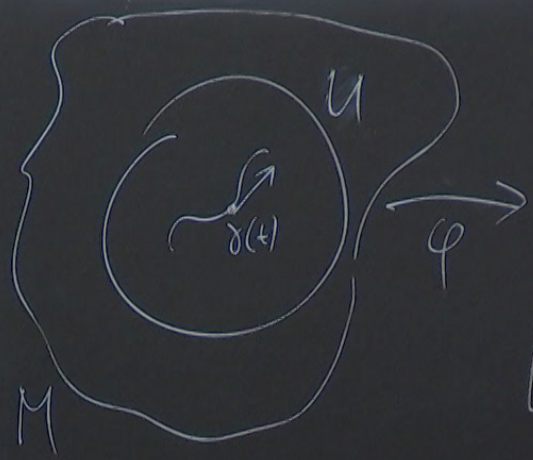
$\dot{\gamma}(t)$ tangent vector at $\gamma(t)$

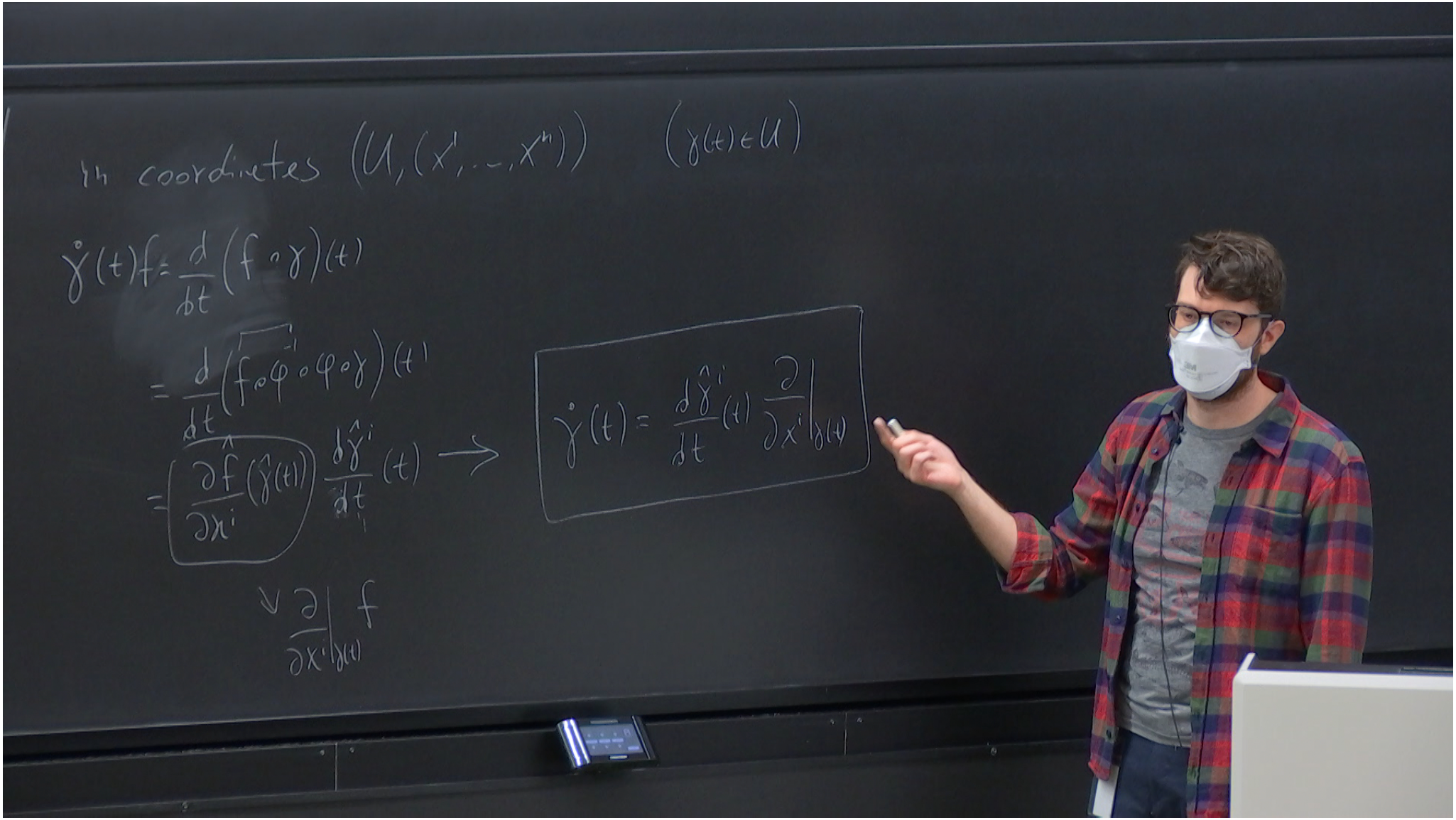
$\in T_{\gamma(t)} M$

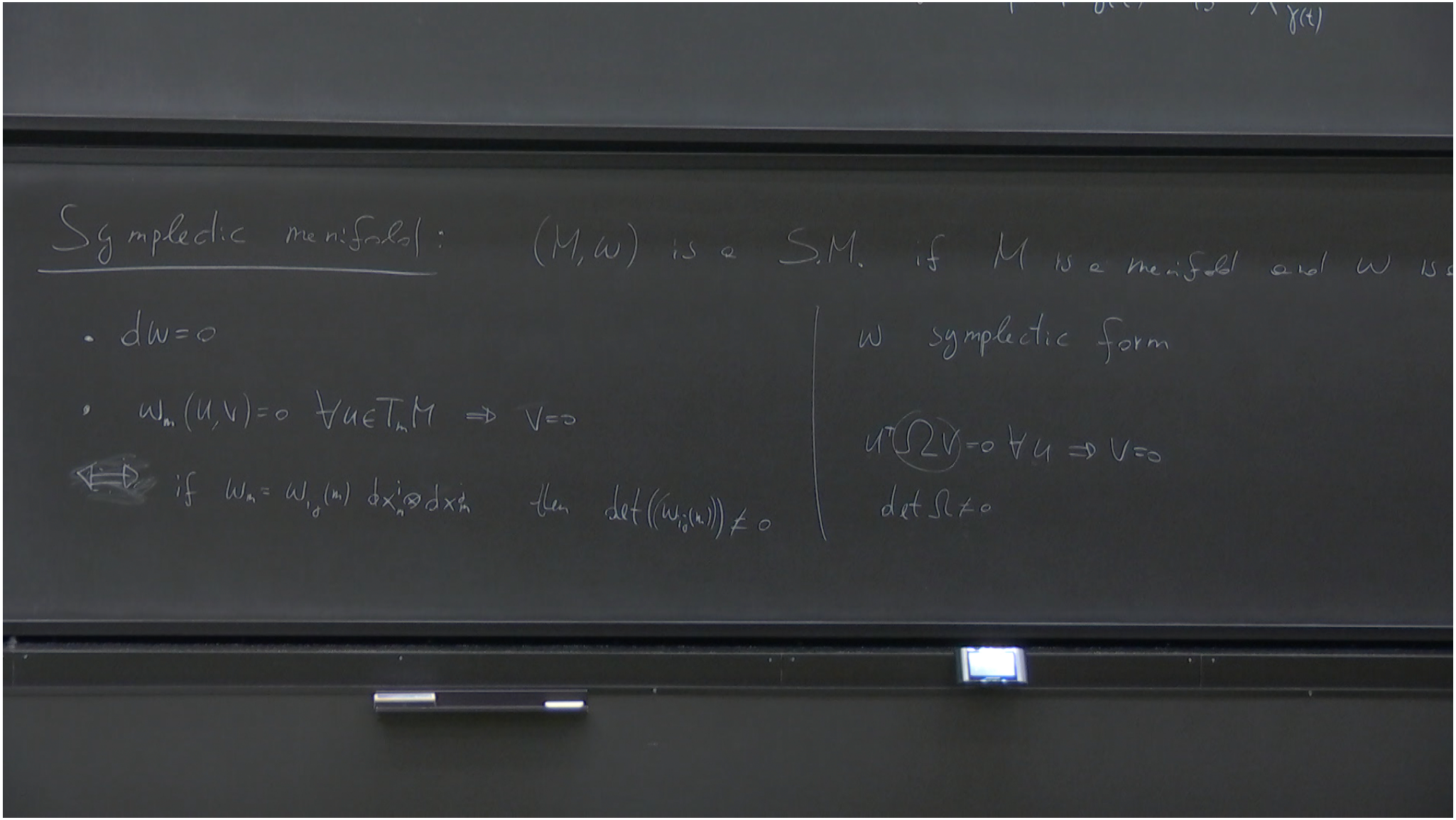
$$\dot{\gamma}(t) f = \frac{d}{dt}(f \circ \gamma)(t)$$

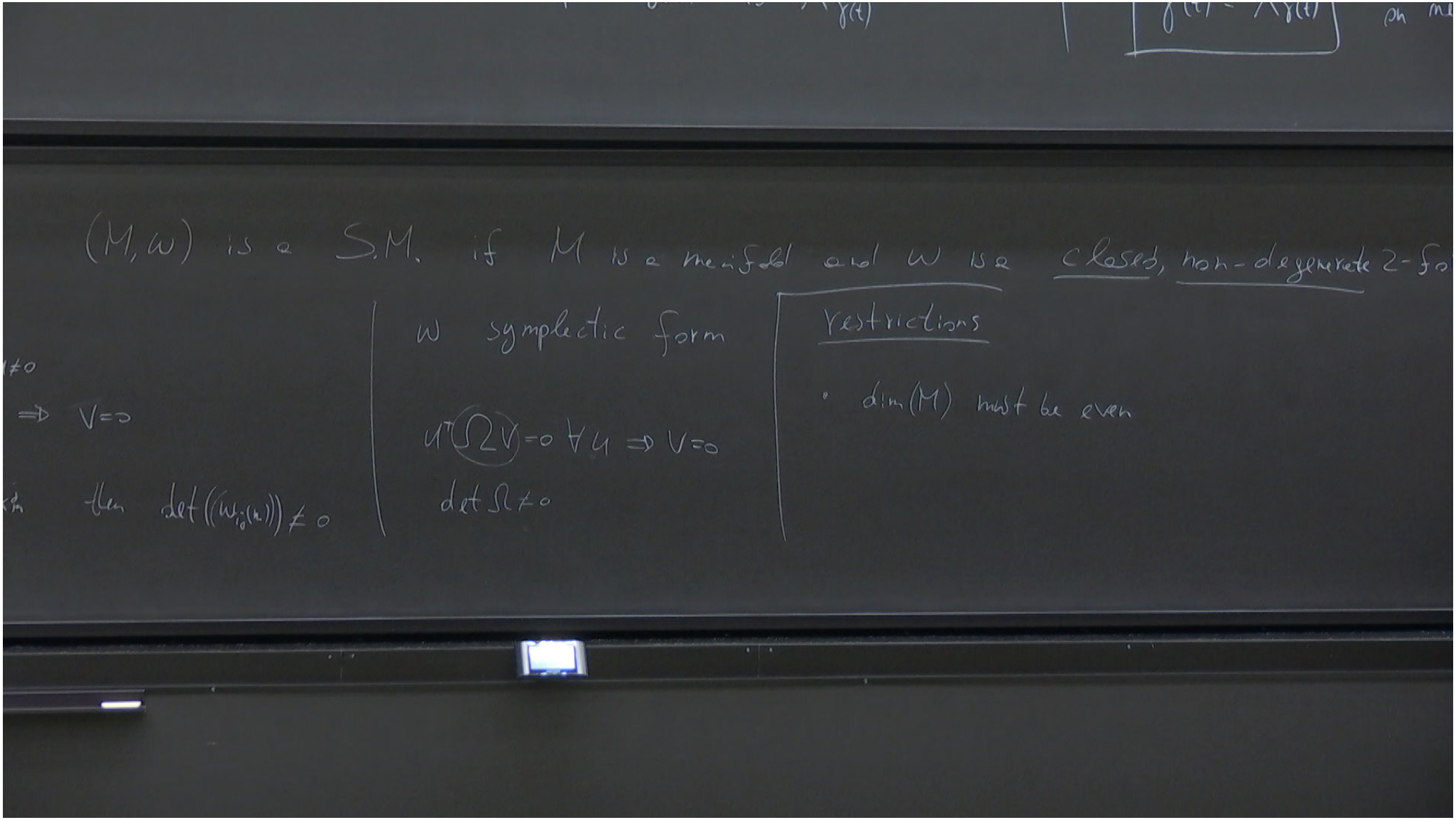
$$\boxed{\dot{\gamma}(t) = X_{\gamma(t)}}$$

1st order ODE on manifold




in coordinates $(U,(x^1,\dots,x^n))$ $(\gamma(t)\in U)$
$\dot\gamma(t)f = \frac{d}{dt}(f\circ\gamma)(t)$
$= \frac{d}{dt}(f\circ\varphi^{-1}\circ\varphi\circ\gamma)(t)$
$= \frac{\partial \hat f}{\partial x^i}(\hat\gamma(t))\frac{d\hat\gamma^i}{dt}(t) \rightarrow$
$\dot\gamma(t) = \frac{d\hat\gamma^i}{dt}(t)\frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$
$\downarrow \frac{\partial}{\partial x^i}\Big|_{\gamma(t)} f$


Symplectic manifold: (M, ω) is a S.M. if M is a manifold and ω is
• dω = 0
• ω_m(u, v) = 0 ∀u ∈ T_m M ⇒ v = 0
⇔ if ω_m = ω_{ij}(m) dx^i_m ⊗ dx^j_m then det((ω_{ij}(m))) ≠ 0
ω symplectic form
u^T Ω v = 0 ∀u ⇒ v = 0
det Ω ≠ 0


(M, ω) is a S.M. if M is a manifold and ω is a closed, non-degenerate 2-fo
ω symplectic form
$u^T \Omega v = 0 \ \forall u \Rightarrow v = 0$
$\det \Omega \neq 0$
Restrictions
• $\dim(M)$ must be even
$\neq 0$
$\Rightarrow v = 0$
then $\det((\omega_{ij}(x))) \neq 0$

is a S.M. if M is a manifold and ω is a closed, non-degenerate 2-form on M

ω symplectic form

$u^T \Omega v = 0 \ \forall u \Rightarrow v = 0$

$\det \Omega \neq 0$

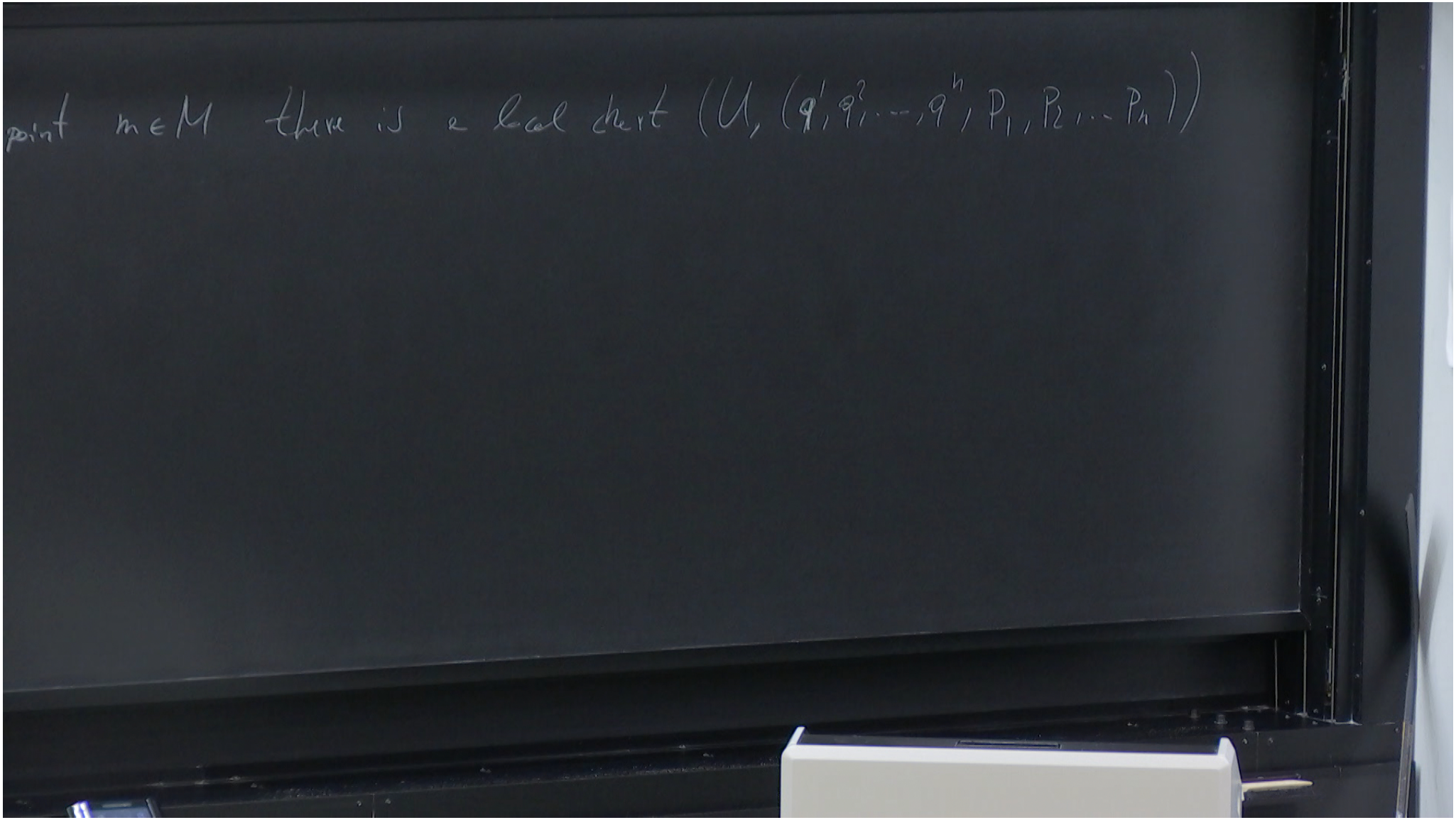
$((\omega_{ij}(x))) \neq 0$

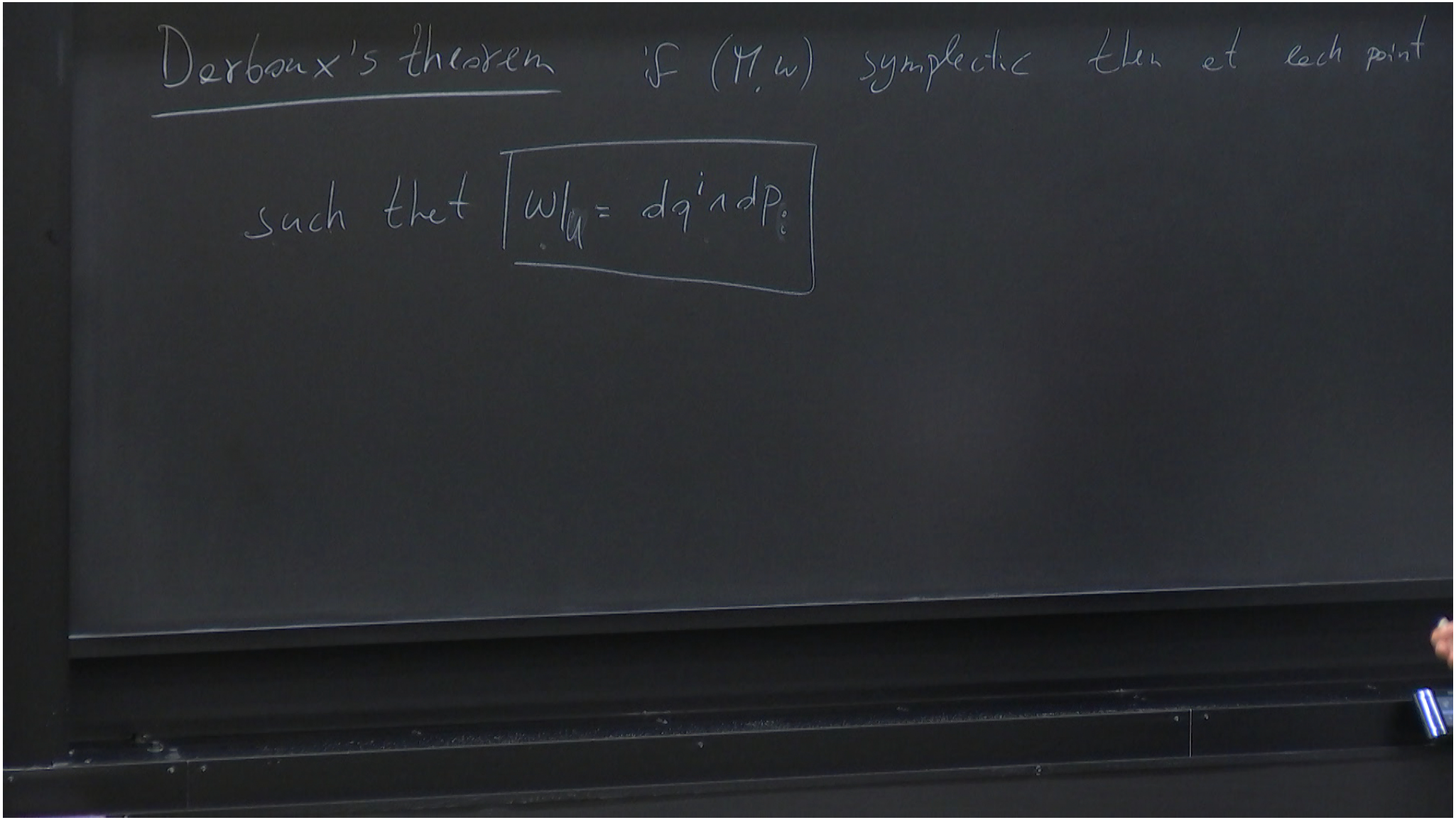
Restrictions

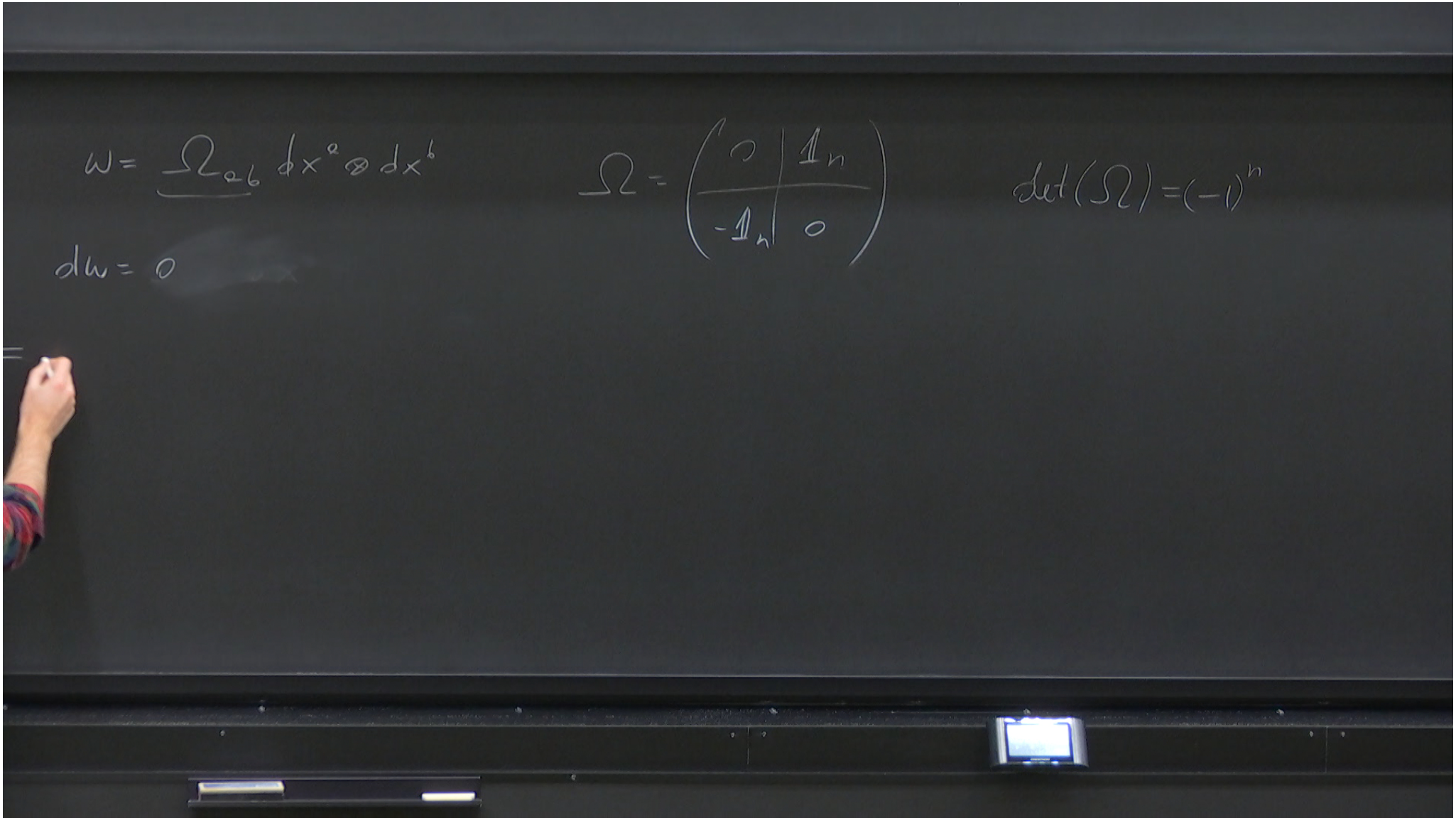
- $\dim(M)$ must be even

$\Omega = (\omega_{ij}) \quad \Omega^t = -\Omega \Rightarrow \det(\Omega^t) = \det(\Omega)$

$(-1)^n \det(\Omega)$


point $m \in M$ there is a local chart $(U, (q^1, q^2, \ldots, q^n, p_1, p_2, \ldots p_n))$


Darboux's theorem if (M, ω) symplectic then at each point
such that $\omega|_U = dq^i \wedge dp_i$


$\omega = \Omega_{ab}\, dx^a \otimes dx^b$
$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$
$\det(\Omega) = (-1)^n$
$d\omega = 0$

Ex $\mathbb{R}^{2n}$ $\omega = \Omega_{ab}\, dx^a \otimes dx^b$ $\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}$ $\det(S$

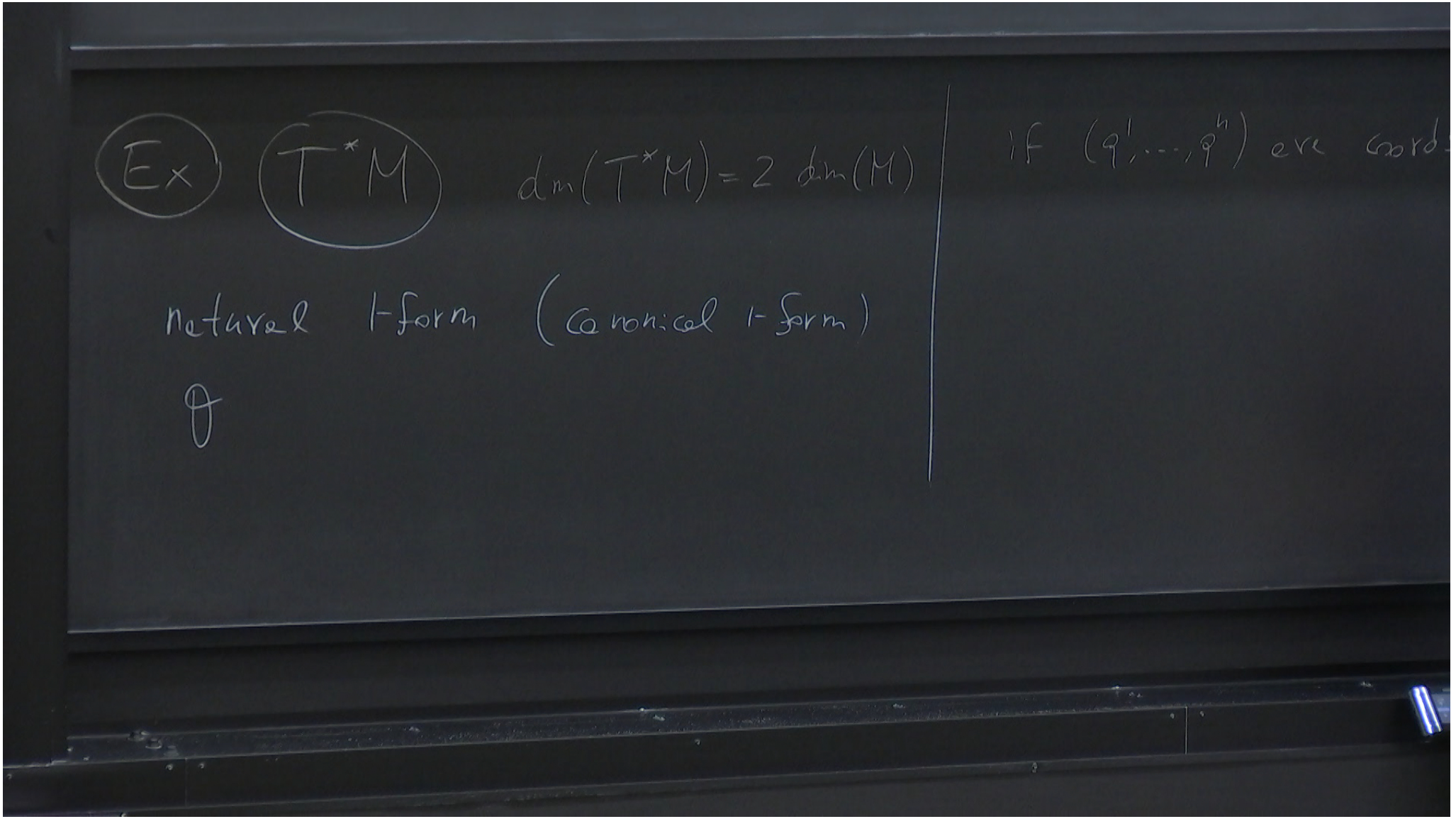
$d\omega = 0$

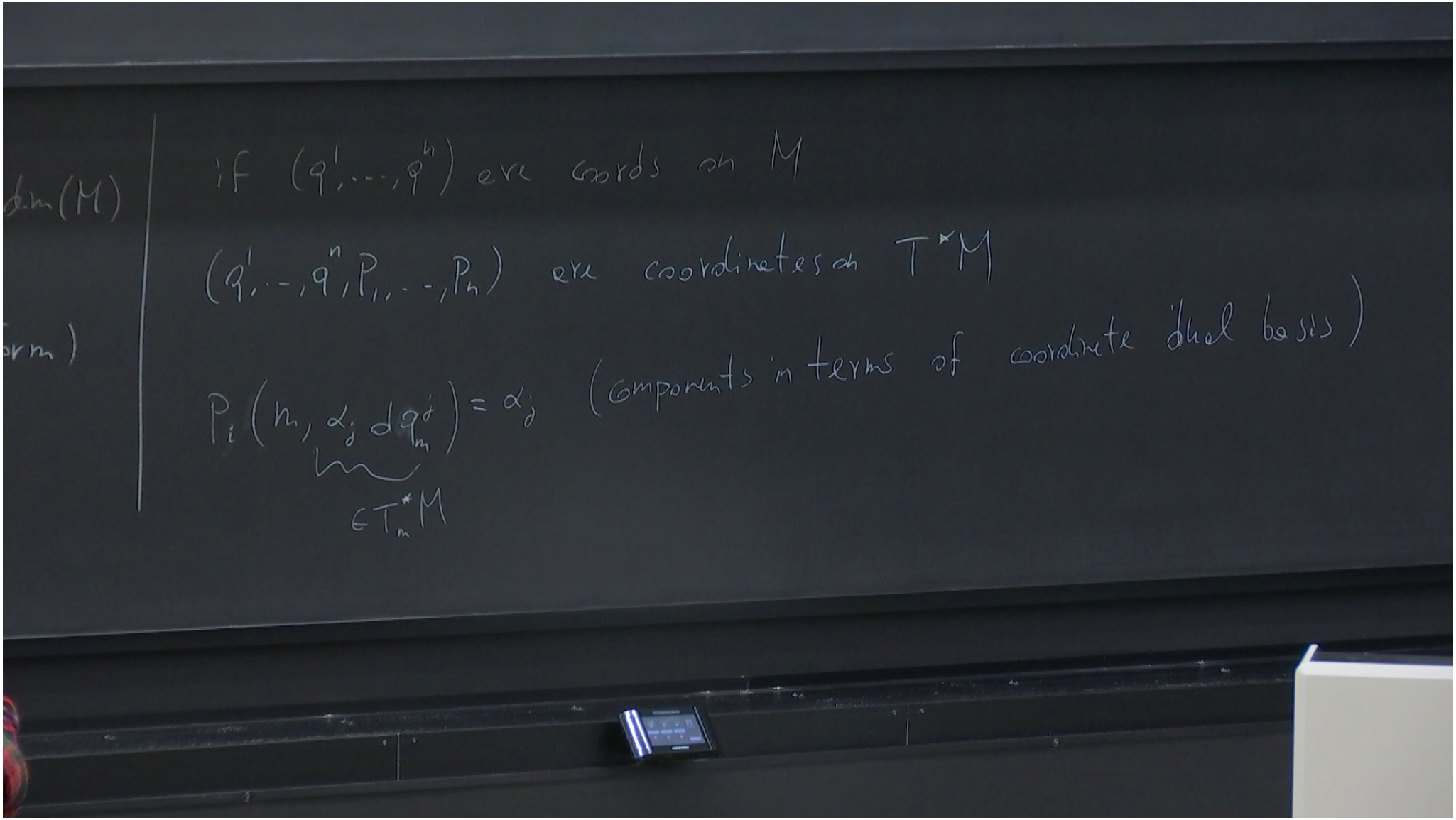
$(x^1, \dots, x^{2n}) = (q^1, \dots, q^n, p_1, \dots, p_n)$

$$\det(\Omega) = 1 \quad \Big| \quad \omega = \sum_{a=1}^{n} \sum_{b=n+1}^{2n} \overset{\delta_{a,b-n}}{\swarrow} dx^a \otimes dx^b - \sum_{a=n+1}^{2n} \sum_{b=1}^{n} \overset{\delta_{a-n,b}}{\swarrow} dx^a \otimes dx^b$$

$$= \sum_{a=1}^{n} \sum_{b=n+1}^{2n} \left(dx^a \otimes dx^b - dx^b \otimes dx^a \right) \delta_{a,b-n}$$

$$= \sum \sum dx^a \wedge dx^b \, \delta_{a,b-n} = dq^i \wedge dp_i$$


Ex
T^*M
$\dim(T^*M) = 2\dim(M)$
if $(q^1, \dots, q^n)$ are coord
natural 1-form (canonical 1-form)
θ


dim(M)
orm)
if $(q^1,\dots,q^n)$ are coords on M
$(q^1,\dots,q^n,P_1,\dots,P_n)$ are coordinates on T^*M
$P_i(m, \alpha_j\, dq^j_m) = \alpha_j$ (components in terms of coordinate dual basis)
$\in T^*_m M$

if $(q^1, \dots, q^n)$ are coords on M

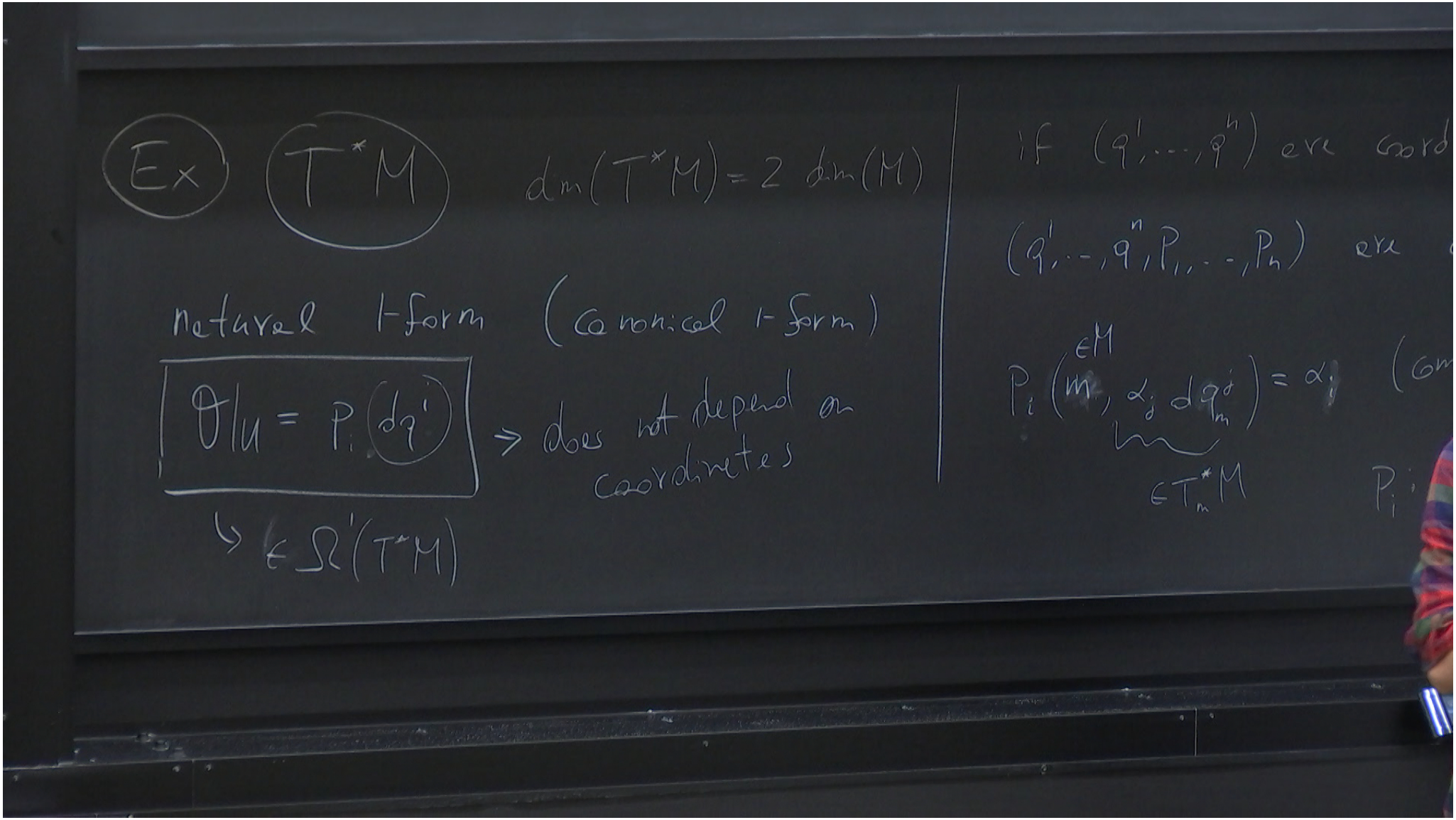
$(q^1, \dots, q^n, P_1, \dots, P_n)$ are coordinates on T^*M

$P_i\left(m, \alpha_j \, dq^j_m\right) = \alpha_i$ (components in terms of coordinate dual basis)

$\in M$

$\alpha_j \, dq^j_m \in T^*_m M$

$P_i : T^*M \to \mathbb{R}$


Ex
T^*M
$\dim(T^*M) = 2\dim(M)$
natural 1-form (canonical 1-form)
$\theta|_U = p_i\,dq^i$
$\Rightarrow$ does not depend on coordinates
$\hookrightarrow \in \Omega^1(T^*M)$
if $(q^1,\dots,q^n)$ are coord
$(q^1,\dots,q^n,p_1,\dots,p_n)$ are
$p_i(m, \alpha_j\,dq^j_m) = \alpha_i$
$m \in M$
$\alpha_j\,dq^j_m \in T^*_m M$

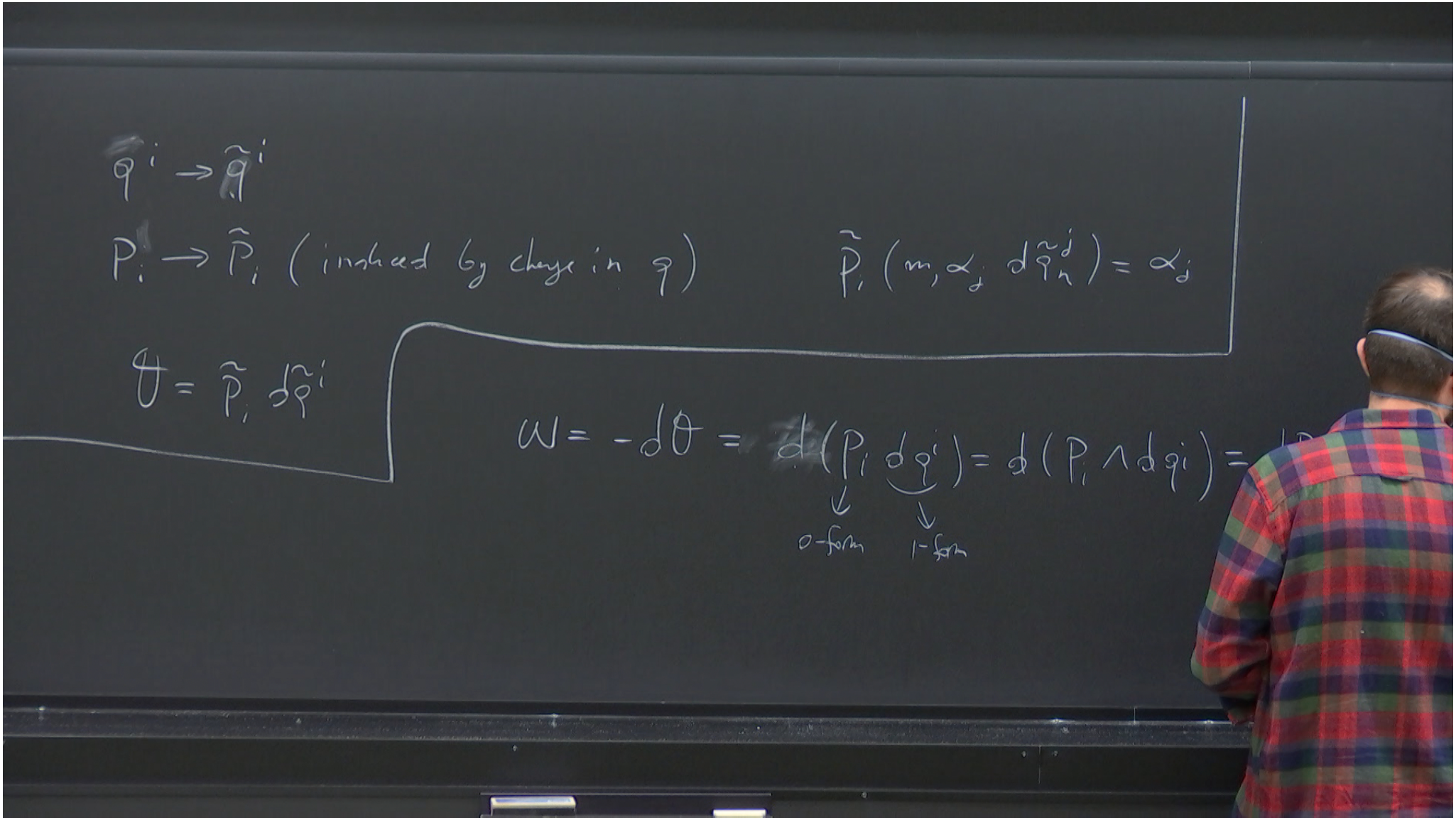
$$\det(\Omega) = 1$$

$$W = \sum_{a=1}^{n} \sum_{b=n+1}^{2n} \underbrace{\delta_{a,b-n}} dx^a \otimes dx^b - \sum_{a=n+1}^{2n} \sum_{b=1}^{n} \underbrace{\delta_{a-n,b}} dx^a \otimes dx^b$$

$$= \sum_{a=1}^{n} \sum_{b=n+1}^{2n} \left(dx^a \otimes dx^b - dx^b \otimes dx^a \right) \delta_{a,b-n}$$

$$= \sum \sum dx^a \wedge dx^b \, \delta_{a,b-n} = \boxed{dq^i \wedge dp_i} \qquad dW = 0$$

$$dq^i \otimes dp_i - dp_i \otimes dq^i$$


$q^i \to \tilde{q}^i$
$P_i \to \tilde{P}_i$ (induced by change in q)
$\tilde{P}_i(m, \alpha_j\, d\tilde{q}^j_n) = \alpha_j$
$\theta = \tilde{P}_i\, d\tilde{q}^i$
$\omega = -d\theta = d(P_i\, dq^i) = d(P_i \wedge dq_i) =$
0-form
1-form

change in q) $\tilde{P}_i\left(m, \alpha_j\, d\tilde{q}^j_n\right) = \alpha_i$

$P_i : (m,$

$$\omega = -d\theta = -d\left(P_i\, dq^i\right) = -d\left(P_i \wedge dq^i\right) = -\left(dP_i \wedge dq^i + P_i \cancel{d^2 q^i}\right) = \boxed{dq^i \wedge dP_i}$$

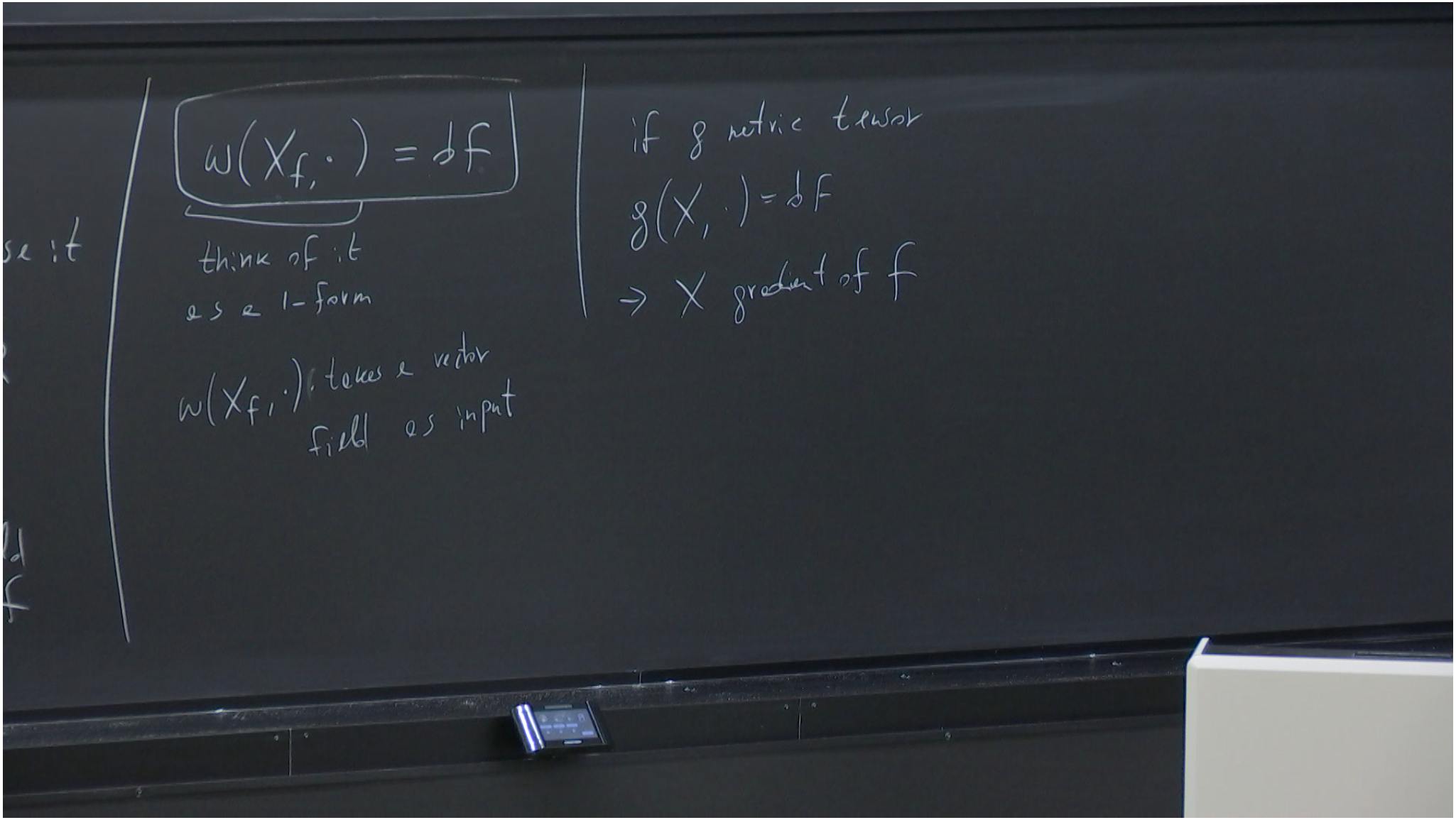
P_i: 0-form dq^i: 1-form

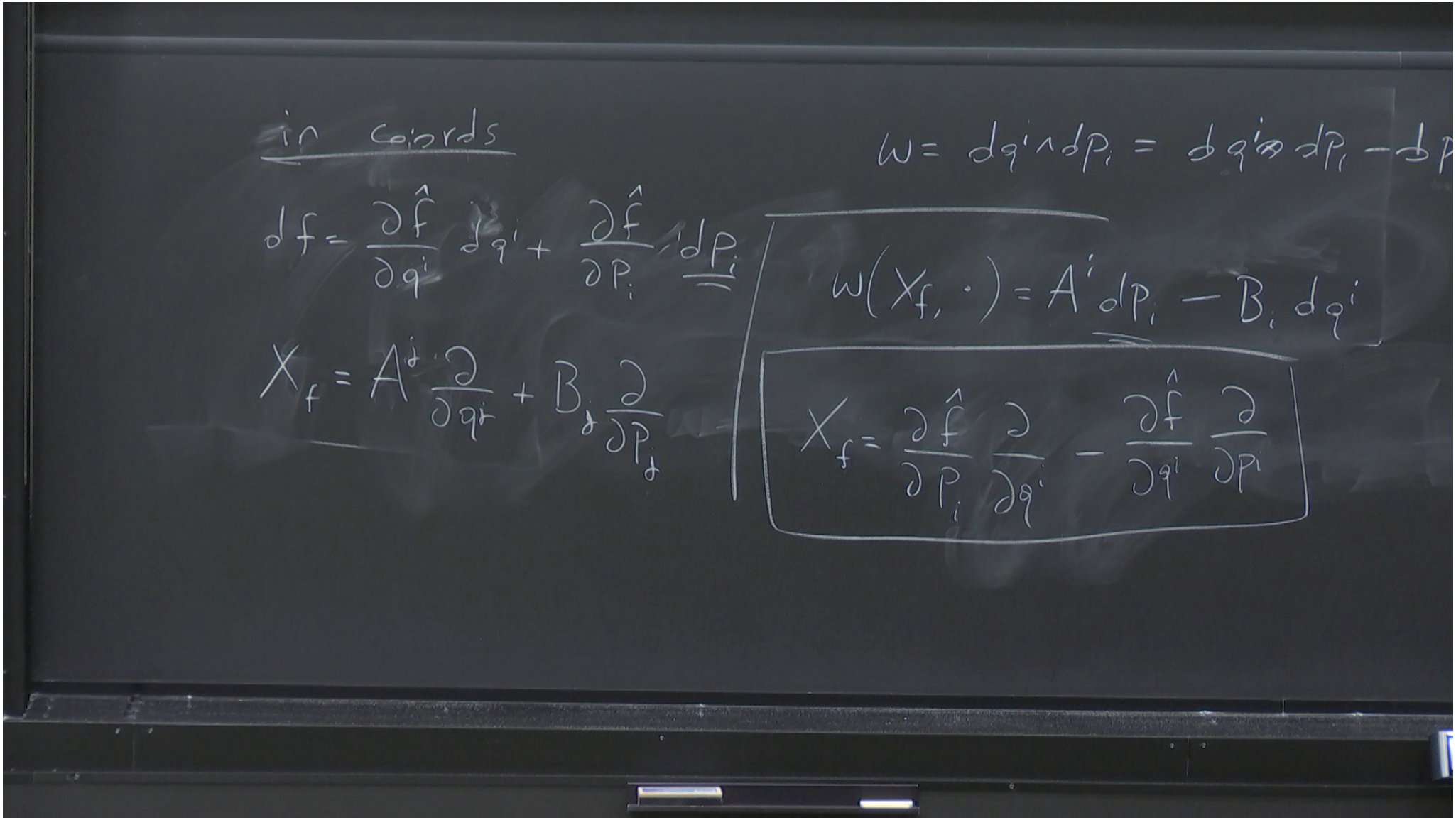
Hamiltonian vector fields

if we have a symplectic form ω, we use it

to associate to a function $f : M \to \mathbb{R}$

a vector field $X_f : M \to TM$


$\omega(X_f, \cdot) = df$
think of it as a 1-form
$\omega(X_f, \cdot)$: takes a vector field as input
if g metric tensor
$g(X, \cdot) = df$
$\rightarrow X$ gradient of f


in coords
$\omega = dq^i \wedge dP_i = dq^i \otimes dP_i - dP$
$df = \frac{\partial \hat{f}}{\partial q^i} dq^i + \frac{\partial \hat{f}}{\partial P_i} dP_i$
$\omega(X_f, \cdot) = A^i dP_i - B_i dq^i$
$X_f = A^i \frac{\partial}{\partial q^i} + B_j \frac{\partial}{\partial P_j}$
$X_f = \frac{\partial \hat{f}}{\partial P_i} \frac{\partial}{\partial q^i} - \frac{\partial \hat{f}}{\partial q^i} \frac{\partial}{\partial P_i}$

$dp_i - \frac{\partial}{\partial p_i} \otimes dq^i$

dq^i

$dq^i\left(\frac{\partial}{\partial q^j}\right) = \delta^i_j$

$dq^i\left(\frac{\partial}{\partial p_j}\right) = 0$

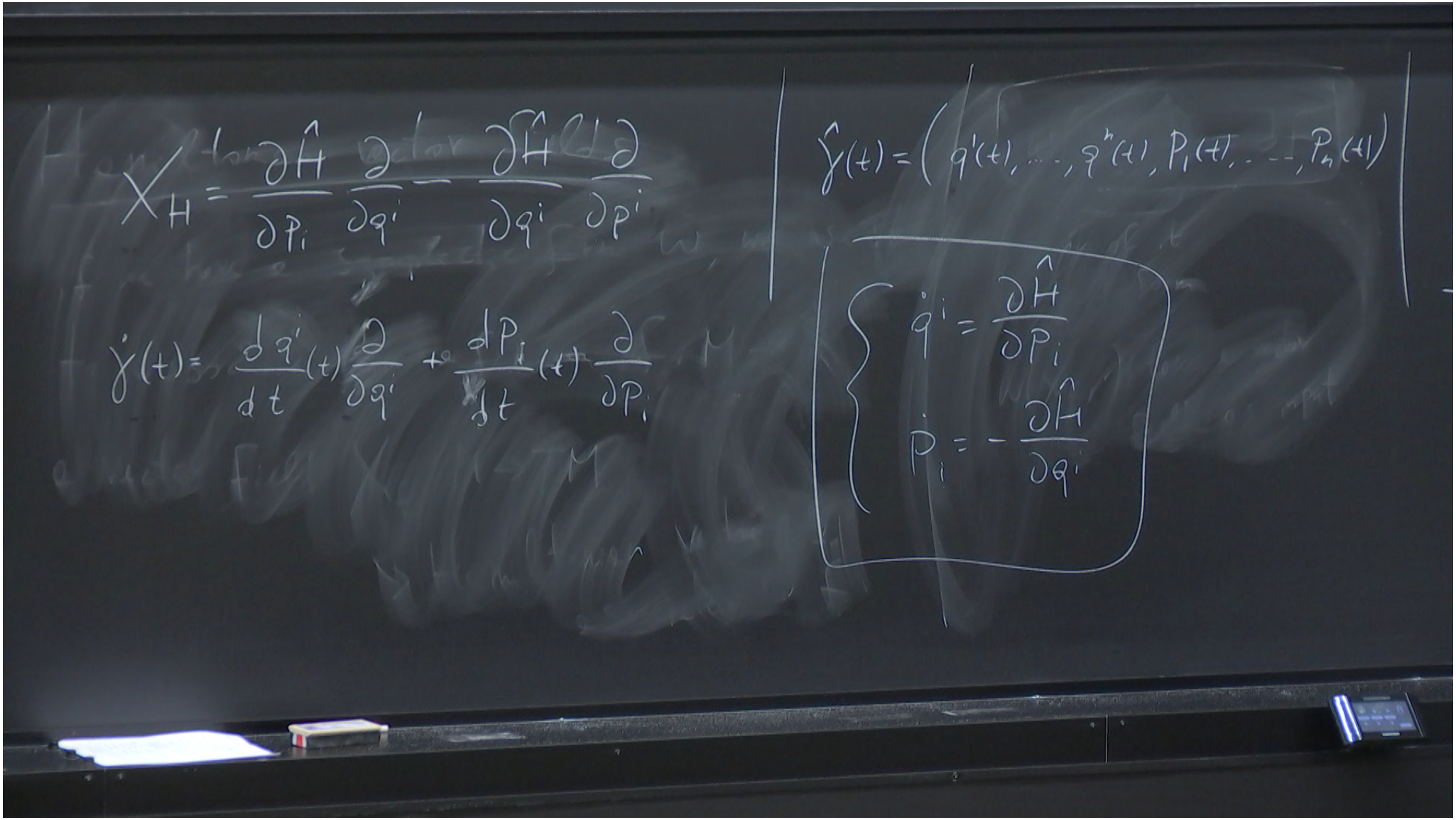
Select a special function: Hamiltonian $H : M \to \mathbb{R}$

$\to X_H$

physical path: integral curve of X_H

$$\dot{\gamma}(t) = X_H|_{\gamma(t)}$$

Hamilton's eq


$X_H = \frac{\partial \hat{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial \hat{H}}{\partial q^i}\frac{\partial}{\partial p^i}$
$\dot{\gamma}(t) = \frac{dq^i}{dt}(t)\frac{\partial}{\partial q^i} + \frac{dp_i}{dt}(t)\frac{\partial}{\partial p_i}$
$\hat{\gamma}(t) = (q^1(t), \dots, q^n(t), p_1(t), \dots, p_n(t))$
$\begin{cases} \dot{q}^i = \frac{\partial \hat{H}}{\partial p_i} \\ \dot{p}_i = -\frac{\partial \hat{H}}{\partial q^i} \end{cases}$