Title: Mathematical Physics Lecture - 230110

Speakers: Giuseppe Sellaroli

Collection: Mathematical Physics (2022/2023)

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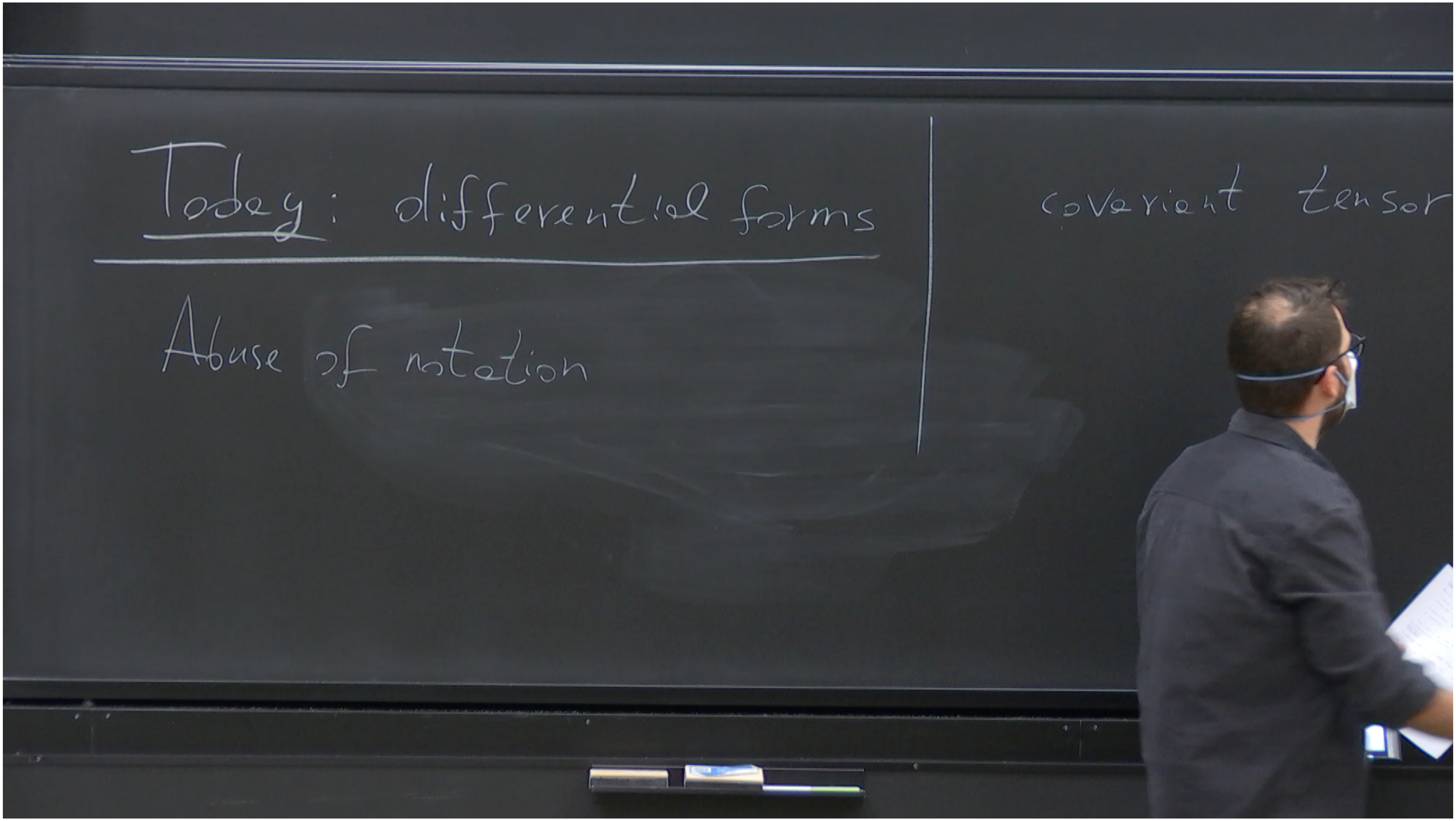
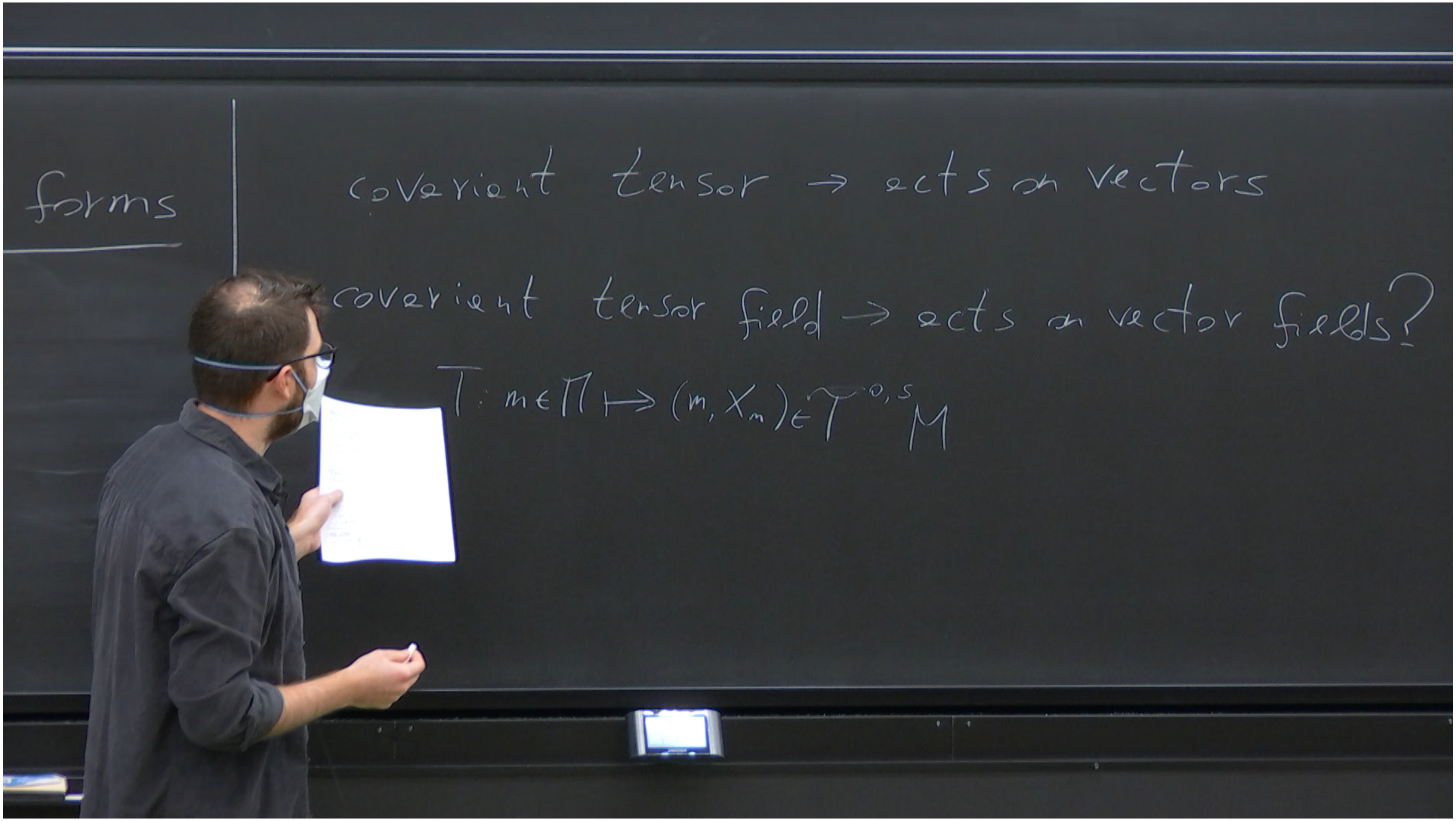


Today: differential forms

Abuse of notation

covariant tensor

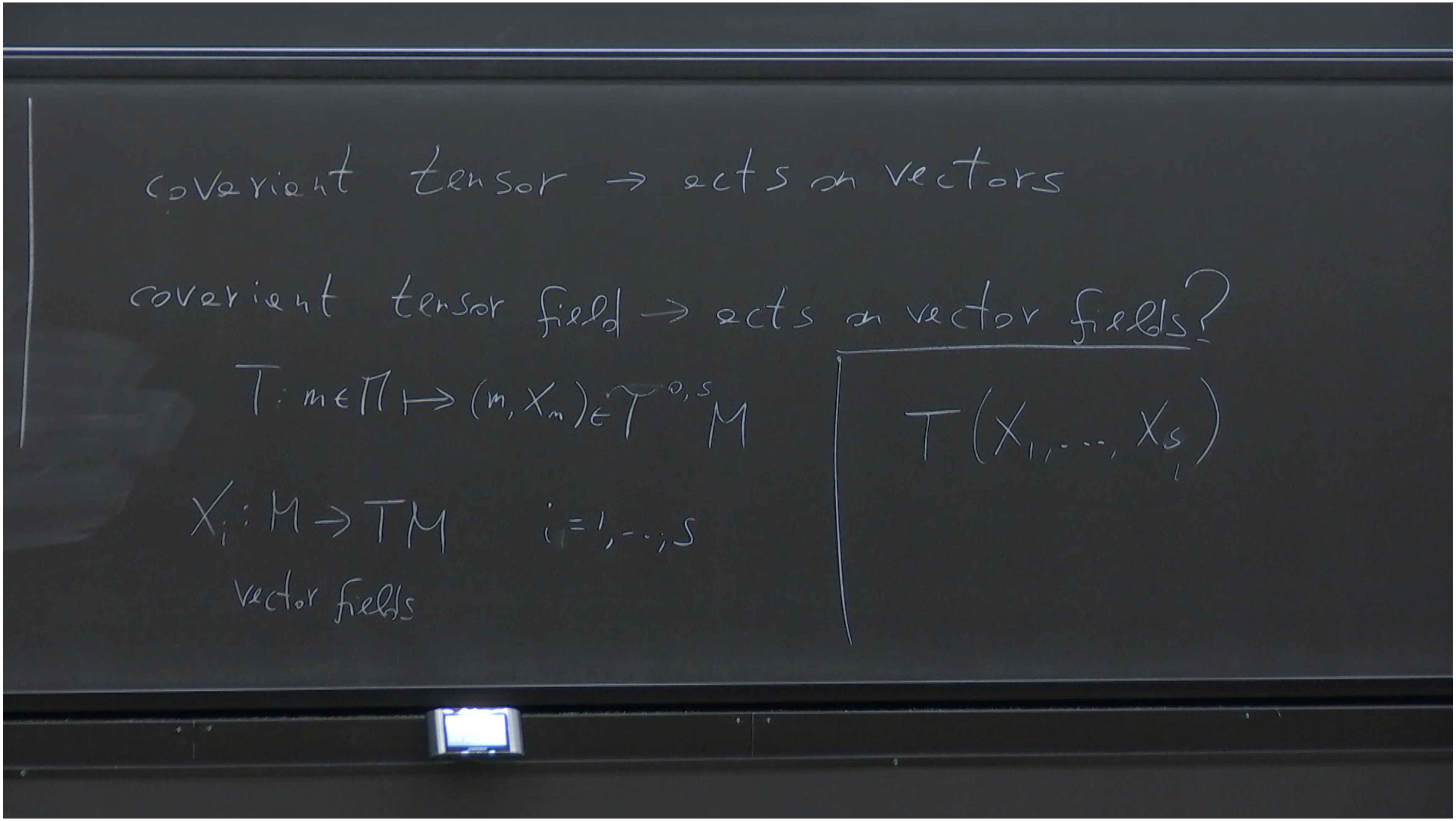


forms

covariant tensor → acts on vectors

covariant tensor field → acts on vector fields?

$T : m \in M \longmapsto (m, X_m) \in T^{0,s}M$



covariant tensor → acts on vectors

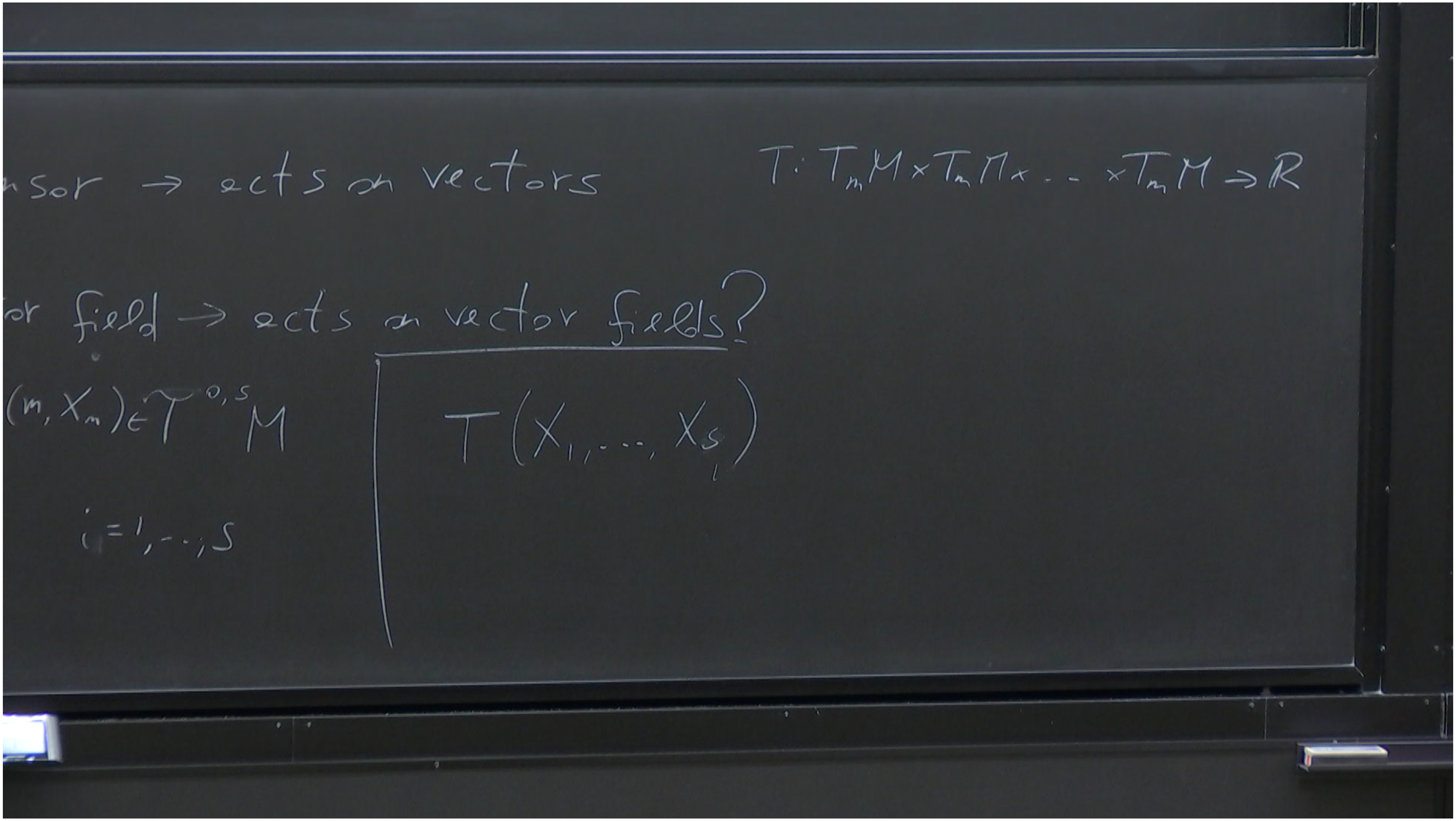
covariant tensor field → acts on vector fields?

$T : m \in M \mapsto (m, X_m) \in T^{0,s} M$

$X_i : M \to TM \qquad i = 1, \dots, s$

vector fields

$T(X_1, \dots, X_s)$



nsor → acts on vectors

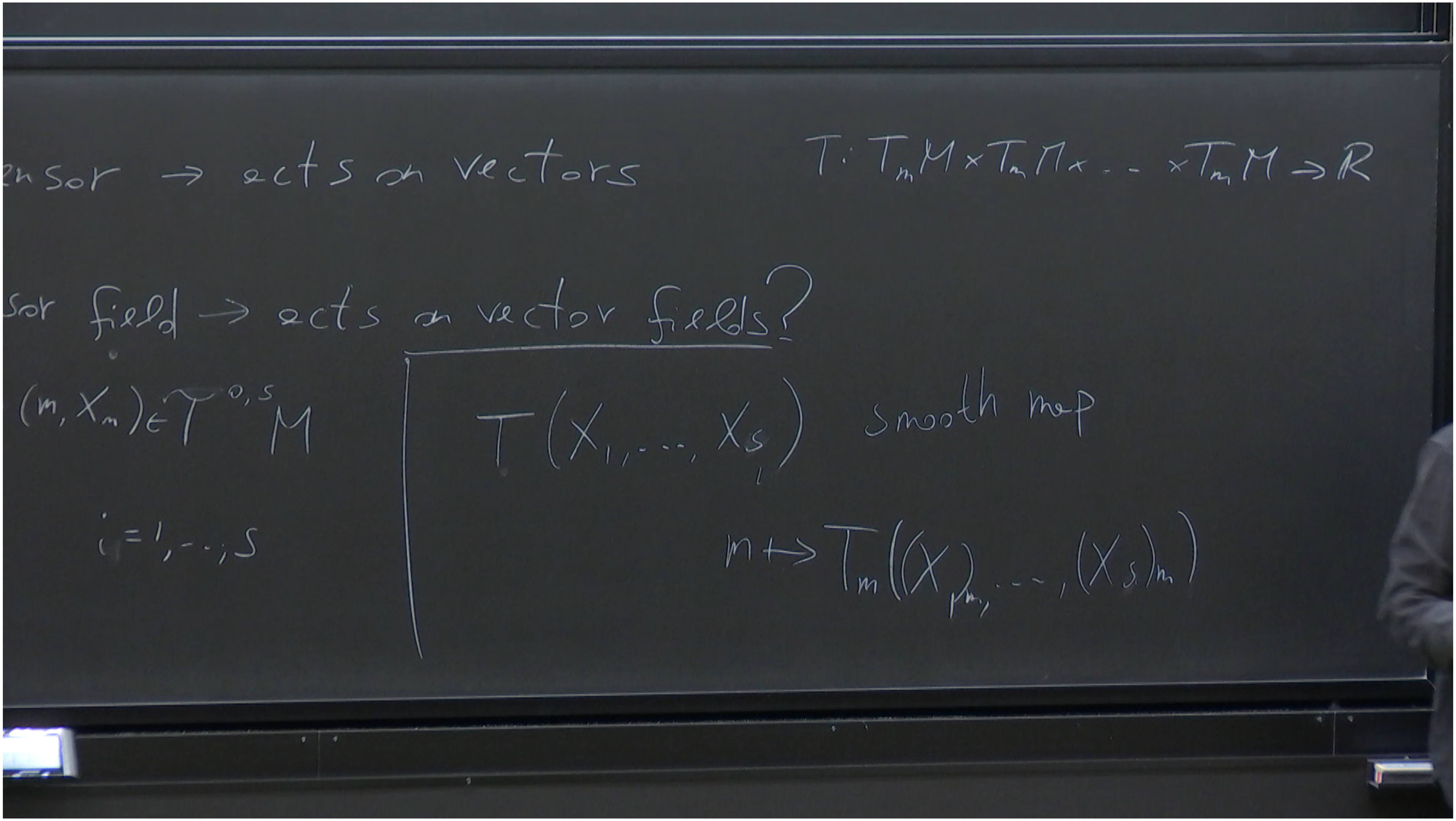
$T : T_mM \times T_mM \times \ldots \times T_mM \to \mathbb{R}$

or field → acts on vector fields?

$(m, X_m) \in T^{0,s}M$

$T(X_1, \ldots, X_s)$

$i = 1, \ldots, s$



ensor → acts on vectors

$T : T_mM \times T_mM \times \dots \times T_mM \to R$

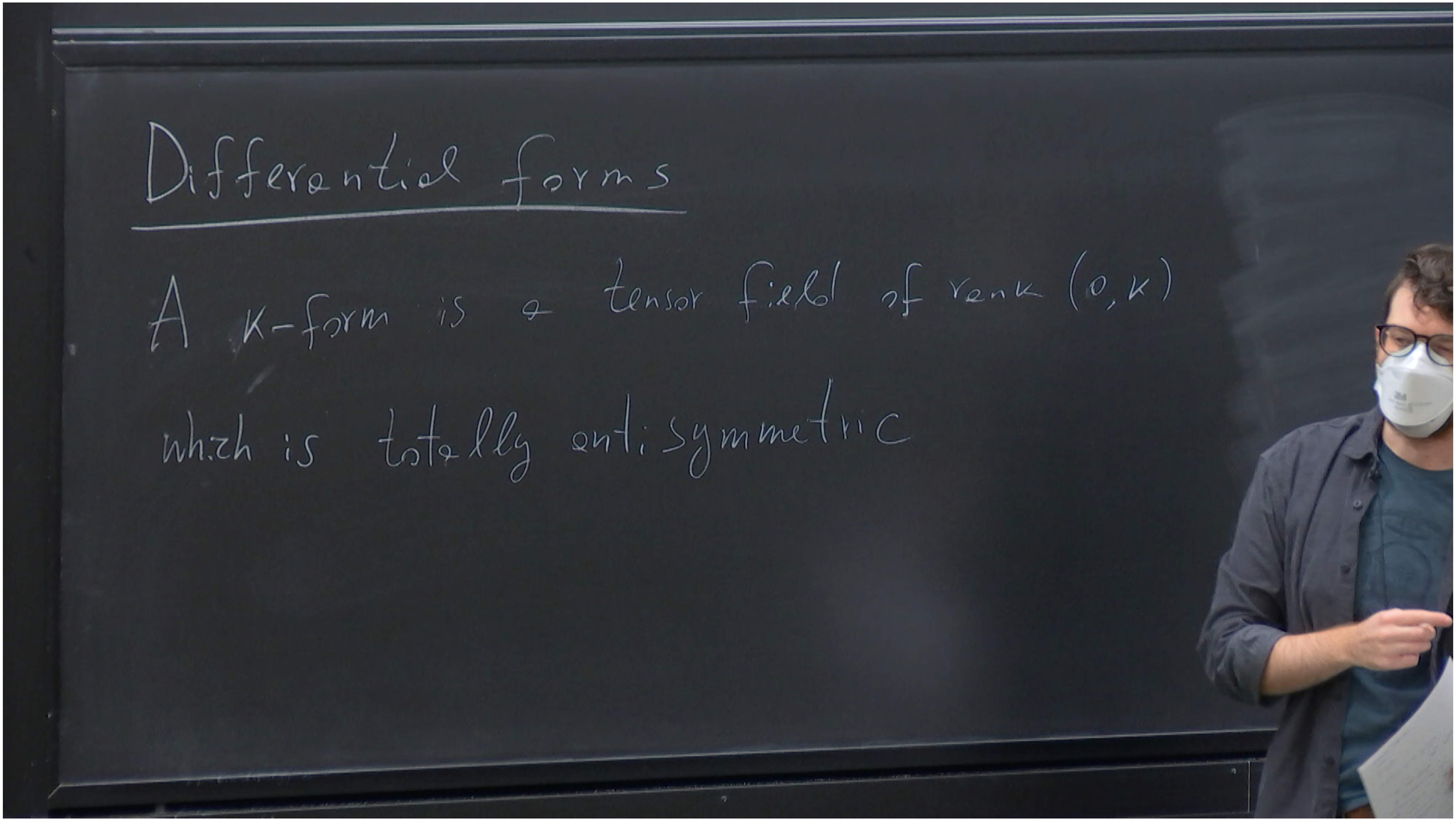
sor field → acts on vector fields?

$(m, X_m) \in T^{0,s}M$

$i = 1, \dots, s$

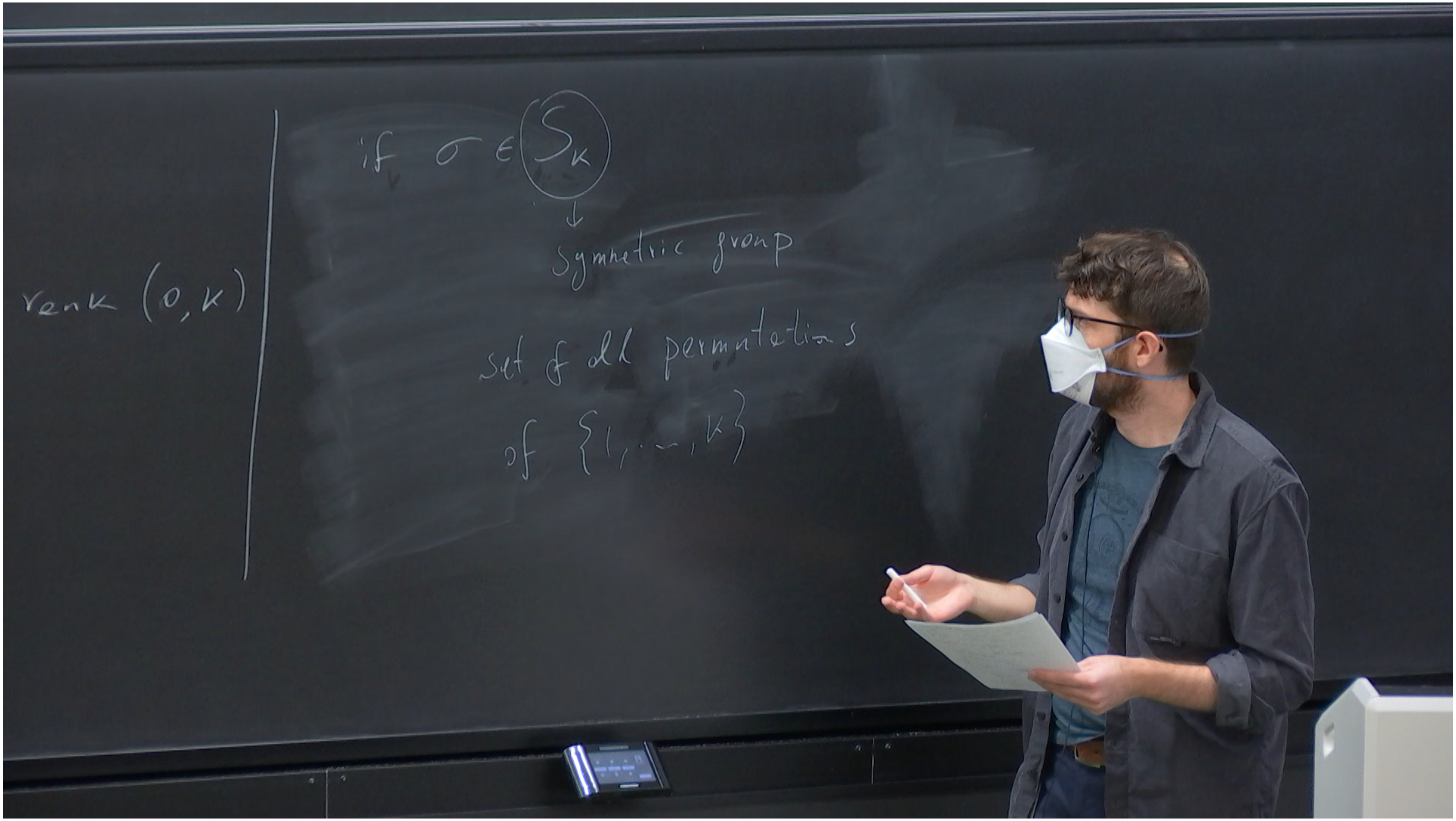
$T(X_1, \dots, X_s)$ smooth map

$m \mapsto T_m((X_1)_m, \dots, (X_s)_m)$



Differential forms

A k-form is a tensor field of rank $(0,k)$ which is totally antisymmetric



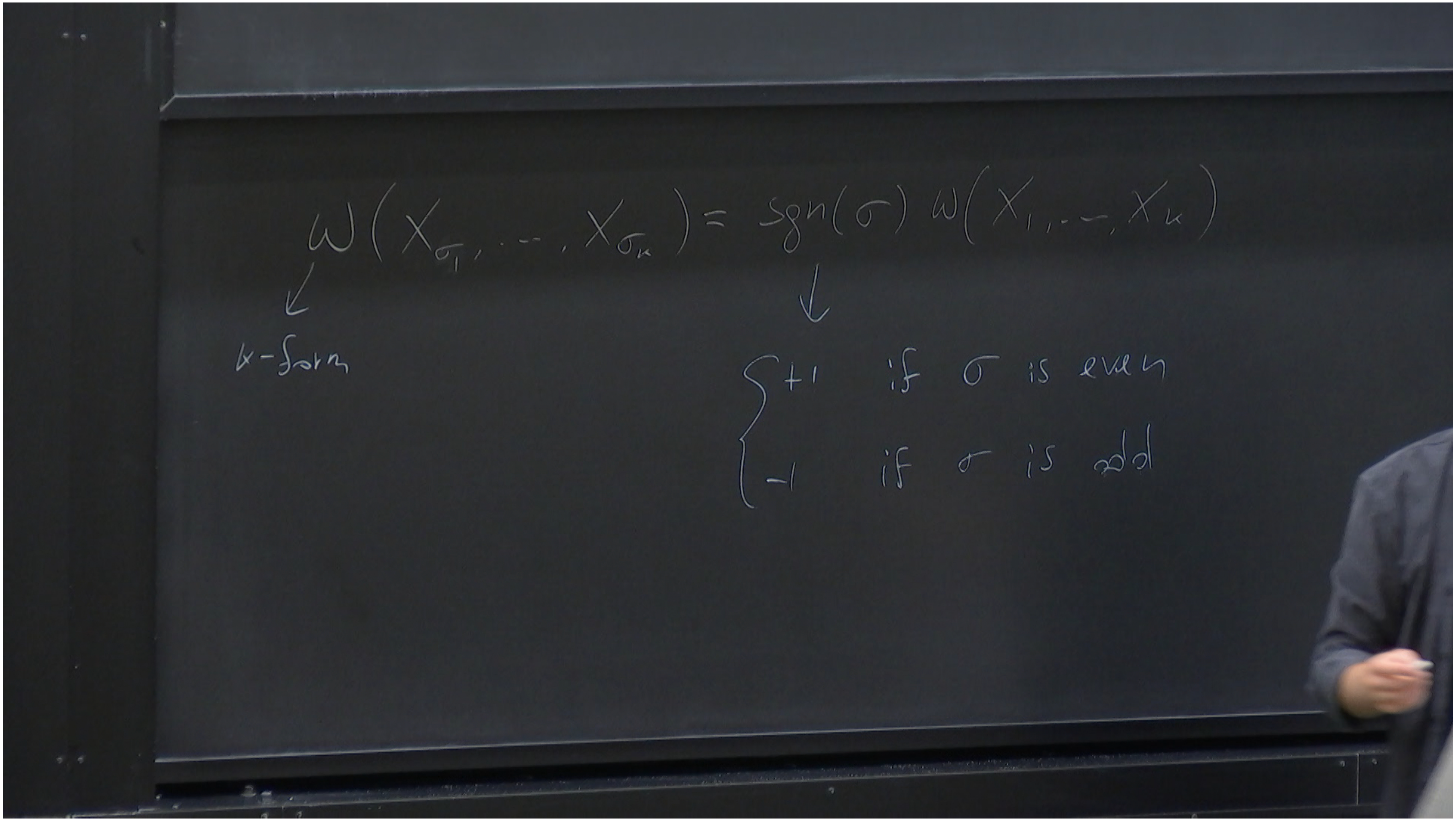
rank $(0,k)$

if $\sigma \in \boxed{S_k}$

$\downarrow$

symmetric group

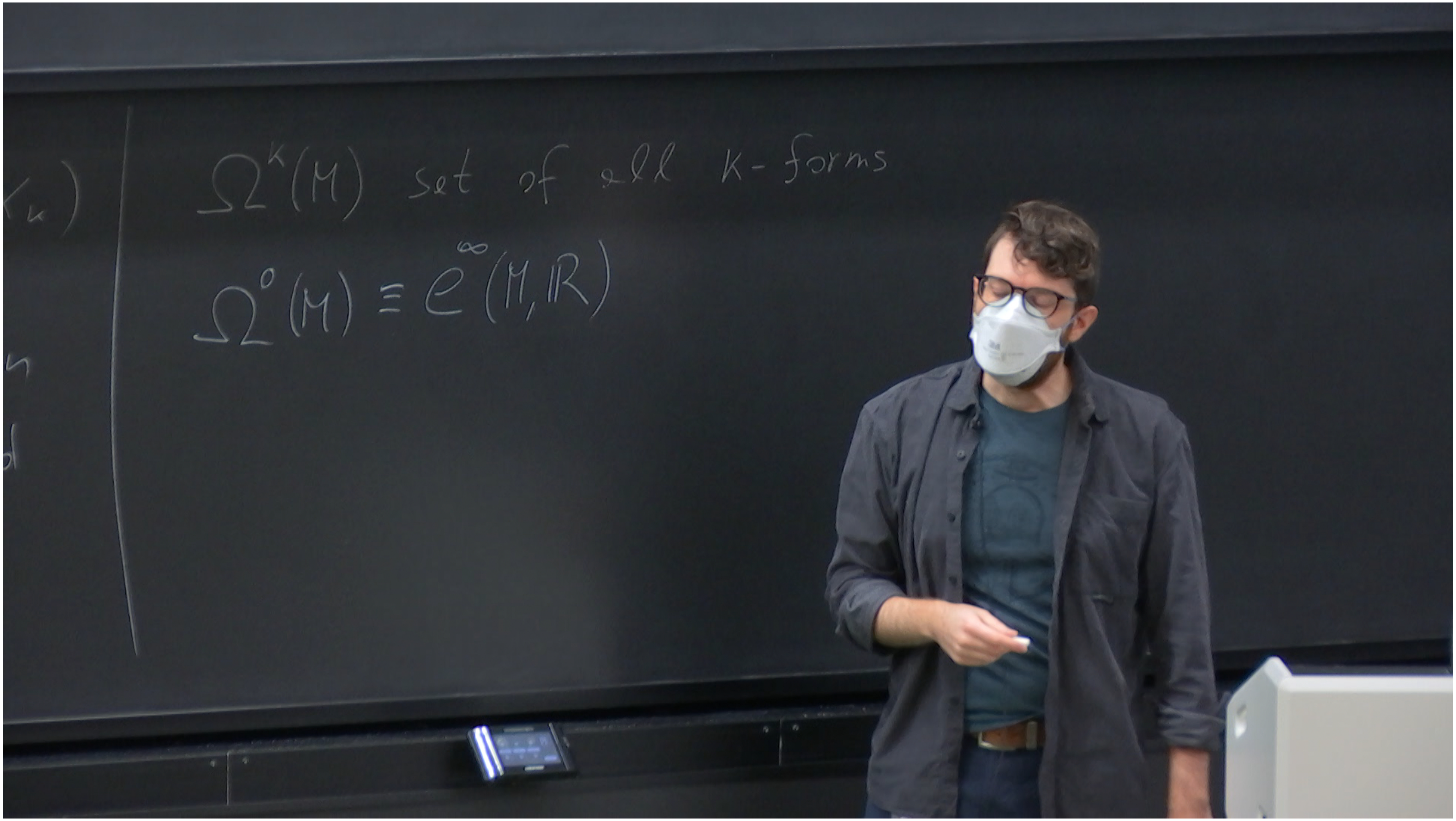
set of all permutations of $\{1,\ldots,k\}$



$$\omega(X_{\sigma_1},\dots,X_{\sigma_k}) = \mathrm{sgn}(\sigma)\,\omega(X_1,\dots,X_k)$$

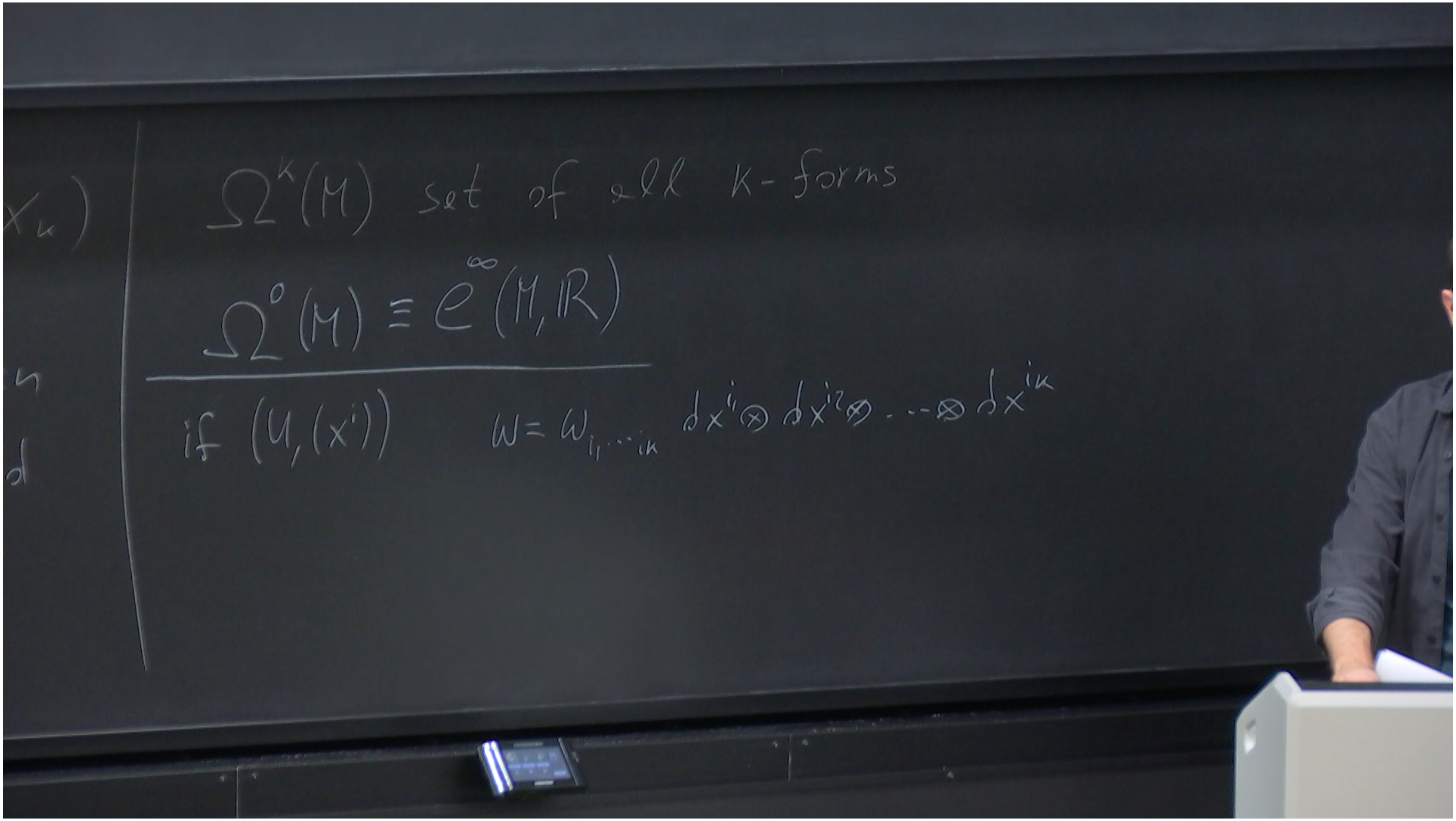
k-form

$$\mathrm{sgn}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$



$\Omega^k(M)$ set of all k-forms

$\Omega^0(M) \equiv \mathcal{C}^\infty(M,\mathbb{R})$



$X_k)$

n

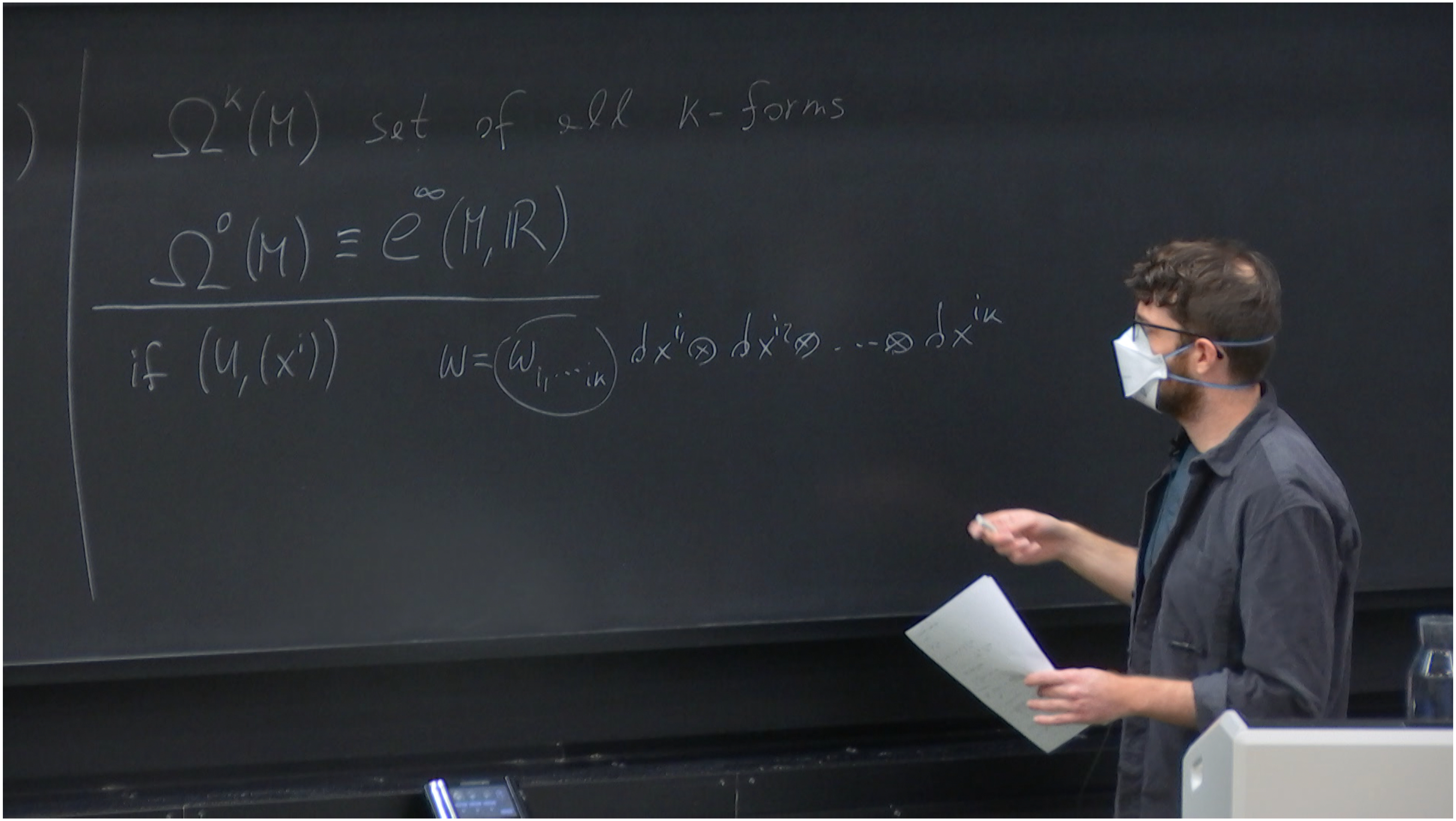
d

$\Omega^k(M)$ set of all k-forms

$\Omega^0(M) \equiv \mathcal{C}^\infty(M,\mathbb{R})$

---

if $(U,(x^i))$ $\quad \omega = \omega_{i_1\dots i_k}\, dx^{i_1} \otimes dx^{i_2} \otimes \dots \otimes dx^{i_k}$



$\Omega^k(M)$ set of all k-forms

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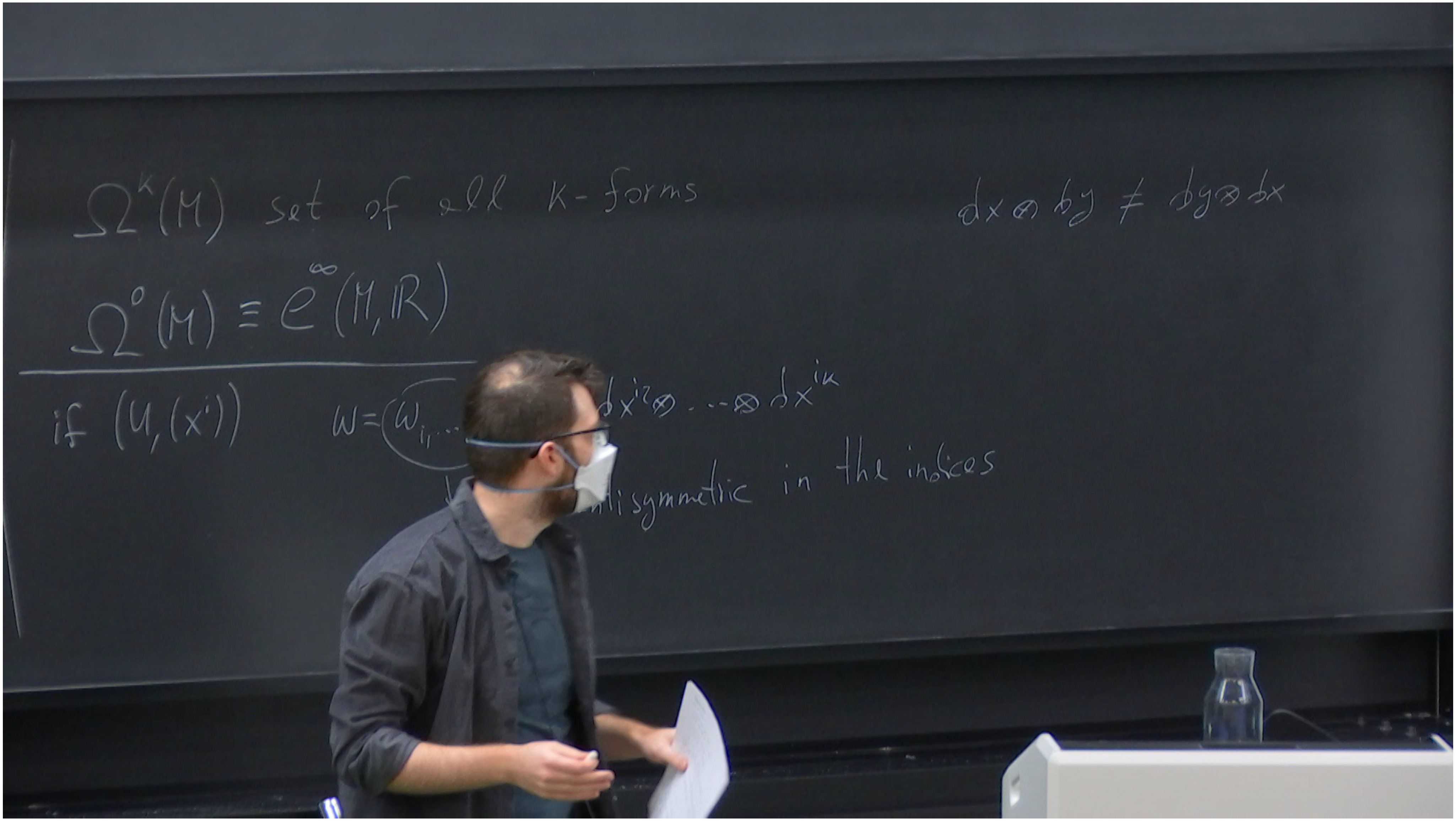

$\Omega^k(M)$ set of all k-forms

$dx \otimes dy \neq dy \otimes dx$

$\Omega^0(M) \equiv \mathcal{C}^\infty(M, \mathbb{R})$

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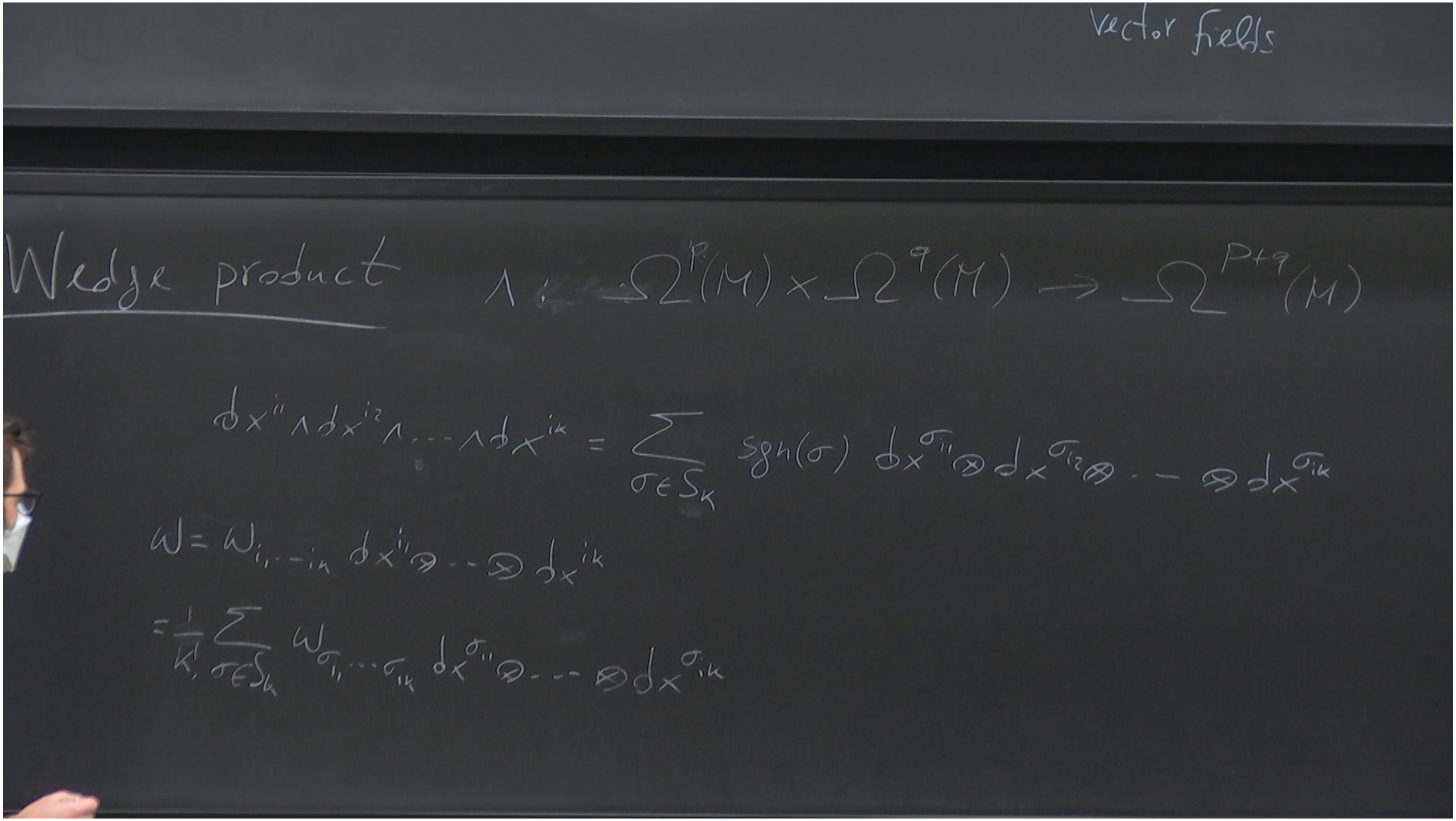
if $(U, (x^i))$ $\omega = \omega_{i_1 \dots}$ $dx^{i_2} \otimes \dots \otimes dx^{i_k}$

ntisymmetric in the indices



vector fields

Wedge product

$$\wedge \quad \Omega^{p}(M) \times \Omega^{q}(M) \to \Omega^{p+q}(M)$$

$$dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, dx^{\sigma_{i_1}} \otimes dx^{\sigma_{i_2}} \otimes \cdots \otimes dx^{\sigma_{i_k}}$$

$$\omega = \omega_{i_1 \cdots i_k}\, dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \omega_{\sigma_{i_1} \cdots \sigma_{i_k}}\, dx^{\sigma_{i_1}} \otimes \cdots \otimes dx^{\sigma_{i_k}}$$

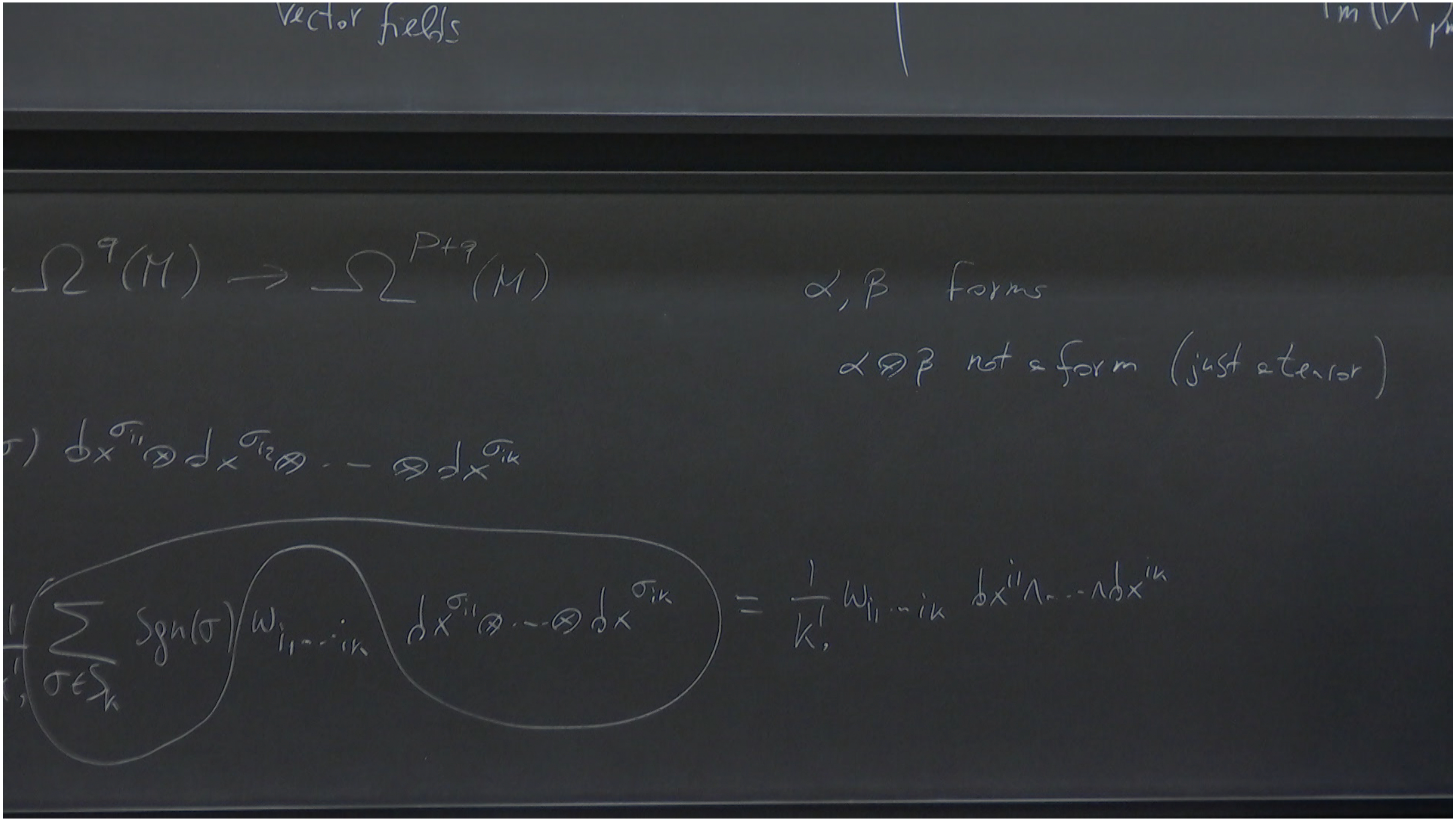
vector fields

dge product $\wedge : \Omega^{p}(M)\times\Omega^{q}(M)\to\Omega^{p+q}(M)$

$$dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k} = \sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\, dx^{\sigma_{i_1}}\otimes dx^{\sigma_{i_2}}\otimes\cdots\otimes dx^{\sigma_{i_k}}$$

$$\omega = \omega_{i_1\ldots i_k}\, dx^{i_1}\otimes\cdots\otimes dx^{i_k}$$

$$= \frac{1}{k!}\sum_{\sigma\in S_k}\omega_{\sigma_{i_1}\ldots\sigma_{i_k}}\, dx^{\sigma_{i_1}}\otimes\cdots\otimes dx^{\sigma_{i_k}} = \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sgn}(\sigma)\,\omega_{i_1\ldots i_k}\, dx^{\sigma_{i_1}}\otimes\cdots\otimes dx^{\sigma_{i_k}}$$



vector fields

$\text{Im}((\Lambda_{p}$

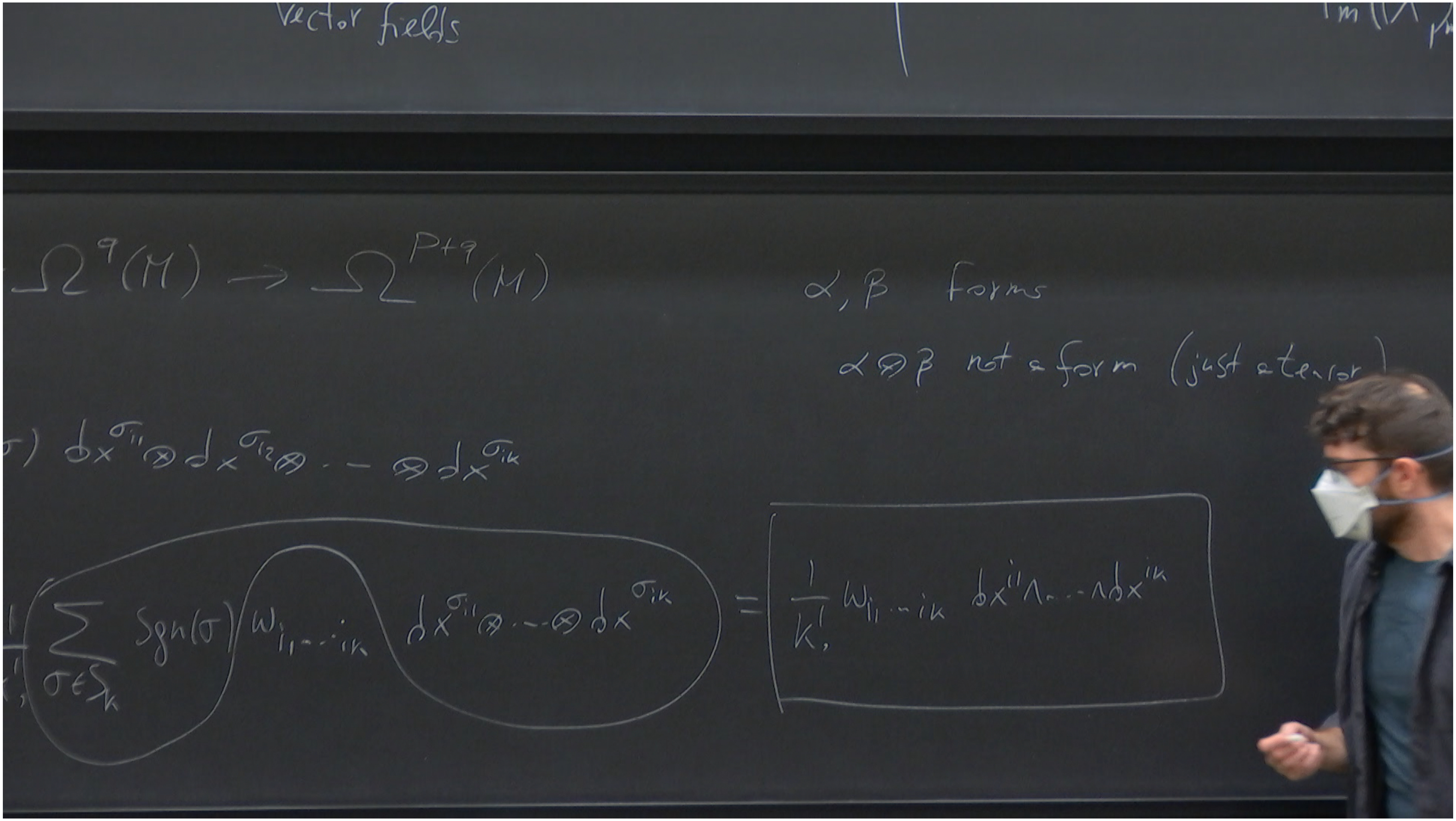
$\Omega^q(M) \to \Omega^{p+q}(M)$

α, β forms

$\alpha \otimes \beta$ not a form (just a tensor)

$\sigma)\, dx^{\sigma_{i_1}} \otimes dx^{\sigma_{i_2}} \otimes \cdots \otimes dx^{\sigma_{i_k}}$

$$\frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\, \omega_{i_1 \cdots i_k}\, dx^{\sigma_{i_1}} \otimes \cdots \otimes dx^{\sigma_{i_k}} = \frac{1}{k!} \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$



vector fields

$\mathrm{Im}((\lambda$

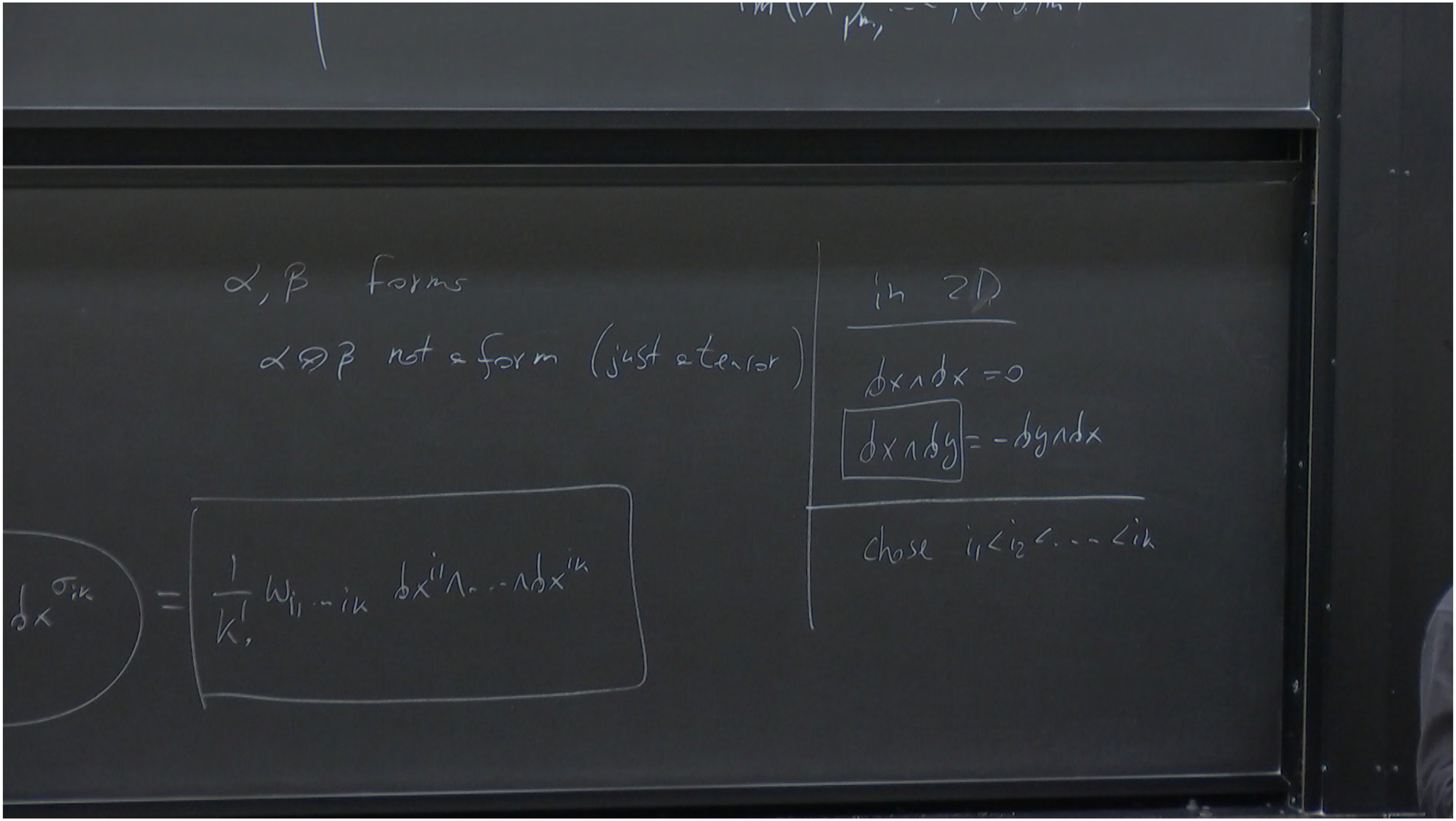
$\Omega^{q}(M) \to \Omega^{p+q}(M)$

α, β forms

α ⊗ β not a form (just a tensor)

$\sigma)\, dx^{\sigma_{i_1}} \otimes dx^{\sigma_{i_2}} \otimes \cdots \otimes dx^{\sigma_{i_k}}$

$$\frac{1}{k!}\left(\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\right) \omega_{i_1 \cdots i_k} \left(dx^{\sigma_{i_1}} \otimes \cdots \otimes dx^{\sigma_{i_k}}\right) = \boxed{\frac{1}{k!}\, \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}}$$



α, β forms

$\alpha \otimes \beta$ not a form (just a tensor)

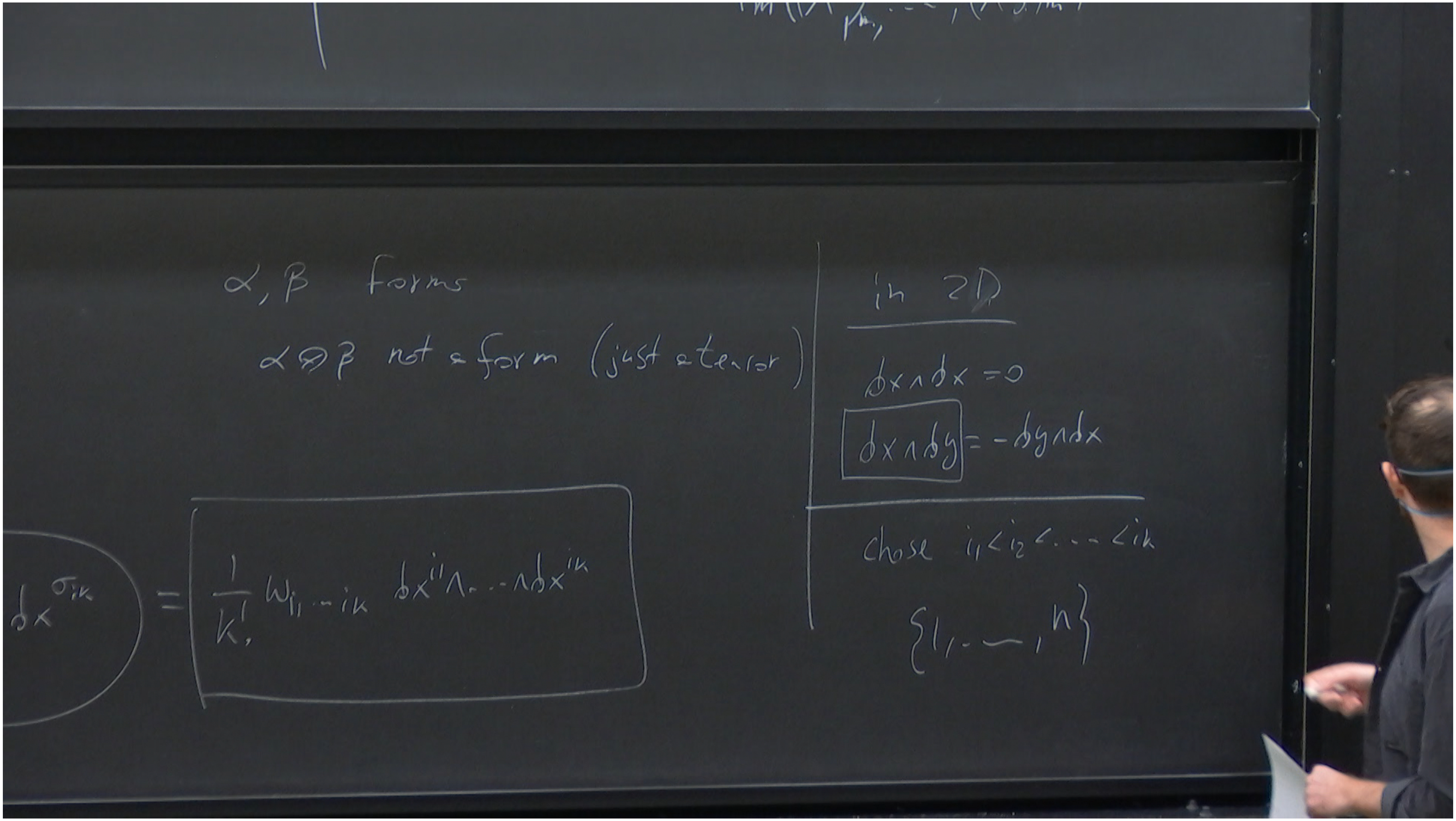
$dx^{\sigma_{ik}} = \boxed{\frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}}$

in 2D

$dx \wedge dx = 0$

$\boxed{dx \wedge dy} = - dy \wedge dx$

chose $i_1 < i_2 < \dots < i_k$



α, β forms

$\alpha \otimes \beta$ not a form (just a tensor)

in 2D

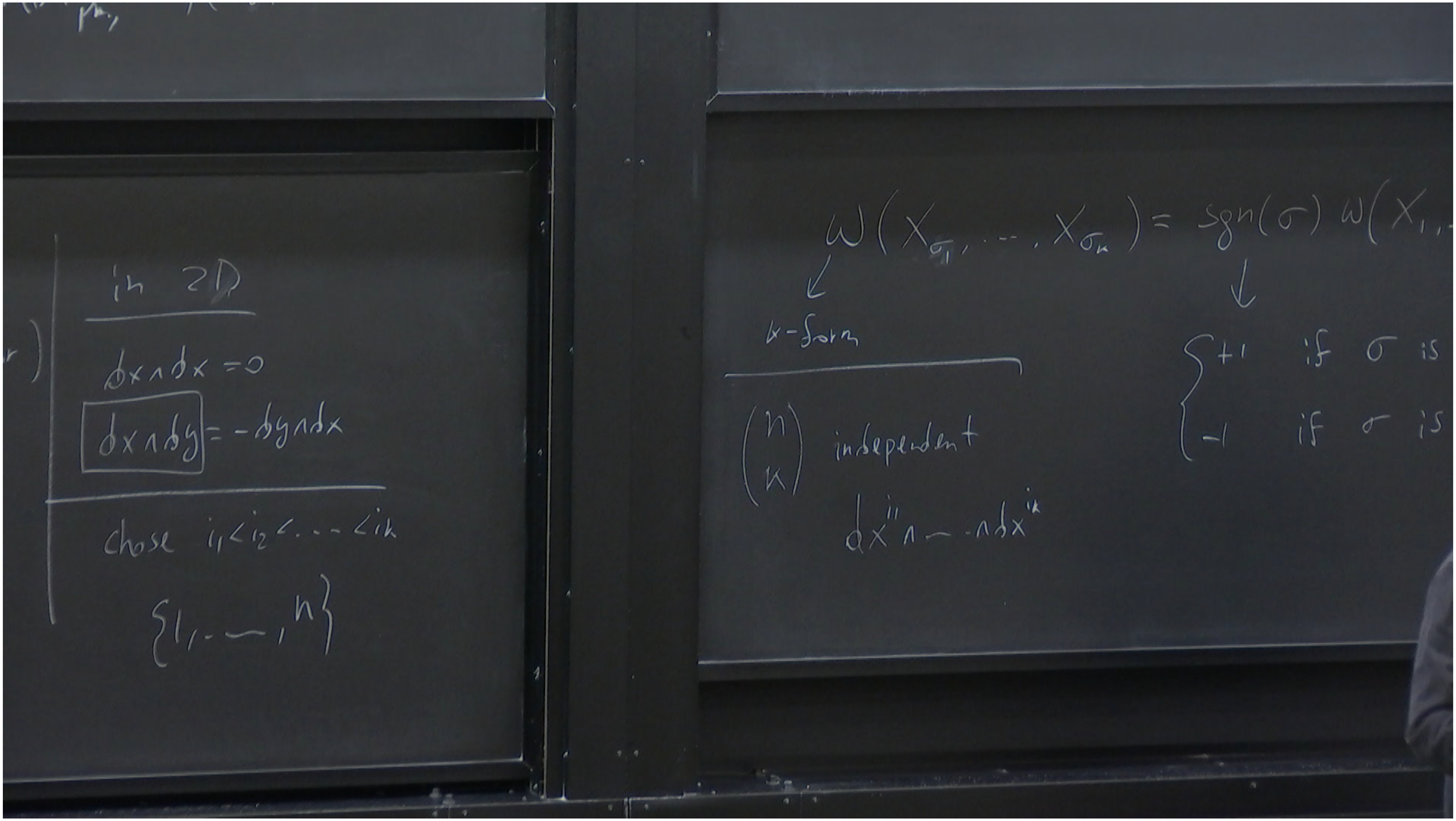
$dx \wedge dx = 0$

$\boxed{dx \wedge dy} = -dy \wedge dx$

chose $i_1 < i_2 < \dots < i_k$

$\{1, \dots, n\}$

$$dx^{\sigma_{ik}} = \boxed{\frac{1}{k!} \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}}$$



in 2D

$dx \wedge dx = 0$

$\boxed{dx \wedge dy} = -dy \wedge dx$

chose $i_1 < i_2 < \dots < i_k$

$\{1, \dots, n\}$

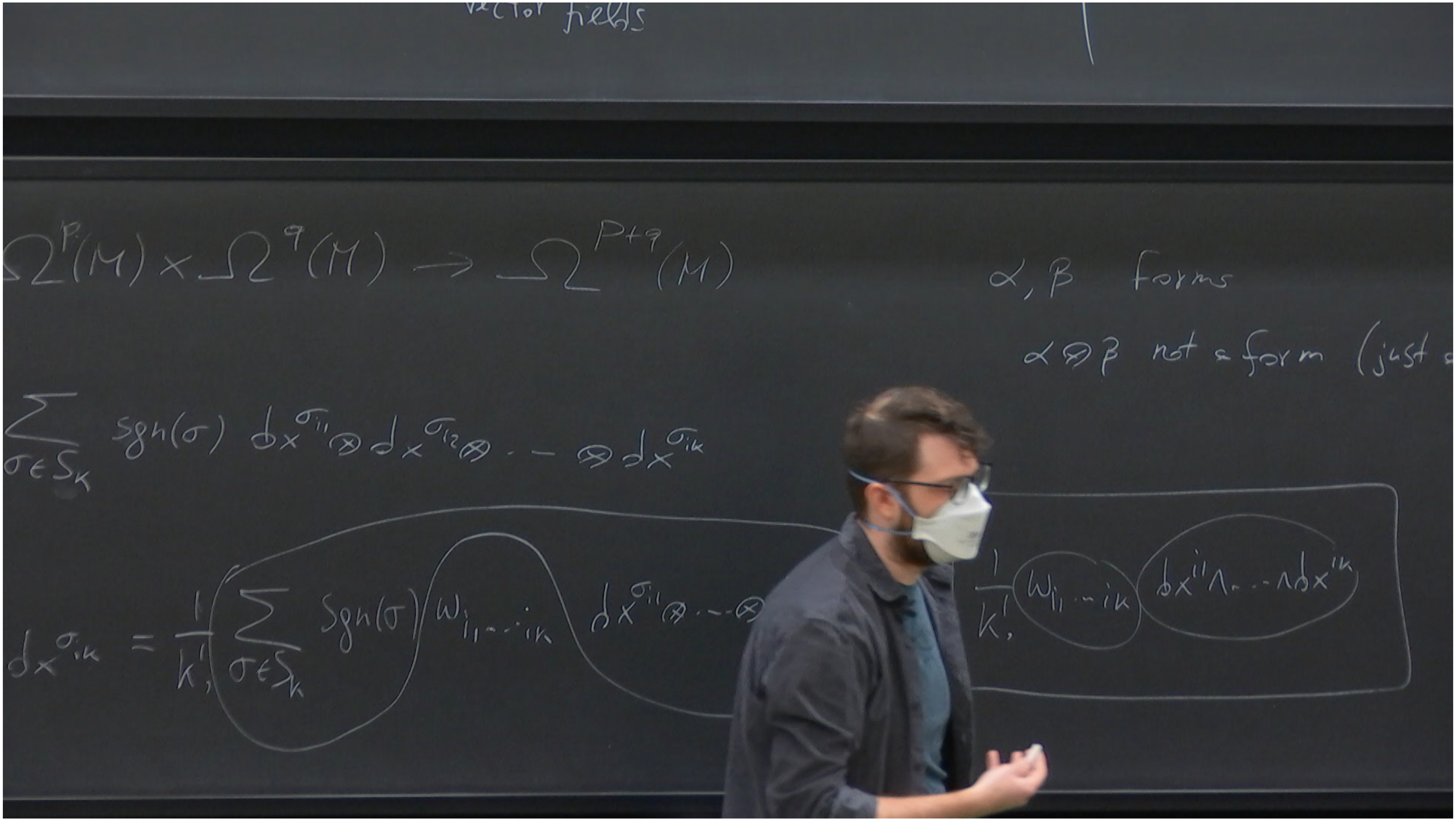
$\omega(X_{\sigma_1}, \dots, X_{\sigma_k}) = \mathrm{sgn}(\sigma)\, \omega(X_1,$

k-form

$\begin{cases} +1 & \text{if } \sigma \text{ is} \\ -1 & \text{if } \sigma \text{ is} \end{cases}$

$\binom{n}{k}$ independent

$dx^{i_1} \wedge \dots \wedge dx^{i_k}$



vector fields

$$\Omega^{p}(M) \times \Omega^{q}(M) \to \Omega^{p+q}(M)$$

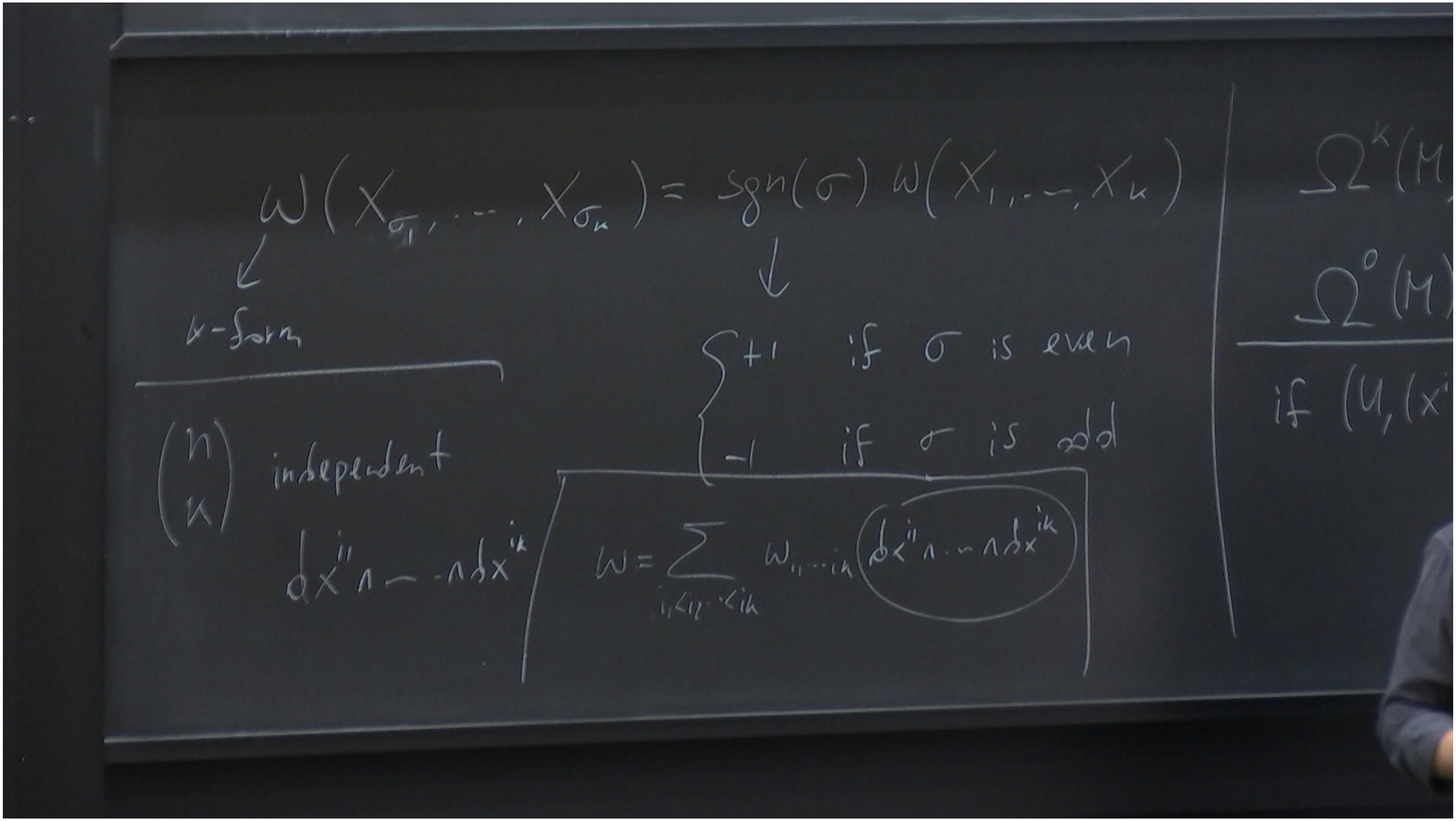
α, β forms

$\alpha \otimes \beta$ not a form (just

$$\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, dx^{\sigma_{i_1}} \otimes dx^{\sigma_{i_2}} \otimes \cdots \otimes dx^{\sigma_{i_k}}$$

$$dx^{\sigma_{i_k}} = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \omega_{i_1 \cdots i_k}\, dx^{\sigma_{i_1}} \otimes \cdots \otimes$$

$$\boxed{\frac{1}{k!}\, \omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}}$$



$$\omega(X_{\sigma_1}, \dots, X_{\sigma_k}) = \mathrm{sgn}(\sigma)\, \omega(X_1, \dots, X_k)$$

↙ k-form

$$\mathrm{sgn}(\sigma) \downarrow \begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

$\binom{n}{k}$ independent

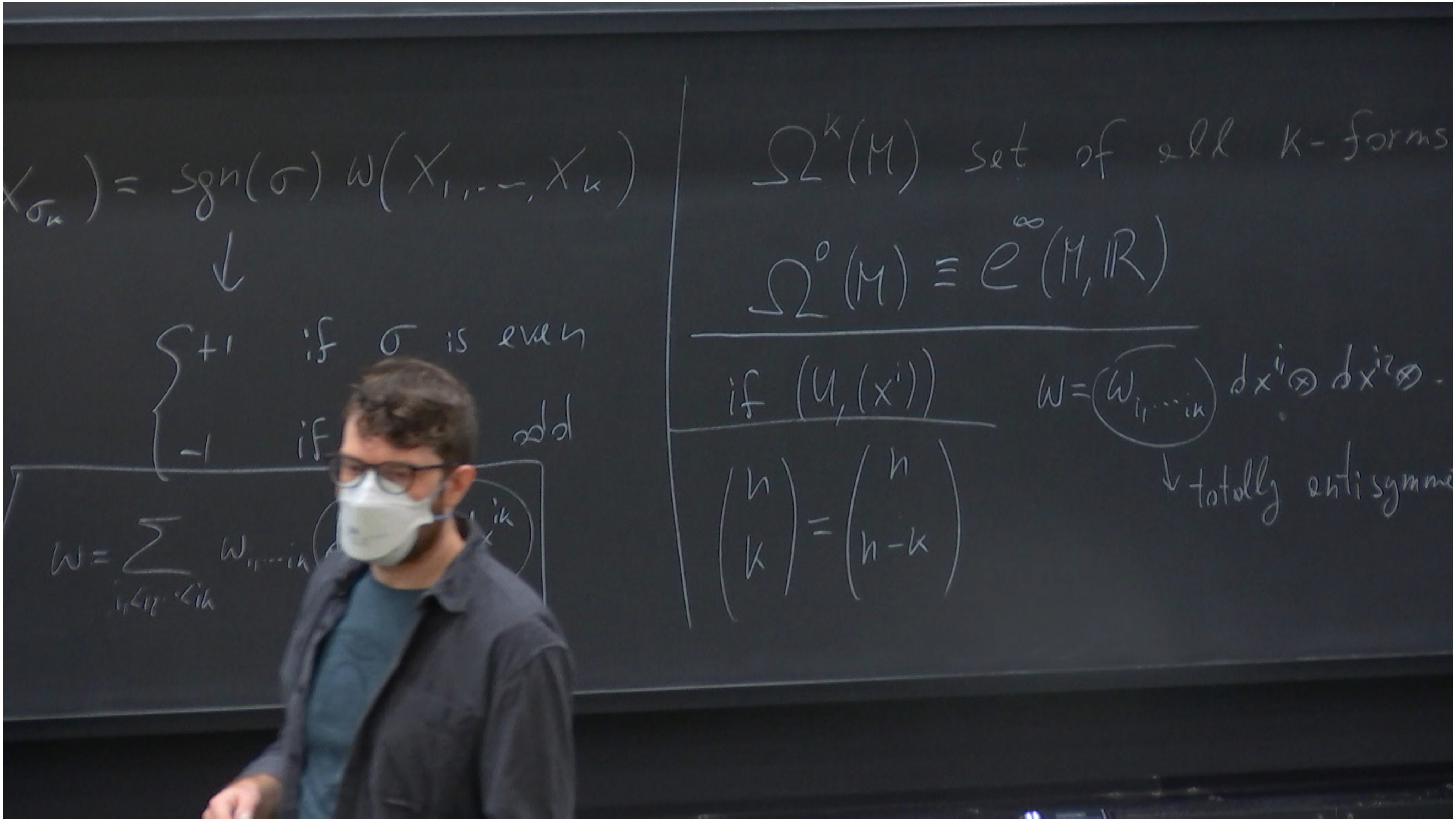
$dx^{i_1} \wedge \dots \wedge dx^{i_k}$

$$\boxed{\omega = \sum_{i_1 < i_2 \dots < i_k} \omega_{i_1 \dots i_k}\, dx^{i_1} \wedge \dots \wedge dx^{i_k}}$$

$\Omega^k(M$

$\Omega^0(M)$

if $(U, (x^$



$X_{\sigma_k}) = \mathrm{sgn}(\sigma)\, \omega(X_1, \dots, X_k)$

↓

$\begin{cases} +1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if} \quad \text{odd} \end{cases}$

$\boxed{\omega = \sum_{i_1<i_2<\dots<i_k} \omega_{i_1\dots i_k} \quad x^{i_k}}$

$\Omega^k(M)$ set of all k-forms

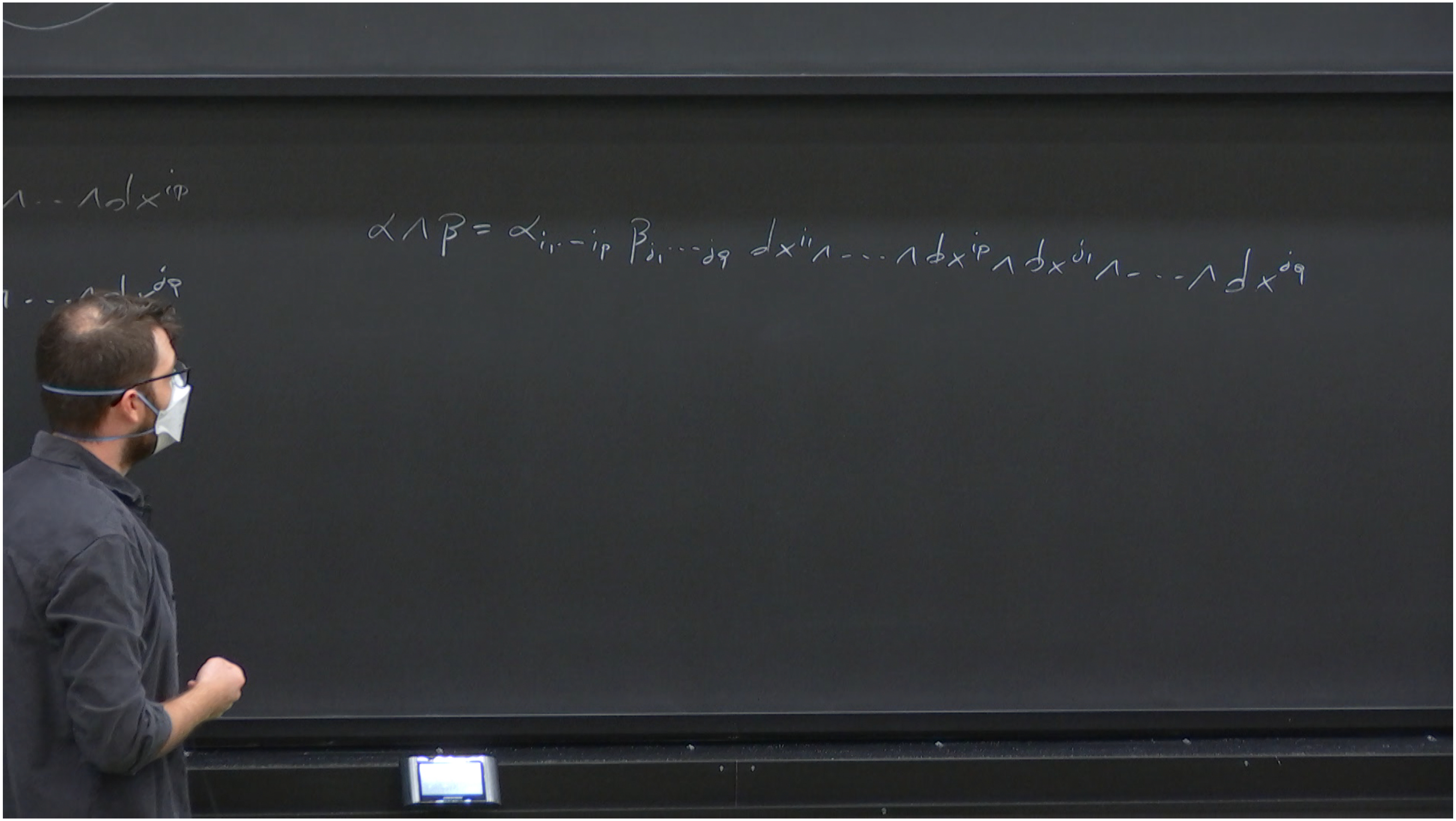
$\Omega^0(M) \equiv C^\infty(M, \mathbb{R})$

if $(U, (x^i))$

$\omega = \omega_{i_1 \dots i_k}\, dx^{i_1} \otimes dx^{i_2} \otimes \dots$

↓ totally antisymme

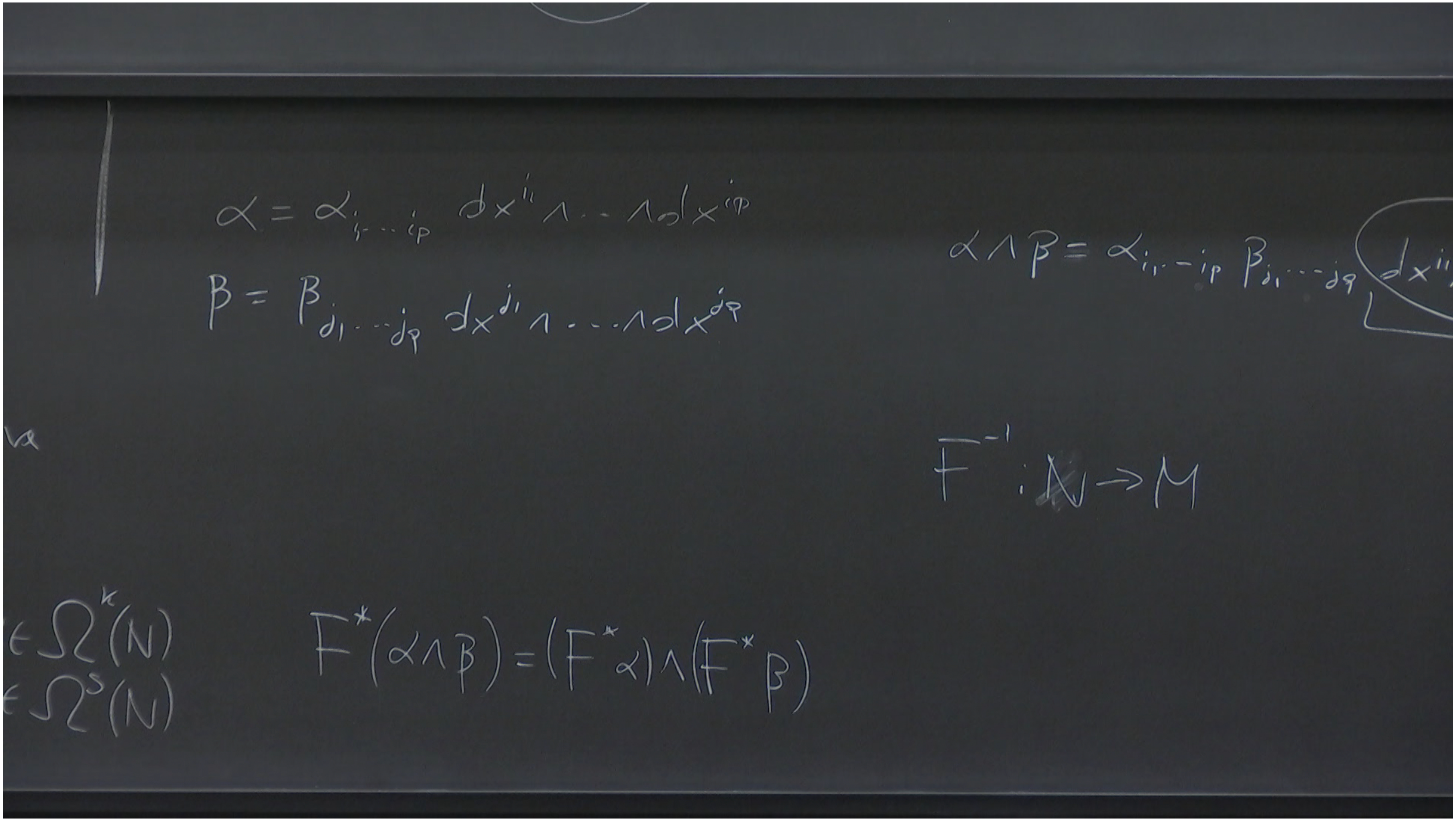
$\binom{n}{k} = \binom{n}{n-k}$



x^{i_p}

x^{j_q}

$$\alpha\wedge\beta=\alpha_{i_1\cdots i_p}\beta_{j_1\cdots j_q}\,\underbrace{dx^{i_1}\wedge\cdots\wedge dx^{i_p}}_{p}\wedge\underbrace{dx^{j_1}\wedge\cdots\wedge dx^{j_q}}_{q}$$



$$\alpha = \alpha_{i_1 \cdots i_p} \, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$\beta = \beta_{j_1 \cdots j_q} \, dx^{j_1} \wedge \cdots \wedge dx^{j_q}$$

$$\alpha \wedge \beta = \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} \, dx^{i_1}$$

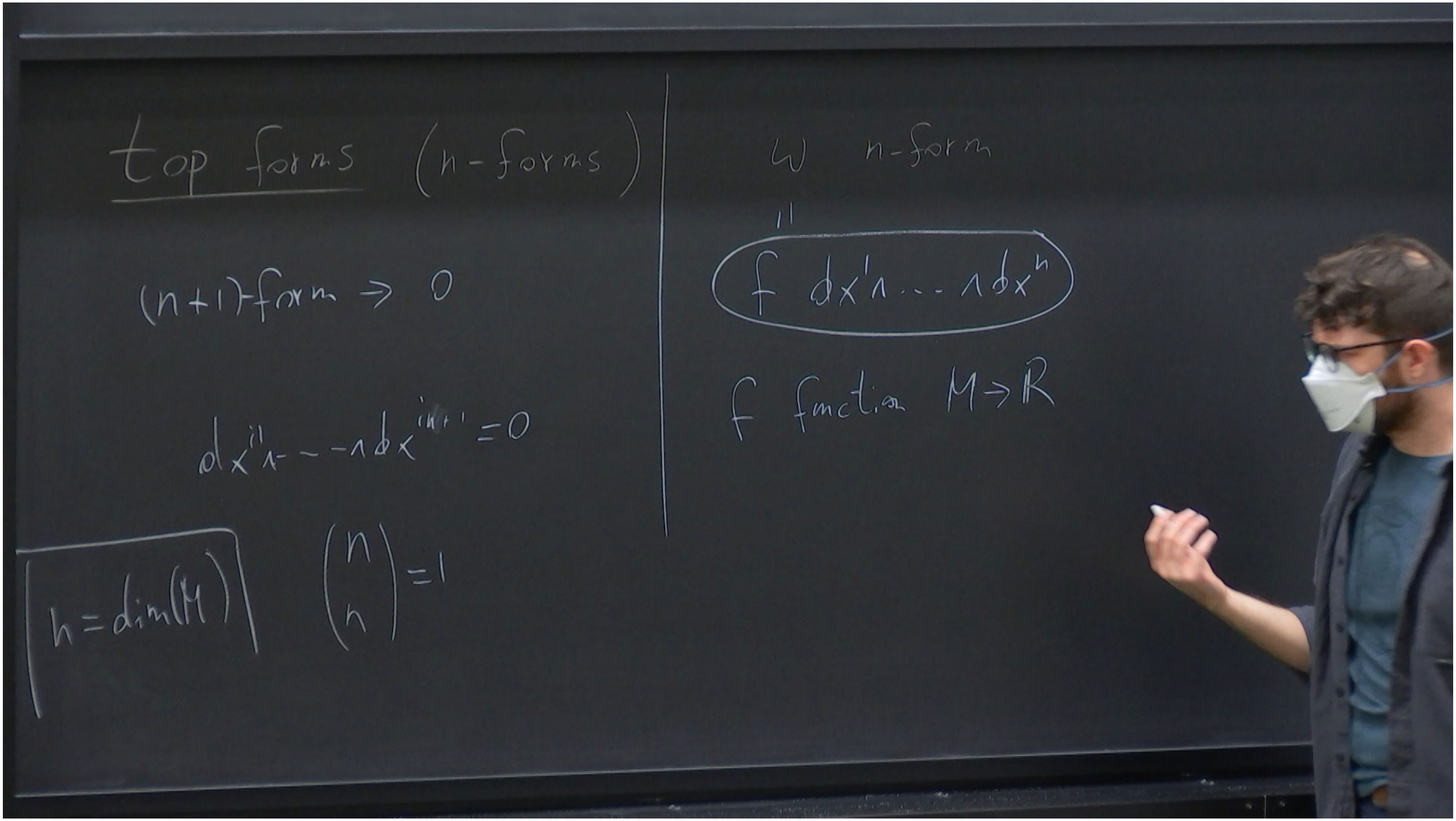
$$F^{-1} : N \to M$$

α

$\in \Omega^k(N)$

$\in \Omega^s(N)$

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta)$$



top forms (n-forms)

(n+1)-form → 0

$$dx^{i_1}\wedge\cdots\wedge dx^{i_{n+1}}=0$$

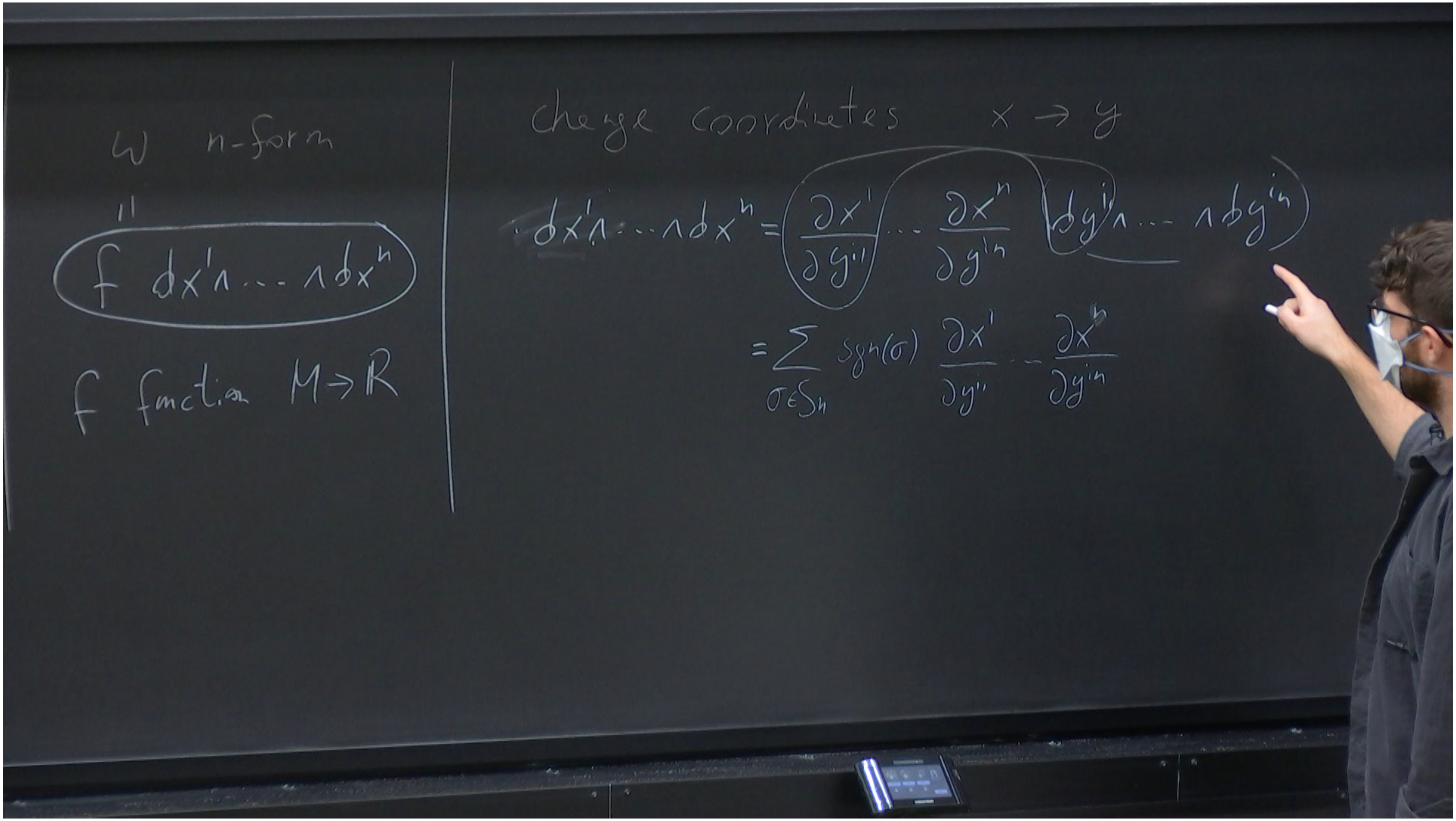
$$\boxed{n=\dim(M)} \qquad \binom{n}{n}=1$$

ω n-form

$\|$

$$\boxed{f\, dx^1\wedge\cdots\wedge dx^n}$$

f function $M\to\mathbb{R}$



ω n-form

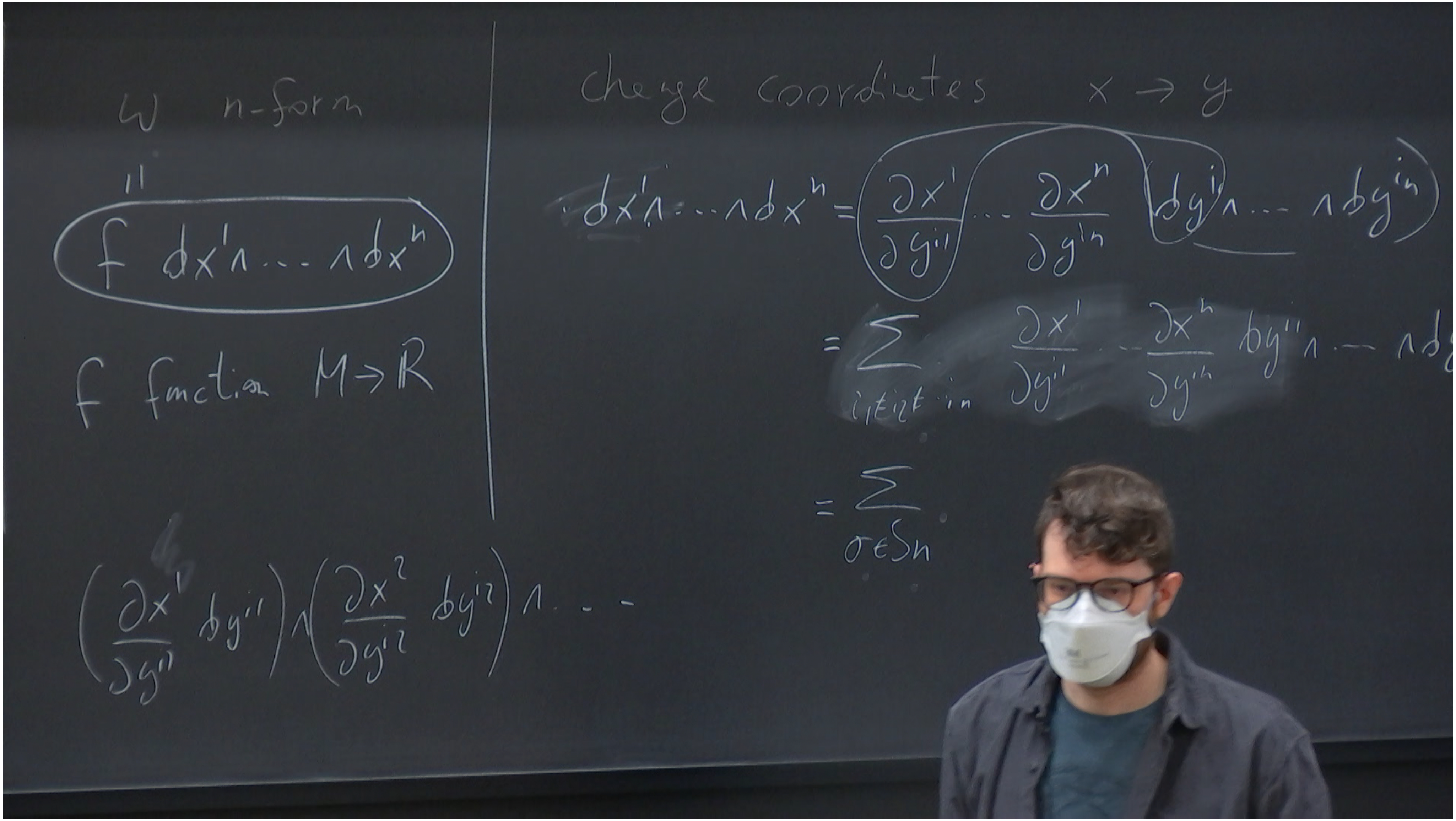
$$\omega = \boxed{f\, dx^1\wedge\dots\wedge dx^n}$$

f function $M\to\mathbb{R}$

change coordinates $x\to y$

$$dx^1\wedge\dots\wedge dx^n = \frac{\partial x^1}{\partial y^{i_1}}\cdots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\dots\wedge dy^{i_n}$$

$$= \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\, \frac{\partial x^1}{\partial y^{i_1}}\cdots\frac{\partial x^n}{\partial y^{i_n}}$$



ω n-form

$\parallel$

$$\boxed{f\, dx^1\wedge\dots\wedge dx^n}$$

f function $M\to\mathbb{R}$

$$\left(\frac{\partial x^1}{\partial y^{i_1}}\, dy^{i_1}\right)\wedge\left(\frac{\partial x^2}{\partial y^{i_2}}\, dy^{i_2}\right)\wedge\dots$$

change coordinates $x\to y$

$$dx^1\wedge\dots\wedge dx^n=\left(\frac{\partial x^1}{\partial y^{i_1}}\dots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\dots\wedge dy^{i_n}\right)$$

$$=\sum_{i_1\neq i_2\neq\dots i_n}\frac{\partial x^1}{\partial y^{i_1}}\dots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\dots\wedge dy$$

$$=\sum_{\sigma\in S_n}$$

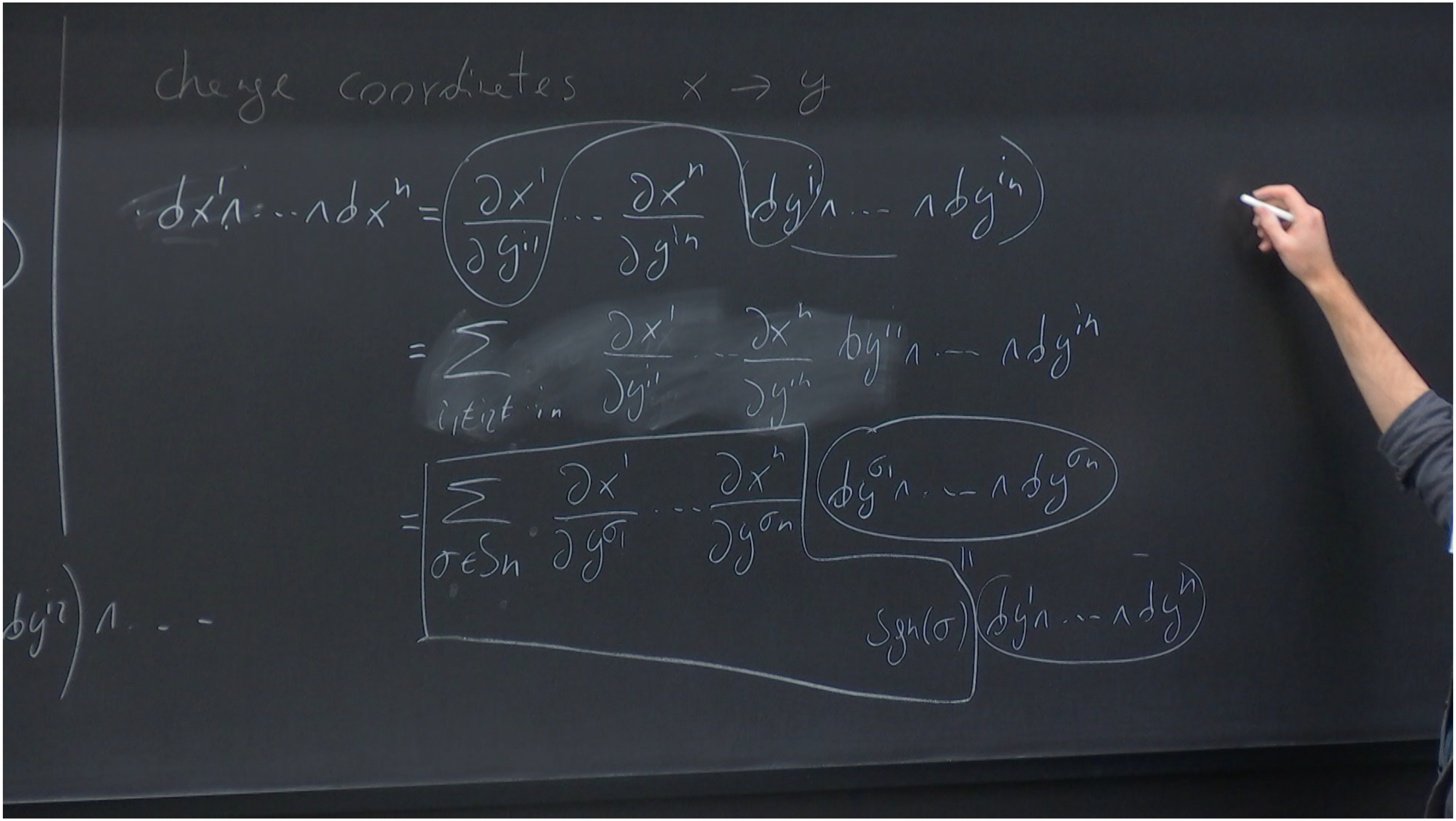
$$= \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_q} \underbrace{dx^{i_1} \wedge \cdots \wedge dx^{i_p}}_{p} \wedge \underbrace{dx^{j_1} \wedge \cdots \wedge dx^{j_q}}_{q}$$

$N \to M$

$$F = F_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$$= \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$dx \wedge dy = dx \otimes dy - dy \otimes dx$$



change coordinates $x \to y$

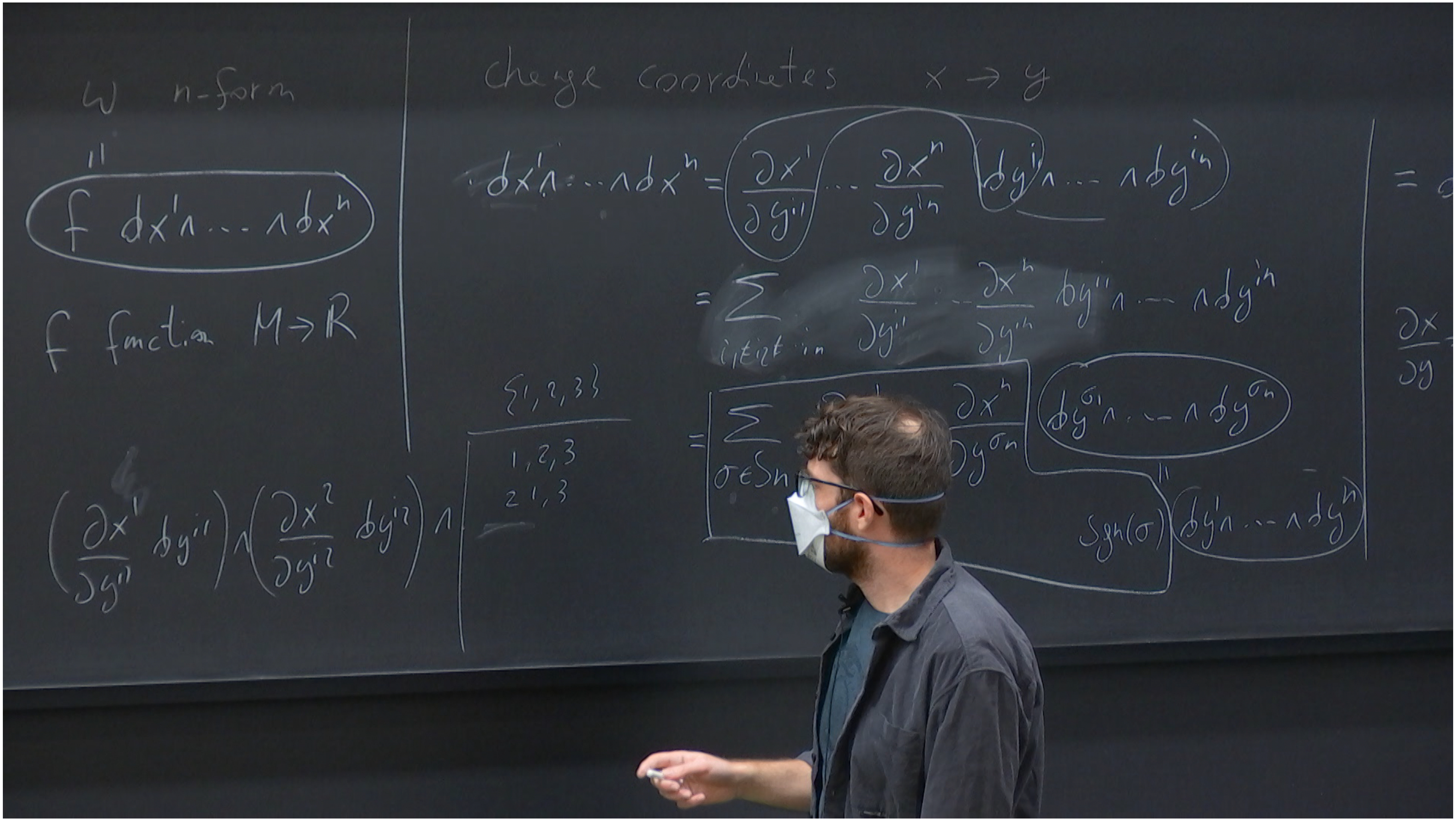
$$dx^1\wedge\cdots\wedge dx^n = \left(\frac{\partial x^1}{\partial y^{i_1}}\cdots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\cdots\wedge dy^{i_n}\right)$$

$$= \sum_{i_1\neq i_2\neq\cdots i_n}\frac{\partial x^1}{\partial y^{i_1}}\cdots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\cdots\wedge dy^{i_n}$$

$$= \boxed{\sum_{\sigma\in S_n}\frac{\partial x^1}{\partial y^{\sigma_1}}\cdots\frac{\partial x^n}{\partial y^{\sigma_n}}\ \mathrm{sgn}(\sigma)}\, dy^{\sigma_1}\wedge\cdots\wedge dy^{\sigma_n}$$

$$dy^{\sigma_1}\wedge\cdots\wedge dy^{\sigma_n} = \mathrm{sgn}(\sigma)\, dy^1\wedge\cdots\wedge dy^n$$

$dy^{i_2})\wedge\cdots$



ω n-form

$$= \boxed{f\, dx^1\wedge\dots\wedge dx^n}$$

f function $M\to\mathbb{R}$

$$\left(\frac{\partial x^1}{\partial y^{i_1}}\, dy^{i_1}\right)\wedge\left(\frac{\partial x^2}{\partial y^{i_2}}\, dy^{i_2}\right)\wedge$$

$\{1,2,3\}$

1, 2, 3
2 1, 3

change coordinates $x\to y$

$$dx^1\wedge\dots\wedge dx^n = \frac{\partial x^1}{\partial y^{i_1}}\cdots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\dots\wedge dy^{i_n}$$

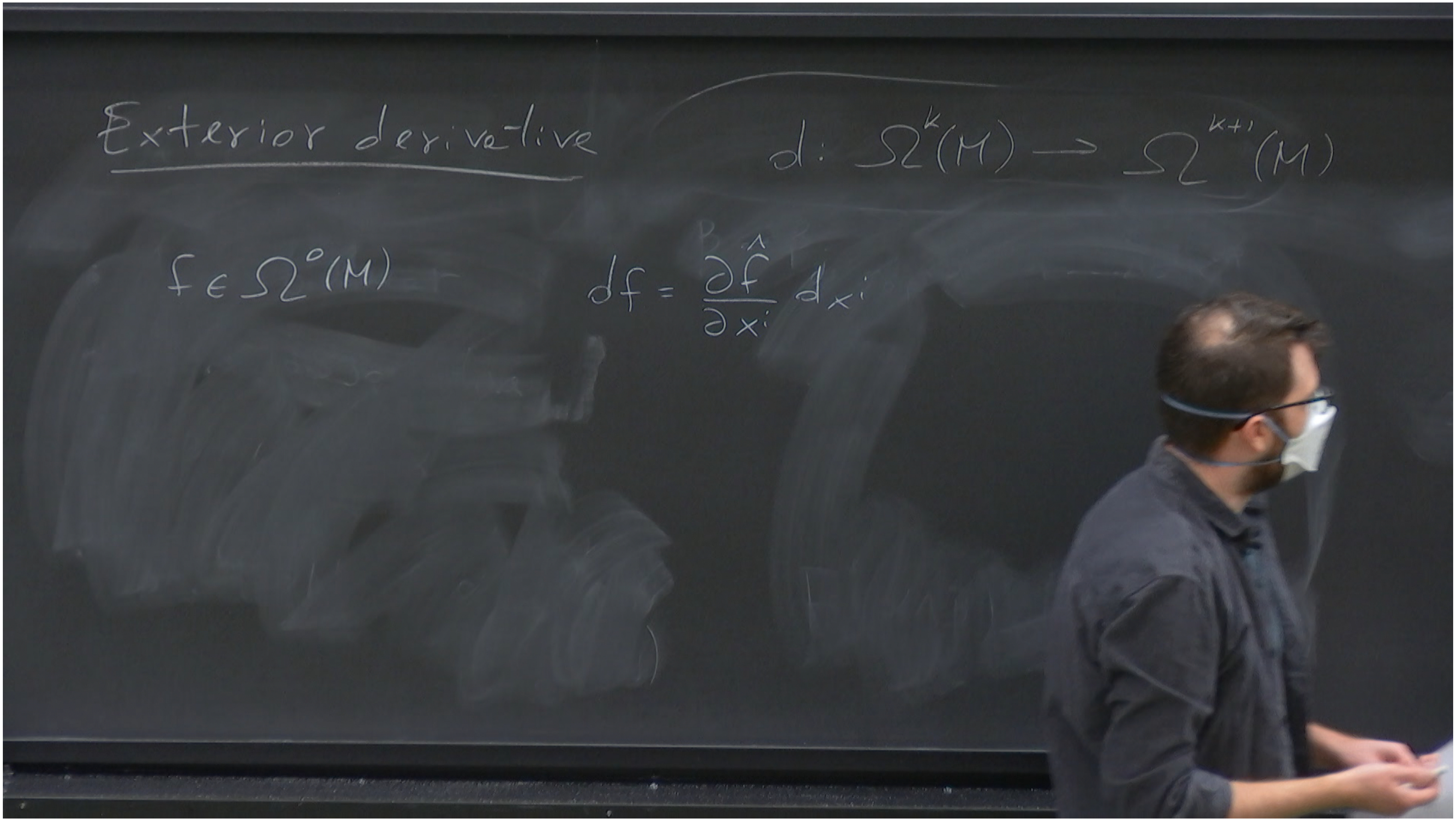
$$= \sum_{i_1\neq i_2\neq\dots i_n} \frac{\partial x^1}{\partial y^{i_1}}\cdots\frac{\partial x^n}{\partial y^{i_n}}\, dy^{i_1}\wedge\dots\wedge dy^{i_n}$$

$$= \sum_{\sigma\in S_n} \frac{\partial x^n}{\partial y^{\sigma_n}}\, dy^{\sigma_1}\wedge\dots\wedge dy^{\sigma_n}$$

$$= \operatorname{sgn}(\sigma)\, dy^1\wedge\dots\wedge dy^n$$

$=$

$\frac{\partial x}{\partial y}$

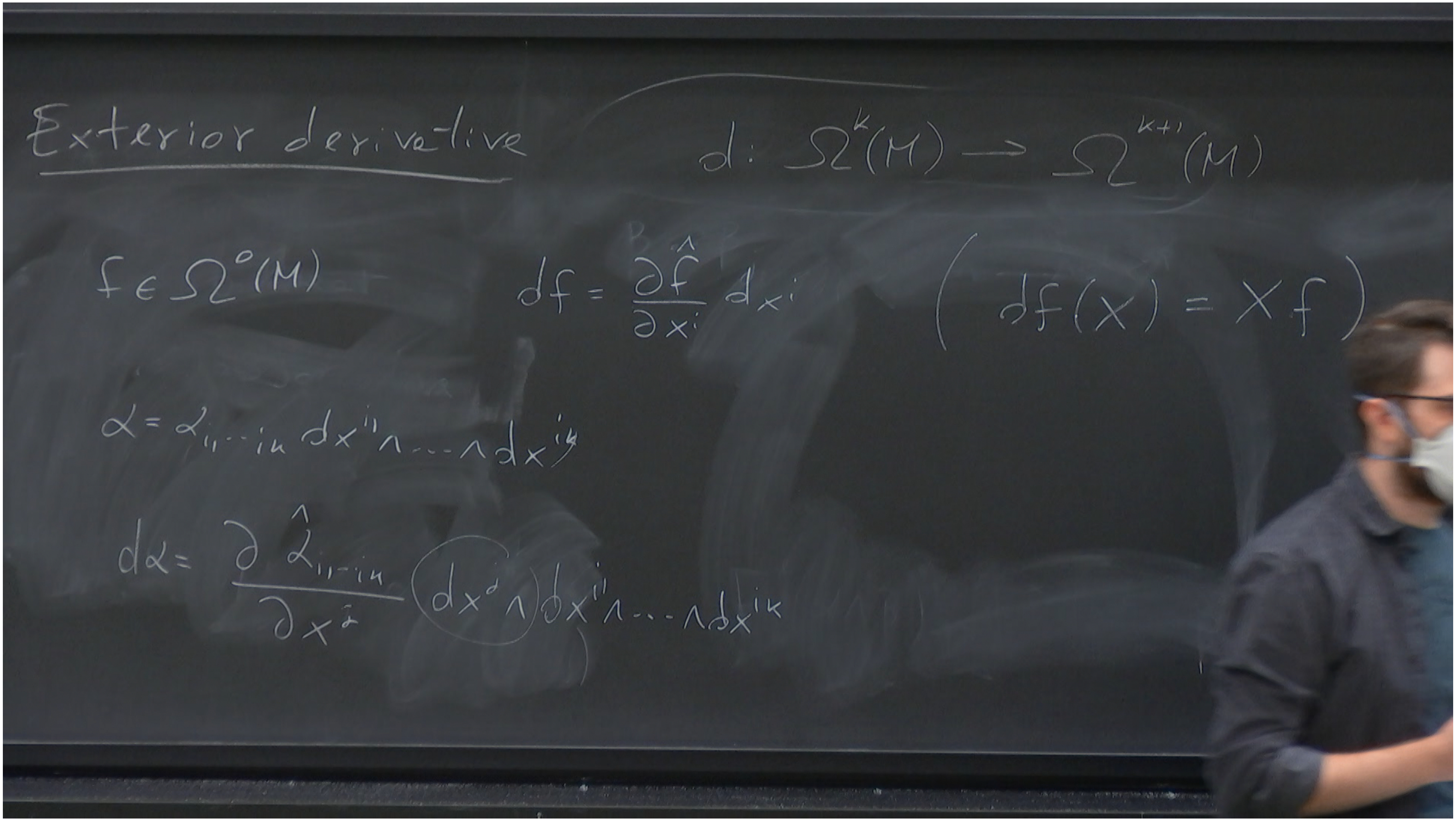


Exterior derivative

$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$

$f \in \Omega^0(M)$

$df = \frac{\partial \hat{f}}{\partial x^i} dx^i$



Exterior derivative

$$d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$$

$f \in \Omega^0(M)$

$$df = \frac{\partial \hat{f}}{\partial x^i} dx^i \qquad \left(df(X) = Xf \right)$$

$$\alpha = \alpha_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$d\alpha = \frac{\partial \hat{\alpha}_{i_1 \cdots i_k}}{\partial x^j} \boxed{dx^j \wedge} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$) \longrightarrow \Omega^{k+1}(M)$$

$$X = X^i \frac{\partial}{\partial x^i}$$

$$\left(df(X) = Xf \right)$$

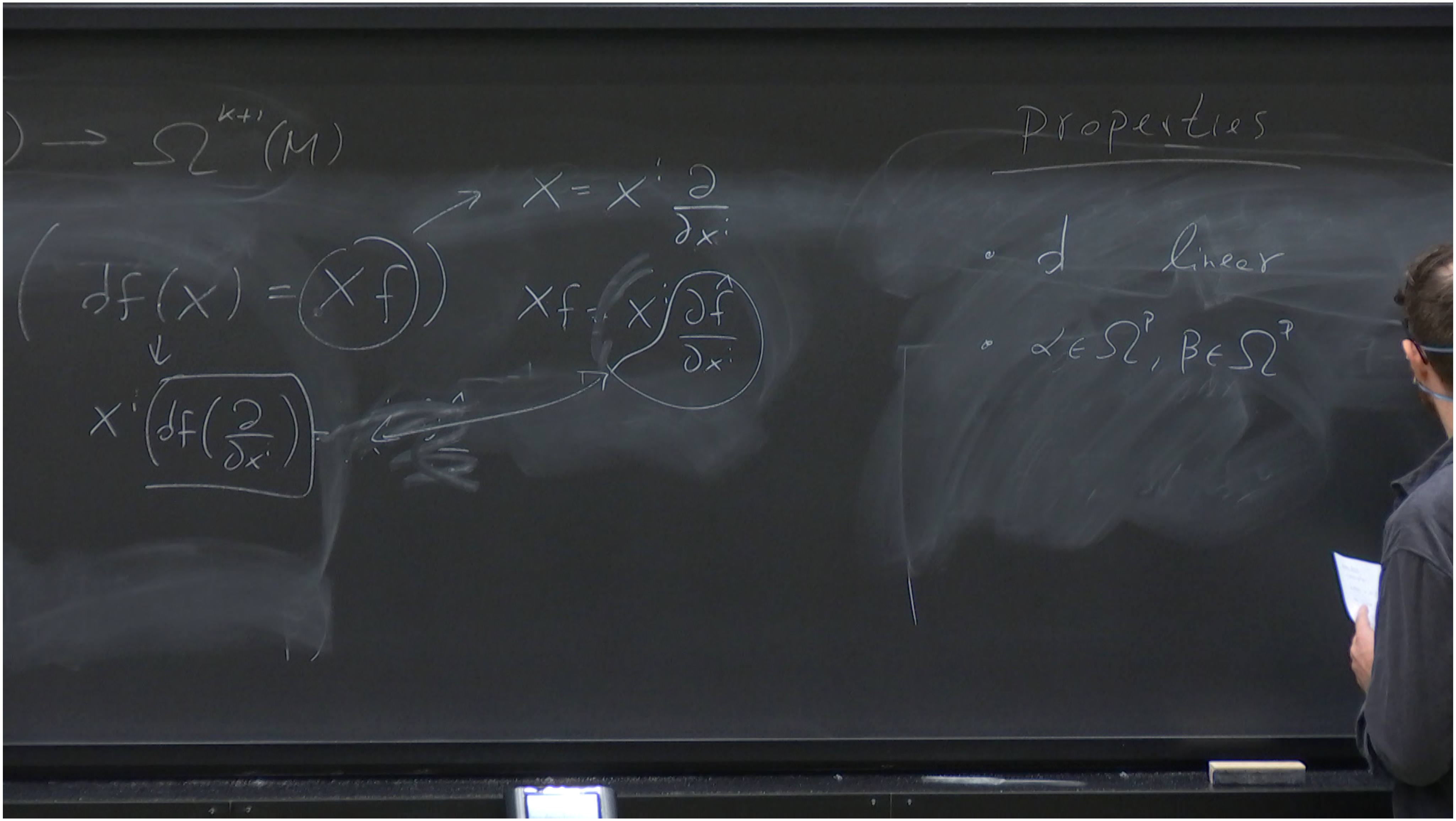
$$\downarrow$$

$$X^i \, \boxed{df\left(\frac{\partial}{\partial x^i}\right)} =$$

$$Xf = X^i \frac{\partial f}{\partial x^i}$$

Properties

- d linear



$) \to \Omega^{k+1}(M)$

$\left(df(X) = \bigcirc\!\!\!\!\!\!Xf \right)$

$X = X^i \frac{\partial}{\partial x^i}$

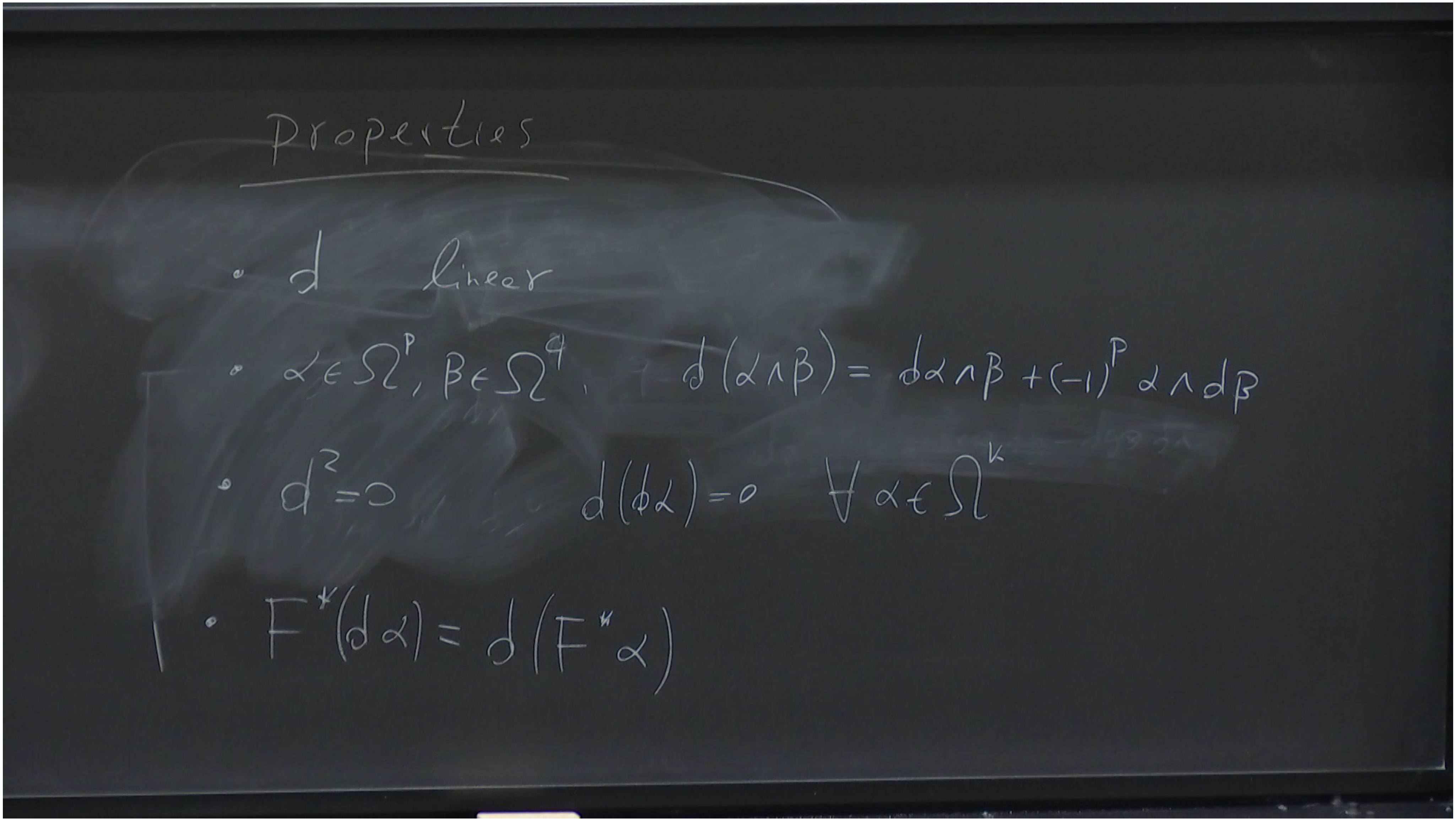
$\downarrow$

$X^i \boxed{df\left(\frac{\partial}{\partial x^i}\right)}$

$Xf = X^i \frac{\partial f}{\partial x^i}$

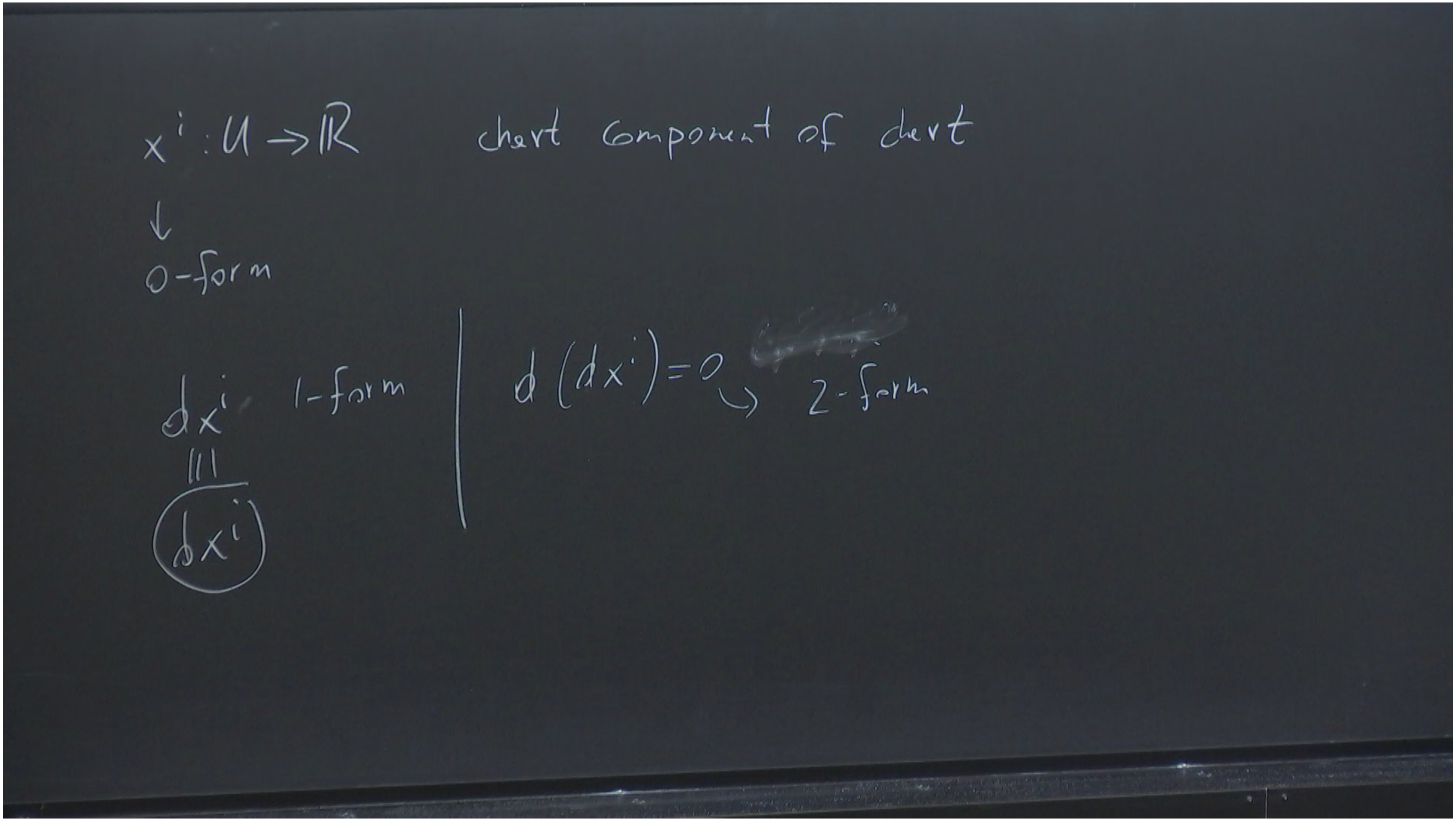
Properties

- d linear
- $\alpha \in \Omega^p, \beta \in \Omega^q$



Properties

- d linear
- $\alpha \in \Omega^p, \beta \in \Omega^q$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$
- $d^2 = 0$ $d(d\alpha) = 0$ $\forall \alpha \in \Omega^k$
- $F^*(d\alpha) = d(F^*\alpha)$



$x^i : U \to \mathbb{R}$ chart component of chart

$\downarrow$

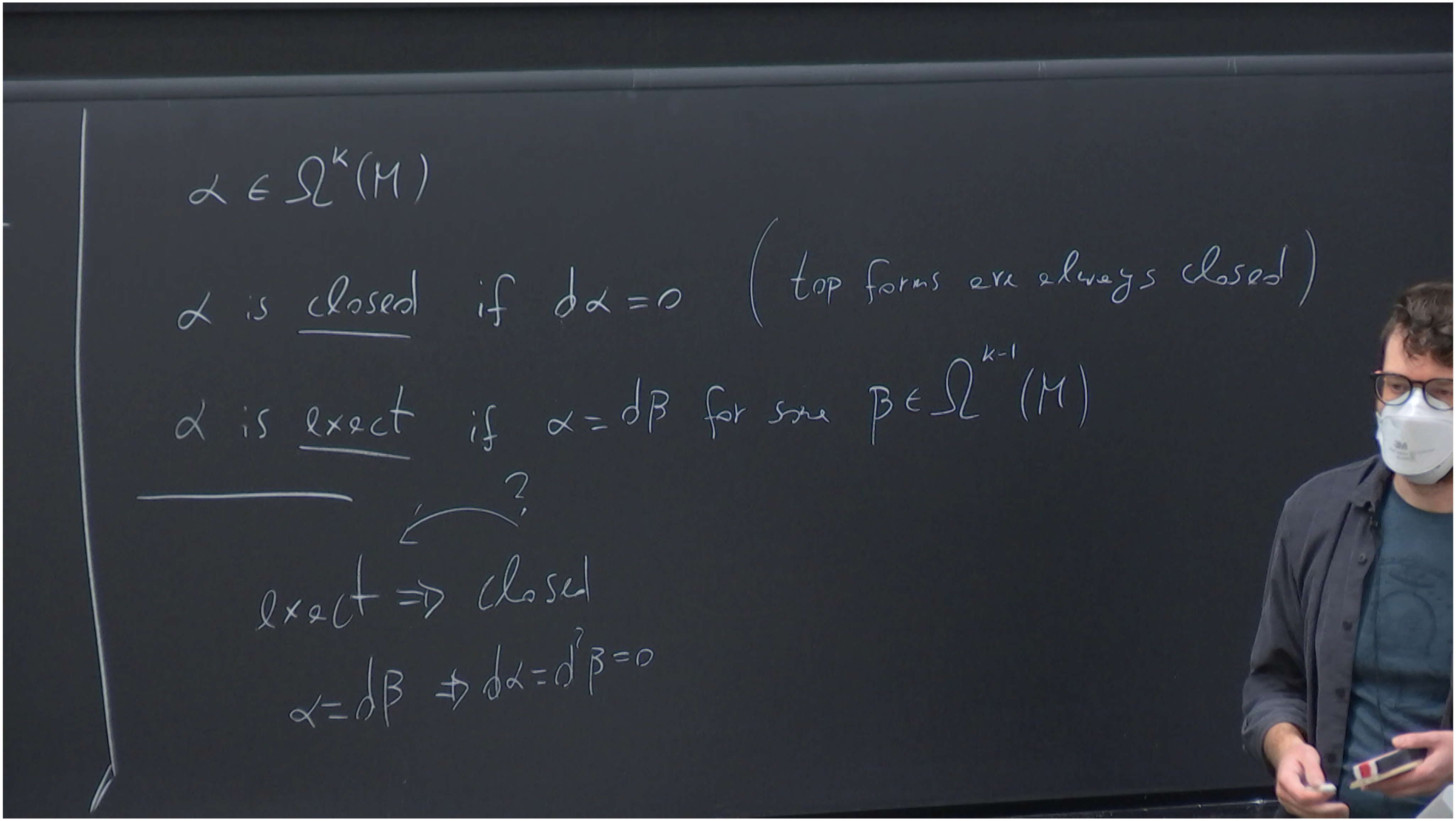
0-form

dx^i 1-form

$|||$

dx^i

$d(dx^i) = 0$ $\hookrightarrow$ 2-form



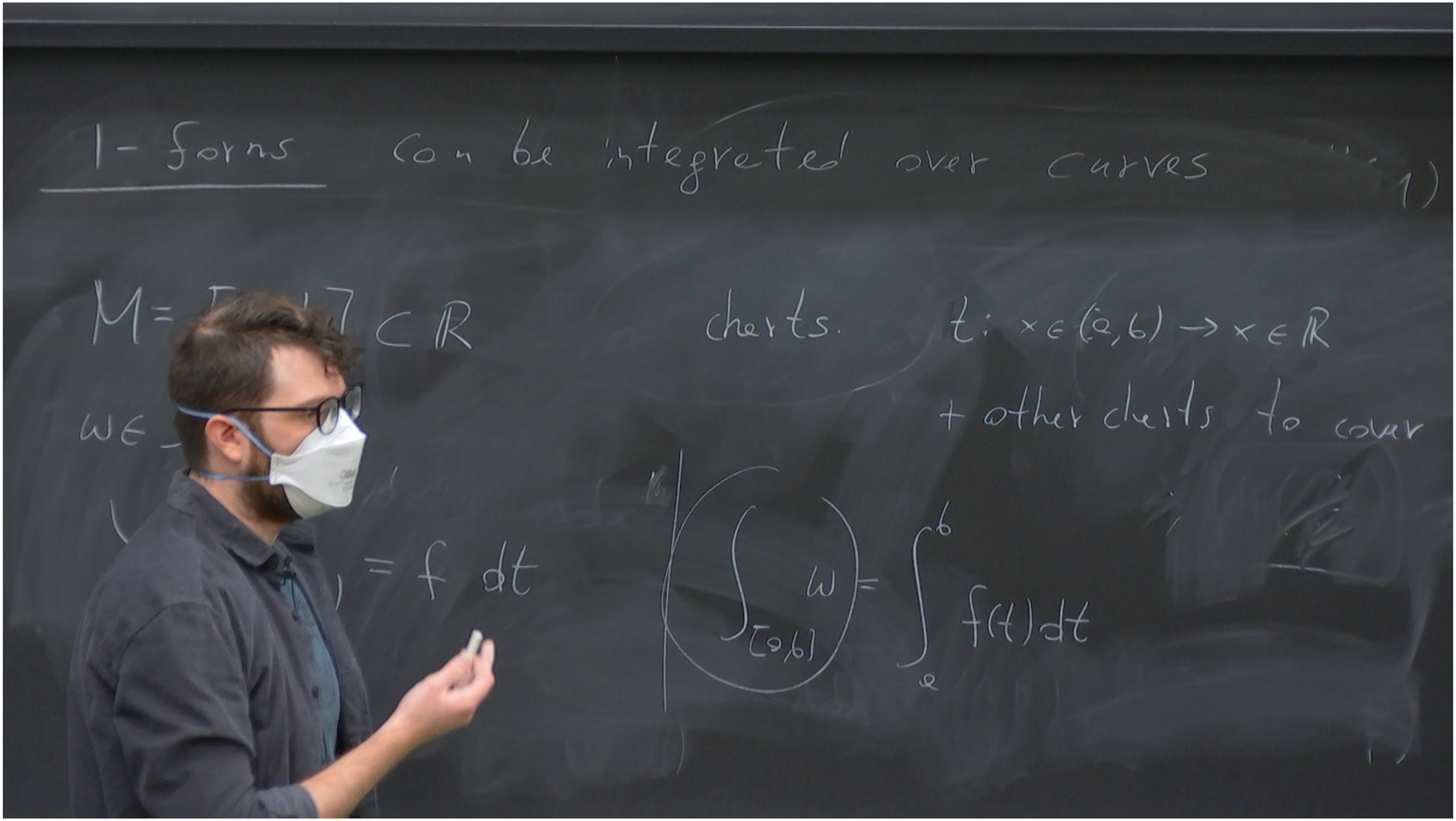
$\alpha \in \Omega^k(M)$

α is closed if $d\alpha = 0$ (top forms are always closed)

α is exact if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$

$$\begin{tikzcd}\text{exact} \arrow[r, Rightarrow] & \text{closed} \arrow[l, bend right, "?"']\end{tikzcd}$$

$\alpha = d\beta \Rightarrow d\alpha = d^2\beta = 0$



1-forms can be integrated over curves 1)

$M =$ $\subset \mathbb{R}$

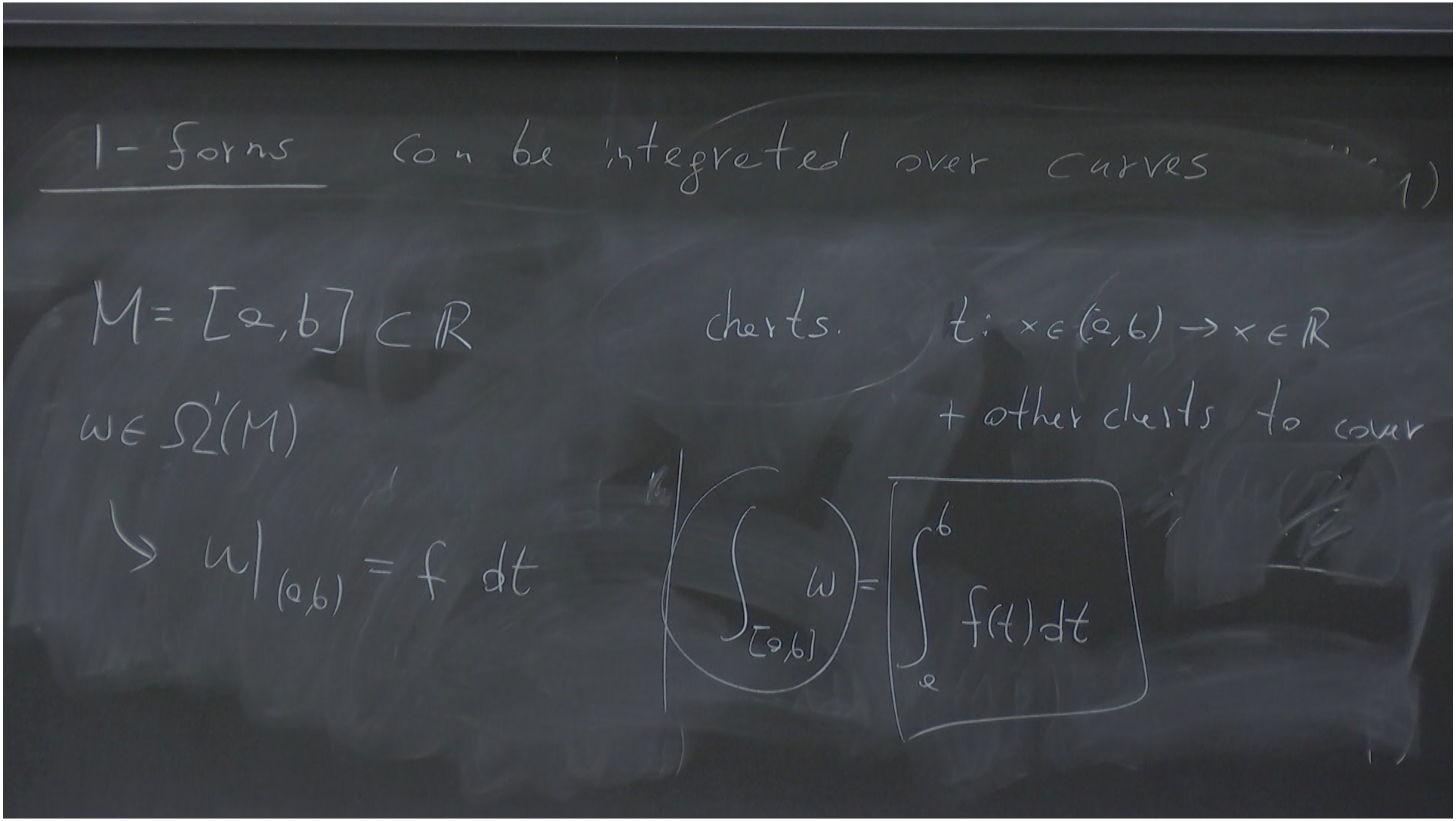
charts. $t : x \in (a,b) \to x \in \mathbb{R}$

+ other charts to cover

$\omega \in$

$= f \; dt$

$$\boxed{\int_{[a,b]} \omega} = \int_a^b f(t)dt$$



1-forms can be integrated over curves

$M = [a,b] \subset \mathbb{R}$

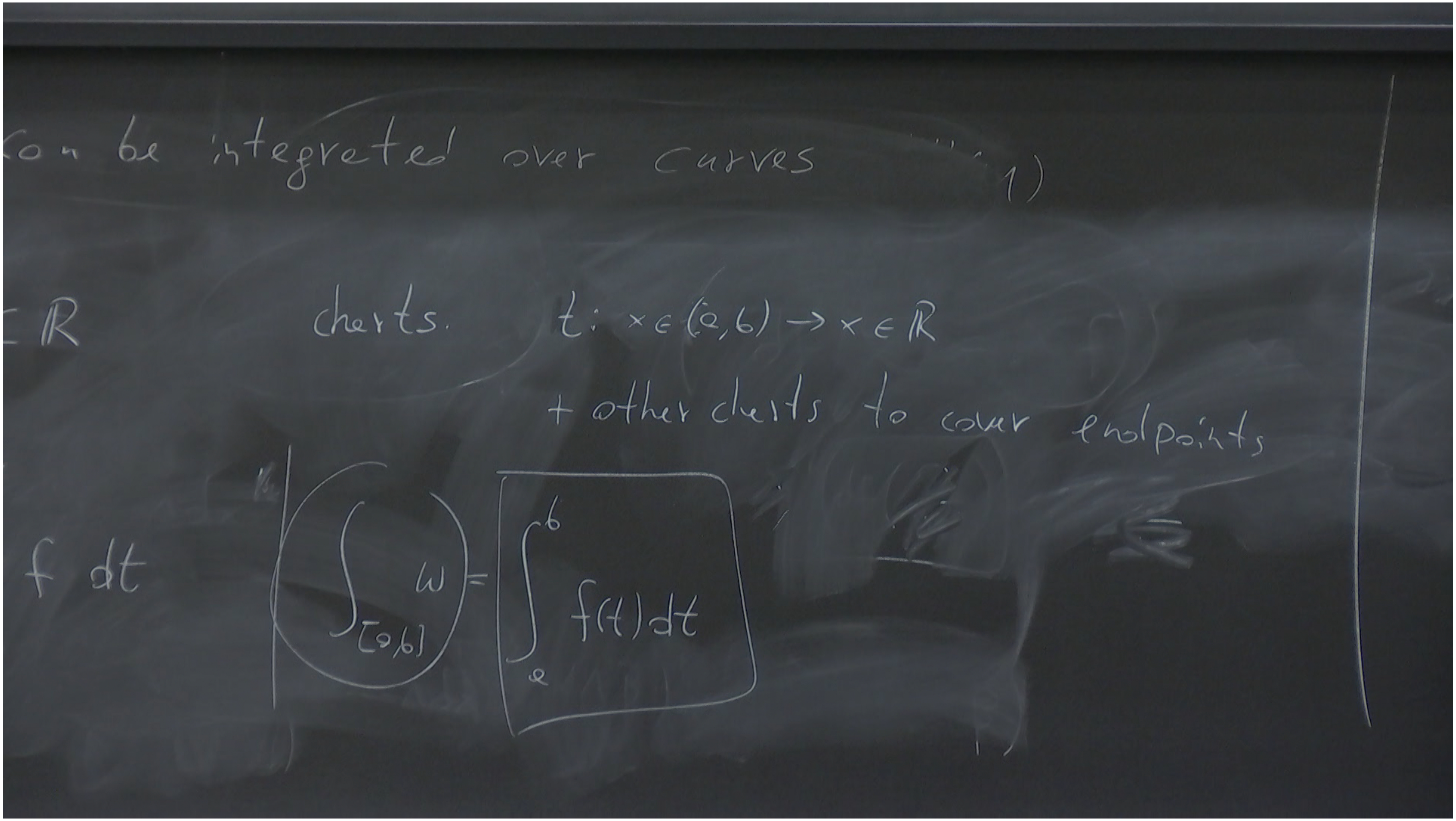
charts. $t : x \in (a,b) \to x \in \mathbb{R}$

+ other charts to cover

$\omega \in \Omega^1(M)$

$\hookrightarrow \omega|_{(a,b)} = f\, dt$

$$\int_{[a,b]} \omega = \boxed{\int_a^b f(t)\,dt}$$

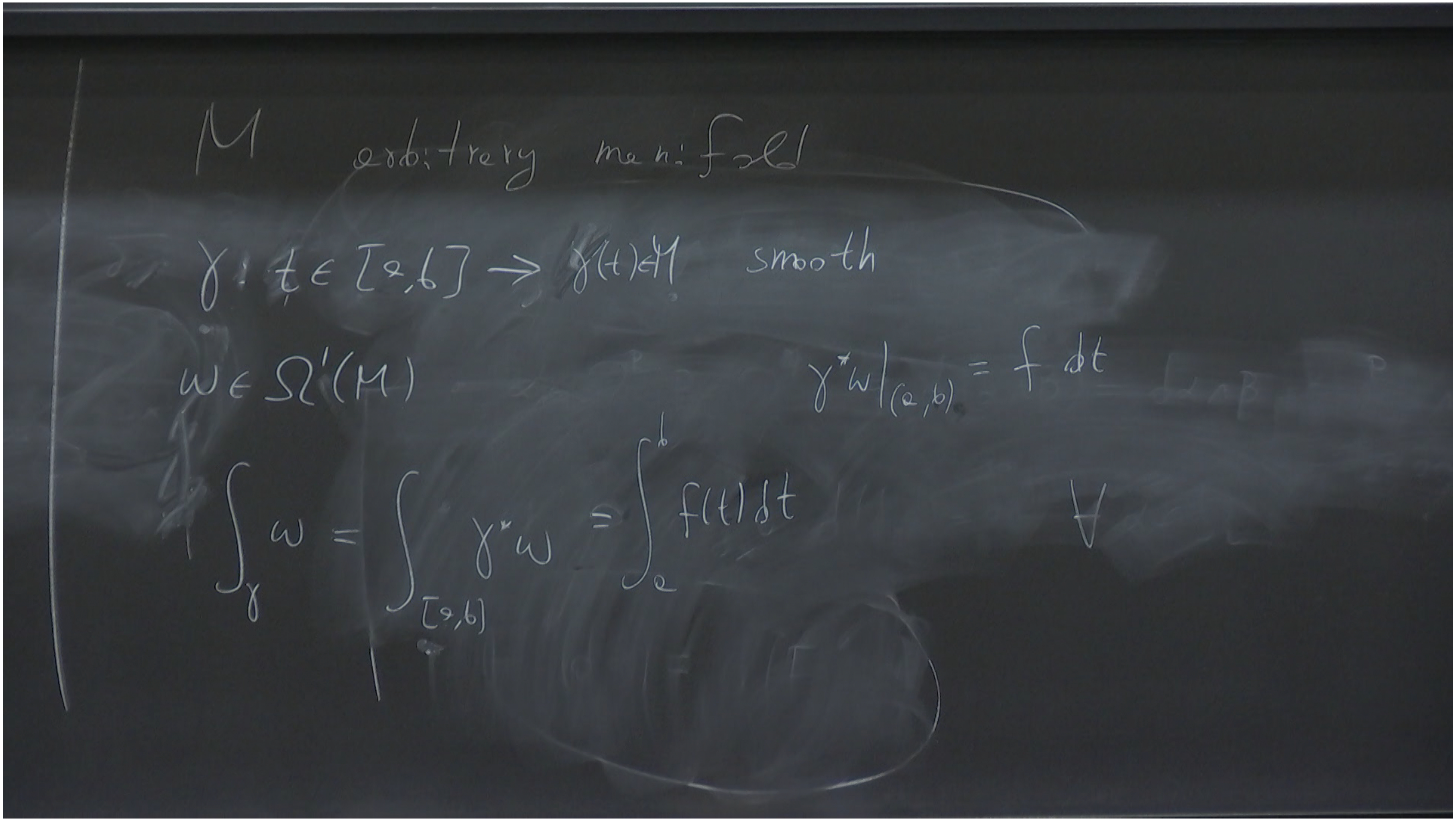


an be integrated over curves 1)

$\in \mathbb{R}$ charts. $t: x \in (a,b) \to x \in \mathbb{R}$

+ other charts to cover endpoints

$f\,dt$ $\int_{[a,b]} \omega = \boxed{\int_a^b f(t)\,dt}$

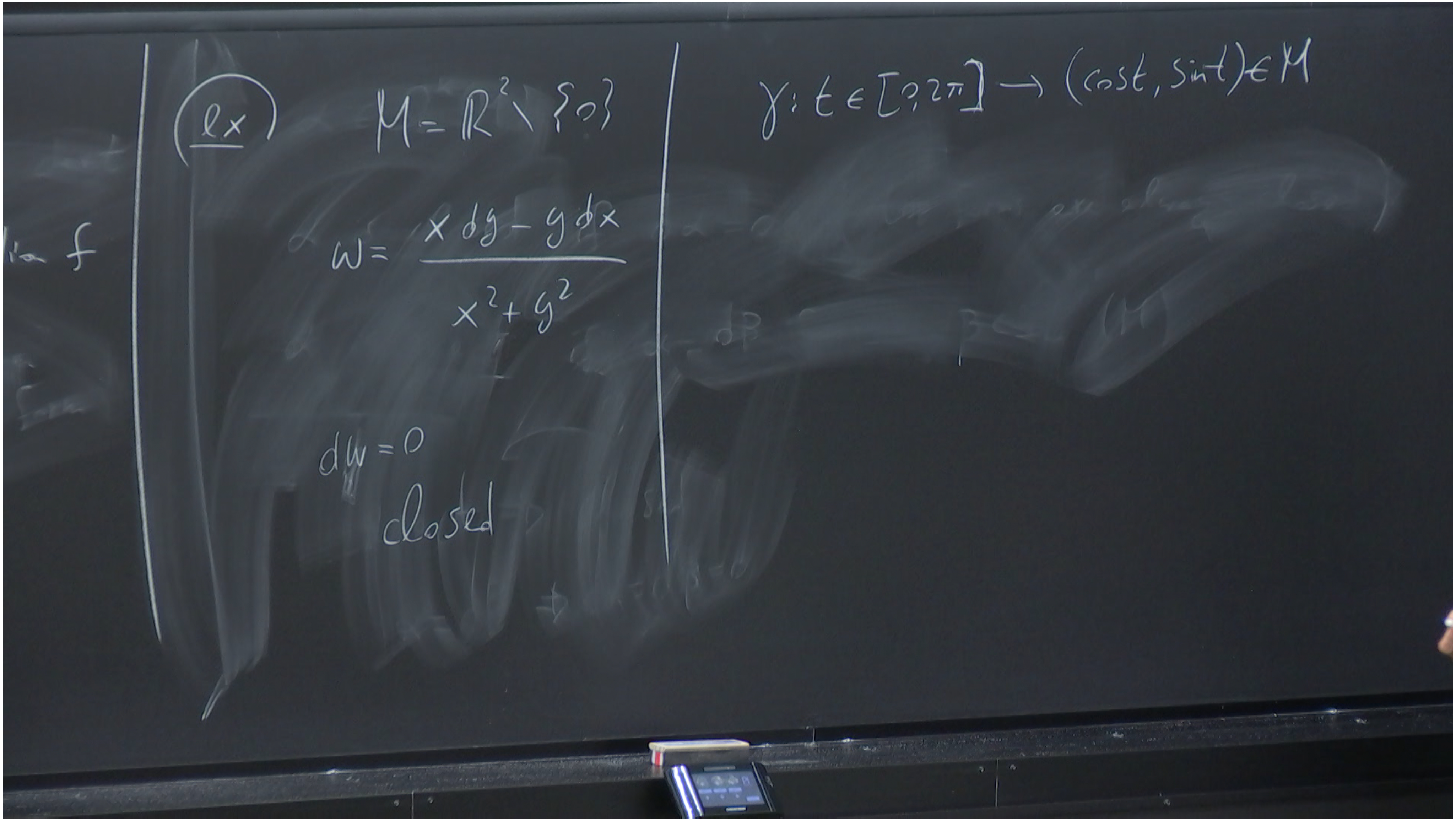


M arbitrary manifold

$\gamma : t \in [a,b] \to \gamma(t) \in M$ smooth

$\omega \in \Omega^1(M)$ $\qquad$ $\gamma^*\omega|_{(a,b)} = f\, dt$

$$\int_\gamma \omega = \int_{[a,b]} \gamma^*\omega = \int_a^b f(t)\,dt \qquad \forall$$



lia f

Ex

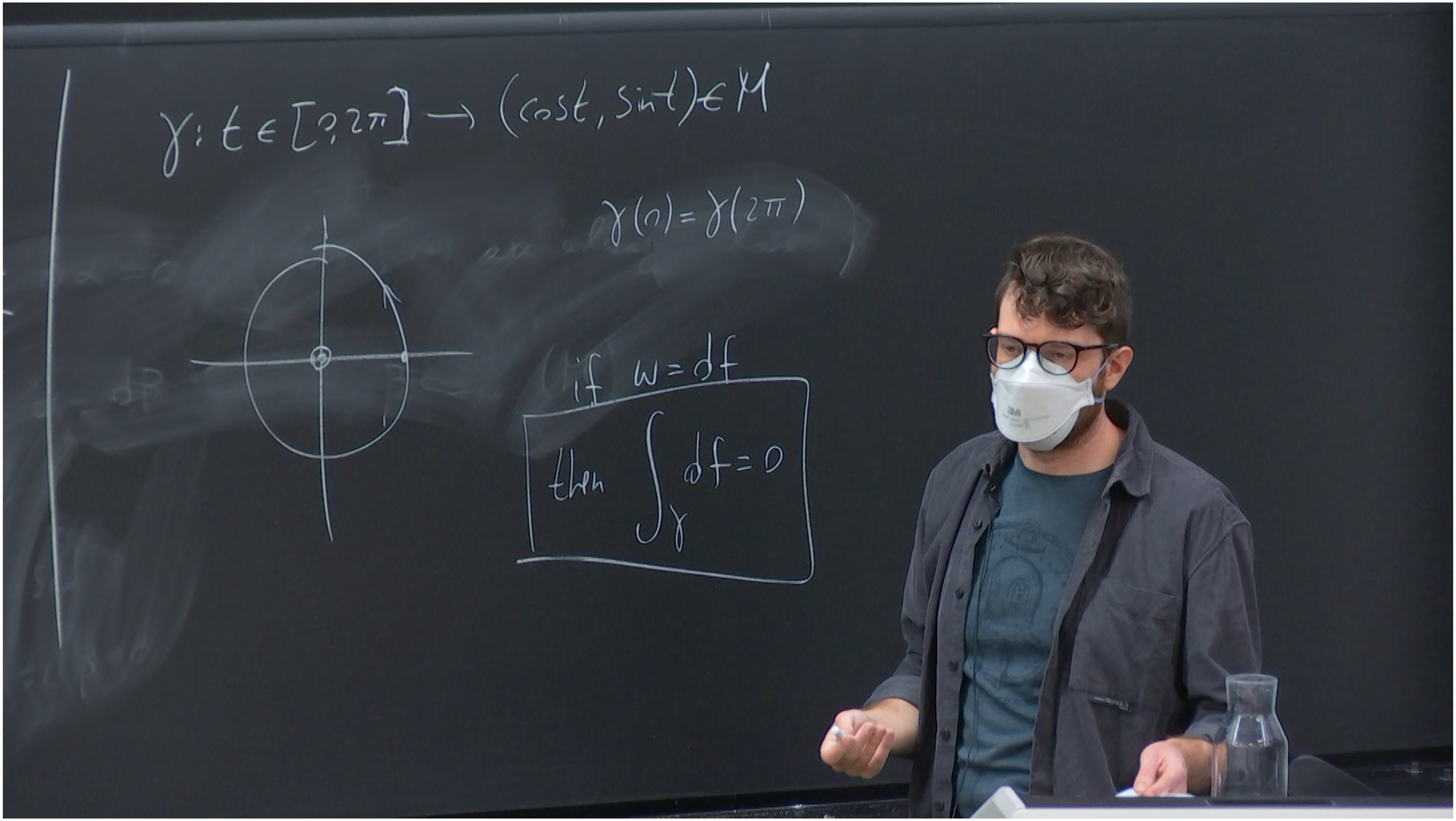
$M = \mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

$d\omega = 0$

closed

$\gamma : t \in [0, 2\pi] \to (\cos t, \sin t) \in M$

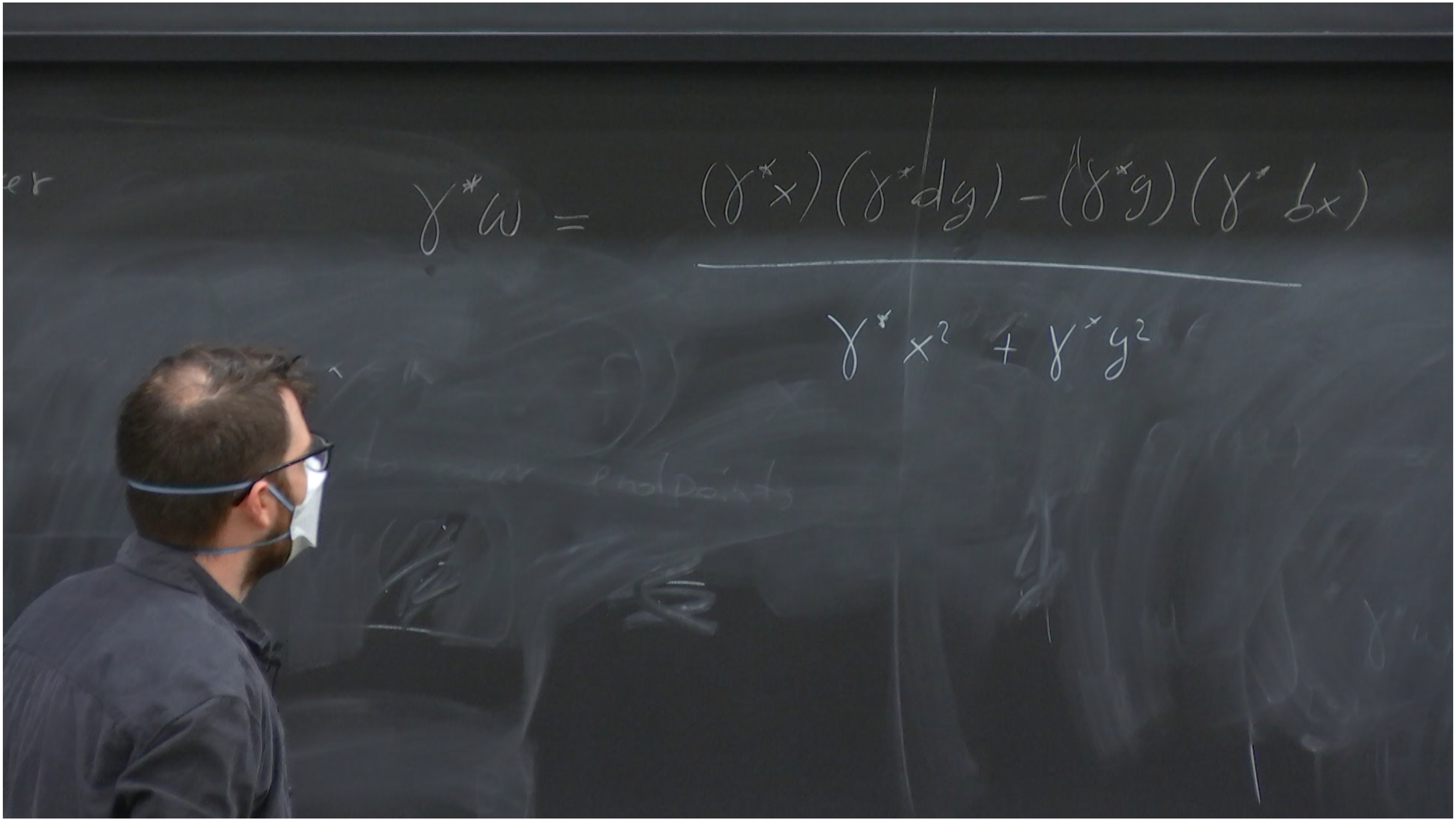


$\gamma : t \in [0,2\pi] \longrightarrow (\cos t, \sin t) \in M$

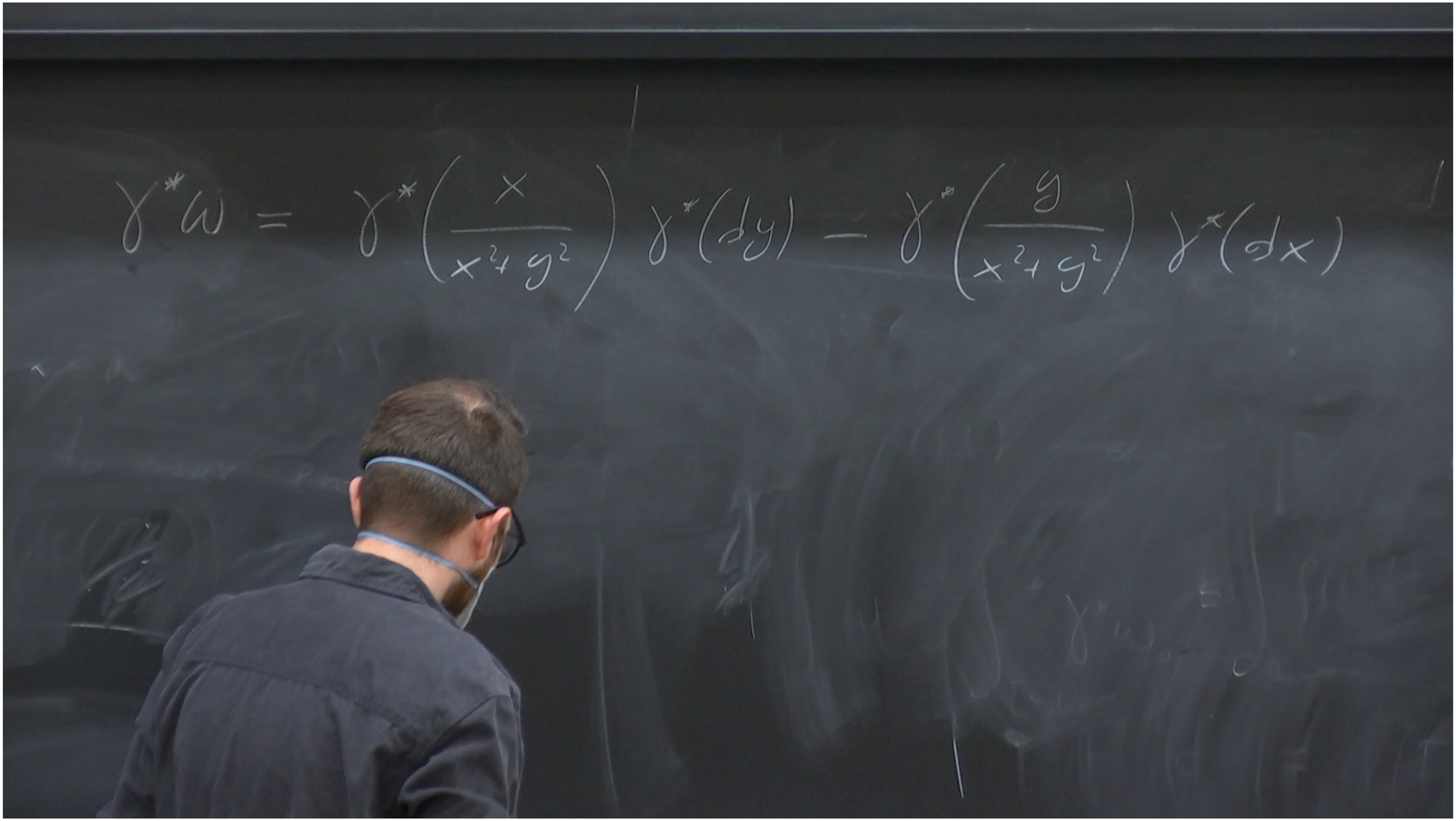
$\gamma(0) = \gamma(2\pi)$

if $\omega = df$

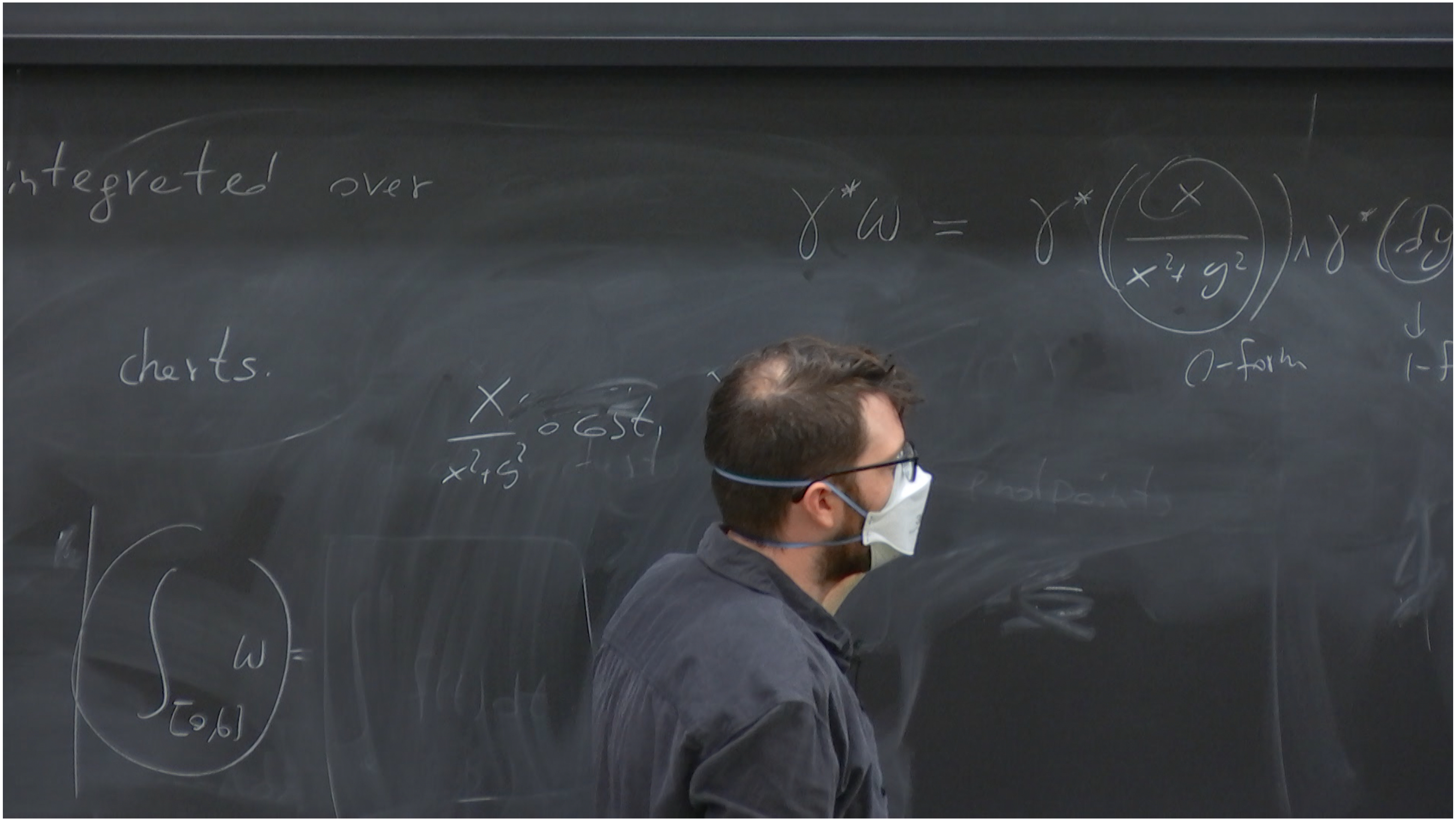
$\boxed{\text{then } \int_\gamma df = 0}$



$$\gamma^* \omega = \frac{(\gamma^* x)(\gamma^* dy) - (\gamma^* y)(\gamma^* dx)}{\gamma^* x^2 + \gamma^* y^2}$$



$$\gamma^*\omega = \gamma^*\left(\frac{x}{x^2+y^2}\right)\gamma^*(dy) - \gamma^*\left(\frac{y}{x^2+y^2}\right)\gamma^*(dx)$$



integrated over

charts.

$$\gamma^*\omega = \gamma^*\left(\frac{x}{x^2+y^2}\right) \wedge \gamma^*(dy$$

0-form

1-f

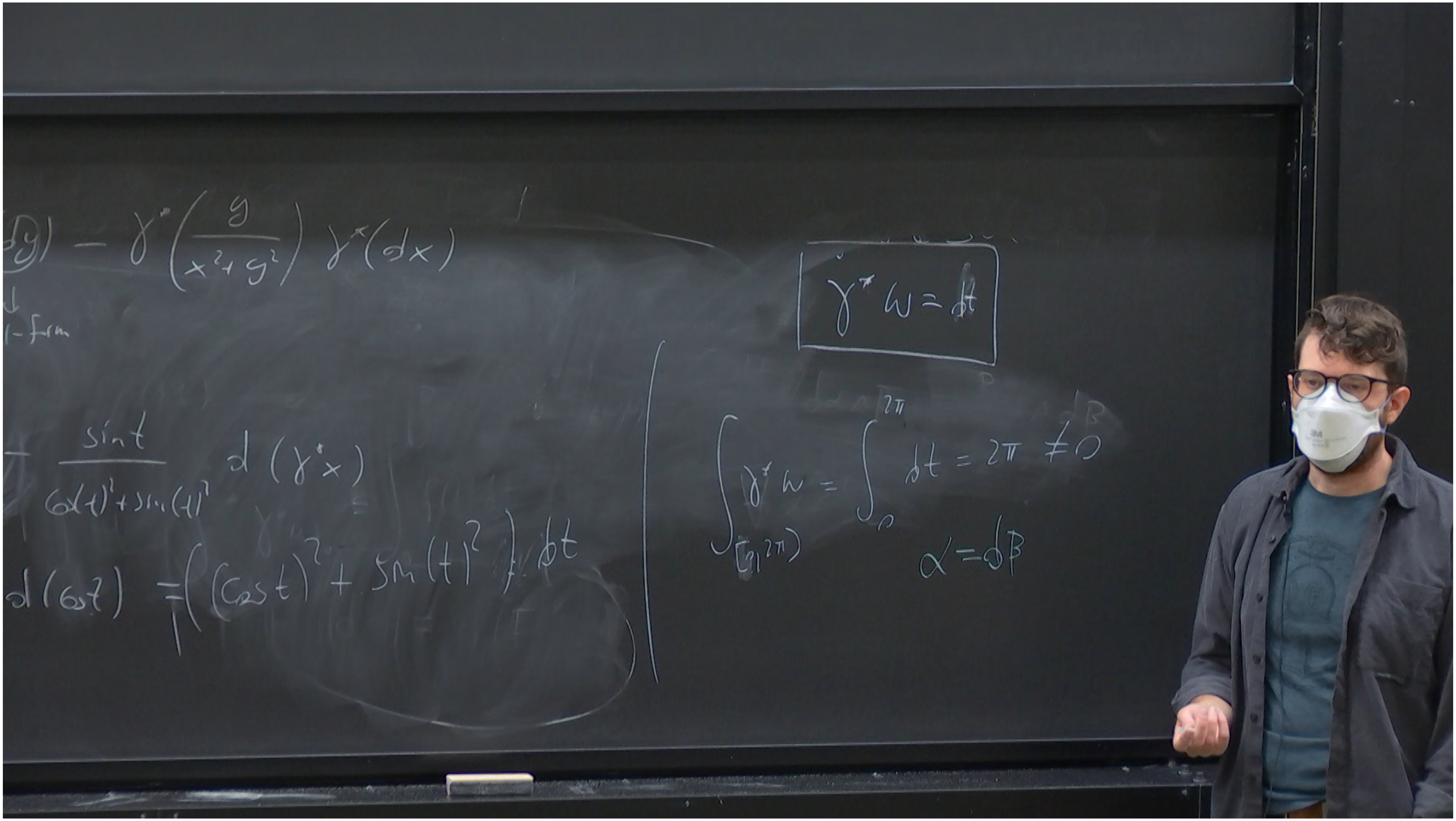
$$\frac{x}{x^2+y^2} \circ \cos t$$

$$\int_{[a,b]} \omega =$$

$$\gamma^*\omega = \gamma^*\underbrace{\left(\boxed{\frac{x}{x^2+y^2}}\right)}_{\text{0-form}}\wedge\gamma^*\underbrace{\left(\boxed{dy}\right)}_{\downarrow\ \text{1-form}} - \gamma^*\left(\frac{y}{x^2+y^2}\right)\gamma^*(dx)$$

$$= \frac{\cos t}{\cos(t)^2+\sin(t)^2}\, d(\gamma^* y) - \frac{\sin t}{\cos(t)^2+\sin(t)^2}\, d(\gamma^* x)$$

$$= \cos t\, d(\sin t) - \sin t\, d(\cos t) = (\cos t)^2 + \sin(t)^2 = 1$$



$$(dy) - \gamma^*\left(\frac{y}{x^2+y^2}\right)\gamma^*(dx)$$

1-form

$$\frac{\sin t}{\cos(t)^2+\sin(t)^2}\, d(\gamma^* x)$$

$$d(\cos t) = \left((\cos t)^2 + \sin(t)^2\right) dt$$

$$\boxed{\gamma^*\omega = dt}$$

$$\int_{[0,2\pi]} \gamma^*\omega = \int_0^{2\pi} dt = 2\pi \neq 0$$

$$\alpha = d\beta$$