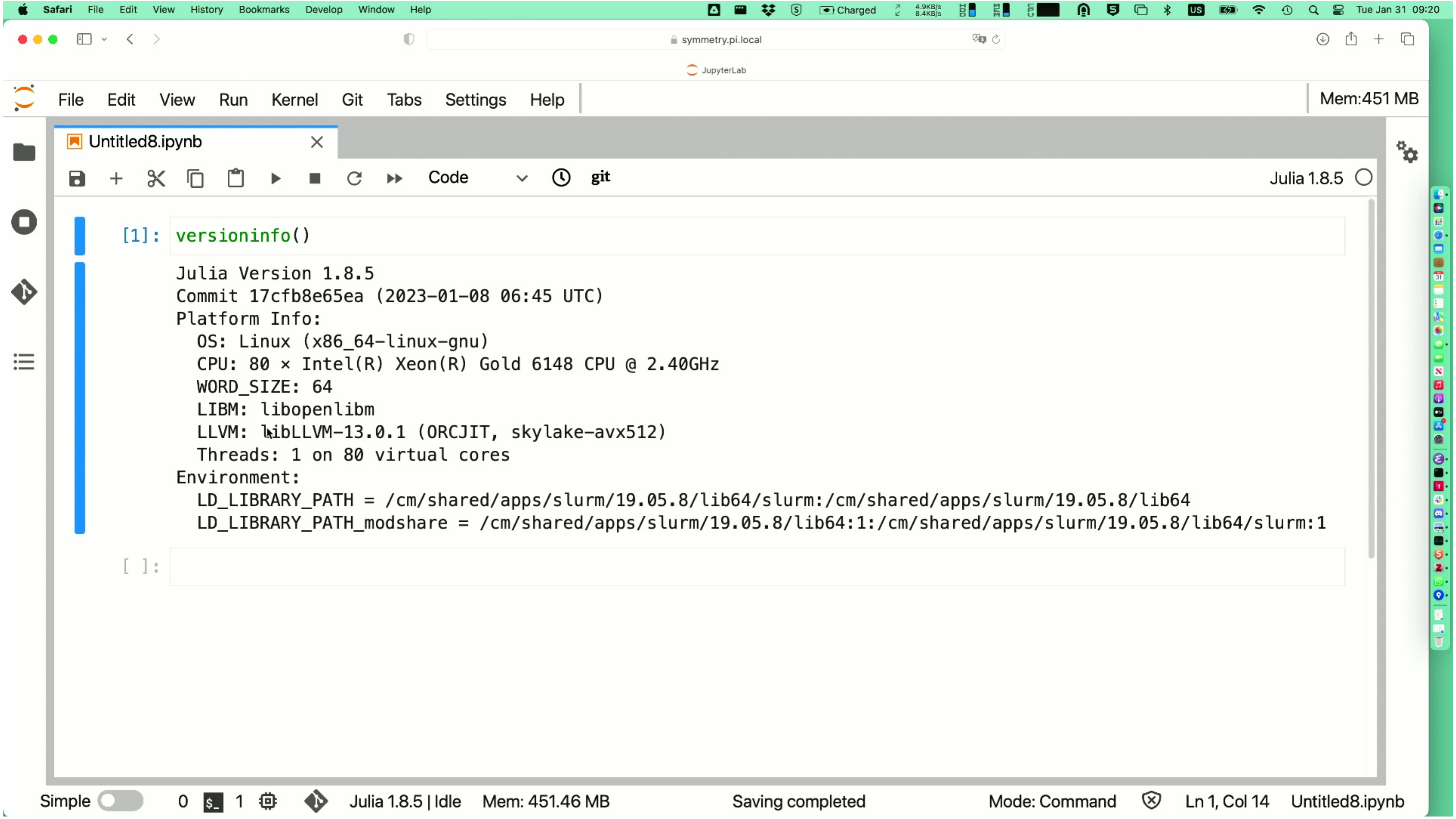
Title: Numerical Methods Lecture - 230131

Speakers: Erik Schnetter

Collection: Numerical Methods (2022/2023)

Date: January 31, 2023 - 9:15 AM

URL: https://pirsa.org/23010008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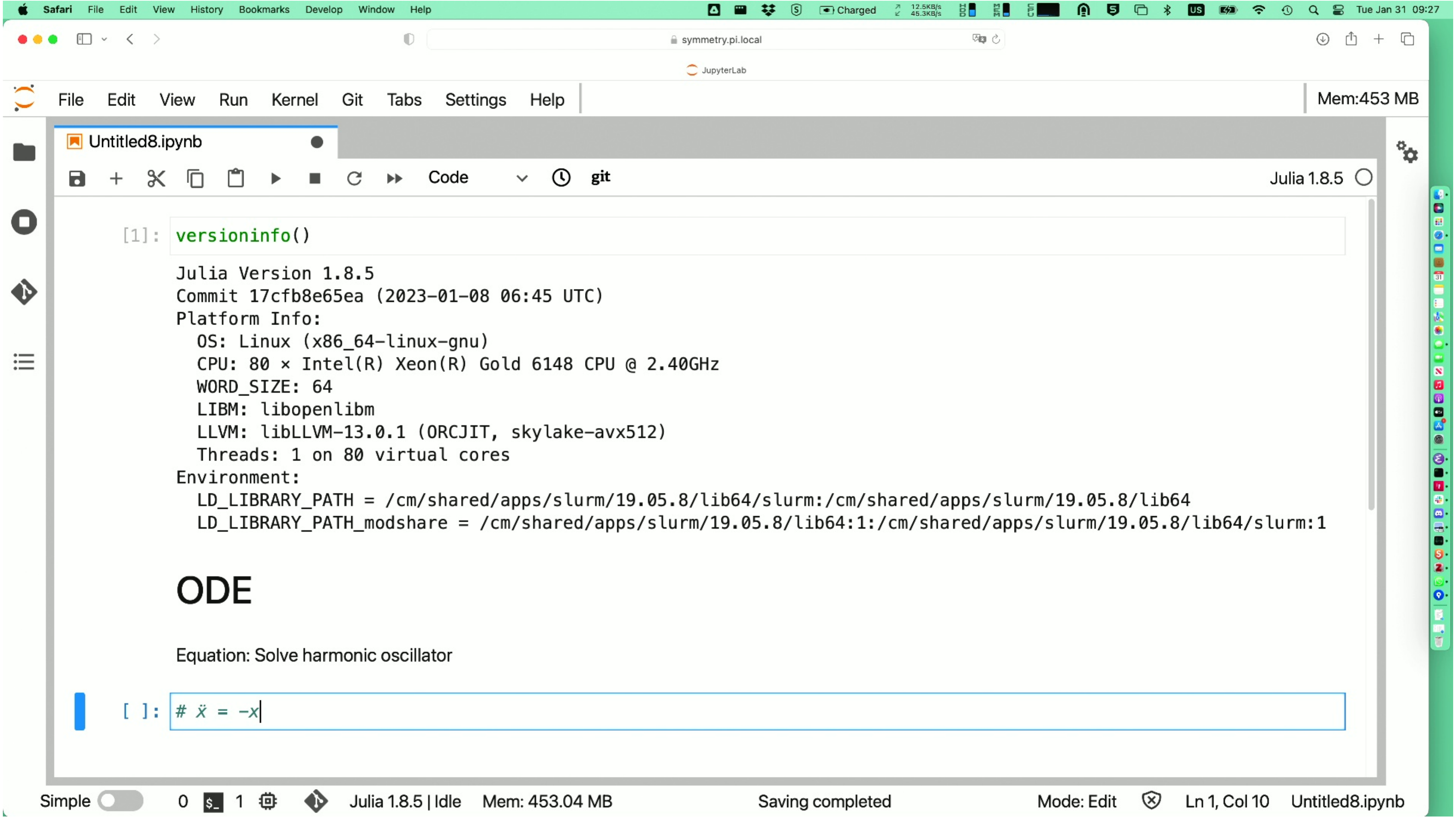
Code
git
Julia 1.8.5

```
[1]: versioninfo()
```

```
Julia Version 1.8.5
Commit 17cfb8e65ea (2023-01-08 06:45 UTC)
Platform Info:
  OS: Linux (x86_64-linux-gnu)
  CPU: 80 × Intel(R) Xeon(R) Gold 6148 CPU @ 2.40GHz
  WORD_SIZE: 64
  LIBM: libopenlibm
  LLVM: libLLVM-13.0.1 (ORCJIT, skylake-avx512)
  Threads: 1 on 80 virtual cores
Environment:
  LD_LIBRARY_PATH = /cm/shared/apps/slurm/19.05.8/lib64/slurm:/cm/shared/apps/slurm/19.05.8/lib64
  LD_LIBRARY_PATH_modshare = /cm/shared/apps/slurm/19.05.8/lib64:1:/cm/shared/apps/slurm/19.05.8/lib64/slurm:1
```

```
[ ]:
```



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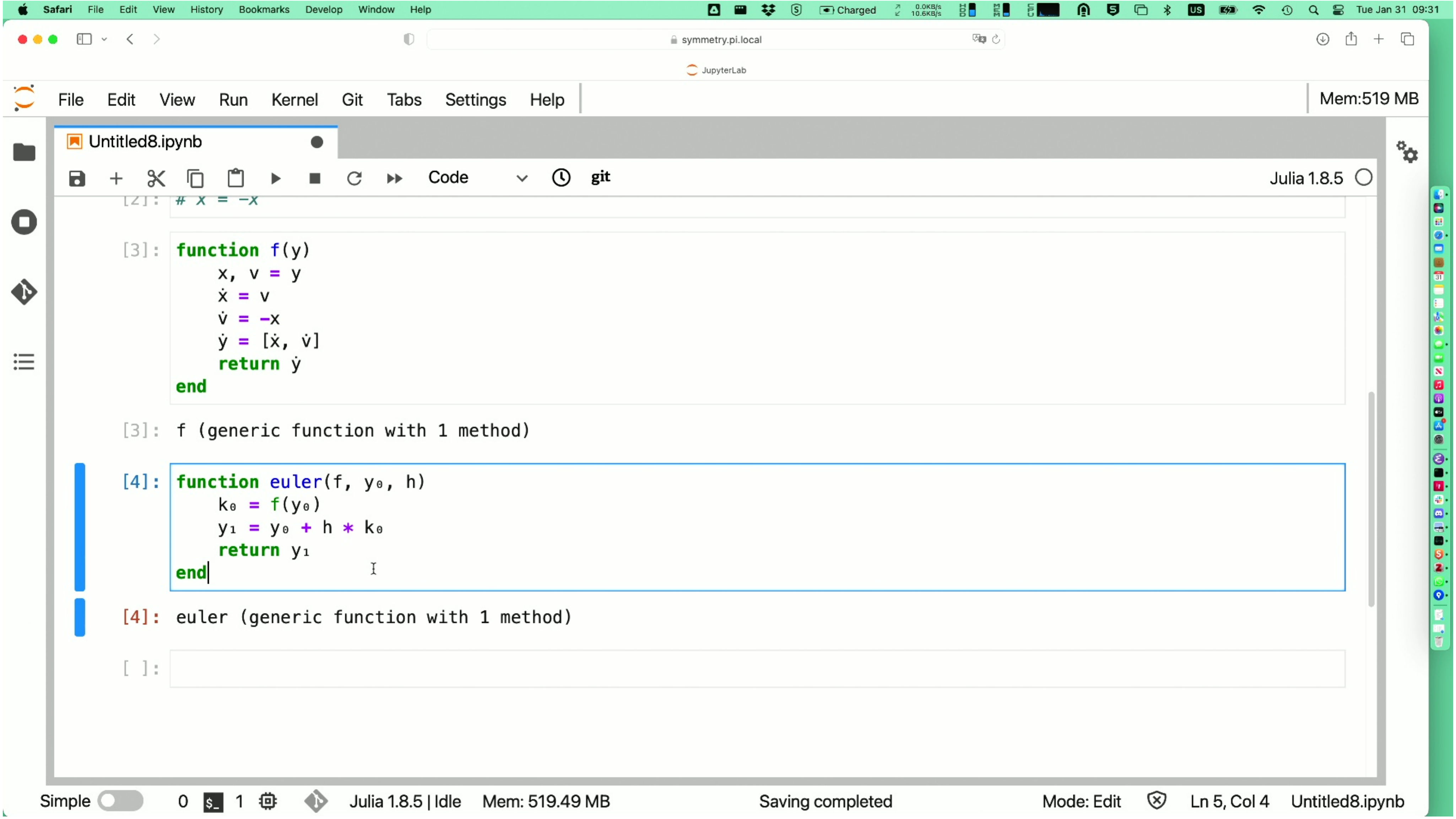
```

ODE

Equation: Solve harmonic oscillator

```
[ ]: # ẍ = -x
```



```
[2]: # x = −x
```

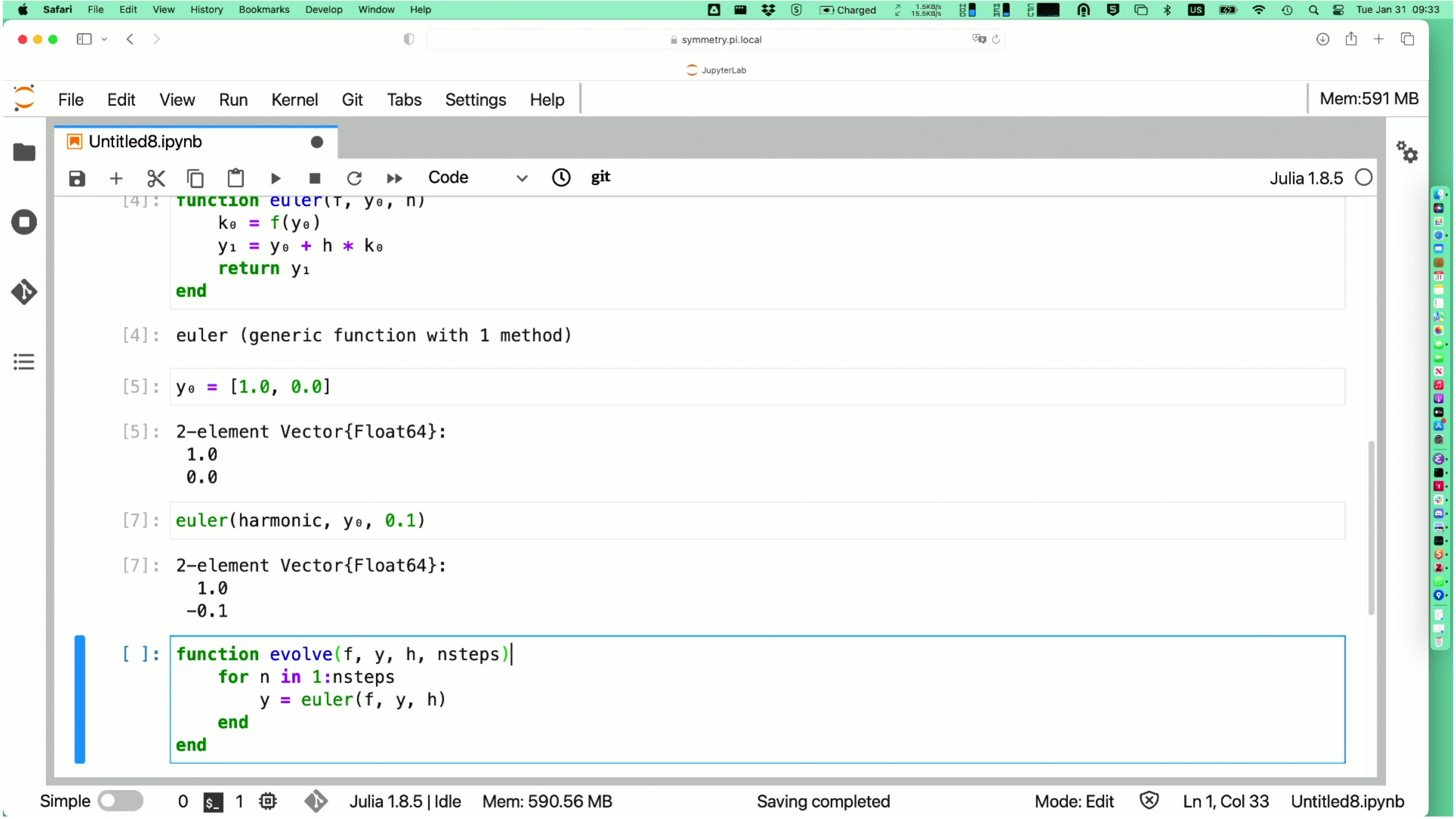
```
[3]: function f(y)
         x, v = y
         ẋ = v
         v̇ = −x
         ẏ = [ẋ, v̇]
         return ẏ
     end
```

```
[3]: f (generic function with 1 method)
```

```
[4]: function euler(f, y₀, h)
         k₀ = f(y₀)
         y₁ = y₀ + h * k₀
         return y₁
     end
```

```
[4]: euler (generic function with 1 method)
```

```
[ ]:
```



```
[4]: function euler(f, y₀, h)
         k₀ = f(y₀)
         y₁ = y₀ + h * k₀
         return y₁
     end

[4]: euler (generic function with 1 method)

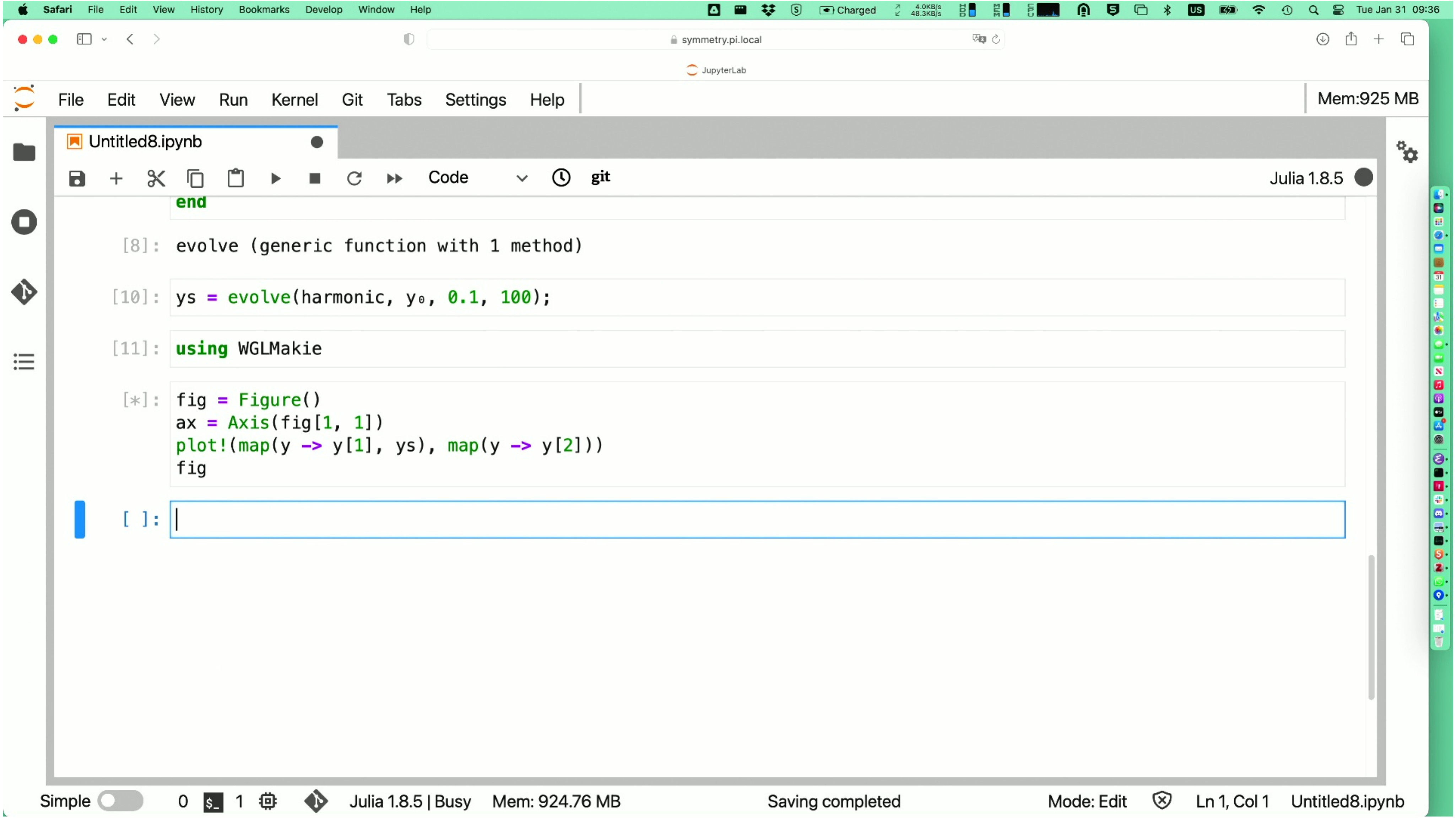
[5]: y₀ = [1.0, 0.0]

[5]: 2-element Vector{Float64}:
      1.0
      0.0

[7]: euler(harmonic, y₀, 0.1)

[7]: 2-element Vector{Float64}:
       1.0
      -0.1

[ ]: function evolve(f, y, h, nsteps)
         for n in 1:nsteps
             y = euler(f, y, h)
         end
     end
```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Code
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Julia 1.8.5

```
    end
```

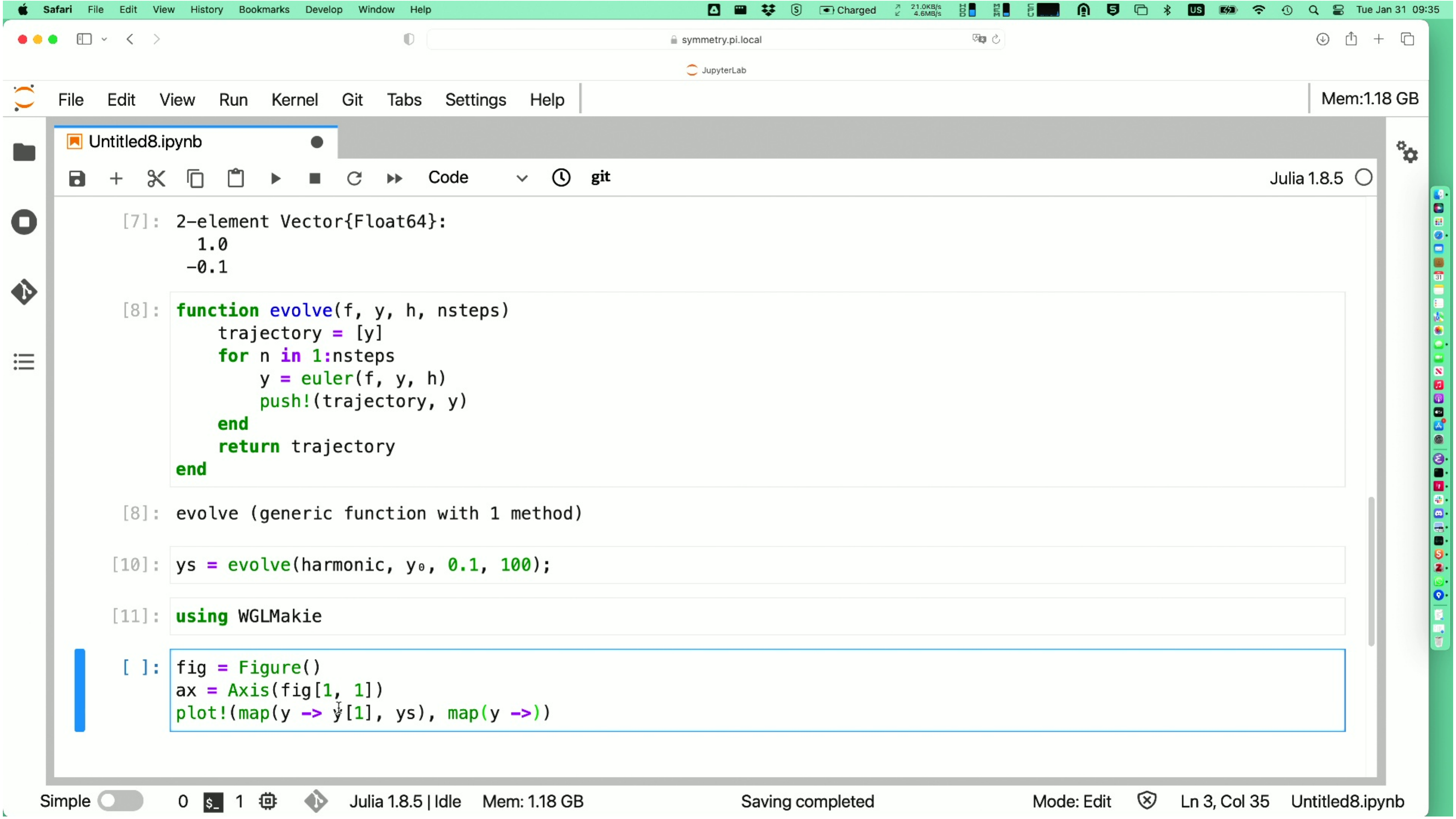
```
[8]: evolve (generic function with 1 method)
```

```
[10]: ys = evolve(harmonic, y₀, 0.1, 100);
```

```
[11]: using WGLMakie
```

```
[*]: fig = Figure()
     ax = Axis(fig[1, 1])
     plot!(map(y -> y[1], ys), map(y -> y[2]))
     fig
```

```
[ ]:
```



```
[7]: 2-element Vector{Float64}:
       1.0
      -0.1

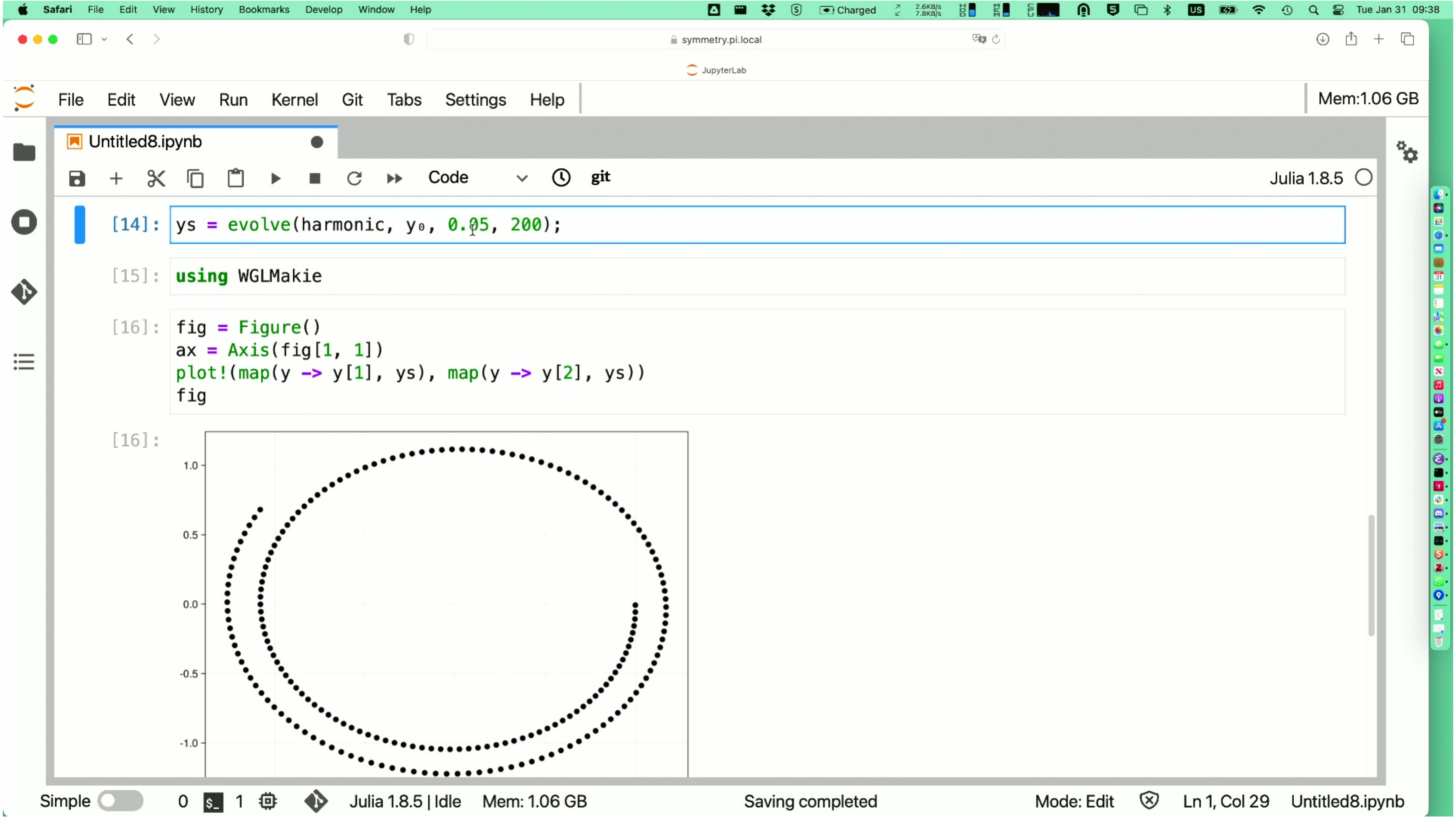
[8]: function evolve(f, y, h, nsteps)
         trajectory = [y]
         for n in 1:nsteps
             y = euler(f, y, h)
             push!(trajectory, y)
         end
         return trajectory
     end

[8]: evolve (generic function with 1 method)

[10]: ys = evolve(harmonic, y₀, 0.1, 100);

[11]: using WGLMakie

[ ]: fig = Figure()
     ax = Axis(fig[1, 1])
     plot!(map(y -> y[1], ys), map(y ->))
```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Code
git
Julia 1.8.5
1.0
0.5
0.0
-0.5
-1.0

[14]:
```
ys = evolve(harmonic, y₀, 0.05, 200);
```

[15]:
```
using WGLMakie
```

[16]:
```
fig = Figure()
ax = Axis(fig[1, 1])
plot!(map(y -> y[1], ys), map(y -> y[2], ys))
fig
```

[16]:

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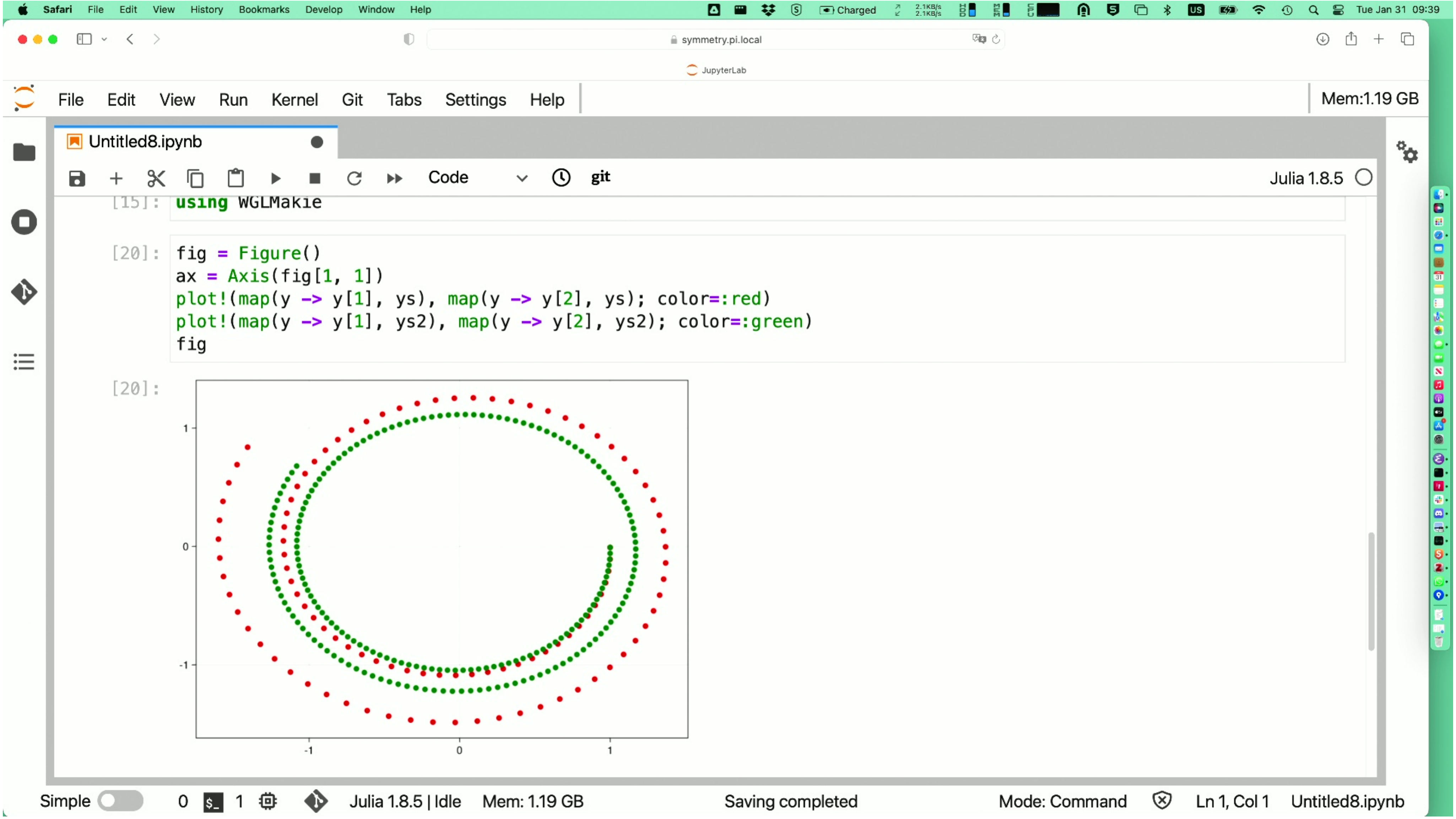
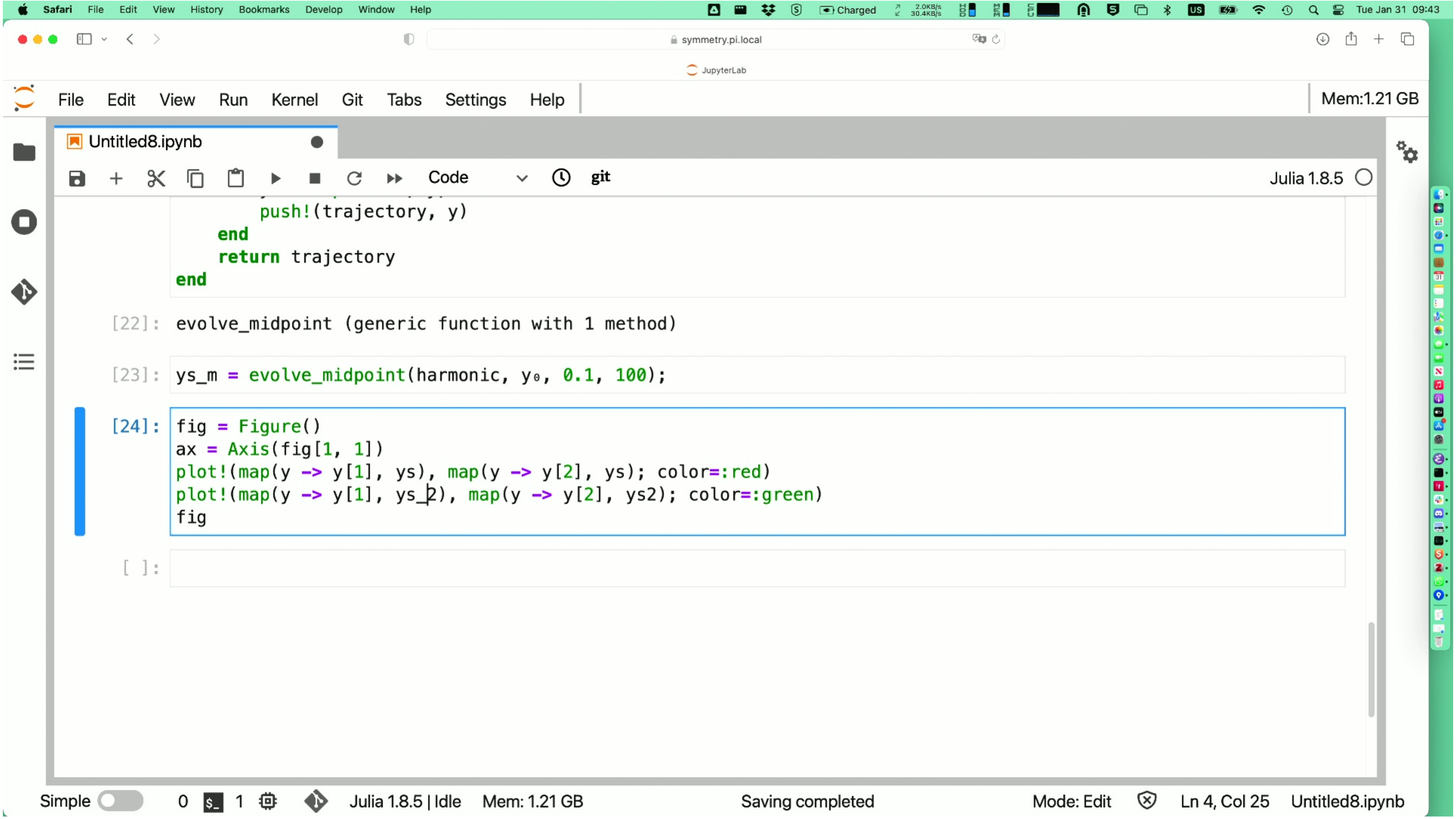


```
[15]: using WGLMakie
```

```
[20]: fig = Figure()
      ax = Axis(fig[1, 1])
      plot!(map(y -> y[1], ys), map(y -> y[2], ys); color=:red)
      plot!(map(y -> y[1], ys2), map(y -> y[2], ys2); color=:green)
      fig
```

[20]:



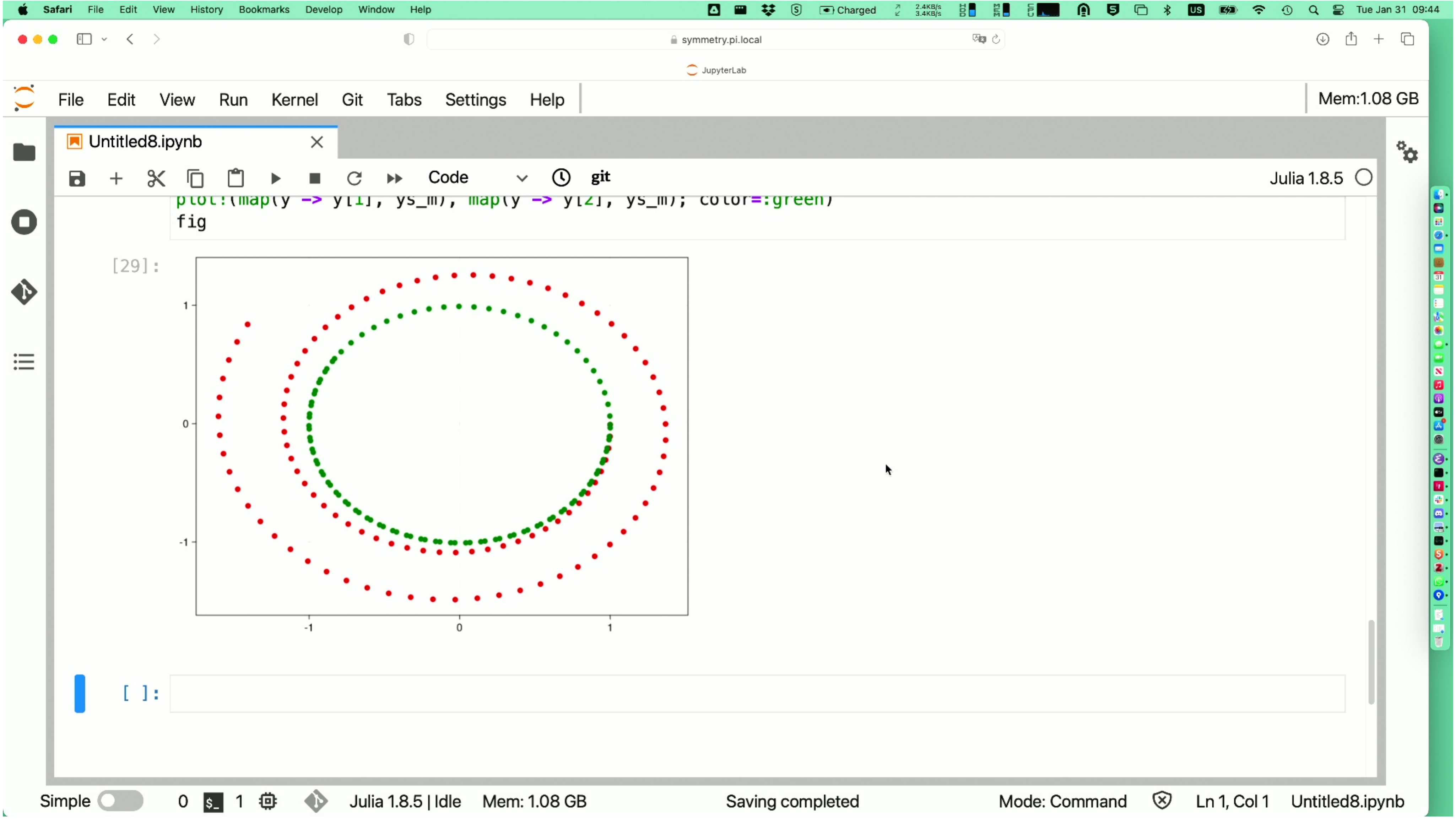
```
        push!(trajectory, y)
    end
    return trajectory
end
```

```
[22]: evolve_midpoint (generic function with 1 method)
```

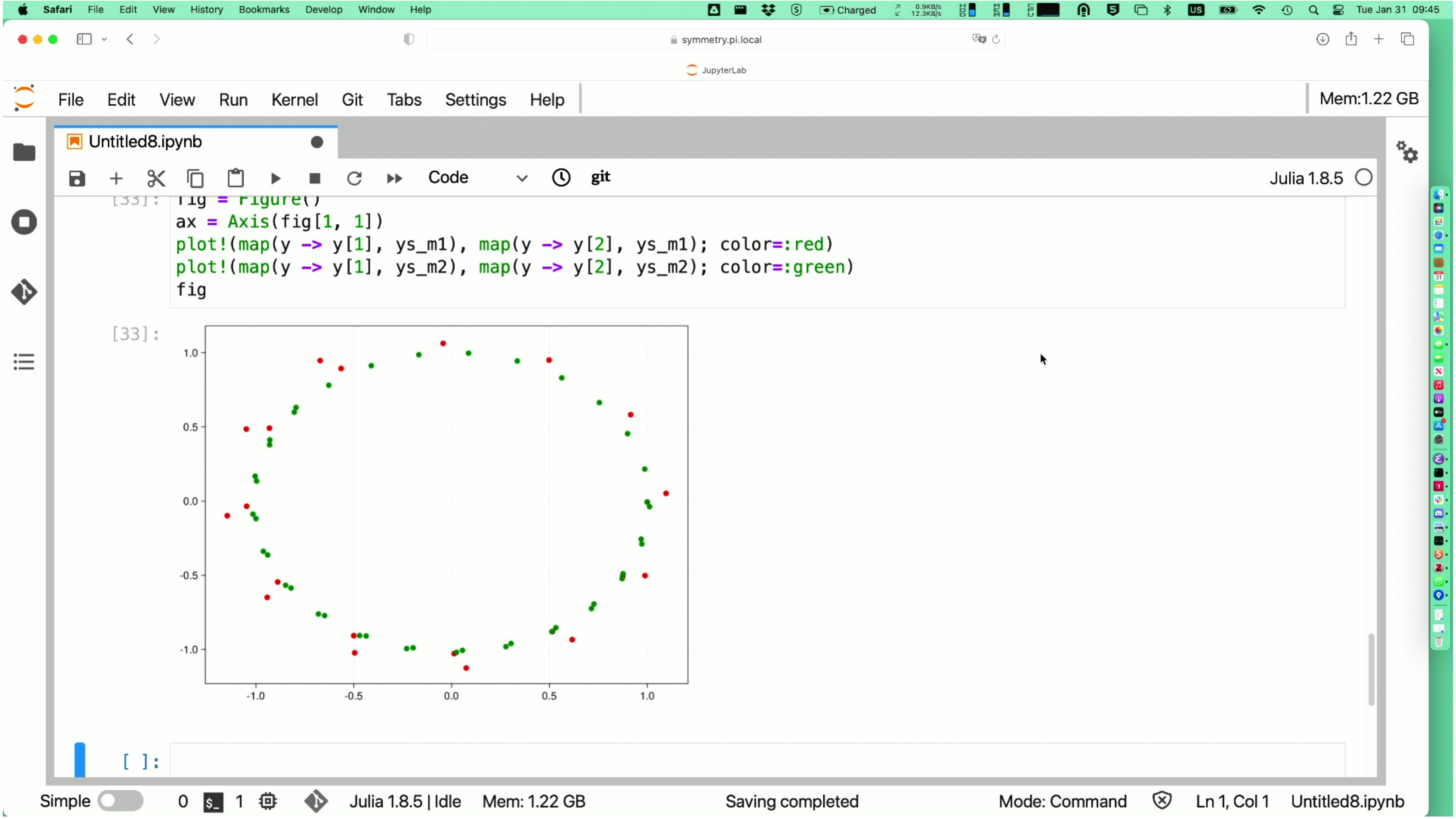
```
[23]: ys_m = evolve_midpoint(harmonic, y₀, 0.1, 100);
```

```
[24]: fig = Figure()
      ax = Axis(fig[1, 1])
      plot!(map(y -> y[1], ys), map(y -> y[2], ys); color=:red)
      plot!(map(y -> y[1], ys_2), map(y -> y[2], ys2); color=:green)
      fig
```

```
[ ]:
```



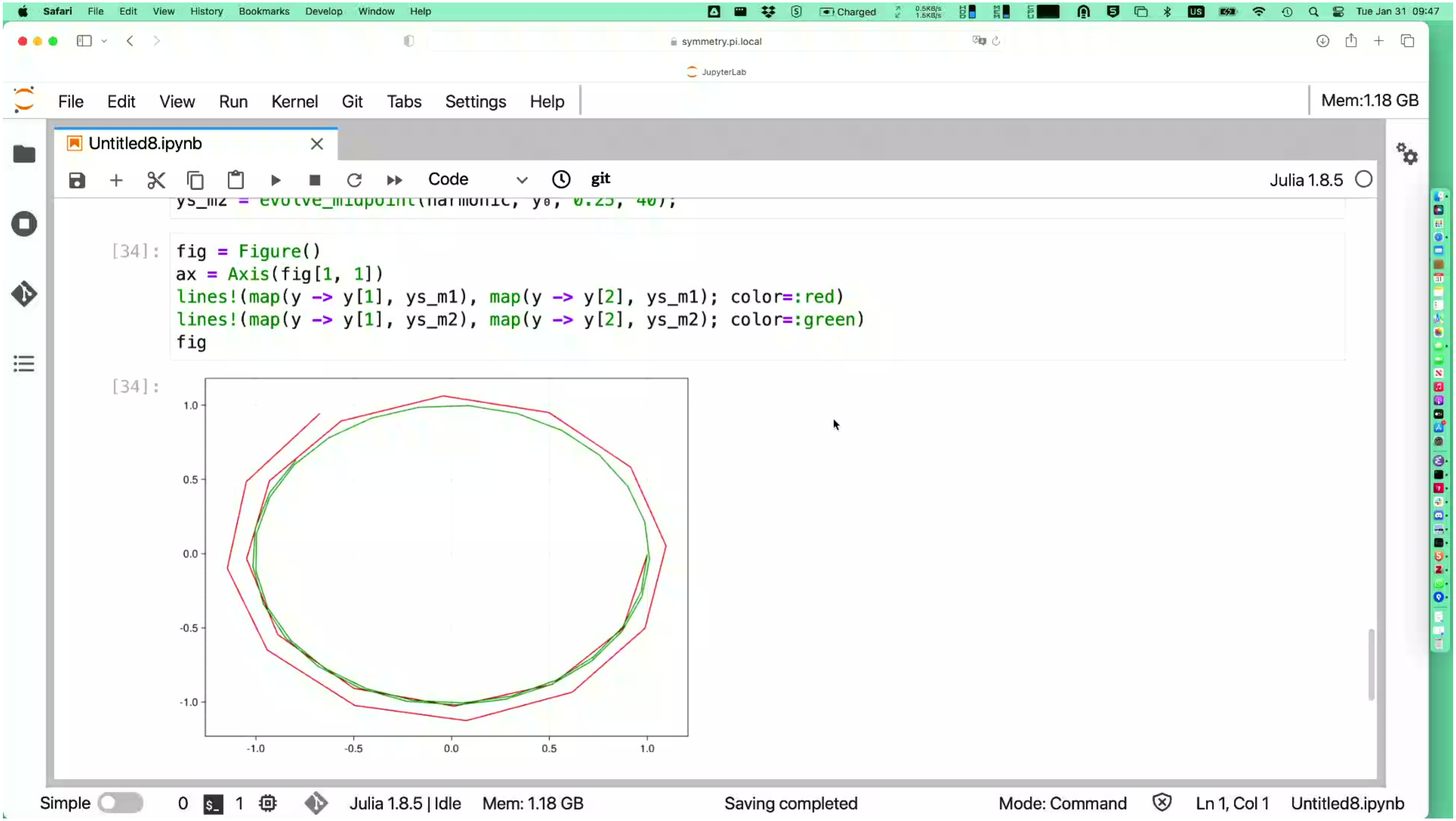
```
plot!(map(y -> y[1], ys_m), map(y -> y[2], ys_m); color=:green)
fig
```



```
[33]: fig = Figure()
      ax = Axis(fig[1, 1])
      plot!(map(y -> y[1], ys_m1), map(y -> y[2], ys_m1); color=:red)
      plot!(map(y -> y[1], ys_m2), map(y -> y[2], ys_m2); color=:green)
      fig
```

```
[33]:
```

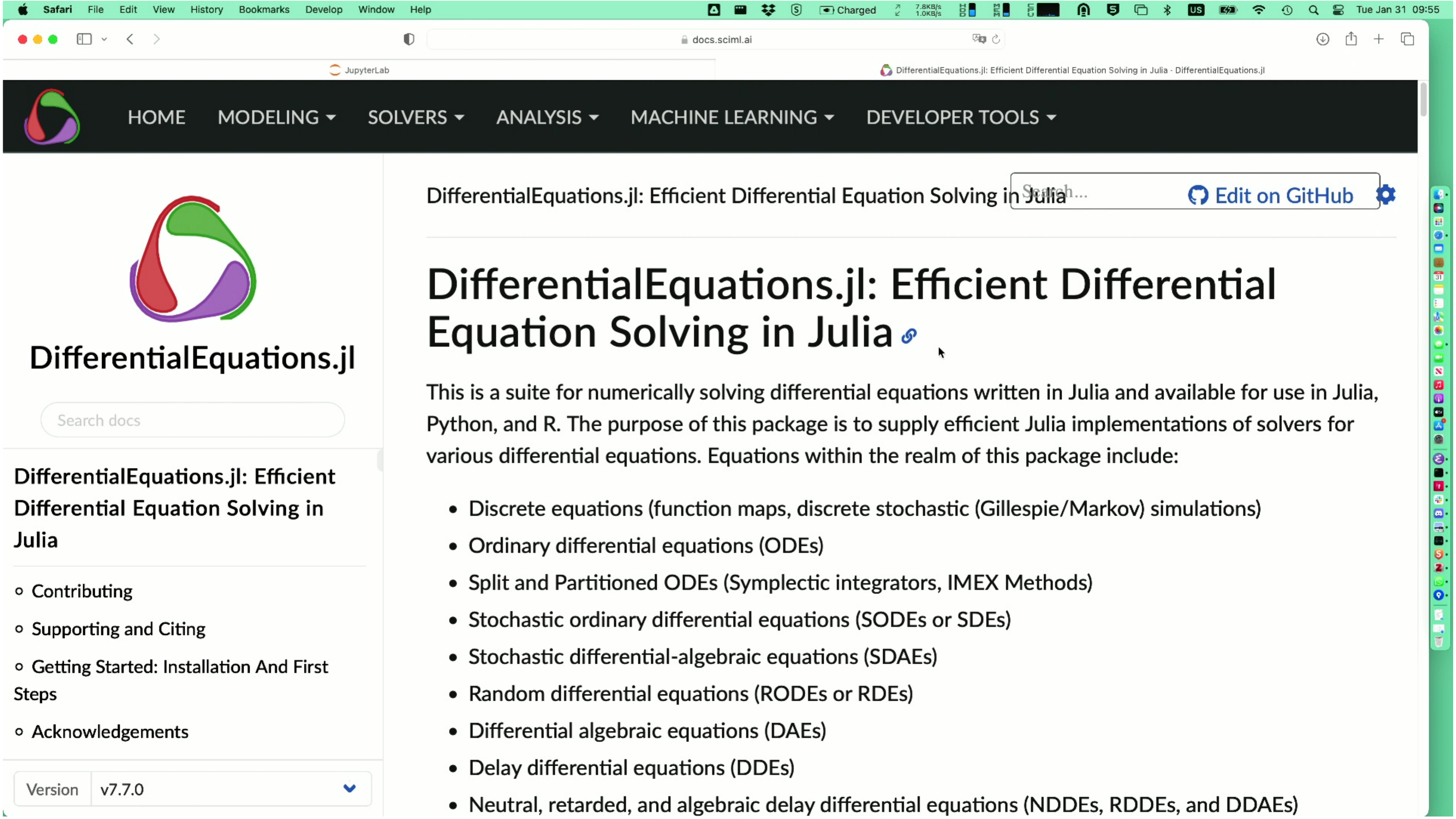
```
[ ]:
```



```
ys_m2 = evolve_midpoint(harmonic, y₀, 0.25, 40);
```

[34]:

```
fig = Figure()
ax = Axis(fig[1, 1])
lines!(map(y -> y[1], ys_m1), map(y -> y[2], ys_m1); color=:red)
lines!(map(y -> y[1], ys_m2), map(y -> y[2], ys_m2); color=:green)
fig
```



DifferentialEquations.jl: Efficient Differential Equation Solving in Julia

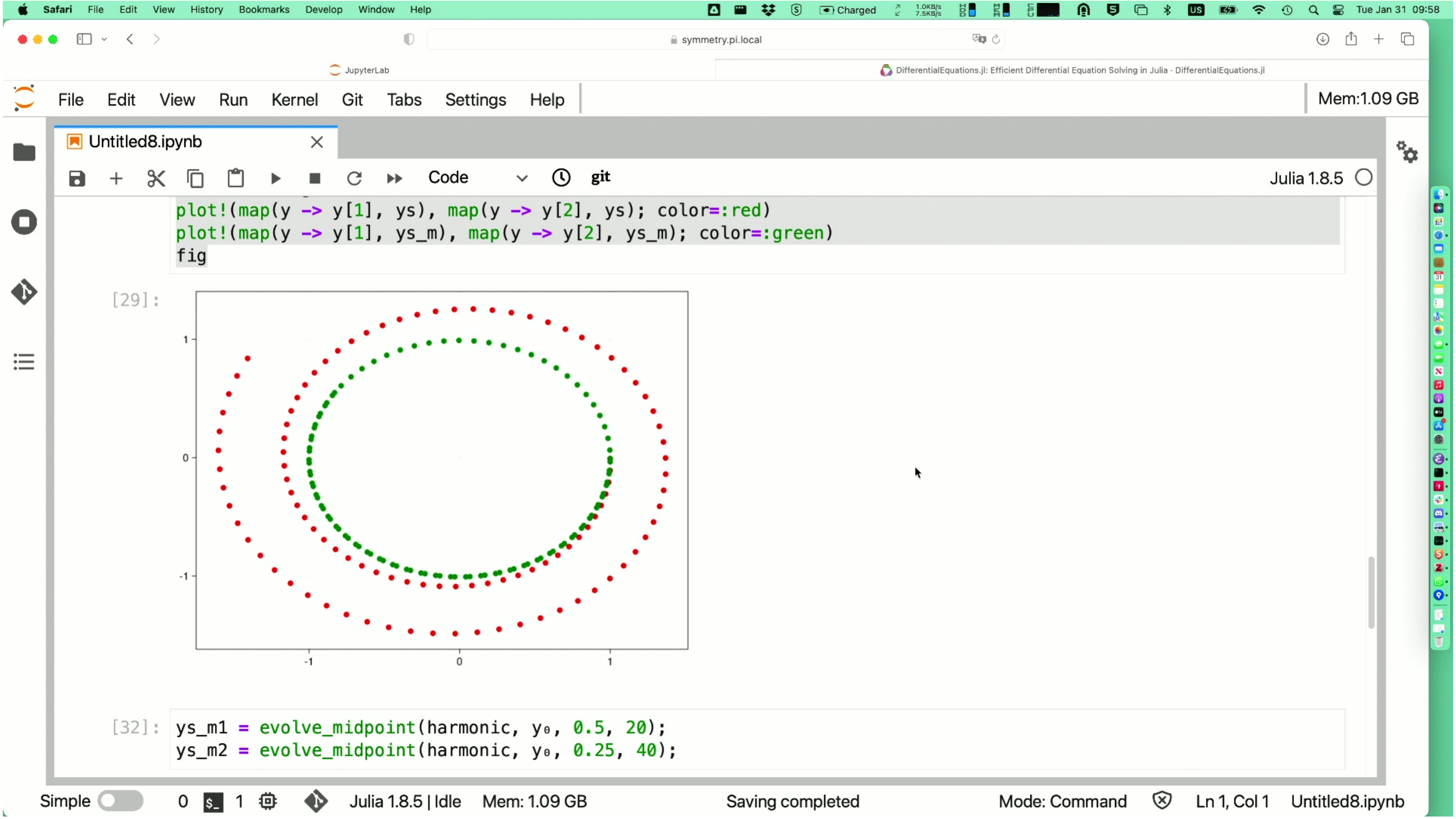
- Contributing
- Supporting and Citing
- Getting Started: Installation And First Steps
- Acknowledgements

DifferentialEquations.jl: Efficient Differential Equation Solving in Julia

DifferentialEquations.jl: Efficient Differential Equation Solving in Julia

This is a suite for numerically solving differential equations written in Julia and available for use in Julia, Python, and R. The purpose of this package is to supply efficient Julia implementations of solvers for various differential equations. Equations within the realm of this package include:

- Discrete equations (function maps, discrete stochastic (Gillespie/Markov) simulations)
- Ordinary differential equations (ODEs)
- Split and Partitioned ODEs (Symplectic integrators, IMEX Methods)
- Stochastic ordinary differential equations (SODEs or SDEs)
- Stochastic differential-algebraic equations (SDAEs)
- Random differential equations (RODEs or RDEs)
- Differential algebraic equations (DAEs)
- Delay differential equations (DDEs)
- Neutral, retarded, and algebraic delay differential equations (NDDEs, RDDEs, and DDAEs)



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Julia 1.8.5

```
plot!(map(y -> y[1], ys), map(y -> y[2], ys); color=:red)
plot!(map(y -> y[1], ys_m), map(y -> y[2], ys_m); color=:green)
fig
```

[29]:

[32]:

```
ys_m1 = evolve_midpoint(harmonic, y₀, 0.5, 20);
ys_m2 = evolve_midpoint(harmonic, y₀, 0.25, 40);
```

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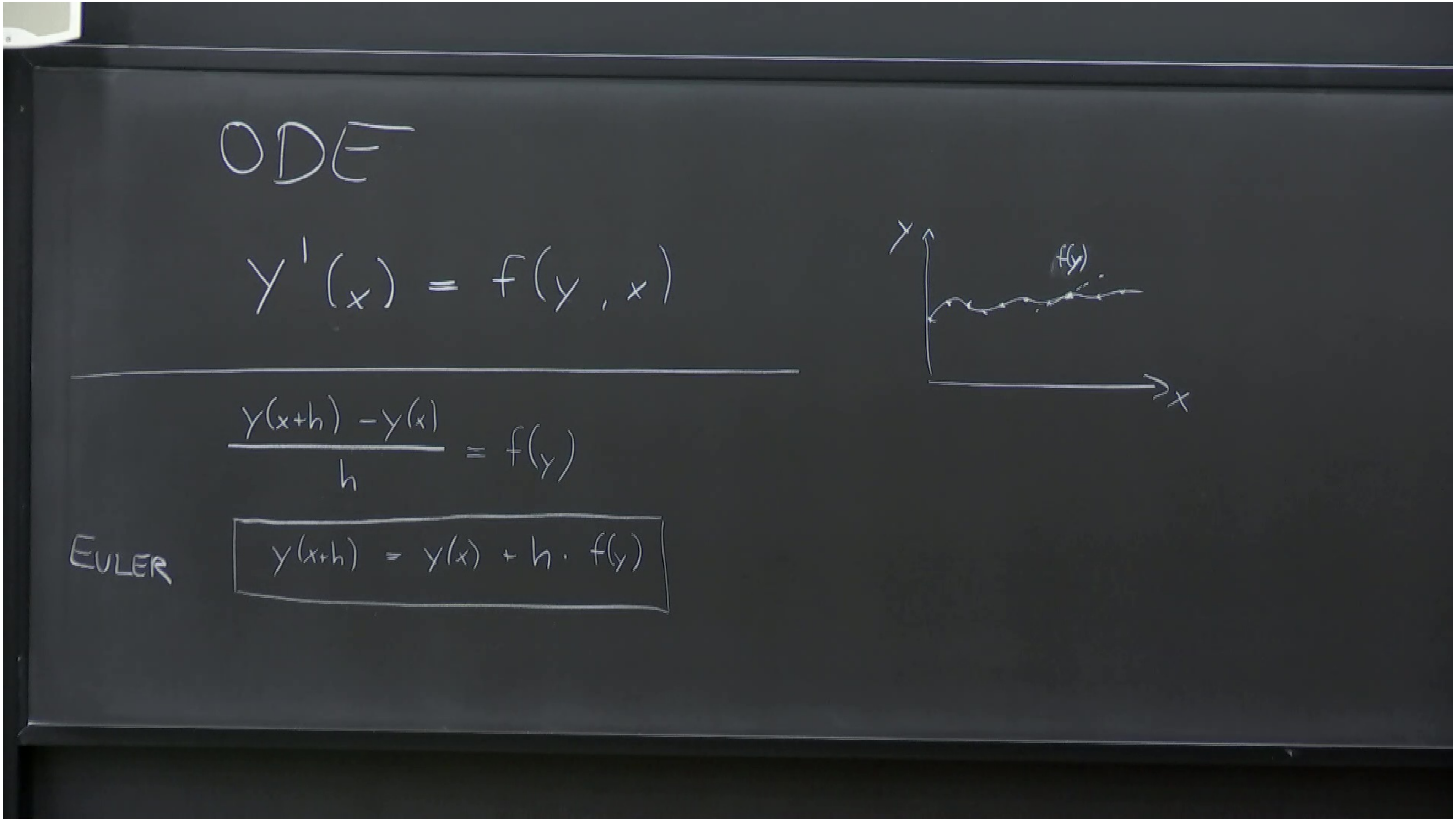
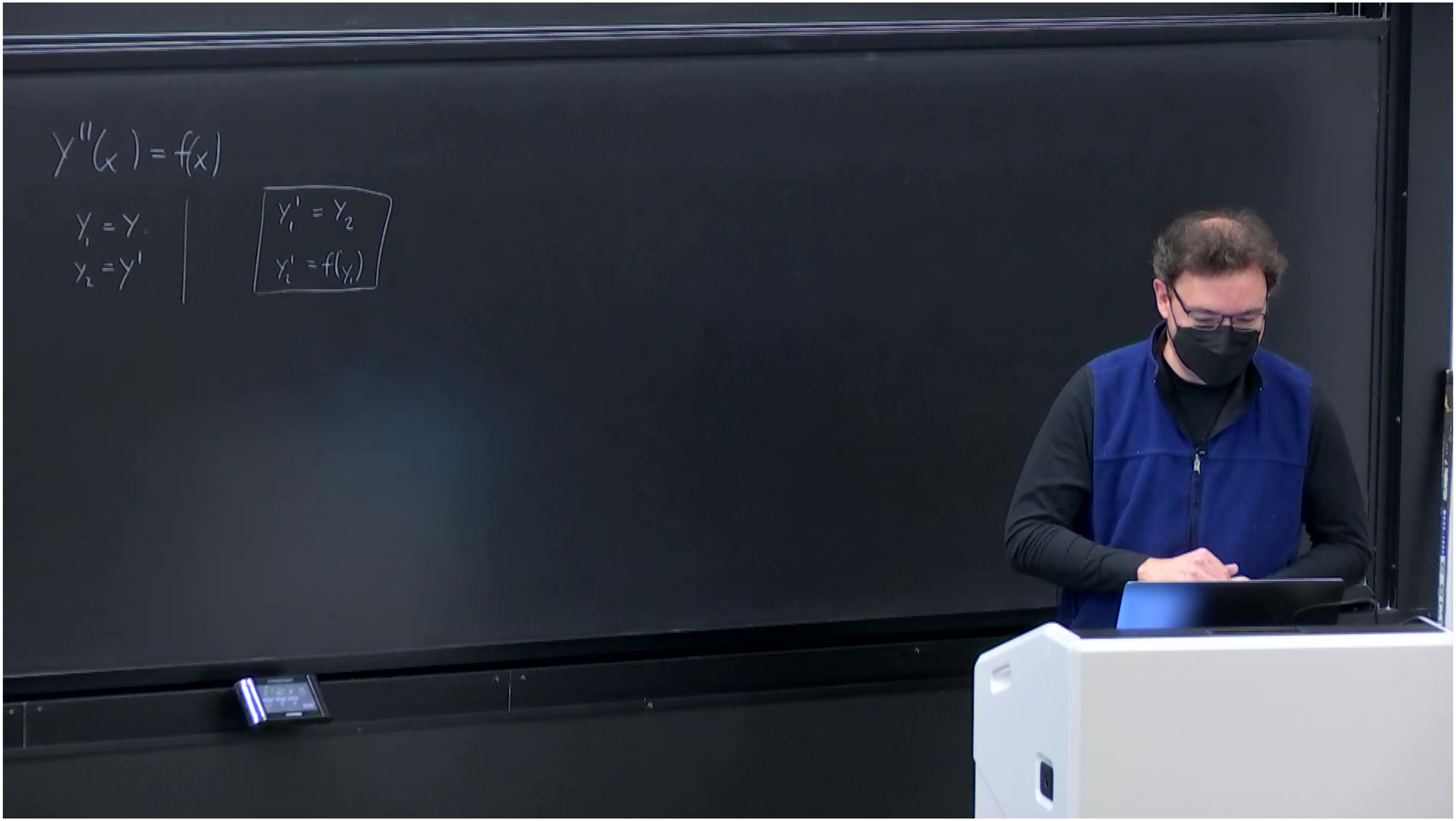


ODE

$$y'(x) = f(y, x)$$

$$\frac{y(x+h) - y(x)}{h} = f(y)$$

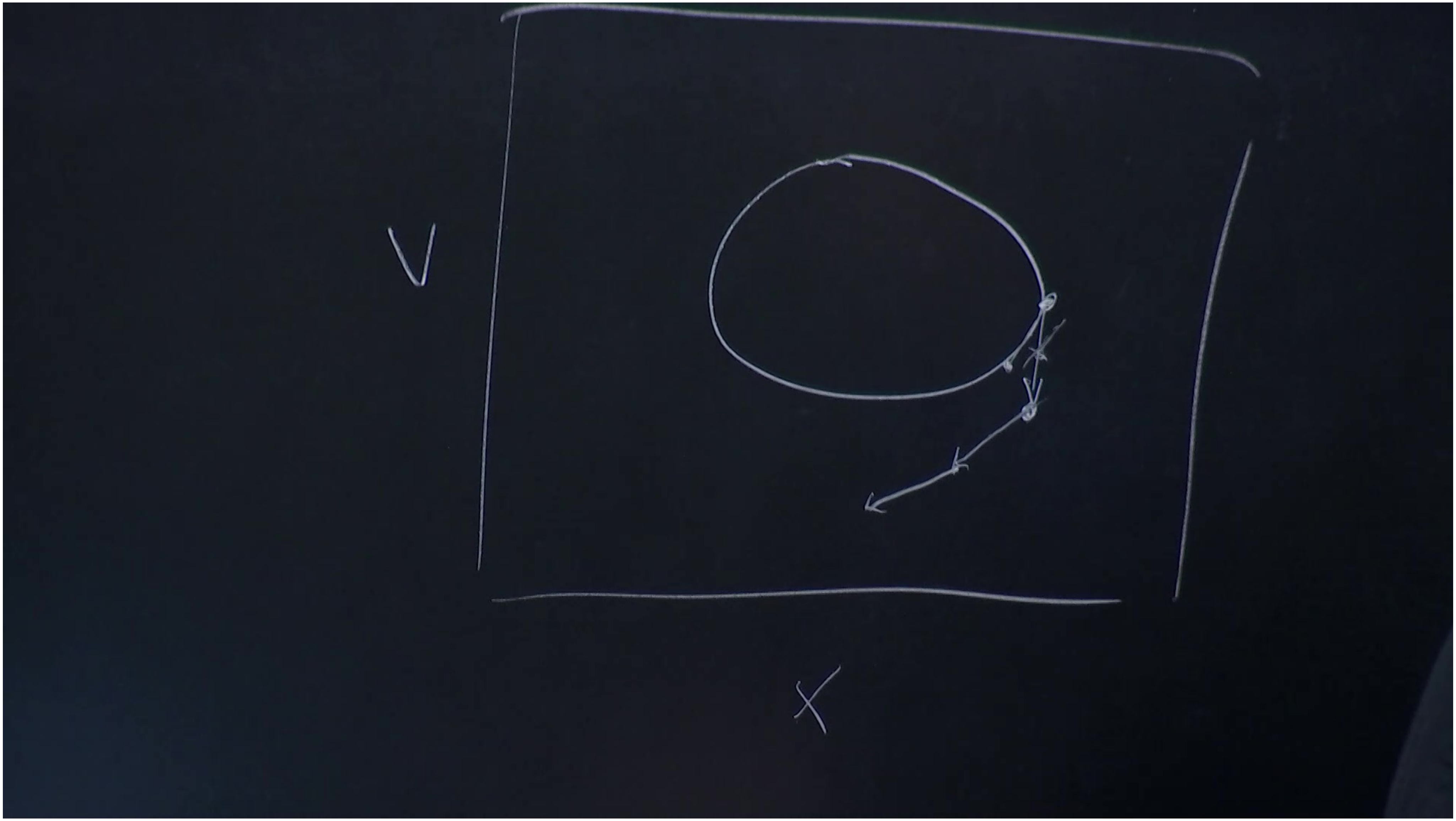
EULER $$\boxed{y(x+h) = y(x) + h \cdot f(y)}$$



$$y''(x) = f(x)$$

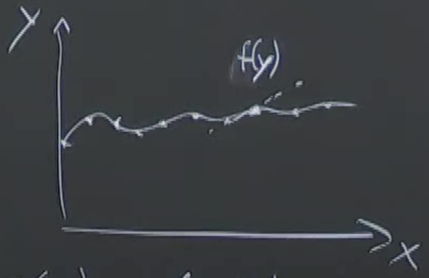
$$y_1 = y$$
$$y_2 = y'$$

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= f(y_1) \end{aligned}}$$


v
x

ODE

$$y'(x) = f(y,x)$$



$$\frac{y(x+h)-y(x)}{h} = f(y)$$

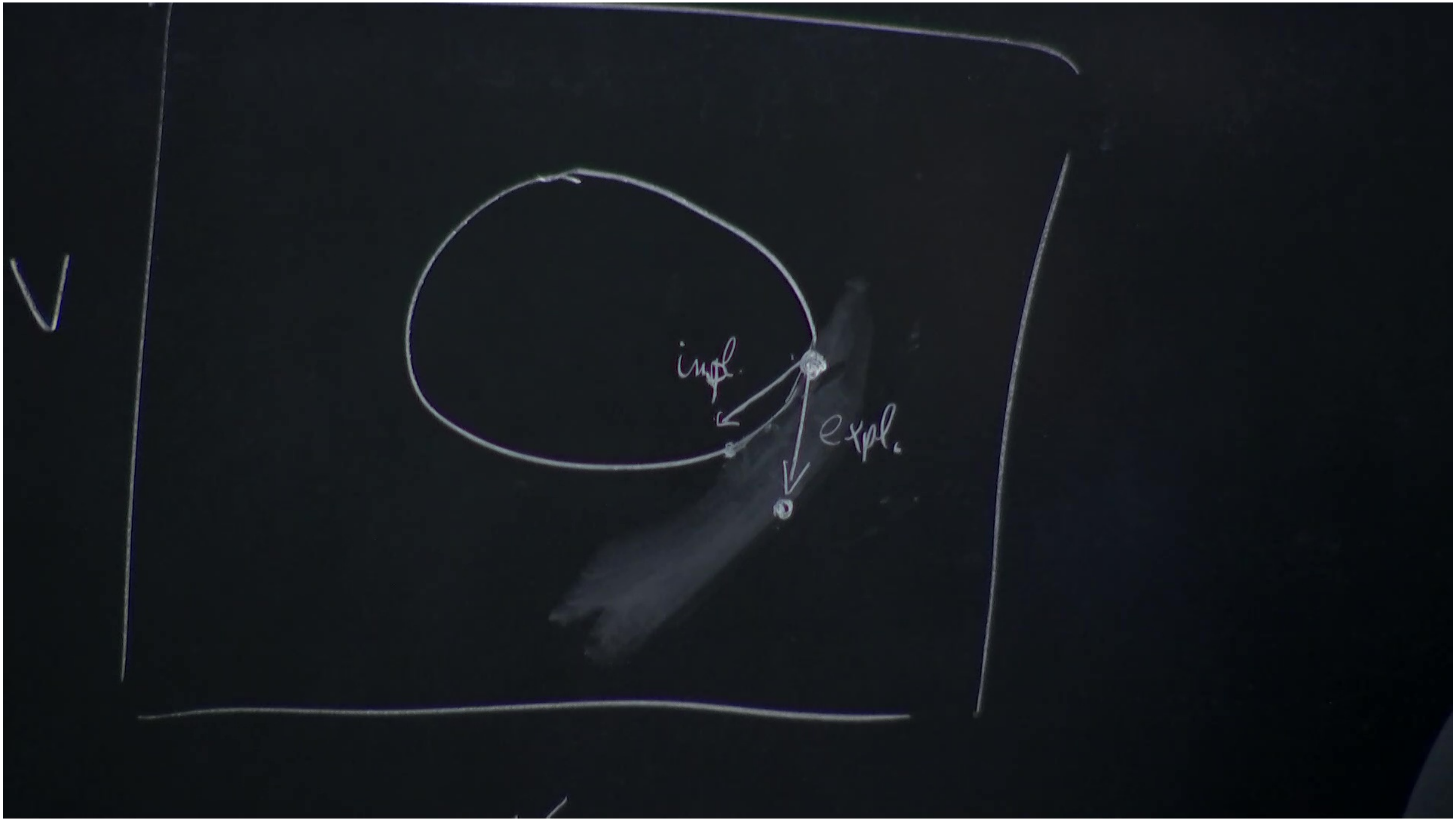
EULER $$\boxed{y(x+h) = y(x) + h\cdot f(y,x)}$$

explicit

$$\frac{y(x)-y(x-h)}{h} = f(y,x)$$

$$\boxed{y(x+h) = y(x) + h\, f(y, x+h)}$$

implicit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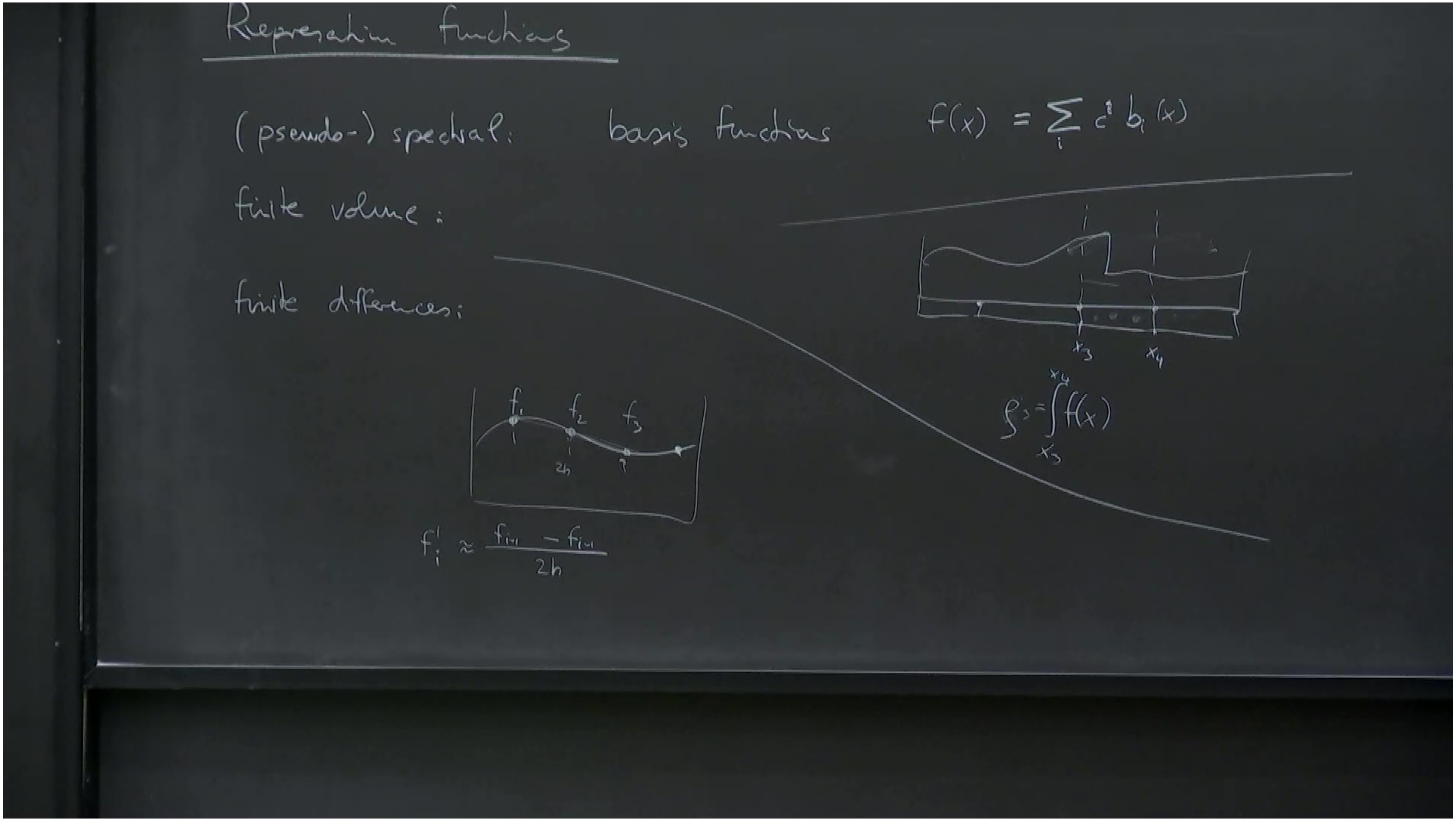


Representation functions

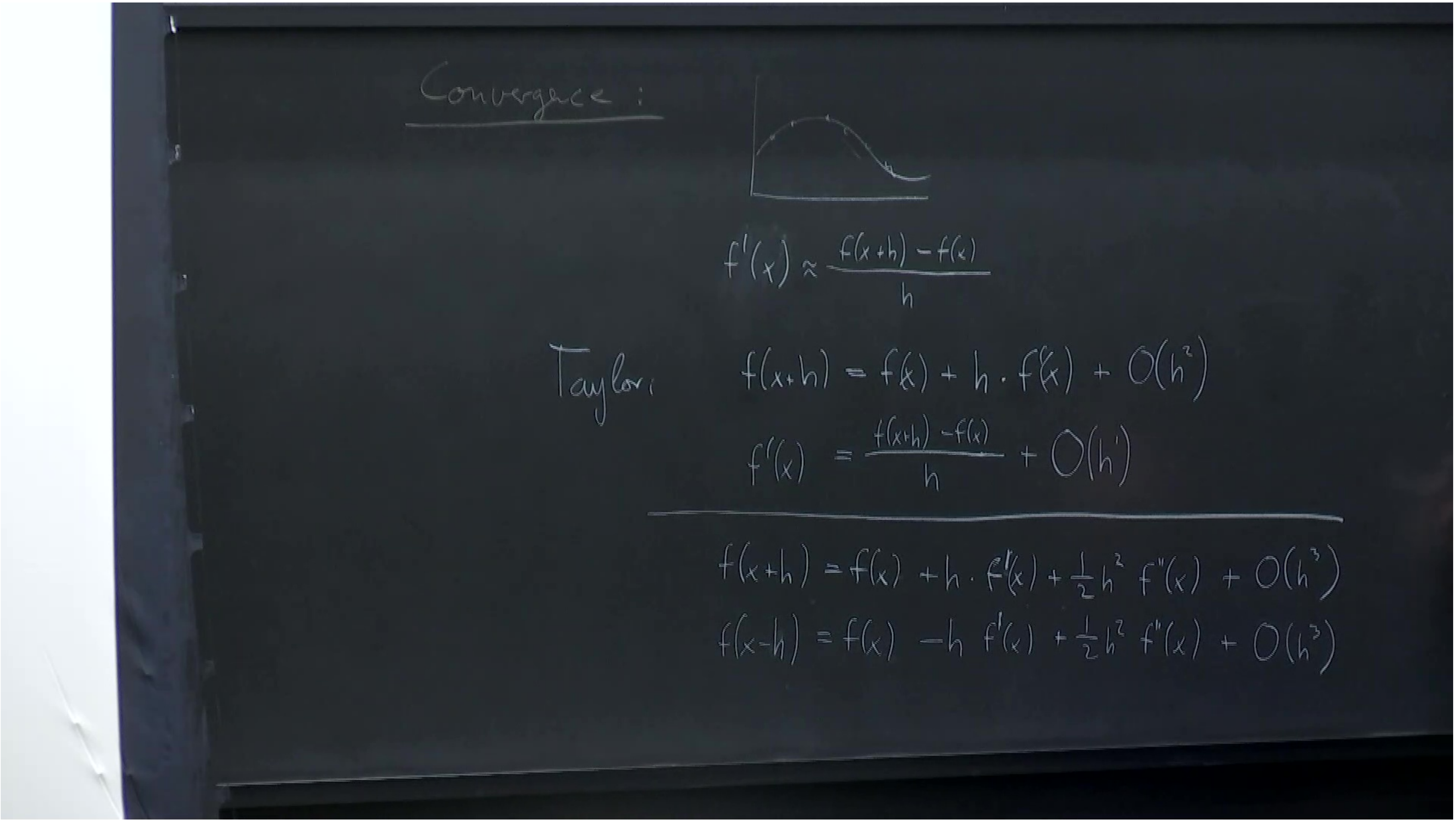
(pseudo-) spectral: basis functions $f(x) = \sum_i c^i b_i(x)$

finite volume:

finite differences:

$\rho_3 = \int_{x_3}^{x_4} f(x)$

$f'_i \approx \frac{f_{i+1} - f_{i-1}}{2h}$



Convergence :

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

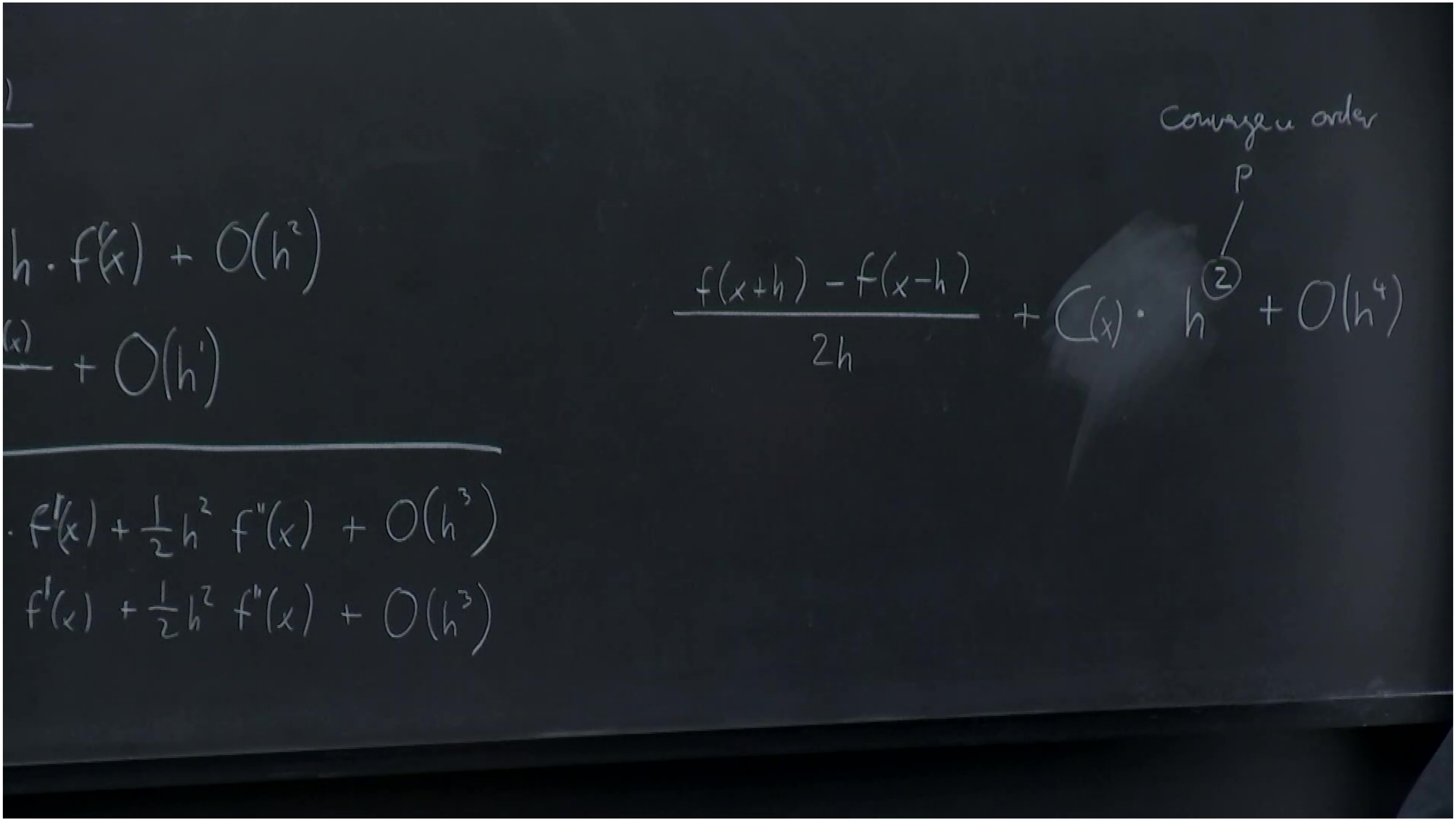
Taylor:

$$f(x+h) = f(x) + h \cdot f'(x) + O(h^2)$$

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h^1)$$

$$f(x+h) = f(x) + h \cdot f'(x) + \frac{1}{2} h^2 f''(x) + O(h^3)$$

$$f(x-h) = f(x) - h f'(x) + \frac{1}{2} h^2 f''(x) + O(h^3)$$



$)$

$h \cdot f'(x) + O(h^2)$

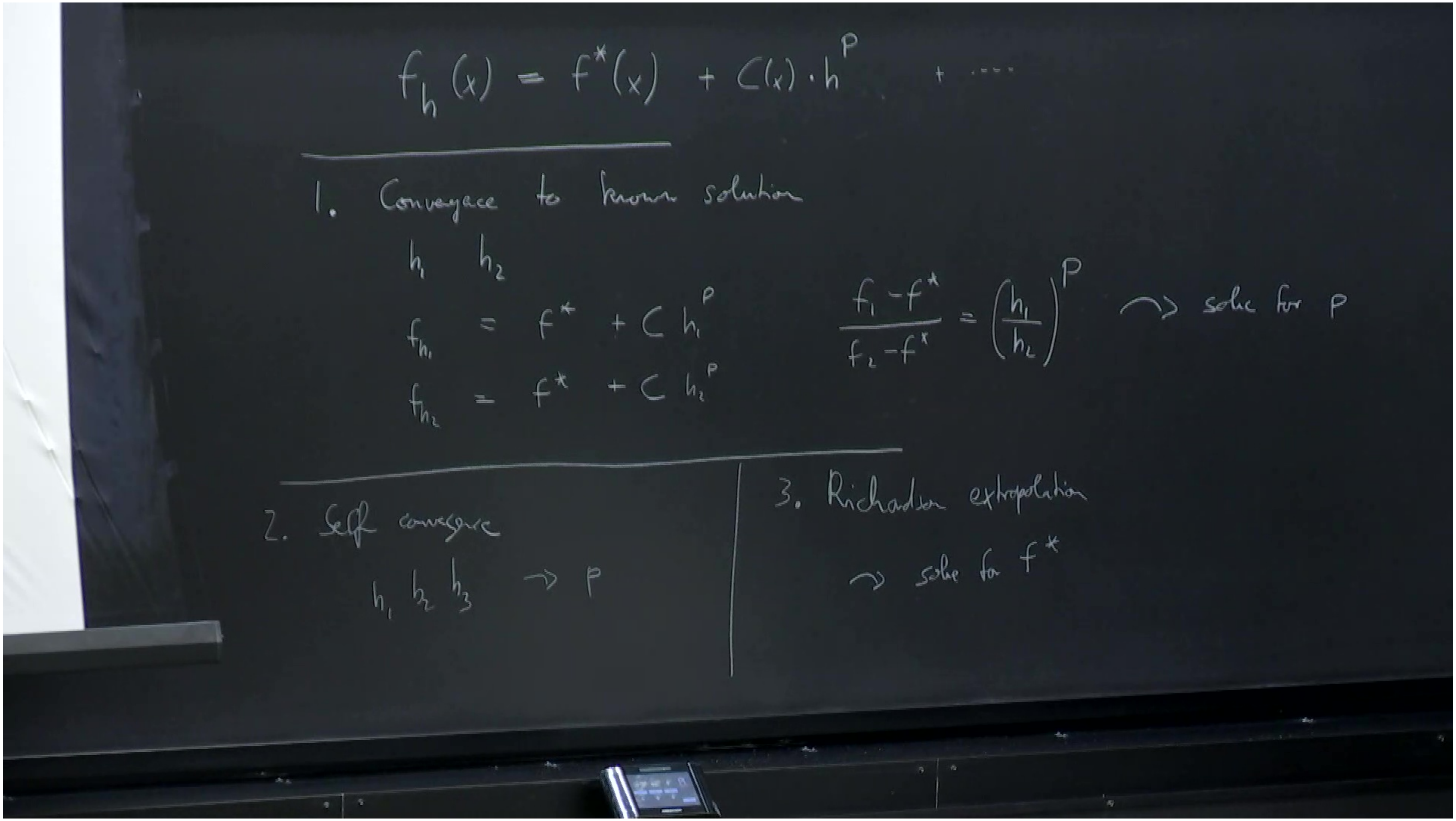
$\frac{(x)}{} + O(h^1)$

$\cdot f'(x) + \frac{1}{2}h^2 f''(x) + O(h^3)$

$f'(x) + \frac{1}{2}h^2 f''(x) + O(h^3)$

Convergence order

$$\frac{f(x+h)-f(x-h)}{2h} + C(x) \cdot h^{\overset{p}{\textcircled{2}}} + O(h^4)$$



$$f_h(x) = f^*(x) + C(x)\cdot h^p + \cdots$$

1. Convergace to known solution

$h_1 \quad h_2$

$f_{h_1} = f^* + C h_1^p$

$f_{h_2} = f^* + C h_2^p$

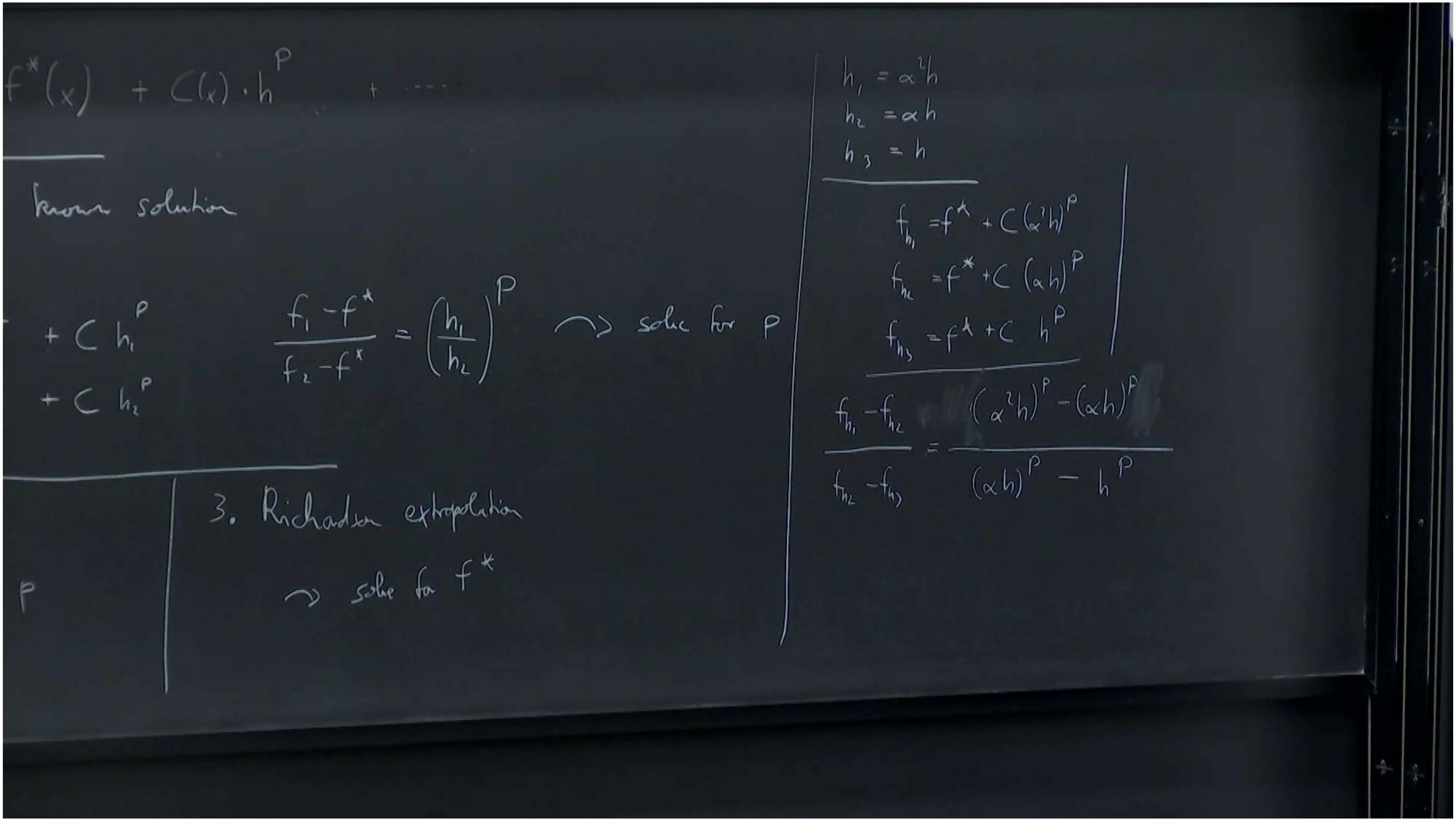
$$\frac{f_1 - f^*}{f_2 - f^*} = \left(\frac{h_1}{h_2}\right)^p \rightsquigarrow \text{solve for } p$$

2. Self convergence

$h_1\ h_2\ h_3 \rightsquigarrow p$

3. Richardson extrapolation

$\rightsquigarrow$ solve for f^*



$f^*(x) \ + C(x) \cdot h^p \quad + \ \cdots$

known solution

$+ C\, h_1^p$

$+ C\, h_2^p$

$$\frac{f_1 - f^*}{f_2 - f^*} = \left(\frac{h_1}{h_2}\right)^p \quad \rightsquigarrow \text{ solve for } p$$

p

3. Richardson extrapolation

$\rightsquigarrow$ solve for f^*

$h_1 = \alpha^2 h$

$h_2 = \alpha h$

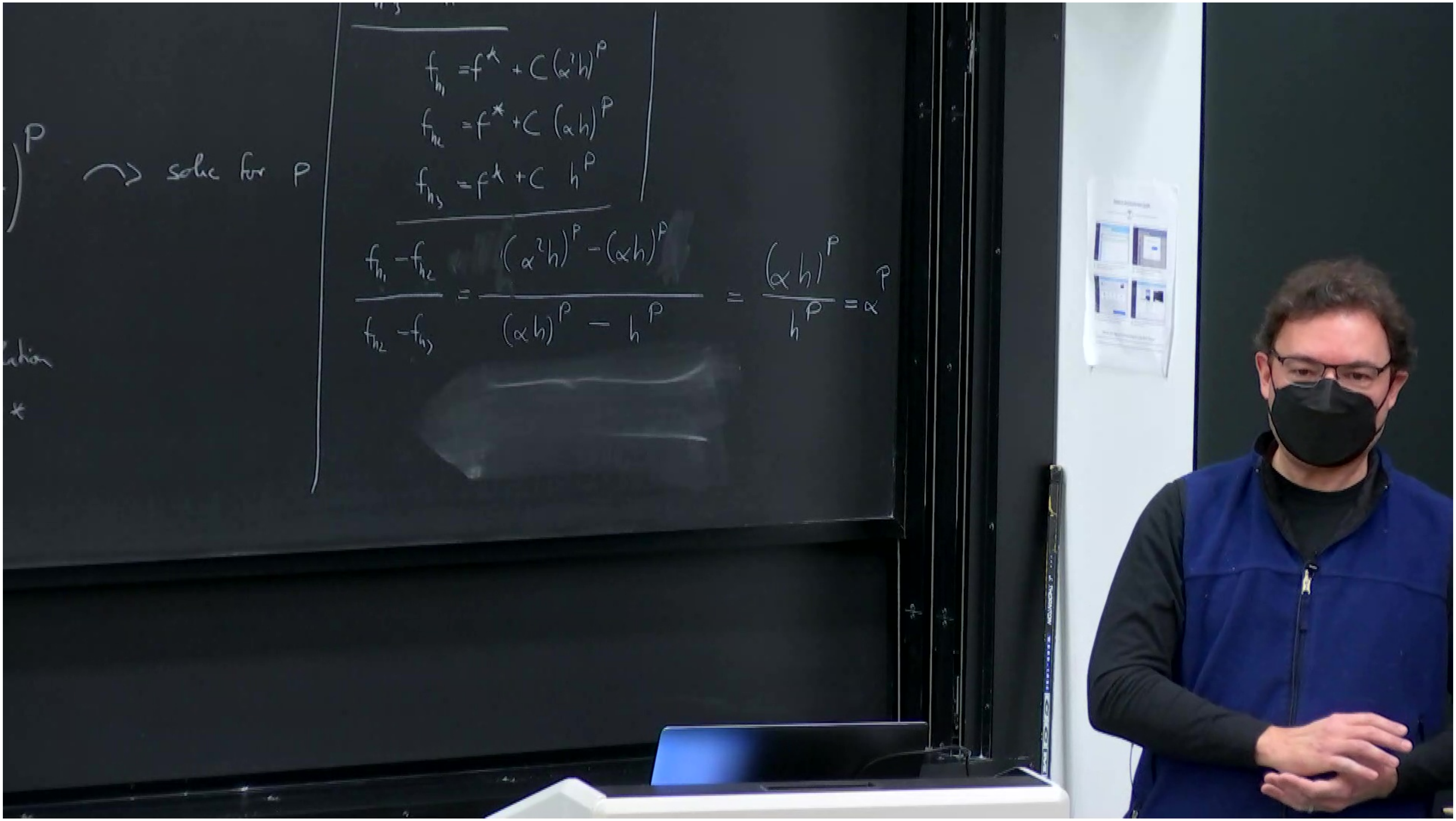
$h_3 = h$

$f_{h_1} = f^* + C(\alpha^2 h)^p$

$f_{h_2} = f^* + C\,(\alpha h)^p$

$f_{h_3} = f^* + C\ h^p$

$$\frac{f_{h_1} - f_{h_2}}{f_{h_2} - f_{h_3}} = \frac{(\alpha^2 h)^p - (\alpha h)^p}{(\alpha h)^p - h^p}$$



$)^p$ ⤳ solve for p

ation

$*$

$$f_{h_1} = f^* + C(\alpha^2 h)^p$$

$$f_{h_2} = f^* + C(\alpha h)^p$$

$$f_{h_3} = f^* + C\ h^p$$

$$\frac{f_{h_1} - f_{h_2}}{f_{h_2} - f_{h_3}} = \frac{(\alpha^2 h)^p - (\alpha h)^p}{(\alpha h)^p - h^p} = \frac{(\alpha h)^p}{h^p} = \alpha^p$$