Title: On dissipation in relativistic fluid theories

Speakers: Alex Pandya

Series: Strong Gravity

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URL: https://pirsa.org/23010001

Abstract: Fluid mechanics has proven to be remarkably successful in describing a wide variety of substances, both familiar and exotic. The latter category includes relativistic fluids, often arising in the most extreme regimes found anywhere in the universe. One such example is the quark-gluon plasma (QGP) formed in collisions of heavy ions, which exists at temperatures hot enough to "melt" hadrons; another is the matter composing neutron stars, whose density is comparable to that of an atomic nucleus. Beyond the surprising fact that the aforementioned substances act as fluids, they share an additional similarity in that they may both be measurably viscous, a feature accounted for in models of the QGP but almost never in neutron star simulations.

In this talk I will overview progress toward the incorporation of dissipative effects such as viscosity into relativistic fluid models of astrophysical systems. I will begin by reviewing the modern inter- pretation of fluid mechanics as a gradient expansion about thermodynamic equilibrium, and will discuss the nuances of constructing a theory compatible with beyond-equilibrium thermodynamics and general relativity. I will then define and motivate a promising new formulation of relativistic dissipative hydrodynamics known as BDNK theory before summarizing recent work toward its application in models of neutron stars.

Zoom link: https://pitp.zoom.us/j/99927210105?pwd=aUJWa0NobWFrT0FHMUhqZmRHWlREdz09

On dissipation in relativistic fluid theories

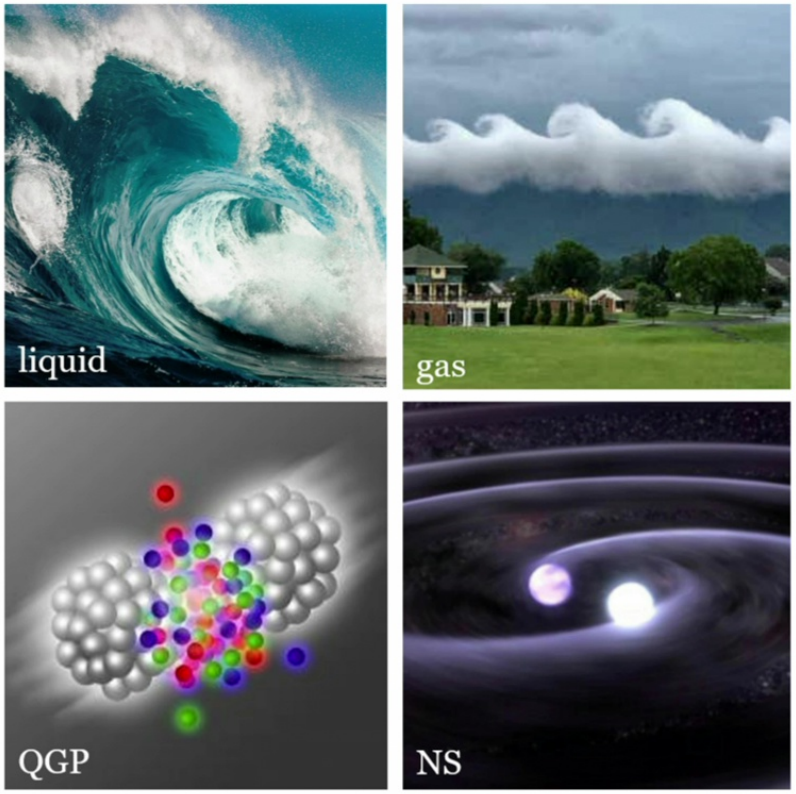
Alex Pandya, Elias R. Most & Frans Pretorius

Strong Gravity Seminar, Perimeter Institute
Jan. 12th, 2023

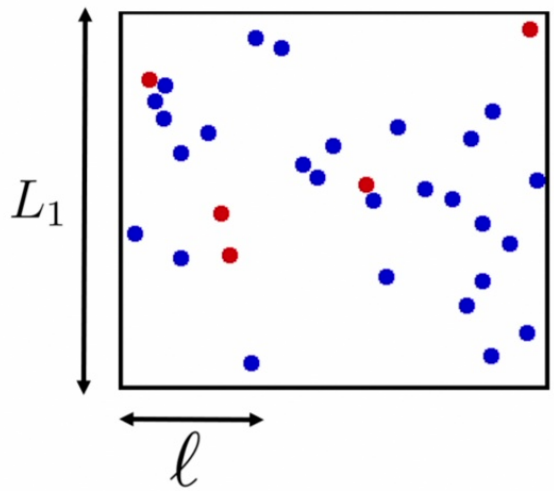
PRINCETON UNIVERSITY

What is a fluid?

- Elementary school definition: *liquids and gases*
  - Works well for terrestrial fluids
- New definition needed for exotic fluids, such as:
  - Quark-gluon plasma (QGP)
  - Neutron star matter
- Why is fluid mechanics so widely applicable?



Microscopic origins of hydrodynamics[1]

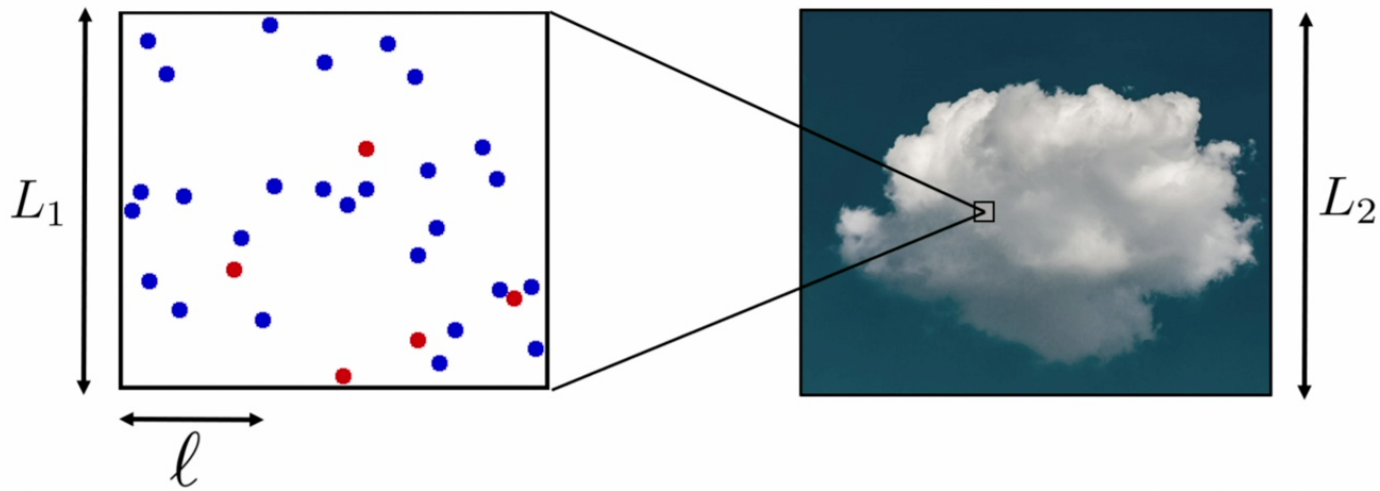


- Consider a simple classical microscopic model for a fluid: ideal gas
- System characterized by two length scales:
  - Mean free path between particle collisions, ℓ
  - Size of system, L_1
- Use them to define *Knudsen number*

$$\mathrm{Kn}_1 \equiv \frac{\ell}{L_1} \sim 1$$

- System has many DOF – each particle has position, momentum, and collides!

Microscopic origins of hydrodynamics[1]



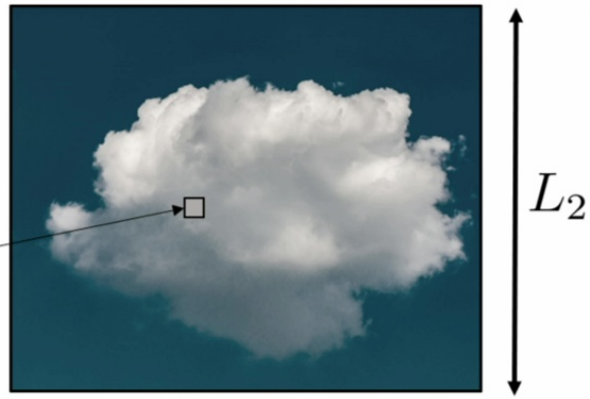
Zoom out to get system on the right.

Microscopic origins of hydrodynamics[1]

- Macroscopic system has same microscopic description (ℓ) but is much larger

$$\mathrm{Kn}_2 \equiv \frac{\ell}{L_2} \ll 1$$

- A fluid element seems almost homogeneous



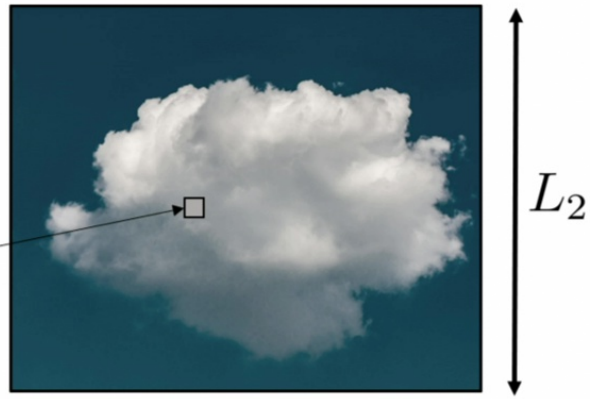
Microscopic origins of hydrodynamics[1]

- Macroscopic system has same microscopic description (ℓ) but is much larger

$$\mathrm{Kn}_2 \equiv \frac{\ell}{L_2} \ll 1$$

- A fluid element seems almost homogeneous
  - Can be described very well by finitely many *bulk* properties

- Solve *hydro* to evolve system state



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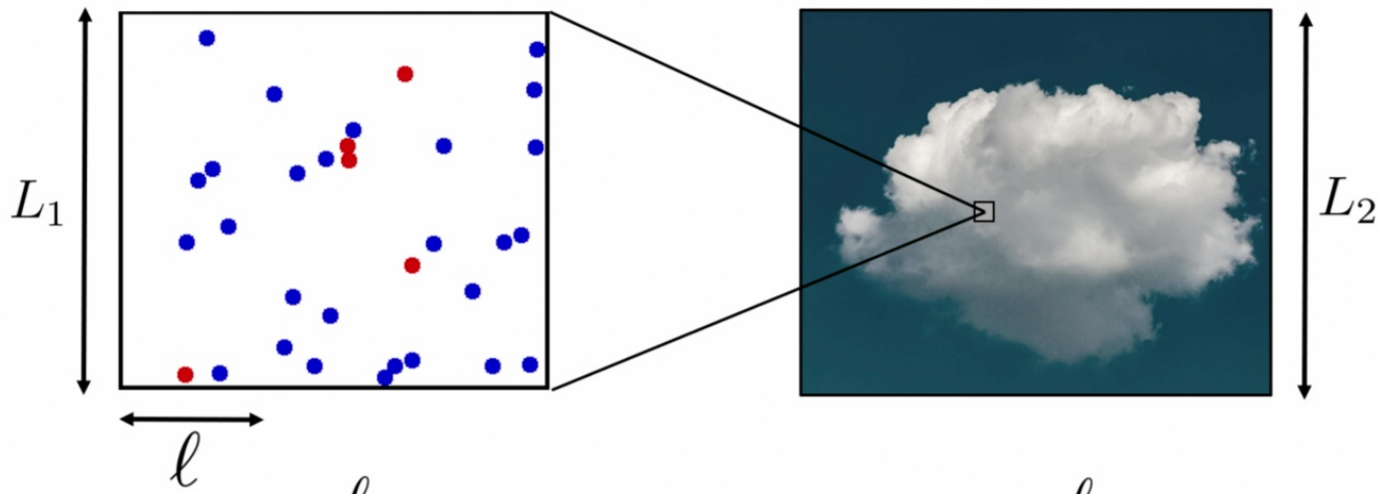


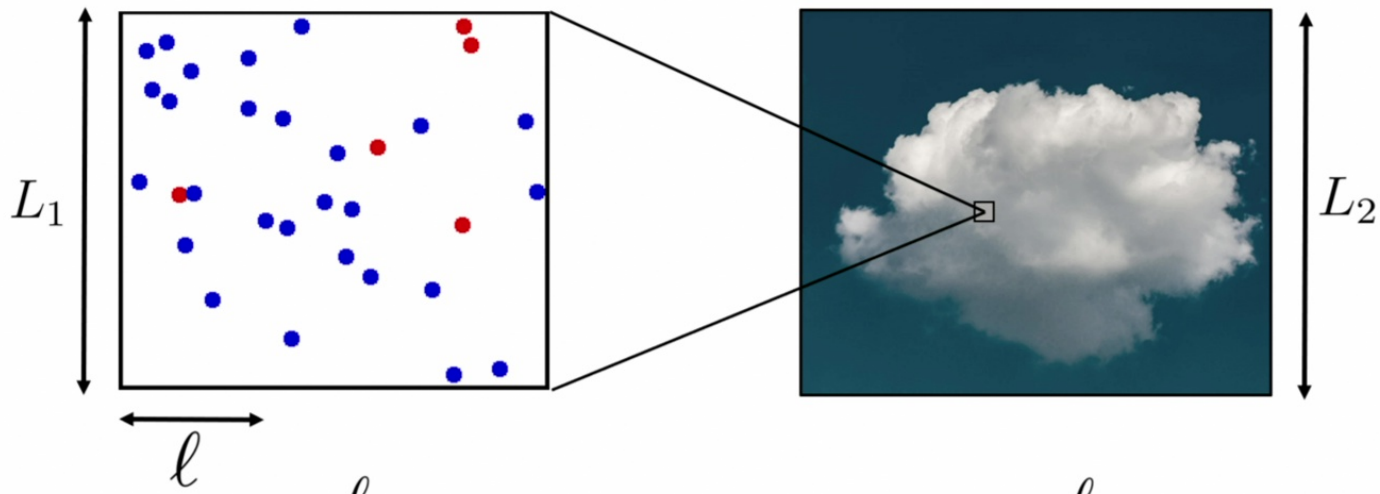
$$\mathrm{Kn}_1 \equiv \frac{\ell}{L_1} \sim 1$$

- Local fluctuations are <u>large</u>
- System has many DOF

$$\mathrm{Kn}_2 \equiv \frac{\ell}{L_2} \ll 1$$

- Local fluctuations are <u>small</u>
- Microscopic DOF average out!

Microscopic origins of hydrodynamics[1]

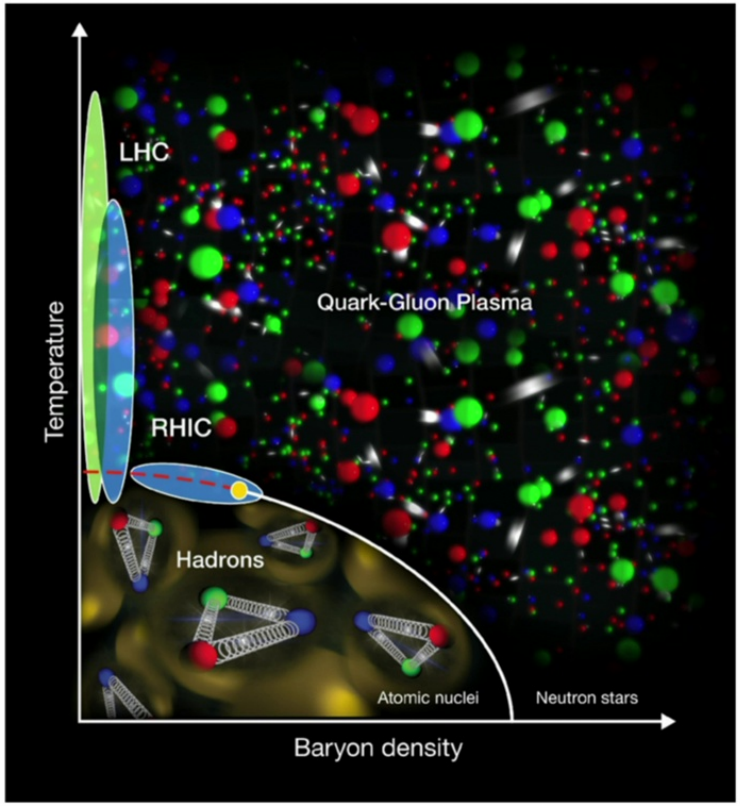


$$\mathrm{Kn}_1 \equiv \frac{\ell}{L_1} \sim 1 \qquad \mathrm{Kn}_2 \equiv \frac{\ell}{L_2} \ll 1$$

Hydro applicability reflects separation of scales.

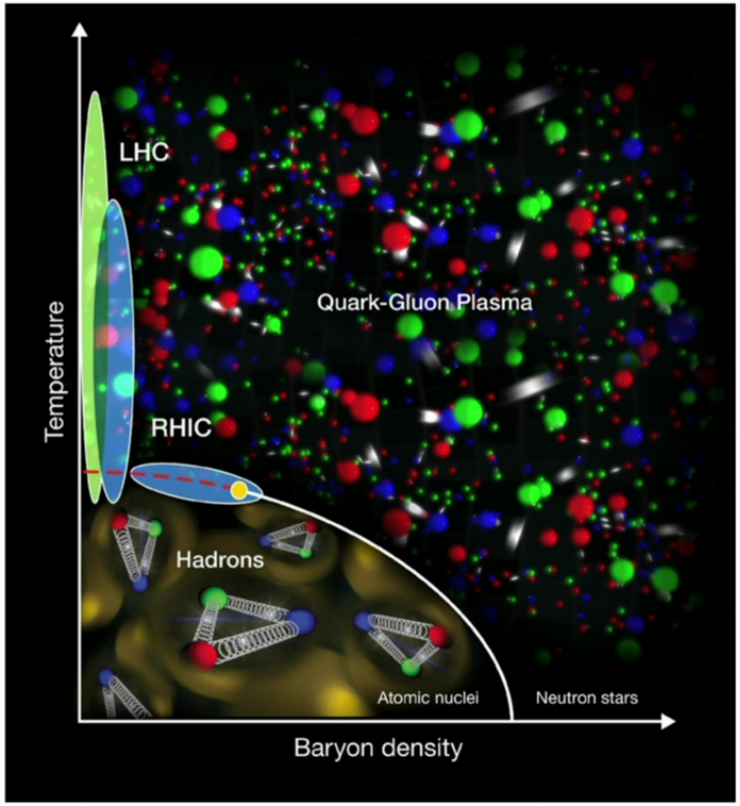
Why study relativistic fluids?

- They probe the most extreme temperatures & densities!

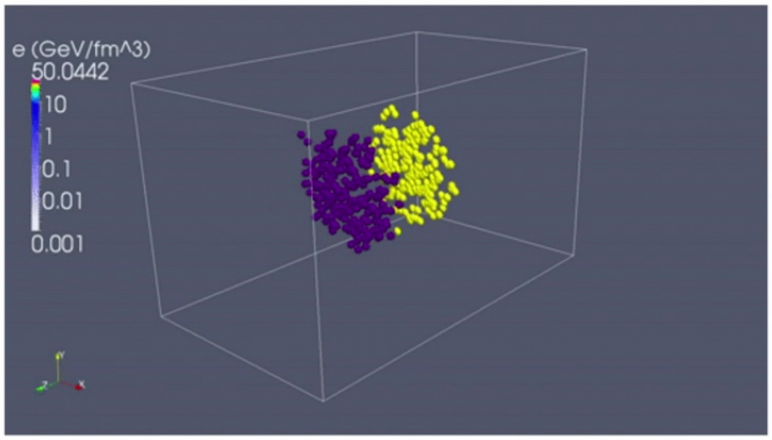


Why study relativistic fluids?

- They probe the most extreme temperatures & densities!
- Microphysics not fully understood in these regimes
- Fluid models can be used to make predictions in lieu of full microphysical theory



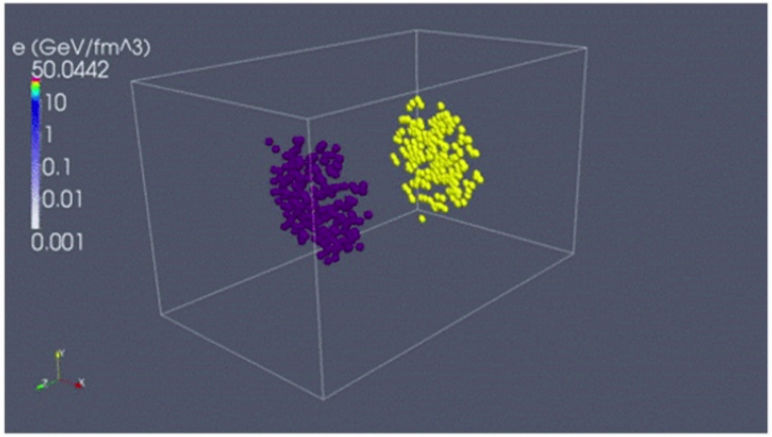
Dissipation in relativistic fluids



C. Shen et al. *Comput. Phys Commun.* **199** 61 (2016)

Shear viscosity needed to fit momentum anisotropy observed in collisions[2]

Dissipation in relativistic fluids



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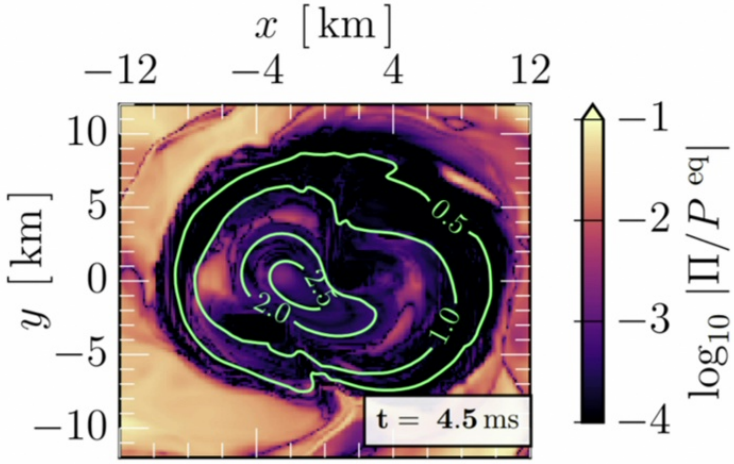


E. R. Most et al. *arXiv:2207.00442* (2022).

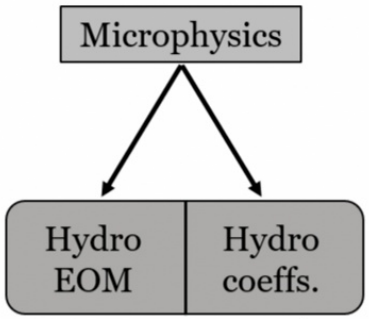
Shear viscosity needed to fit momentum anisotropy observed in collisions[2]

Bulk viscosity may also be dynamically relevant in BNS mergers[3]

Formulating a fluid theory

Two approaches:

1. Average short-wavelength modes out of microscopic theory to get hydro
   - *NR examples*: Hilbert (1912); Chapman-Enskog (1917); Burnett (1935); Grad (1949)



Chapman-Enskog expansion:

- Start with Boltzmann equation, expand f in Kn:

$$f(\boldsymbol{x}, \boldsymbol{p}) \sim f^{(0)} + \mathrm{Kn} f^{(1)} + \mathrm{Kn}^2 f^{(2)} + \ldots$$

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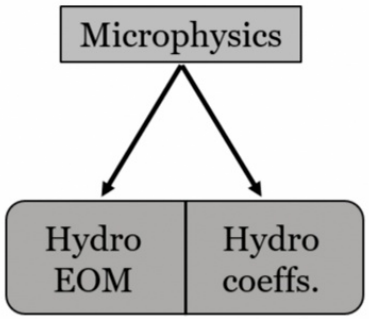

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- Taking moments then yields hydro:
  - Euler equations (no dissipation)
  - Navier-Stokes equations (leading order dissipation)
  - Burnett and super-Burnett theories

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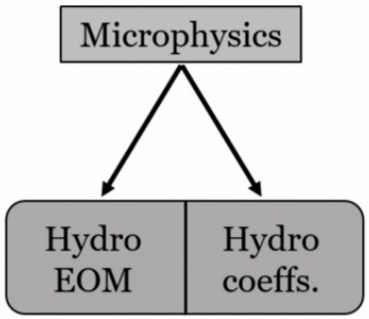
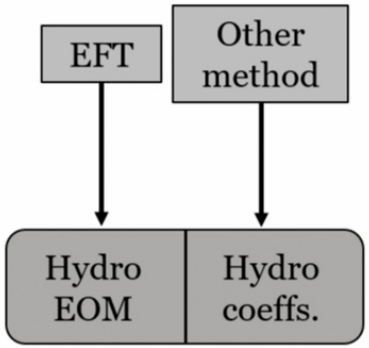
   - *NR examples*: Hilbert (1912); Chapman-Enskog (1917); Burnett (1935); Grad (1949)
   - *Rel. example*: Denicol, Niemi, Molnár, Rischke (DNMR, 2012)

2. Use effective field theory (EFT) to get hydro; determine coefficients by other method (matching, experiment, ...)





Formulating a fluid theory

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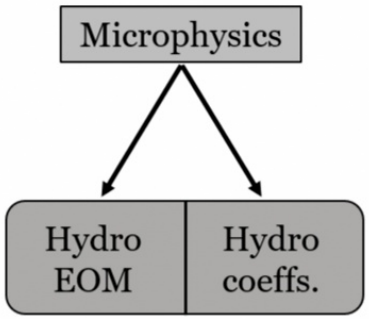
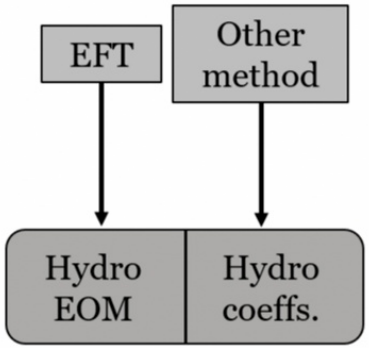
2. Use effective field theory (EFT) to get hydro; determine coefficients by other method (matching, experiment, ...)
   - *Rel. examples*: Müller, Israel-Stewart (MIS, 1970s); Bemfica, Disconzi, Noronha, Kovtun (BDNK, 2018-)





The EFT approach

Based in two principles:

1. Conservation laws provide equations of motion

$$\nabla_a T^{ab} = 0$$
$$\nabla_a J^a = 0$$

2. Thermodynamics provides parameterization

$$T^{ab} = T^{ab}(u^a, \epsilon, n)$$
$$J^a = J^a(u^a, \epsilon, n)$$

"hydro variables" are not all unique, e.g. $\epsilon, n \to T, \mu$

Simplifying assumption: conformal symmetry

Specialize to conformal fluids at zero chemical potential, so:

$$g_{ab}T^{ab} = 0$$

and

$$\boxed{T^{ab}(u^a, \epsilon, n) \to T^{ab}(u^a, \epsilon)}$$

$$J^a(u^a, \epsilon, n) \to J^a(u^a, n)$$

so we will focus on stress-energy evolution and ignore J^a.

The gradient expansion

- Goal: define $T^{ab}(u^a, \epsilon)$
- EFT approach includes linear combination of all possible terms
  - Including deriv. terms (i.e. $\nabla_c u^c$), there are infinitely many!
- Make two assumptions:
  1. We are 'near equilibrium' (so we can still use eq. parameters u^a, ϵ)
  2. Local fluctuations are small (so derivative terms are small)
- Stratify conserved currents by order of *gradient* terms:

$$T^{ab} = \mathcal{O}(\nabla^0) + \mathcal{O}(\nabla^1) + \ldots$$

Zeroth order: equilibrium fluid mechanics

- Begin with the general decomposition:

$$T^{ab} = \mathcal{E} u^a u^b + \mathcal{P} \Delta^{ab} + \mathcal{Q}^a u^b + \mathcal{Q}^b u^a + \mathcal{T}^{ab}$$

- T_0^{ab} must be a zero-derivative function of ϵ, u^a, so
  - $\mathcal{E}, \mathcal{P}$ contain: any scalar function of ϵ
  - $\mathcal{Q}^a$ contains: N/a
  - $\mathcal{T}^{ab}$ contains: N/a
- Rest-frame arguments give *perfect fluid*:

$$T_0^{ab} = \epsilon u^a u^b + P(\epsilon) \Delta^{ab}$$

Zeroth order: equilibrium fluid mechanics

- Perfect fluid stress-energy tensor is:

$$T_0^{ab} = \epsilon u^a u^b + P(\epsilon)\Delta^{ab}$$

- EOM given by $\nabla_a T^{ab} = 0$ are the *relativistic Euler equations*
  - Most common relativistic fluid theory for astrophysics applications!

- Zeroth order hydro implies insensitivity to gradient effects, e.g.
  - Viscosity – transfer of *momentum* due to velocity *gradients*

First order: *dissipative* fluid mechanics

- Begin with the general decomposition:

$$T^{ab} = \mathcal{E} u^a u^b + \mathcal{P} \Delta^{ab} + \mathcal{Q}^a u^b + \mathcal{Q}^b u^a + \mathcal{T}^{ab}$$

First order: *dissipative* fluid mechanics

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$$T^{ab} = \mathcal{E} u^a u^b + \mathcal{P} \Delta^{ab} + \mathcal{Q}^a u^b + \mathcal{Q}^b u^a + \mathcal{T}^{ab}$$

- T_1^{ab} can now contain first-order gradients of ϵ, u^a, so
  - $\mathcal{E}, \mathcal{P}$ contain: $\epsilon, P(\epsilon)\,, u^c \nabla_c \epsilon, \nabla_c u^c$
  - $\mathcal{Q}^a$ contains: $\Delta^{ac} \nabla_c \epsilon, u^c \nabla_c u^a$
  - $\mathcal{T}^{ab}$ contains: σ^{ab}

First order: *dissipative* fluid mechanics

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  - $\mathcal{Q}^a$ contains: $\Delta^{ac} \nabla_c \epsilon, u^c \nabla_c u^a$
  - $\mathcal{T}^{ab}$ contains: σ^{ab}
- *General* linear combination gives (conformal) <u>BDNK theory</u>
  - Linear combo → "transport" coefficients, not all of which are independent due to thermodynamics

BDNK theory

- BDNK theory has the form:

$$T^{ab} = \mathcal{E} u^a u^b + \mathcal{P} \Delta^{ab} + \mathcal{Q}^a u^b + \mathcal{Q}^b u^a + \mathcal{T}^{ab}$$

where:

$$\mathcal{E} = 3\mathcal{P} = \epsilon + \tau_\epsilon \left[\boxed{u^c \nabla_c \epsilon} + (\epsilon + P) \boxed{\nabla_c u^c} \right]$$

$$\mathcal{Q}^a = \tau_Q \left[(\epsilon + P) \boxed{u^c \nabla_c u^a} + \frac{1}{3} \boxed{\Delta^{ac} \nabla_c \epsilon} \right]$$

$$\mathcal{T}^{ab} = -2\eta \boxed{\sigma^{ab}}$$

Note that we have the five unique gradient terms from before

BDNK theory

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$$\mathcal{T}^{ab} = -2\boxed{\eta}\sigma^{ab}$$

We have three unique "transport coefficients"

Hydro variables outside of equilibrium

- Consider energy density seen by observer co-moving w. flow:

$$u_a u_b T^{ab} = \mathcal{E} = \underbrace{\epsilon}_{\text{Equilibrium energy density}} + \underbrace{\tau_\epsilon \left[u^c \nabla_c \epsilon + (\epsilon + P) \nabla_c u^c \right]}_{\text{Beyond-equilibrium correction}}$$

- Dissipation ⇒ theory can equilibrate ⇒ theory possesses states which are not in equilibrium!
- Note that $\mathcal{E}$ is a physical observable, ⇒ so is right-hand side
  - Outside of equilibrium, ϵ is not a physical observable on its own!
  - Consistent with fact that the hydro variables have no first principles microscopic definition outside equilibrium

Field redefinitions

- Consider energy density seen by observer co-moving w. flow:

$$u_a u_b T^{ab} = \mathcal{E} = \epsilon + \boxed{\tau_\epsilon}\left[u^c \nabla_c \epsilon + (\epsilon + P)\nabla_c u^c\right]$$

- Consider field redefinition for ϵ of the form:

$$\epsilon' \equiv \epsilon + X\left[u^c \nabla_c \epsilon + (\epsilon + P)\nabla_c u^c\right]$$

- Notice that $\nabla_a \epsilon = \nabla_a \epsilon' + \mathcal{O}(\nabla^2)$, so:

$$\mathcal{E}(\epsilon') = \epsilon' + \boxed{(\tau_\epsilon - X)}\left[u^c \nabla_c \epsilon' + \left(\epsilon' + P(\epsilon')\right)\nabla_c u^c\right] + \mathcal{O}(\nabla^2)$$

- So field redefinitions of ϵ are equivalent to changing τ_ϵ !

"Hydrodynamic frame"

- BDNK theory has three transport coefficients: $\tau_\epsilon, \tau_Q, \eta$
- Of these, τ_ϵ, τ_Q change under field redefinitions of ϵ, u^a
- The other coefficient, η, is invariant; also has physical interpretation as the *shear viscosity*
- Appears τ_ϵ, τ_Q only serve to set "hydrodynamic frame": non-unique beyond-equilibrium definitions of ϵ, u^a

Hydro frames & pathologies

- Consider energy density seen by observer co-moving w. flow:

$$u_a u_b T^{ab} = \mathcal{E} = \epsilon + \tau_\epsilon \left[u^c \nabla_c \epsilon + (\epsilon + P) \nabla_c u^c \right]$$

- Specialize to flat spacetime, zero flow velocity, Cartesian coords

$$\mathcal{E} = \epsilon + \tau_\epsilon \frac{\partial \epsilon}{\partial t} \implies \frac{\partial \epsilon}{\partial t} = \frac{1}{\tau_\epsilon} (\mathcal{E} - \epsilon)$$

- This is a *relaxation equation* forcing $\epsilon \to \mathcal{E}$, i.e. equilibration!
- Notice that:
  - If $\tau_\epsilon < 0$, fluid runs away from equilibrium; thermo instability
  - In the limit $\tau_\epsilon \to 0$ it equilibrates instantly... violating causality?

Nuances of beyond-equilibrium hydro

Summary:

1. Outside of equilibrium, ϵ, u^a do not have unique definitions
2. Precise defn depends on *transport coeffs* → "hydro frame"
3. In conformal BDNK there are 3: τ_ϵ, τ_Q, η
   a. Of these, τ_ϵ, τ_Q change w field redefinitions of ϵ, u^a
   b. η is invariant, describes shear viscosity (physical effect)
   c. Choice of τ_ϵ, τ_Q determines *causality* and *stability* of solutions!

Historical aside:

- Original formulations of first-order relativistic diss. hydro (Eckart, Landau-Lifshitz; 1940s) did not account for 3.c., chose hydro frames with $\tau_\epsilon = 0$
- Theories shown to be acausal and badly unstable (water at STP unstable w timescale 10^{-34} s)

Motivation for using BDNK theory

- BDNK theory accounts for hydro frame, rigorously proves:
  - causality,
  - strong hyperbolicity,
  - local well-posedness,
  - (linear) stability of thermodynamic equilibrium.

- Similar proofs do not exist in general for other theories, almost all of which use Eckart/Landau frames ($\sim \tau_\epsilon = 0$)

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- <u>Goal</u>: apply BDNK to astrophysics problems (e.g. BNS mergers)

Roadmap

Initial nonlinear study (conformal) → Develop suitable numerical scheme → Generalize to realistic microphysics → Solve BDNK with realistic microphysics + gravity

AP & F. Pretorius, *Phys. Rev. D,* **104** (2021)

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **105** (2022)

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **106** (2022)

In progress

BNS merger!

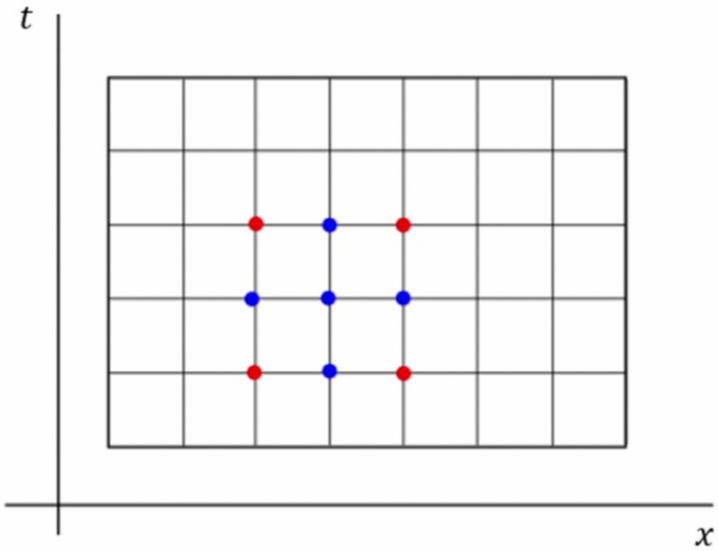
Future work

AP & F. Pretorius, *Phys. Rev. D*, **104**, (2021)

Initial nonlinear study – Setup

Setup:

- (1+1)D numerical comparison of ideal hydro, BDNK, toy-model MIS
- conformal fluid, flat spacetime, planar symmetry
- Simple finite difference method used for viscous terms



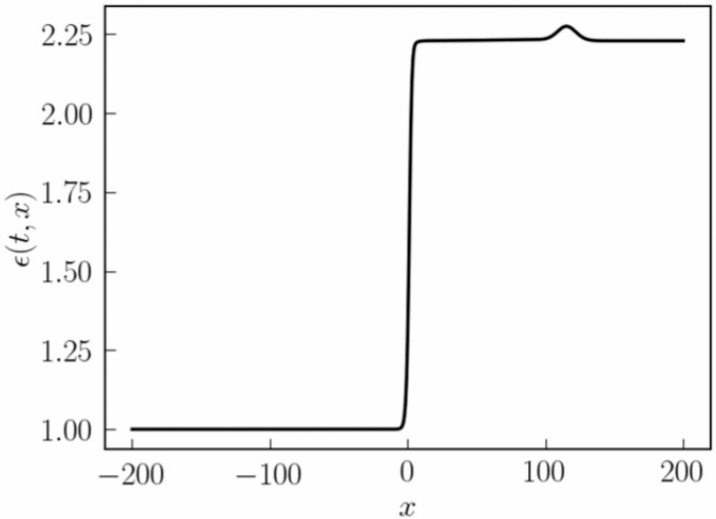
Derivatives approximated using truncations of pointwise Taylor series expansion.

AP & F. Pretorius, *Phys. Rev. D*, **104**, (2021)

Initial nonlinear study – Results

Results:

1. Numerical method is fine for smooth flows…
2. Solutions are 'sensible': agreement with MIS, perfect fluid as $\eta \to 0$
3. Shockwave solutions exist for good choices of hydro frame*

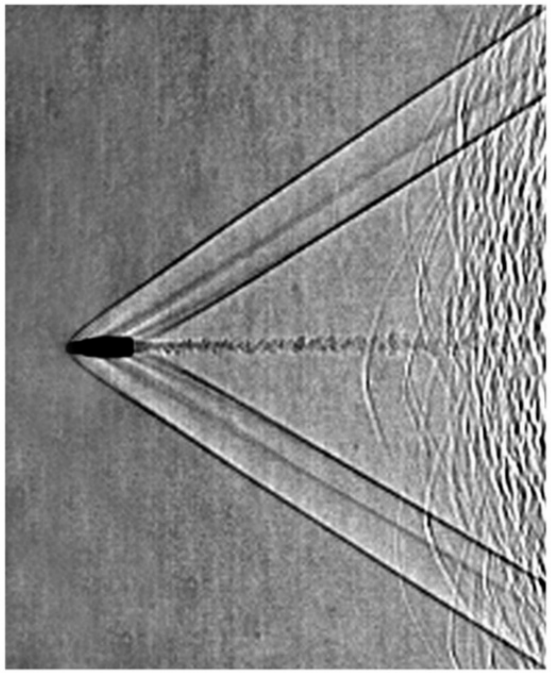


ID approximating shockwave in rest frame; late-time solution is t-independent

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **105** (2022)

Numerical challenge: shocks

- Fluids generically form *shocks* when flow speed exceeds local sound speed
  - Characterized by sharp variation in flow



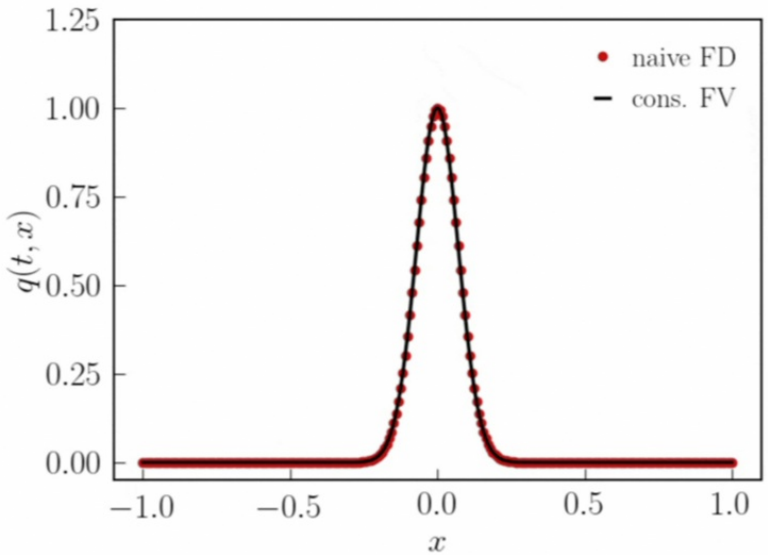
Why don't 'general purpose' methods work?

- Simple (1+1)D conservation law:

$$\frac{\partial q}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(q^2) = 0$$

- Forms shocks dynamically

- Movie: Gaussian ID in q



Standard finite difference method in red, conservative finite volume in black

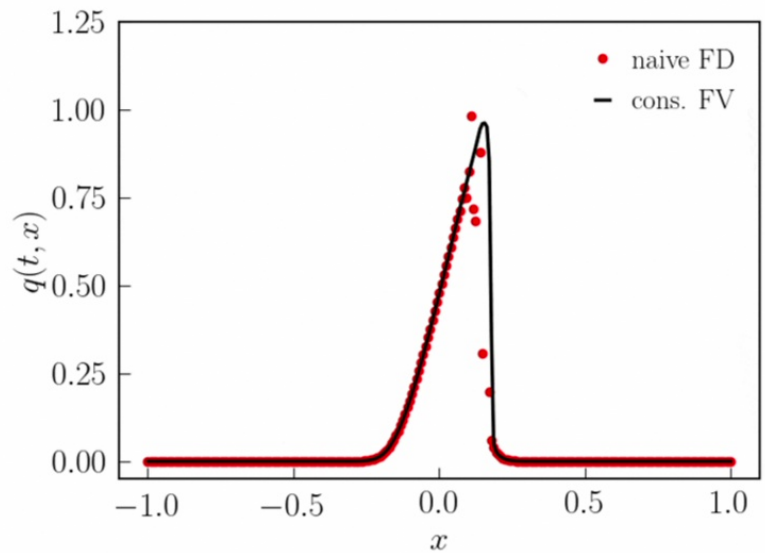
Why don't 'general purpose' methods work?

- Simple (1+1)D conservation law:

$$\frac{\partial q}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}(q^2) = 0$$

- Forms shocks dynamically

- Movie: Gaussian ID in q
  - FDM: spurious oscillations!
  - FVM: stable, resolves shock



Standard finite difference method in red, conservative finite volume in black

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **105** (2022)

BDNK shock-capturing method

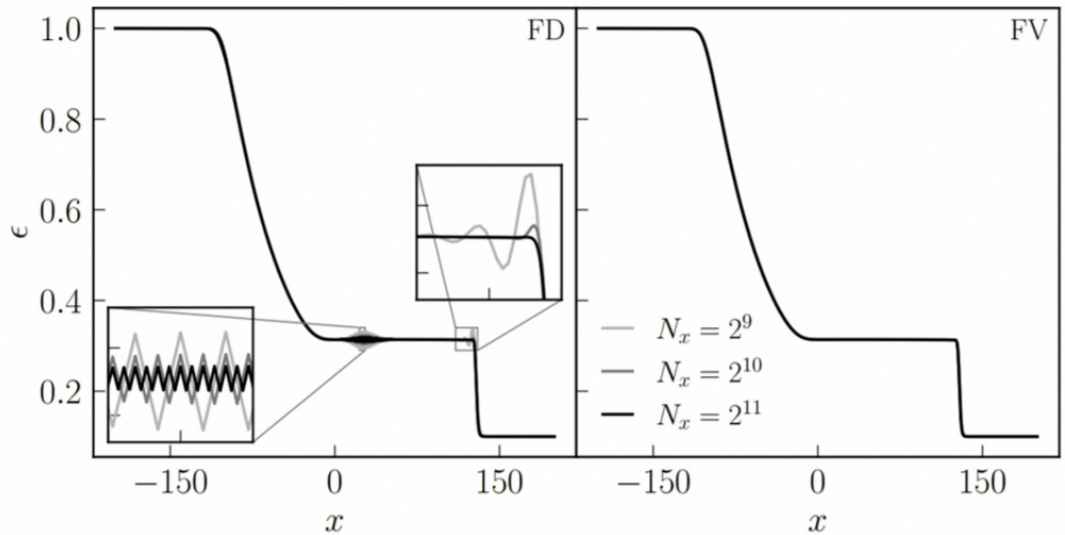
- Developed high-res. shock-capturing FV method for BDNK

- The method:
  - is flux-conservative
  - uses a non-oscillatory reconstruction (of the primitive variables)
  - reduces to a 'modern' perfect fluid solver in inviscid limit

- Tested the method on a variety of problems in flat spacetime, Cartesian coordinates, conformal fluid

AP, E. R. Most, F. Pretorius, *Phys. Rev. D*, **105** (2022)

Code tests: 1D shock tube



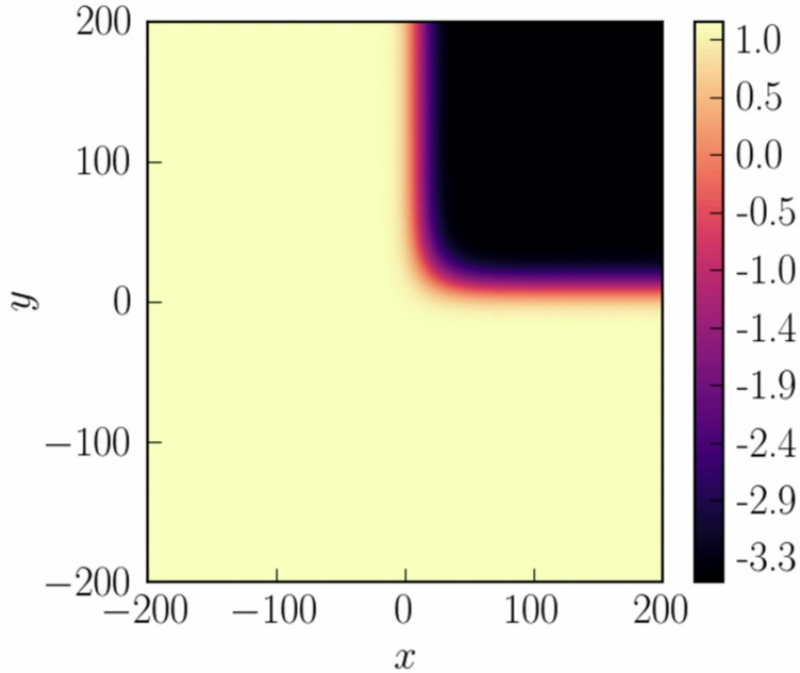
FV method performs significantly better about discontinuities.

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **105** (2022)

Code tests: 2D shockwave

Test 2: 2D oblique shockwave

- Flow extruded from lower-left region into upper-right at high speeds
- Forms high energy density region bounded by curved shock



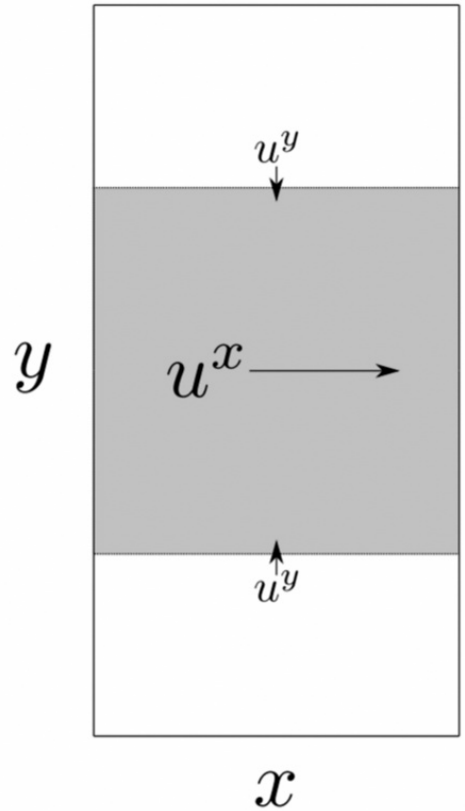
Color: log of energy density

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **105** (2022)

Code tests: inviscid limit

Test 3: Kelvin-Helmholtz instability

- Astrophysical jet flowing through ambient medium
- Kelvin-Helmholtz unstable; forms vortices
- Vortex size strongly viscosity dependent



AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **106** (2022)

Beyond conformal fluids

- Conformal fluid was a nice starting point, as it:
  - Decouples stress-energy tensor from particle current
  - Fixes EOS to $P(\epsilon) = \epsilon/3$
  - Has only 1 legitimate dissipative effect: shear viscosity

- Unfortunately, simplicity comes at a cost:
  - Conformal model lacks *bulk viscosity* and heat conduction!

- For this reason, we generalized to non-conformal BDNK based in "gamma-law" ideal gas EOS: $P = (\Gamma - 1)(\epsilon - mn)$

AP, E. R. Most, F. Pretorius, *Phys. Rev. D,* **106** (2022)

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- For this reason, we generalized to non-conformal BDNK based in "gamma-law" ideal gas EOS: $P = (\Gamma - 1)(\epsilon - mn)$
  - Should be suitable for first studies of astro systems

Coupling to gravity – work in progress

- All work so far has been in flat spacetime!

- Current work: BDNK fluid, coupled to gravity in spherical sym.

- Challenges:
  1. Covariant derivatives introduce *metric derivative terms* into T^{ab}

$$\text{e.g. heat flux} \sim \nabla_a u^b = \partial_a u^b + \boxed{\Gamma^b_{ac}} u^c$$

  2. These terms also cause regularity problems at the origin

Future work

- Groundwork almost complete for BNS merger problem
  - Approach: write BDNK fluid module for existing binary merger code
  - Study: impact of dissipative effects (like bulk viscosity!) on GW signal

- Afterward:
  - Many possible applications to modeling QGP
  - Extension to include EM fields (relativistic dissipative *magneto*hydrodynamics) exists but has yet to be applied

Summary

- BDNK theory is a promising approach for incorporating first-order dissipative effects into relativistic fluid models

- We have made some progress toward this goal:
  1. Investigated nonlinear properties of (conformal) BDNK
  2. Developed HRSC FV numerical scheme for BDNK
  3. Computed suitable class of non-conformal BDNK hydro frames
  4. Coupled BDNK to gravity (work in progress)

- Many extensions possible due to wide applicability of hydro