

Title: Analysis of the superdeterministic Invariant-set theory in a hidden-variable setting

Speakers: Indrajit Sen

Series: Quantum Foundations

Date: December 21, 2022 - 11:00 AM

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Abstract: Superdeterminism has received a recent surge of attention in the foundations community. A particular superdeterministic proposal, named Invariant-set theory, appears to bring ideas from several diverse fields (eg. number theory, chaos theory etc.) to quantum foundations and provides a novel justification for the choice of initial conditions in terms of state-space geometry. However, the lack of a concrete hidden-variable model makes it difficult to evaluate the proposal from a foundational perspective.

In this talk, I will critically analyse this superdeterministic proposal in three steps. First, I will show how to build a hidden-variable model based on the proposal's ideas. Second, I will analyse the properties of the model and show that several arguments that appear to work in the proposal (on counter-factual measurements, non-commutativity etc.) fail when considered in the model. Further, the model is not only superdeterministic but also nonlocal, $\$|\psi\$$ -ontic and contains redundant information in its bit-string. Third, I will discuss the accuracy of the model in representing the proposal. I will consider the arguments put forward to claim inaccuracy and show that they are incorrect. My results lend further support to the view that superdeterminism is unlikely to solve the puzzle posed by the Bell correlations.

Based on the papers:

1. I. Sen. "Analysis of the superdeterministic Invariant-set theory in a hidden-variable setting." Proc. R. Soc. A 478.2259 (2022): 20210667.
2. I. Sen. "Reply to superdeterminists on the hidden-variable formulation of Invariant-set theory." arXiv:2109.11109 (2021).

Zoom link: <https://pitp.zoom.us/j/99415427245?pwd=T3NOWUxKTENnMThRVEd3ZTRzU3ZKZz09>

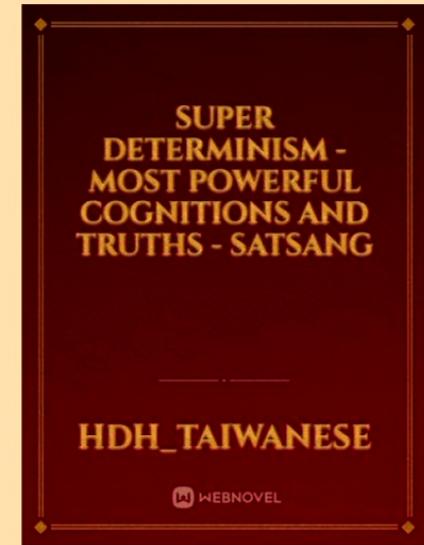
Analysis of the superdeterministic Invariant set theory in a hidden-variable setting

Indrajit Sen

Institute for Quantum Studies, Chapman University
Clemson University

Dec 21, 2022





Introduction



Introduction



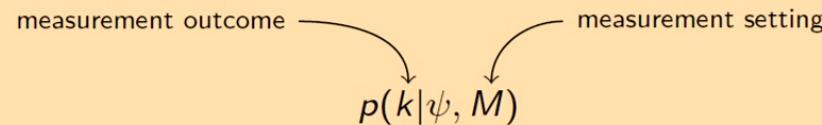
Why study superdeterminism?

Nonlocality, nonseparability, preferred foliation of spacetime, no-signalling, noncontextuality, ψ -onticity, exponential complexity of state space...

The assumption

$$p(k|\psi, M)$$

measurement outcome measurement setting



The assumption

$$p(k|\psi, M) = \int p(k, \lambda|\psi, M) d\lambda = \int d\lambda p(k|\psi, M, \lambda) \rho(\lambda|\psi, M)$$

Measurement Independence: $\rho(\lambda|\psi, M) = \rho(\lambda|\psi, M')$.

The assumption

measurement outcome —————→
measurement setting —————→

$$p(k|\psi, M) = \int p(k, \lambda|\psi, M) d\lambda = \int d\lambda p(k|\psi, M, \lambda) \rho(\lambda|\psi, M)$$

Measurement Independence: $\rho(\lambda|\psi, M) = \rho(\lambda|\psi, M')$.

Assumed in nearly all the major no-go theorems in quantum foundations.

Superdeterminism

Determinism

Measurement dependence

Superdeterminism

Determinism
Measurement dependence
Correlation between λ and the setting mechanisms



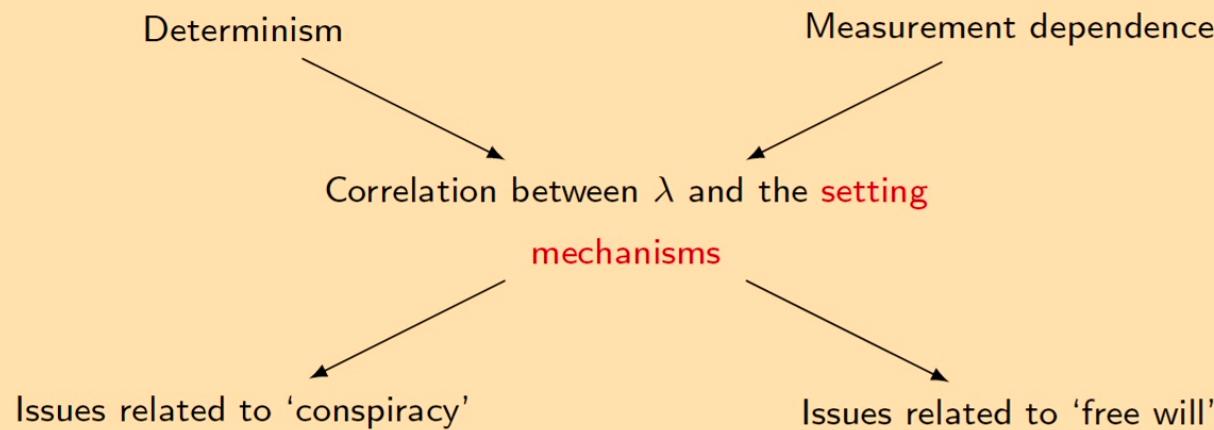
Superdeterminism

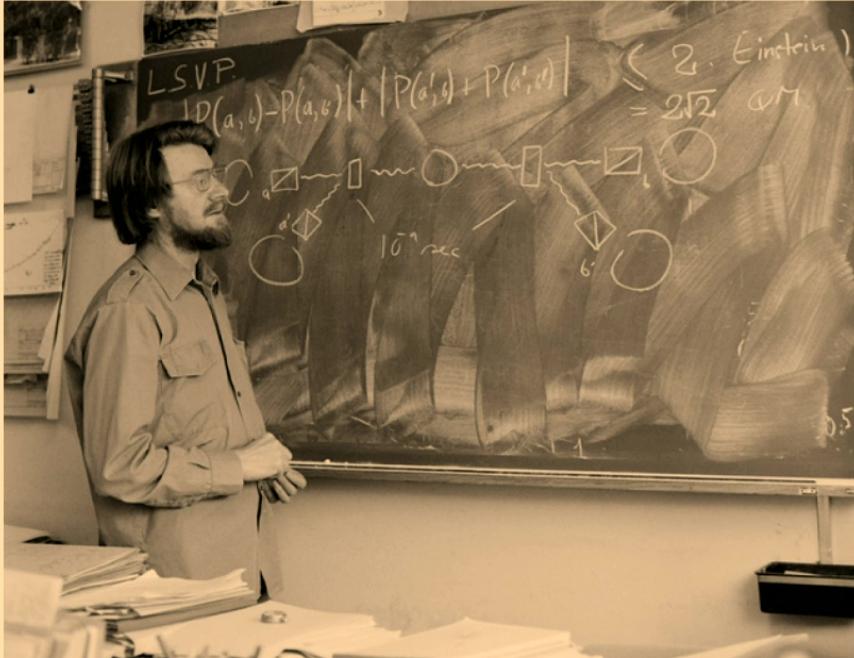
Determinism Measurement dependence

Correlation between λ and the setting
mechanisms



Superdeterminism





"A theory may appear in which such conspiracies inevitably occur, and these conspiracies may then seem more digestible than the non-localities of other theories. When that theory is announced I will not refuse to listen, either on methodological or other grounds."

²J. S. Bell, Epistemol. Lett. 1977, 15, Republished in Dialectica, 1985, 85-1

Invariant-set proposal

T.N. Palmer A local deterministic model of quantum spin measurement. Proc. R. Soc. Lond. A **1995**, 451, 585–608



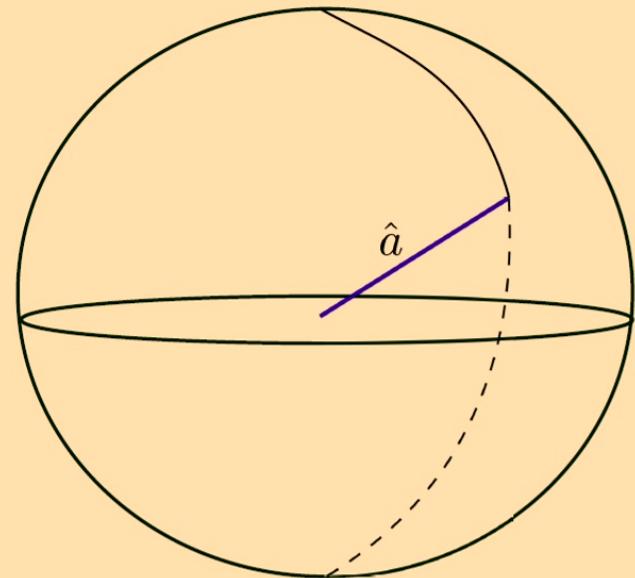
T. N. Palmer Discretization of the Bloch sphere, fractal invariant sets and Bell's theorem. Proc. R. Soc. A. **2020**, 476, 20190350

Invariant-set proposal

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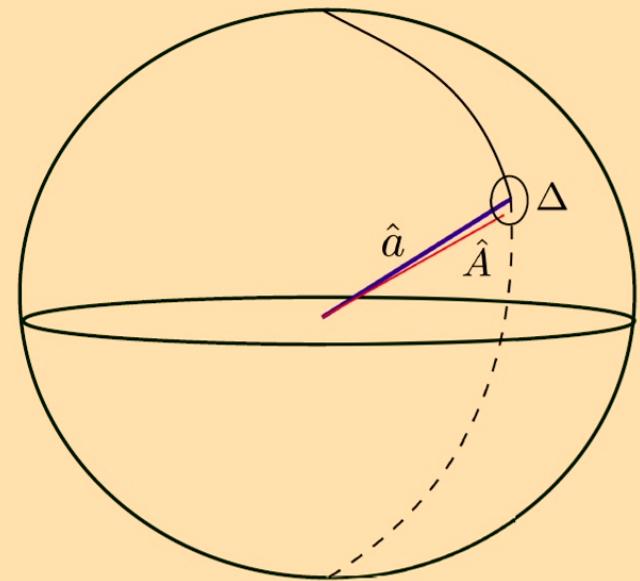
"The Proposal"

Sketch of the proposal



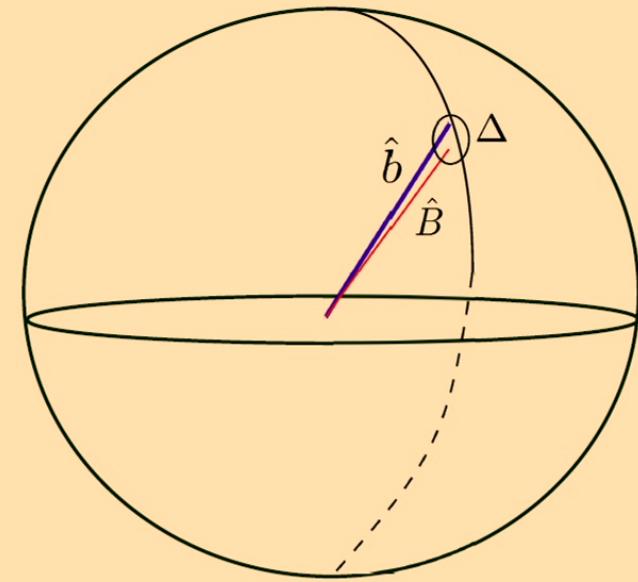
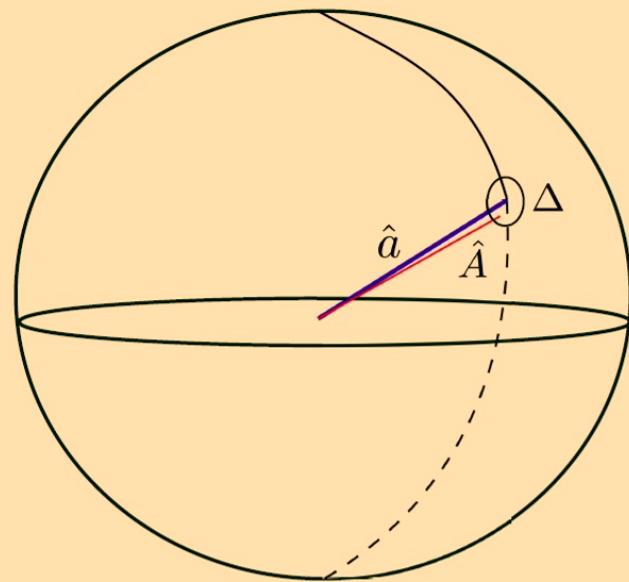
Orientation of a Stern-Gerlach apparatus

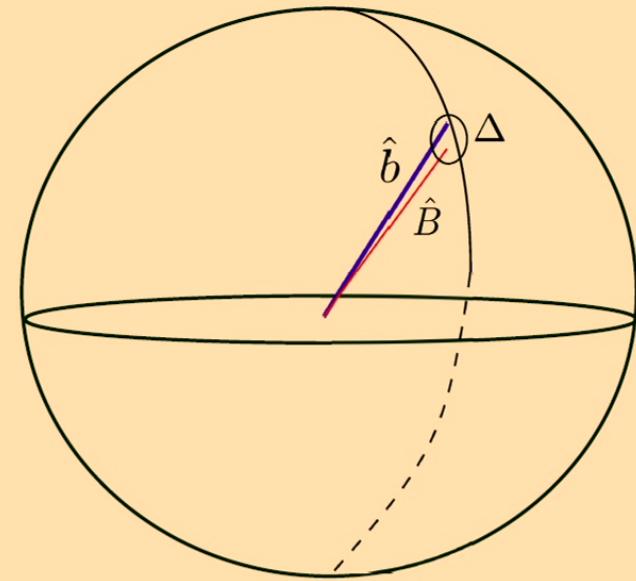
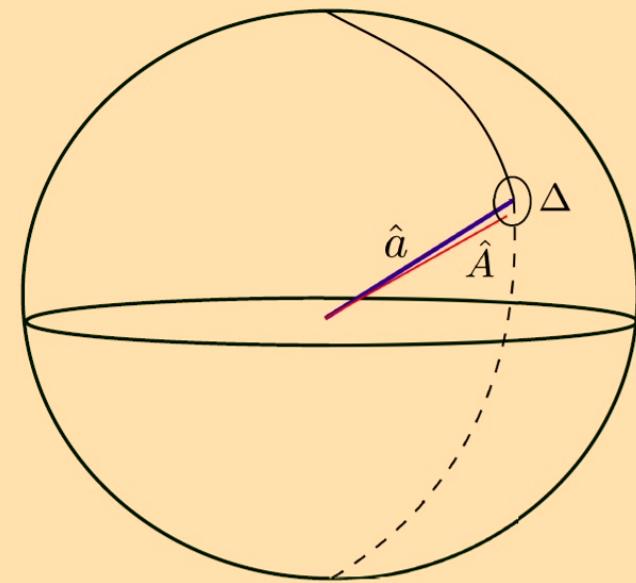
$\hat{a} \rightarrow$ experimentally-selected measurement setting



Orientation of a Stern-Gerlach apparatus

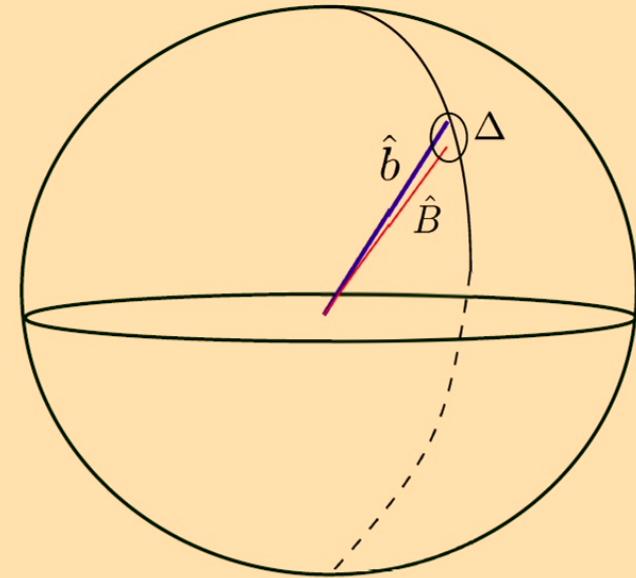
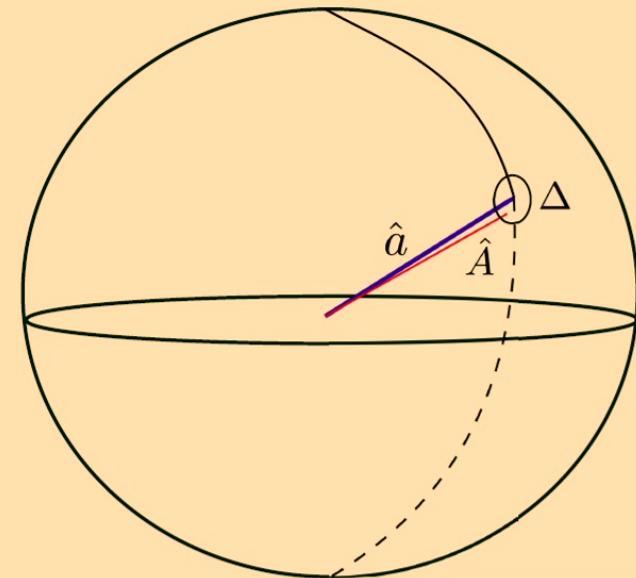
\hat{a} → experimentally-selected measurement setting
 \hat{A} → exact setting
 $|\hat{A} - \hat{a}| < \Delta$ → minimum measurable distance





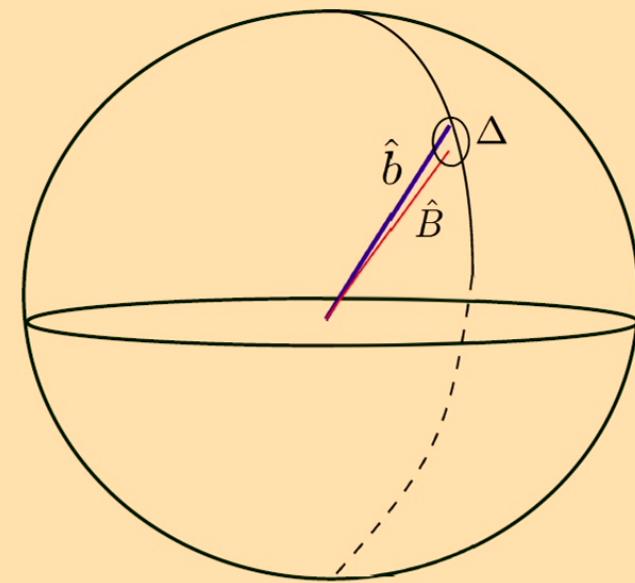
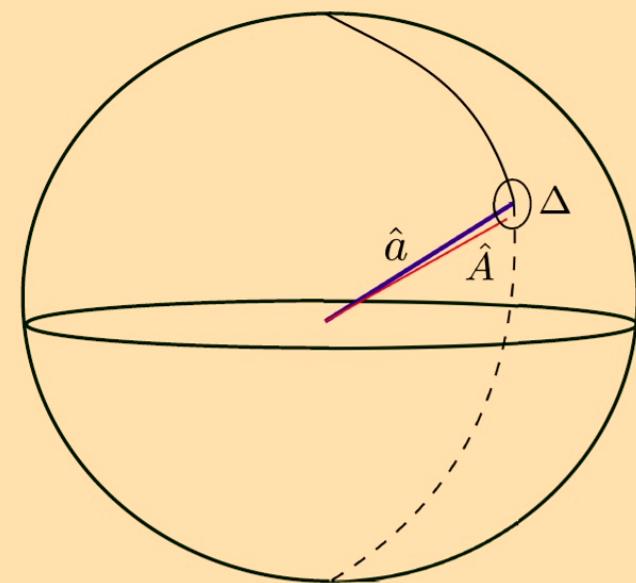
$$\hat{A} \cdot \hat{B} = 1 - \frac{2n - 1}{N/2}$$

$n \in \{1, 2, \dots N/2\}$, $N \rightarrow$ very large even constant.



$$\hat{A} \cdot \hat{B} = 1 - \frac{2n-1}{N/2} \rightarrow \text{"rationality constraint"}$$

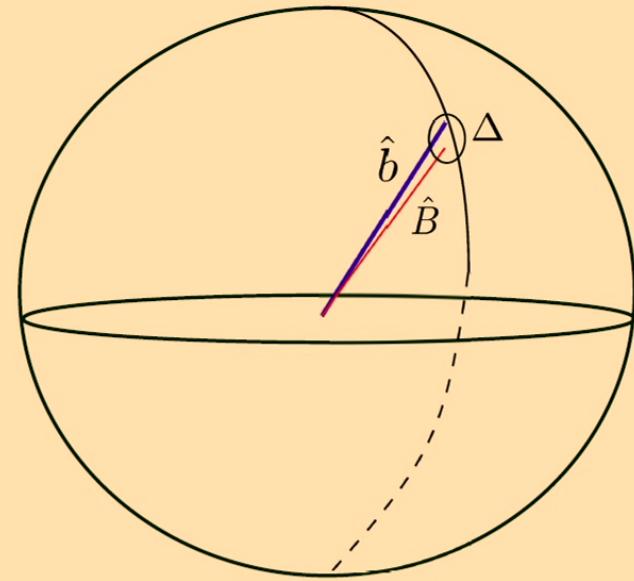
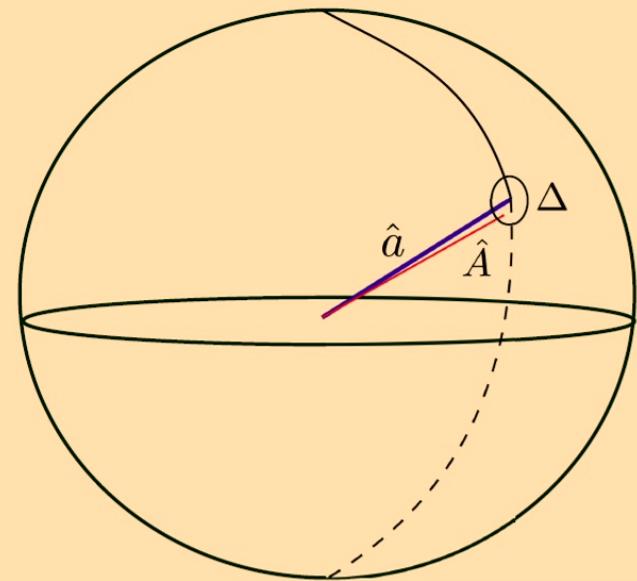
$$L_{1 \times N}(|+\rangle_{\hat{A}}, \hat{B}) = [+1 - 1 + 1 \dots - 1]_{1 \times N}$$



$$\hat{A} \cdot \hat{B} = 1 - \frac{2n-1}{N/2} \rightarrow \text{"rationality constraint"}$$

$L_{1 \times N}(|+\rangle_{\hat{A}}, \hat{B}) \rightarrow \text{"bit string"}$

counter-factual measurements, non-commutativity, local causality, measurement independence...



$$\hat{A} \cdot \hat{B} = 1 - \frac{2n-1}{N/2}$$

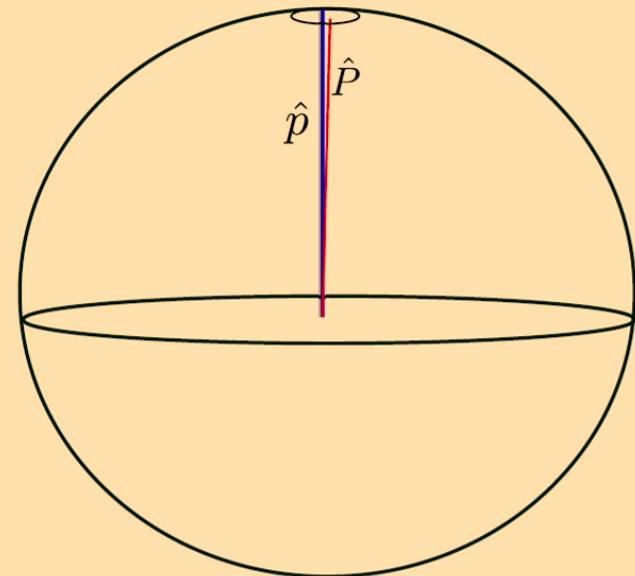
\rightarrow "rationality constraint"

not realistic

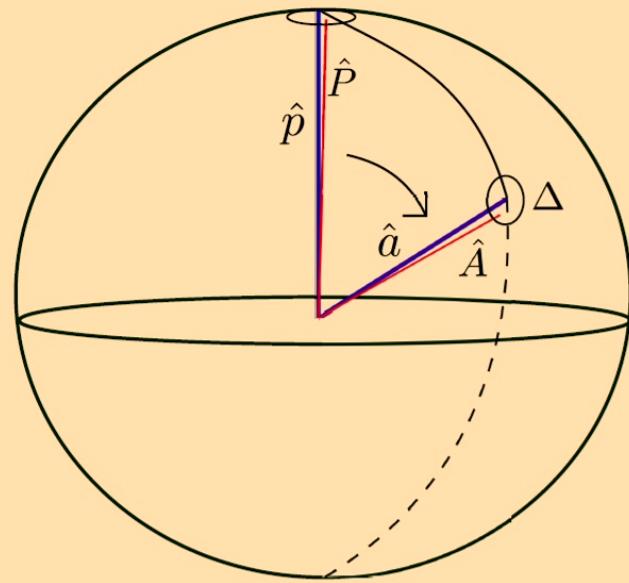
$L_{1 \times N}(|+\rangle_{\hat{A}}, \hat{B}) \rightarrow$ "bit string"

hidden-variable model ?

Model for single spin-1/2 particles



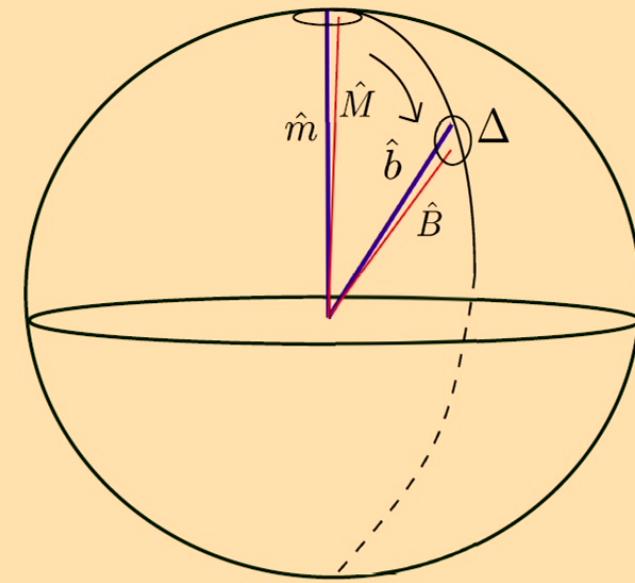
Preparation end



Exact preparation $|+\rangle_{\hat{A}}$

$$\hat{A} \equiv \hat{A}(\hat{P}, \hat{p}, \hat{a})$$

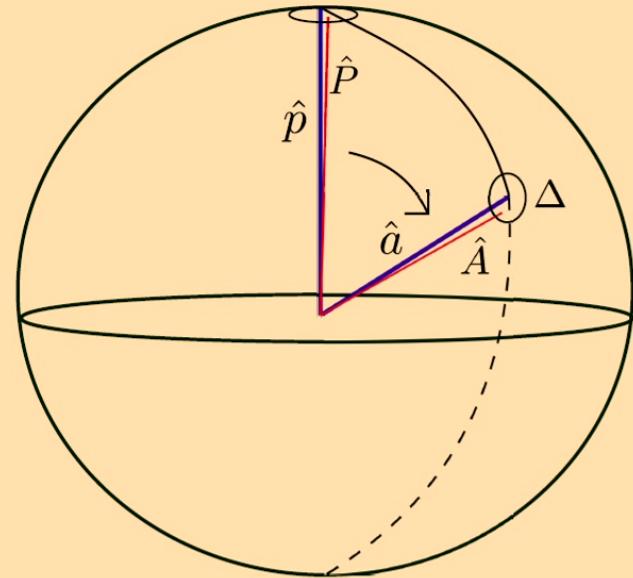
$$|\hat{A}(\hat{P}, \hat{p}, \hat{a}) - \hat{a}| < \Delta \quad \forall \hat{P}, \hat{p}, \hat{a}$$



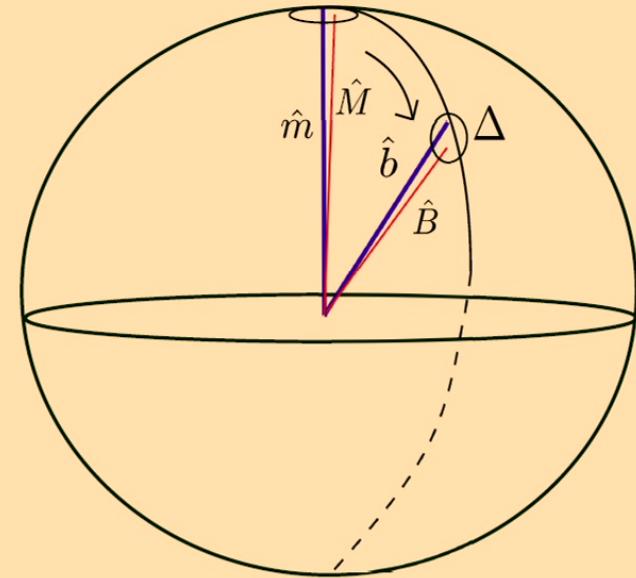
Exact measurement basis $\{|+\rangle_{\hat{B}}, |-\rangle_{\hat{B}}\}$

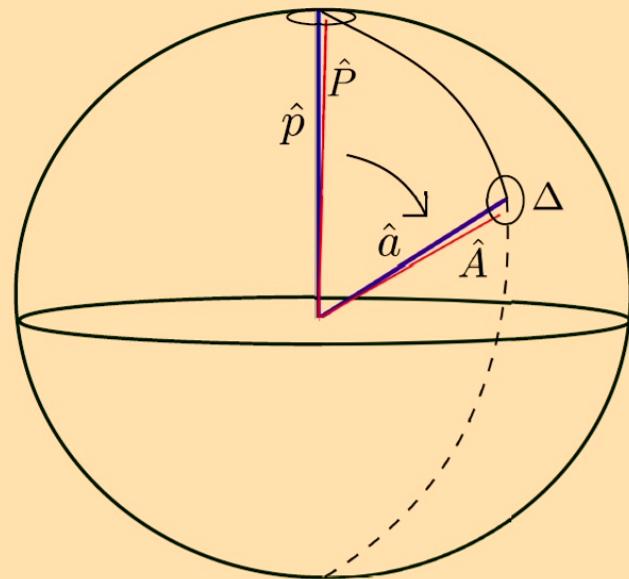
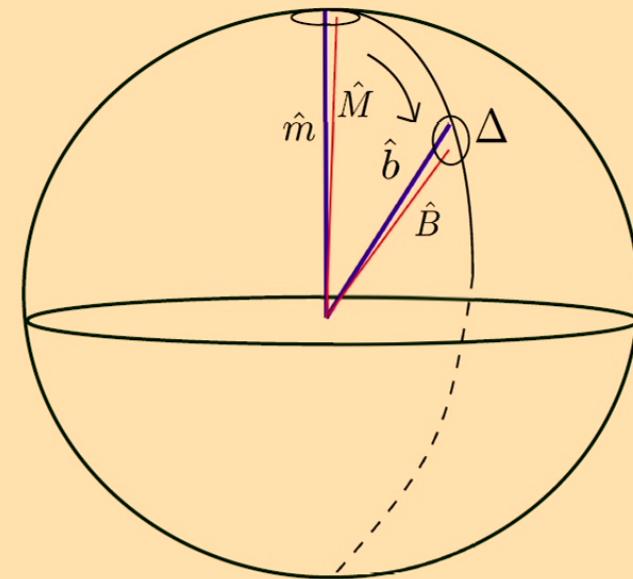
$$\hat{B} \equiv \hat{B}(\hat{M}, \hat{m}, \hat{b})$$

$$|\hat{B}(\hat{M}, \hat{m}, \hat{b}) - \hat{b}| < \Delta \quad \forall \hat{M}, \hat{m}, \hat{b}$$

Exact preparation $|+\rangle_{\hat{A}}$

$$|+\rangle_{\hat{A}} = \cos(\theta_{AB}/2)|+\rangle_{\hat{B}} + e^{i\phi_{AB}} \sin(\theta_{AB}/2)|-\rangle_{\hat{B}}, \cos\theta_{AB} = \hat{A} \cdot \hat{B}$$

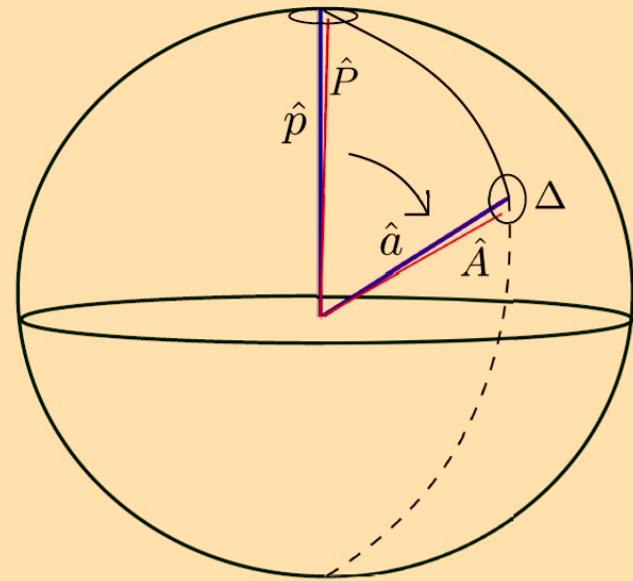
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Exact preparation $|+\rangle_{\hat{A}}$ Exact measurement basis $\{|+\rangle_{\hat{B}}, |-\rangle_{\hat{B}}\}$

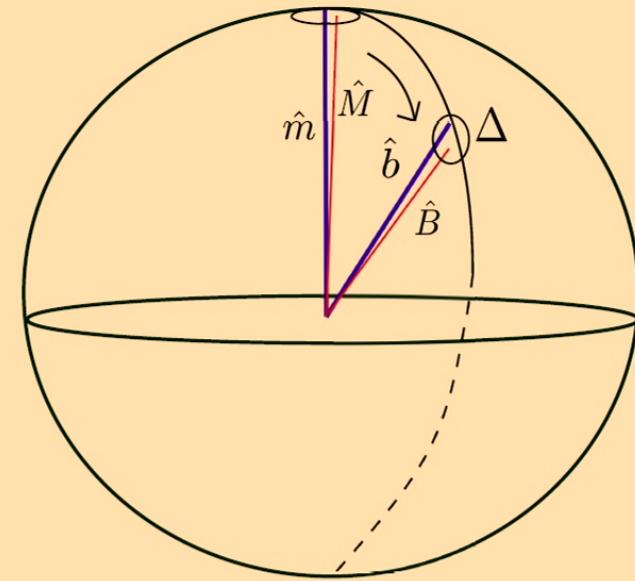
$$|+\rangle_{\hat{A}} = \cos(\theta_{AB}/2)|+\rangle_{\hat{B}} + e^{i\phi_{AB}} \sin(\theta_{AB}/2)|-\rangle_{\hat{B}}, \cos \theta_{AB} = \hat{A} \cdot \hat{B}$$

$L_{1 \times N}(|+\rangle_{\hat{A}}, \hat{B}) \rightarrow N \cos^2(\theta_{AB}/2) + 1$ elements, $N \sin^2(\theta_{AB}/2) - 1$ elements.

integer Ordering depends on ϕ_{AB} integer



Exact preparation $|+\rangle_{\hat{A}}$



Exact measurement basis $\{|+\rangle_{\hat{B}}, |-\rangle_{\hat{B}}\}$

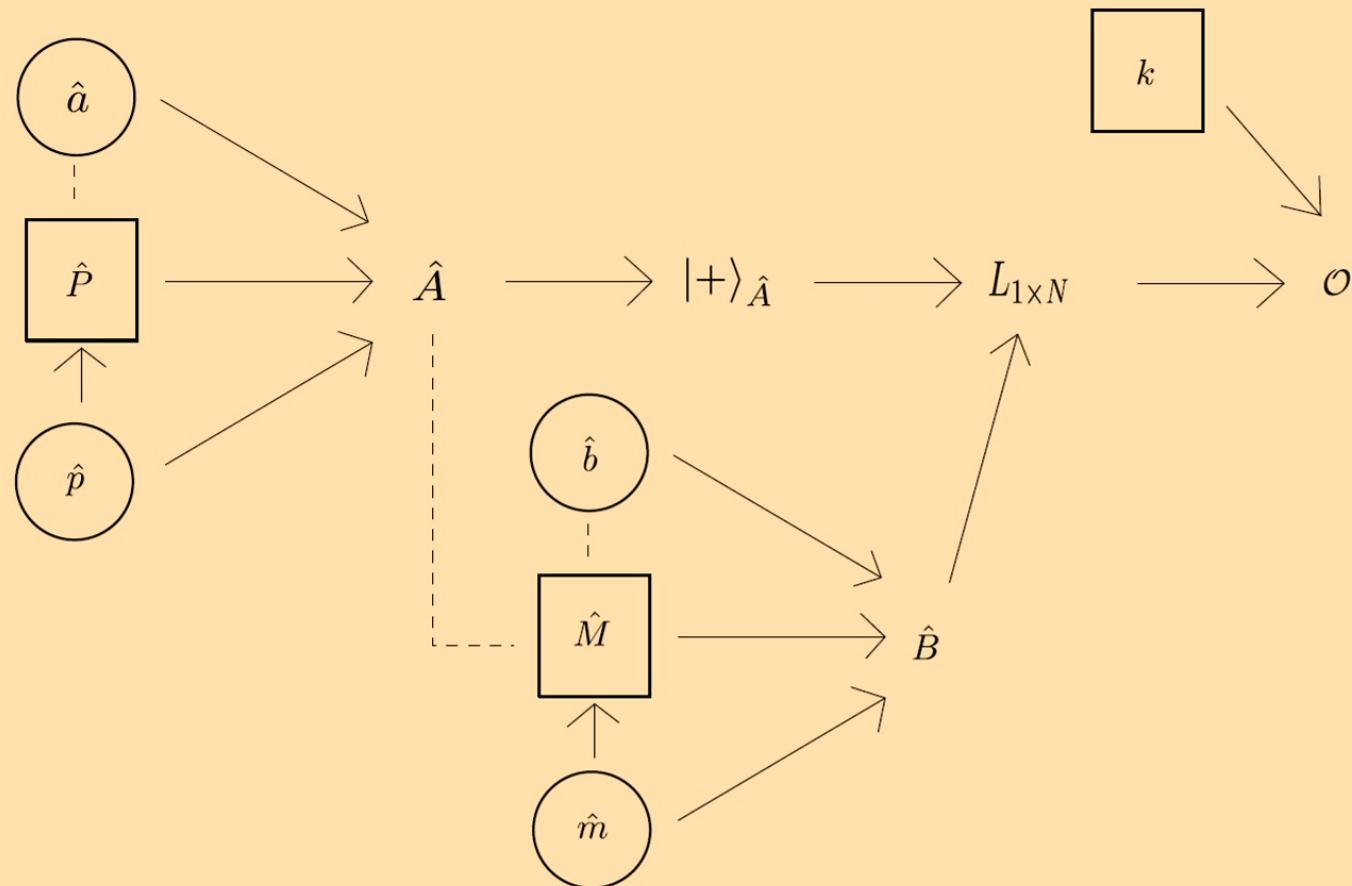
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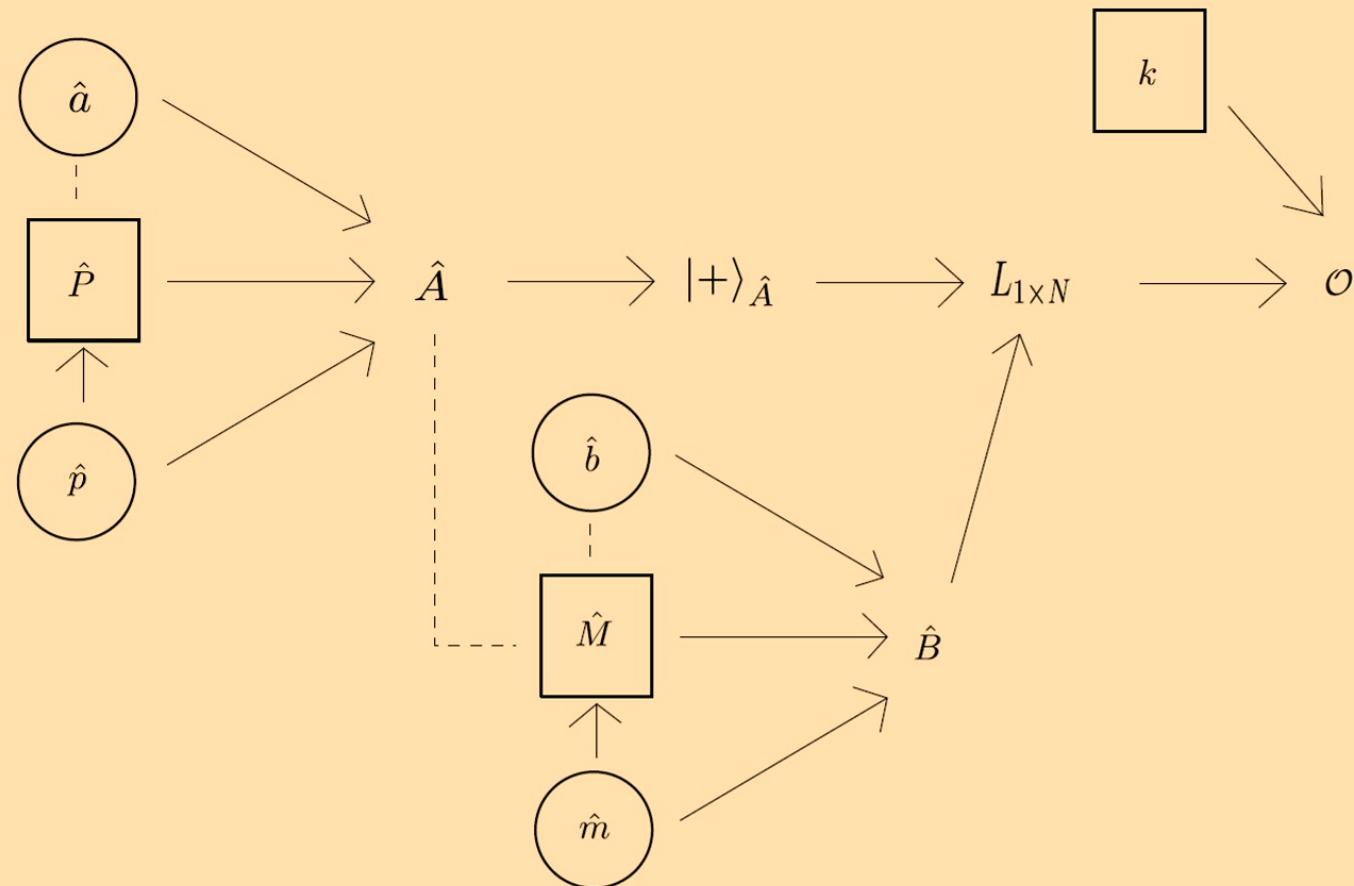
$L_{1 \times N}(|+\rangle_{\hat{A}}, \hat{B}) \rightarrow N \cos^2(\theta_{AB}/2) + 1$ elements, $N \sin^2(\theta_{AB}/2) - 1$ elements.

integer Ordering depends on ϕ_{AB} integer

$$\text{Outcome } \mathcal{O} = L_{1,k}(|+\rangle_{\hat{A}}, \hat{B})$$

position on bit string





Ontology

meas. experimental parameter
 prep. experimental parameter

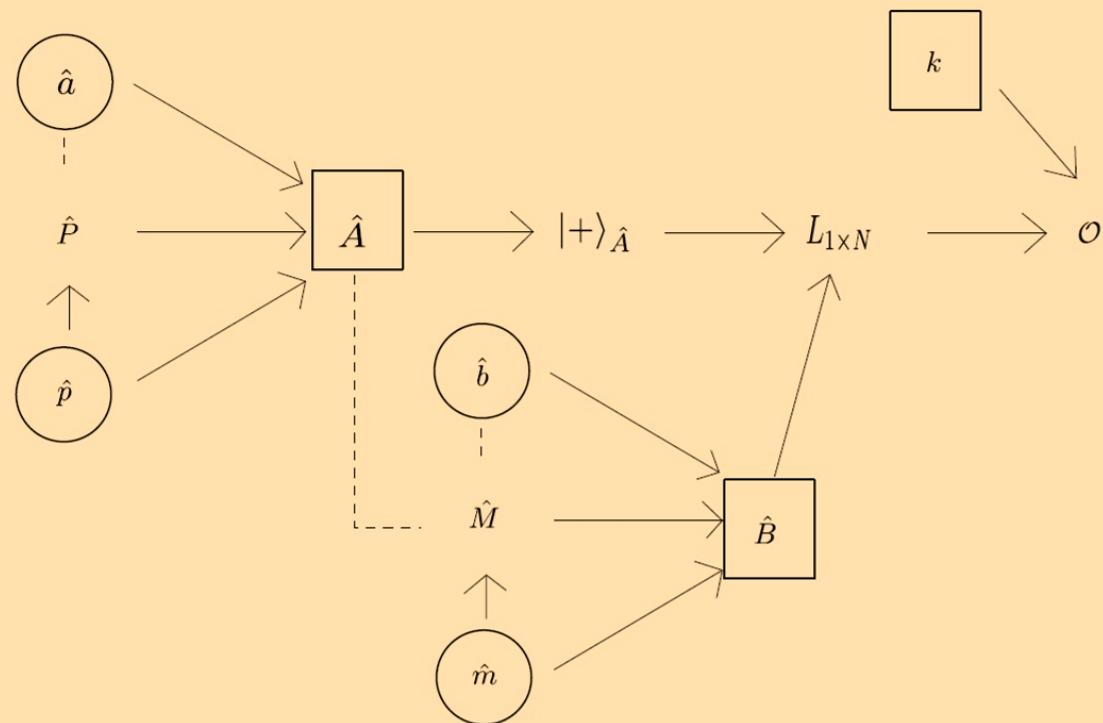
hidden variable

$$\mathcal{O}((\hat{p}, \hat{a}), (\hat{m}, \hat{b}), \mu) = \mathcal{O}(\hat{A}(\hat{P}, \hat{p}, \hat{a}), \hat{B}(\hat{M}, \hat{m}, \hat{b}), k) = L_{1,k}(|+\rangle_{\hat{A}(\hat{P}, \hat{p}, \hat{a})}, \hat{B}(\hat{M}, \hat{m}, \hat{b}))$$

Define $\mu \equiv (\hat{P}, \hat{M}, k)$

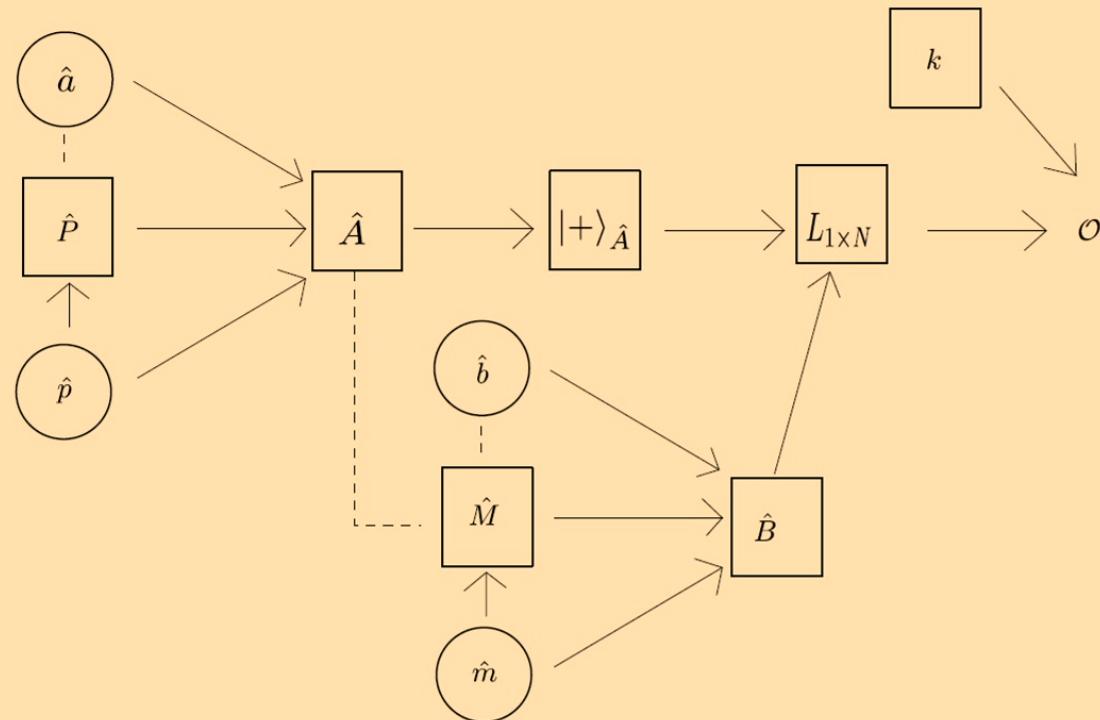
hidden variable associated with measurement device

$$\mathcal{O}\left((\hat{p}, \hat{a}), (\hat{m}, \hat{b}), \mu\right) = \mathcal{O}(\hat{A}(\hat{P}, \hat{p}, \hat{a}), \hat{B}(\hat{M}, \hat{m}, \hat{b}), k) = L_{1,k}(|+\rangle_{\hat{A}}(\hat{P}, \hat{p}, \hat{a}), \hat{B}(\hat{M}, \hat{m}, \hat{b}))$$



λ (along with measurement settings) completely specifies the individual measurement outcome.

Therefore, $\hat{P}, \hat{M}, \hat{A}, \hat{B}, |+\rangle_{\hat{A}}, L_{1 \times N}, k$ have ontological status.



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Consequences:

1. Model is ψ -ontic.
2. Bit string is ontological.

Therefore, $\hat{P}, \hat{M}, \hat{A}, \hat{B}, |+\rangle_{\hat{A}}, L_{1 \times N}, k$ have ontological status.

Consequences:

1. Model is ψ -ontic.
2. Bit string is ontological \Rightarrow model has redundant information.

Does the model reproduce quantum mechanics?

Consider an ensemble of runs for which $\hat{p}, \hat{a}, \hat{m}, \hat{b}$ are constant.

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Consider an ensemble of runs for which $\hat{p}, \hat{a}, \hat{m}, \hat{b}$ are constant.

Define

$$1. \rho(\hat{P}|\hat{a}, \hat{p}) : |\hat{a}' - \hat{a}| = |\int \rho(\hat{P}|\hat{a}, \hat{p})\hat{A}(\hat{P}, \hat{p}, \hat{a})d\Omega_P - \hat{a}| < \Delta$$

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$$2. p(\hat{M}_i|\hat{A}, \hat{b}, \hat{m}) : |\hat{b}' - \hat{b}| = |\sum_i p(\hat{M}_i|\hat{A}, \hat{b}, \hat{m})\hat{B}(\hat{M}_i, \hat{m}, \hat{b}) - \hat{b}| < \Delta$$

measurement dependence, discreteness

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$$2. p(\hat{M}_i|\hat{A}, \hat{b}, \hat{m}) : |\hat{b}' - \hat{b}| = \left| \sum_i p(\hat{M}_i|\hat{A}, \hat{b}, \hat{m}) \hat{B}(\hat{M}_i, \hat{m}, \hat{b}) - \hat{b} \right| < \Delta$$

$$3. p(k) = 1/N$$

measurement dependence, discreteness

The model predicts the following expectation value of outcomes:

$$\begin{aligned} \sum_{k=1}^N \sum_i^\alpha \int \rho(\hat{P}|\hat{p}, \hat{a}) p(\hat{M}_i|\hat{A}, \hat{b}, \hat{m}) p(k) \mathcal{O}(\hat{A}(\hat{P}, \hat{p}, \hat{a}), \hat{B}(\hat{M}_i, \hat{m}, \hat{b}), k) d\Omega_P \\ = \hat{a}' \cdot \hat{b}' \end{aligned}$$

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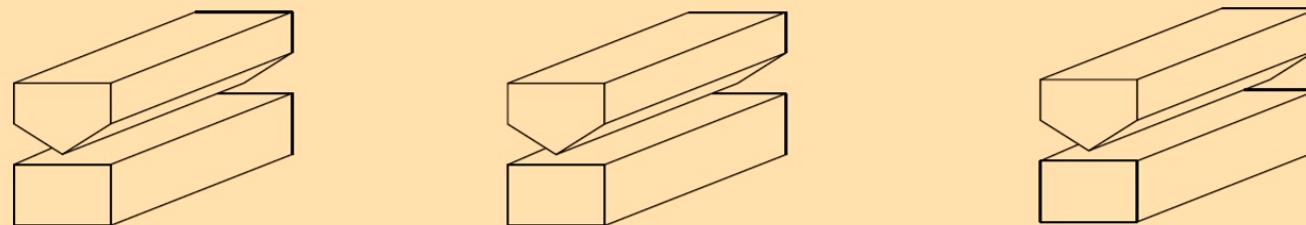
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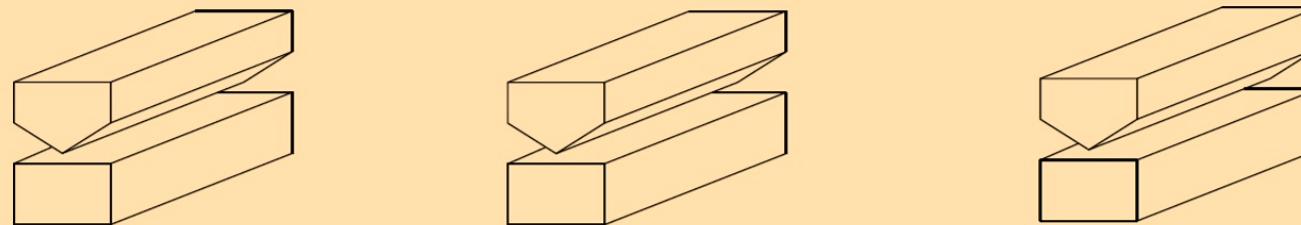
Does the model provide insight into counterfactual reasoning?



$$\hat{a} \longrightarrow \hat{c} \longrightarrow \hat{b} \quad ?$$

X

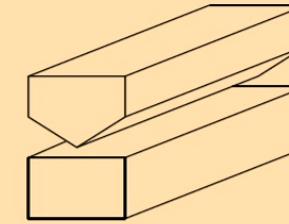
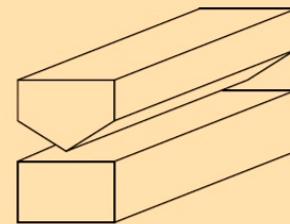
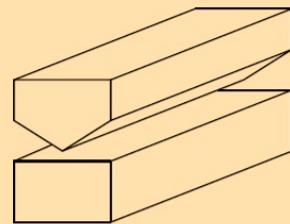
Does the model provide insight into counterfactual reasoning?



$$\hat{a} \longrightarrow \hat{b} \longrightarrow \hat{c}$$

$$\hat{m}_1, \hat{M}_1 \longrightarrow \hat{m}_2, \hat{M}_2 \longrightarrow \hat{m}_3, \hat{M}_3$$

Does the model provide insight into counterfactual reasoning?



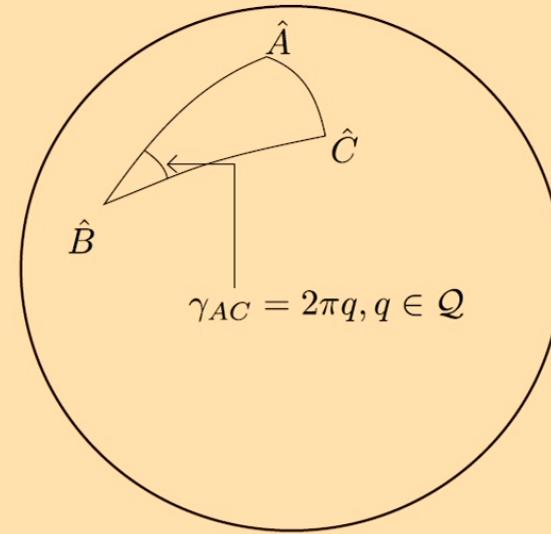
$$\hat{a} \longrightarrow \hat{b} \longrightarrow \hat{c}$$

$$\hat{m}_1, \hat{M}_1 \longrightarrow \hat{m}_2, \hat{M}_2 \longrightarrow \hat{m}_3, \hat{M}_3$$

$$\hat{A}(\hat{m}_1, \hat{M}_1, \hat{a}) \longrightarrow \hat{B}(\hat{m}_2, \hat{M}_2, \hat{b}) \longrightarrow \hat{C}(\hat{m}_3, \hat{M}_3, \hat{c})$$

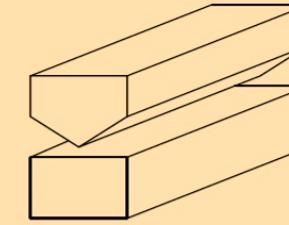
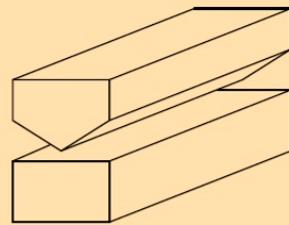
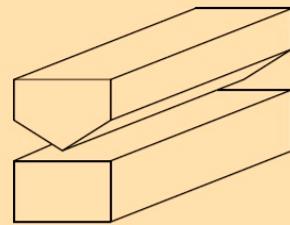
$$\begin{aligned}\hat{A} \cdot \hat{B} &\in \mathcal{Q} \\ \hat{B} \cdot \hat{C} &\in \mathcal{Q} \\ \gamma_{AC}/2\pi &\in \mathcal{Q}\end{aligned}$$

$$\Rightarrow \hat{A} \cdot \hat{C} \notin \mathcal{Q}$$



$$\Delta(ABC)$$

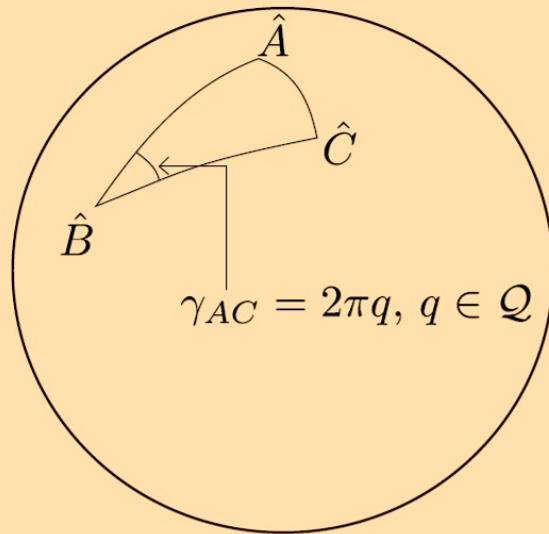
$$\hat{a} \rightarrow \hat{b} \rightarrow \hat{c}$$



$$\hat{a} \longrightarrow \hat{c} \longrightarrow \hat{b} \quad ?$$

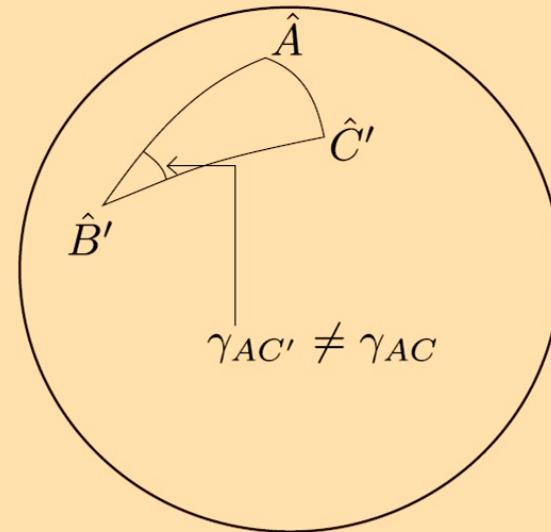
$$\hat{m}_1, \hat{M}_1 \longrightarrow \hat{m}_2, \hat{M}_2 \longrightarrow \hat{m}_3, \hat{M}_3$$

$$\hat{A}(\hat{m}_1, \hat{M}_1, \hat{a}) \longrightarrow \hat{C}'(\hat{m}_2, \hat{M}_2, \hat{c}) \longrightarrow \hat{B}'(\hat{m}_3, \hat{M}_3, \hat{b})$$



$$\Delta(ABC)$$

Case I: $\hat{a} \rightarrow \hat{b} \rightarrow \hat{c}$



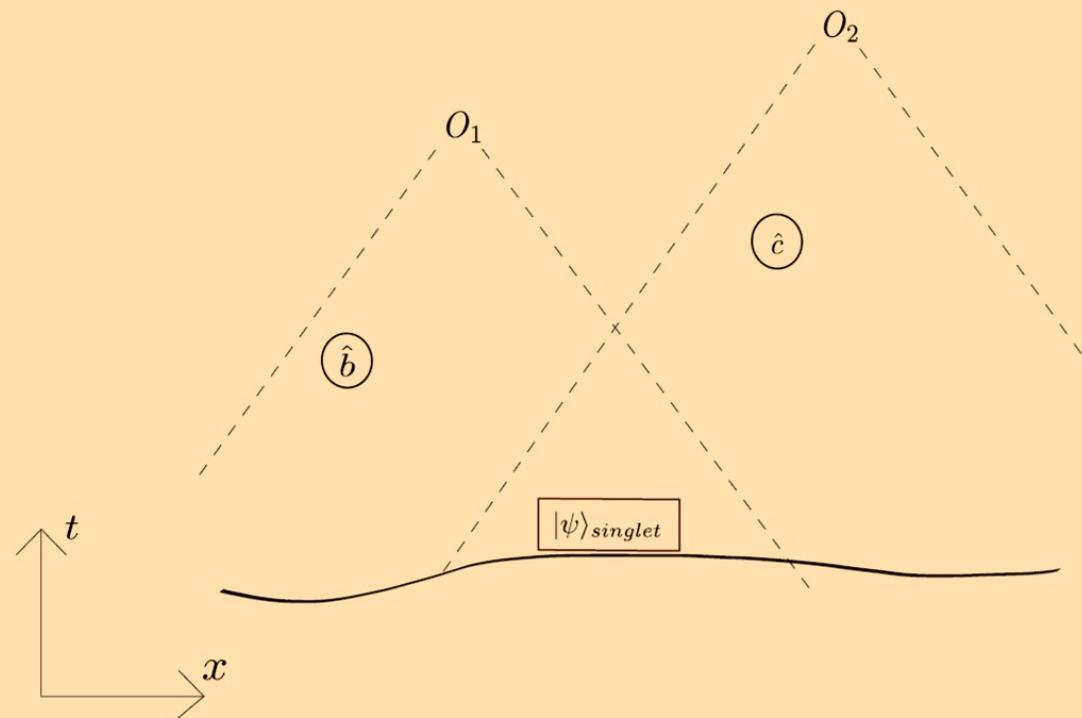
$$\Delta(AB'C')$$

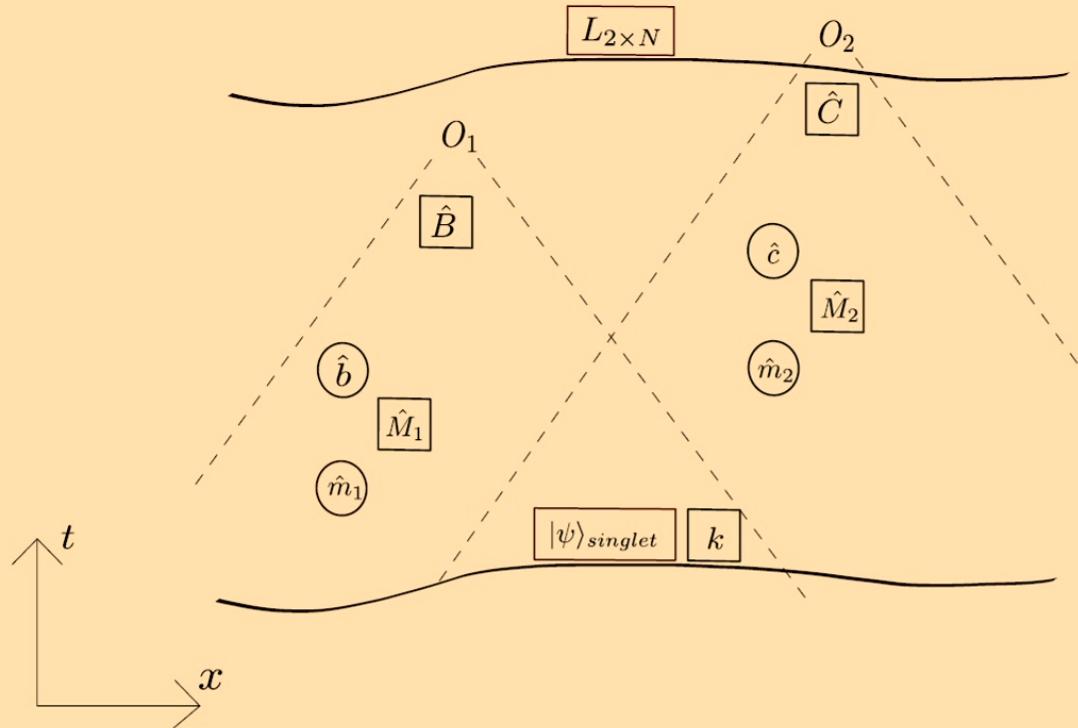
Case II (counterfactual): $\hat{a} \rightarrow \hat{c} \rightarrow \hat{b}$

“..if U is a universe in which $\hat{a} \rightarrow \hat{b} \rightarrow \hat{c}$ is performed on a particular particle... the counterfactual universe U' where $\hat{a} \rightarrow \hat{c} \rightarrow \hat{b}$ is performed on the same particle cannot lie on I_U .¹

¹T. Palmer, Proc. R. Soc. A 2020, 476, 20190350.

Extending the model to Bell scenario





$$\hat{B} \cdot \hat{C} = 1 - \frac{4n}{N}, \text{ where } n \in \{1, 2, \dots, N/2\}$$

Single particle

$$\mu \equiv (\hat{P}, \hat{M}, k)$$

$$O(\mu, \hat{p}, \hat{a}, \hat{b}, \hat{m}) = L_{1,k}(|+\rangle_{\hat{A}}, \hat{B}) \quad O_2(\mu, \hat{b}, \hat{m}_1, \hat{c}, \hat{m}_2) = L_{2,k}(|\psi\rangle_{singlet}, \hat{B}, \hat{C})$$

Entangled singlet pair

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$$\mathcal{O}_1(\mu) = L_{1,k}(|\psi\rangle_{singlet})$$

$$L_{2 \times N}(|\psi\rangle_{singlet}, \hat{B}, \hat{C}) = \begin{bmatrix} \overbrace{+1.....+1}^{N/2} & \overbrace{+1....+1}^{N/2} \\ \underbrace{+1....+1}_n & \underbrace{-1....-1}_{N/2-n} \\ \underbrace{-1.....-1}_n & \underbrace{+1.....+1}_{N/2-n} \end{bmatrix}$$

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nonlocal

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arbitrary $\hat{B} \cdot \hat{C} = 1 - \frac{4n}{N}$, where $n \in \{1, 2, \dots, N/2\}$
 discrete, using $|\psi(t)\rangle$

The model predicts the expectation value of outcomes to be

$$\sum_{k=1}^N \sum_i^\alpha \int \rho(\hat{M}_1|\hat{b}, \hat{m}_1) p(\hat{M}_{2i}|\hat{c}, \hat{m}_2, \hat{B}) p(k) \mathcal{O}_1(\mu) \mathcal{O}_2(\mu, \hat{b}, \hat{m}_1, \hat{c}, \hat{m}_2) d\Omega_{M_1}$$
$$= -\hat{b}' \cdot \hat{c}' \sim -\hat{b} \cdot \hat{c}$$

ψ -onticity

"...the bit-string is not the ontic state of the model, it is an ensemble of ontic states."⁴

$$L_{1 \times N}(|+\rangle_{\hat{A}}, \hat{B}) \neq |+\rangle_{\hat{A}}$$

⁴ J. Hance et al., arXiv:2108.08144 2021.

ψ -onticity

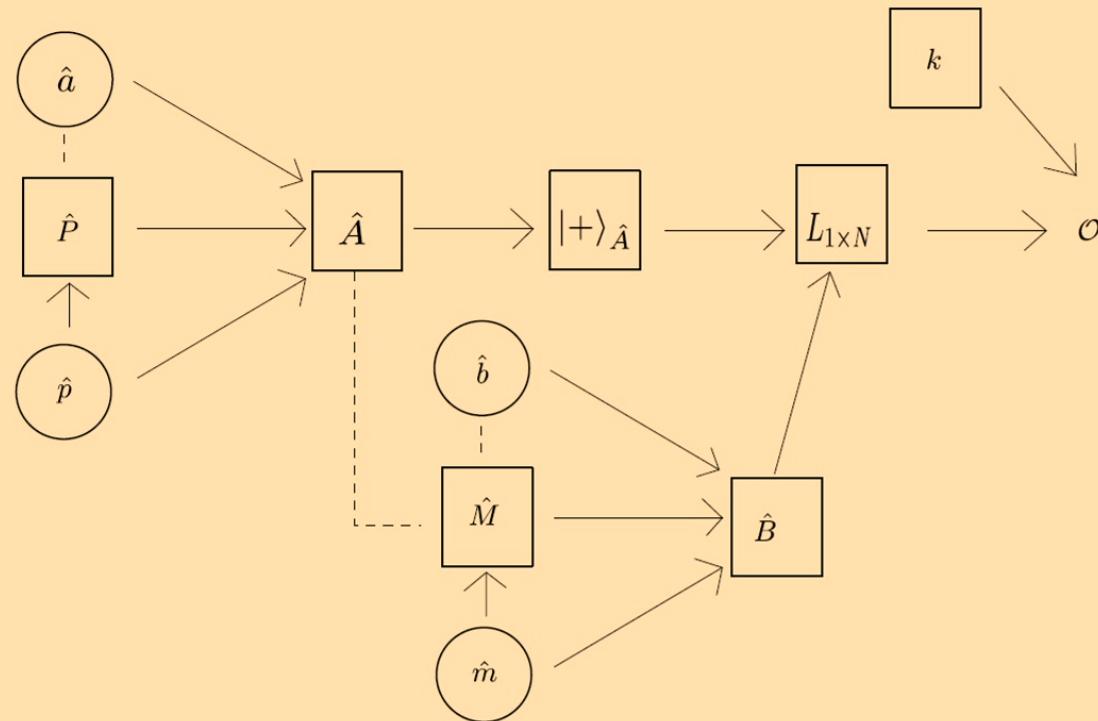
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λ (along with measurement settings) completely specifies the individual measurement outcome.

Therefore, $\hat{P}, \hat{M}, \hat{A}, \hat{B}, |+\rangle_{\hat{A}}, L_{1 \times N}, k$ have ontological status.



Local causality

"A theory will be said to be locally causal if the probabilities attached to values of local beables in a space-time region 1 are unaltered by specification of values of local beables in a space-like separated region 2, when what happens in the backward light cone of 1 is already sufficiently specified, for example by a full specification of local beables in a space-time region 3 [located in the backward light cone of 1 or in the overlap of backward light cones of 1 and 2]."¹⁰

¹⁰ J. S. Bell in *Speakable and unspeakable in quantum mechanics: Collected papers on quantum philosophy*, Cambridge Univ. Press, 2004.

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