Title: Equivariant Higher Berry classes and chiral states

Speakers: Nikita Sopenko

Series: Quantum Matter

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Abstract: I will talk about the generalization of Berry classes for quantum lattice spin systems. It defines invariants of topologically ordered states or families thereof. In particular, its equivariant version for 2d gapped states gives the zero-temperature Hall conductance and its various generalizations. I will also discuss the construction of chiral states realizing the topological order associated with a unitary rational vertex operator algebra for which these invariants are non-trivial

Zoom link: https://pitp.zoom.us/j/99910103969?pwd=VlVHVGRiV29iVEFyTXZyR3ovMkRaQT09

Equivariant Berry classes and 2d chiral states

Nikita Sopenko

California Institute of Technology

Perimeter Institute, December 5, 2022

based on arXiv: 2201.01327 with Anton Kapustin and work in progress

Motivation

General goal: describe the space $\mathcal{X}$ of topologically ordered states of quantum many-body systems at zero temperature $T = 0$.

Assumptions:

- By many-body systems, we mean a lattice spin system with local interaction
- Often assumed to be gapped or some analog of thereof.
- Equivalence relation: evolution by a local Hamiltonian and addition of disentangled degrees of freedom.

Belief: There is a class of states of lattice systems with a continuous limit described by a unitary TQFT. All the information must be encoded in the entanglement structure of the state.

A mathematical definition of such a class is yet to be understood.

Modest goal: classify invertible states of matter. [A. Kitaev]

Topological invariants of states

We can probe the space of states from a given class by considering families parameterized by some manifold $M \to \mathcal{X}$ and construct their invariants.

By topological invariants of a family of states, we mean locally computable quantities or objects, which do not depend on the location of computation and do not change if we continuously deform the family.

If $M = \mathrm{pt}$, it gives an invariant of the topological phase = connected component of the space $\mathcal{X}$.

We can guess what sort of invariants we should expect from the corresponding effective field theory and try to define it microscopically.

Berry classes for 0d systems

Let M be a smooth manifold parameterizing a family $\psi : M \to \mathcal{X}_0$ of 0d states. We can pick a connection $\mathsf{G} \in \Omega^1(M, \mathfrak{D})$ such that

$$d\langle \mathcal{A} \rangle_\psi = \langle D\mathcal{A} \rangle_\psi := \langle d\mathcal{A} + [\mathsf{G}, \mathcal{A}] \rangle_\psi. \tag{1}$$

If $\mathsf{F} = d\mathsf{G} + \frac{1}{2}[\mathsf{G}, \mathsf{G}]$, then $\frac{1}{2\pi i}[\langle \mathsf{F} \rangle_\psi] \in H^2(M, \mathbb{R})$ defines the usual Berry class that is quantized.

The space of all 0d states has the homotopy type of $\mathbb{CP}^\infty$. The only non-vanishing homotopy group $\pi_2(\mathbb{CP}^\infty) = \mathbb{Z}$ is responsible for the existence and the quantization of the Berry class.

Generalization to many-body systems with locality

For many-body systems, one has to impose locality by allowing only deformations that can be performed by a local Hamiltonian. Even if we restrict to such G, the average $\langle \mathrm{F} \rangle_\psi$ is not defined in the thermodynamic limit.

Intuition: Suppose at long distances our d-dimensional lattice system is described by a trivial (d+1)-dimensional QFT on Σ. If the parameters are a slowly varying function $\phi : \Sigma \to M$, the partition function $Z(\Sigma, \phi)$ may have a topological term

$$Z \sim \exp(i \int_\Sigma \phi^* B),$$

where B is a $(d+1)$-form gauge field on M. We should be able to compute the cohomology class of $[dB]$.

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What is the class of states we should consider? How to define families of states preserving locality?

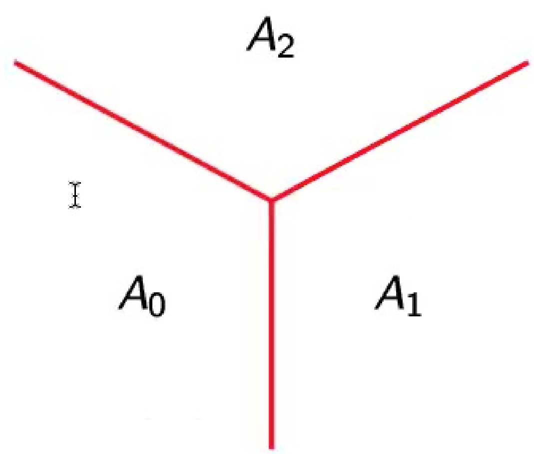
Local Hamiltonians

By a local Hamiltonian in this talk we mean a Hamiltonian which is a sum of uniformly bounded traceless almost local terms (with rapidly decaying tails).

Notation:
$\mathfrak{D}$ - the space of local Hamiltonians;
$\mathfrak{D}^{\psi}$ - the space of local Hamiltonians preserving the state ψ, i.e. $\langle [\mathrm{H}, ...] \rangle_{\psi} = 0$;
$\mathfrak{D}^{\psi}_{A}$ - the space of local Hamiltonians preserving the state ψ localized on A;
$\mathfrak{D}^{\psi}_{A_1 \dots A_n}$ - the space of local Hamiltonians preserving the state ψ localized on the intersection of $A_1, \dots, A_n$.

The key property

The condition on ψ: Let $\mathsf{H} \in \mathfrak{D}^{\psi}$. We require that it is possible to find a decomposition $\mathsf{H} = \sum_k \mathsf{H}_{A_k}$ such that $\mathsf{H}_{A_k} \in \mathfrak{D}^{\psi}_{A_k}$. We also require that it is true for lower-dimensional cones.



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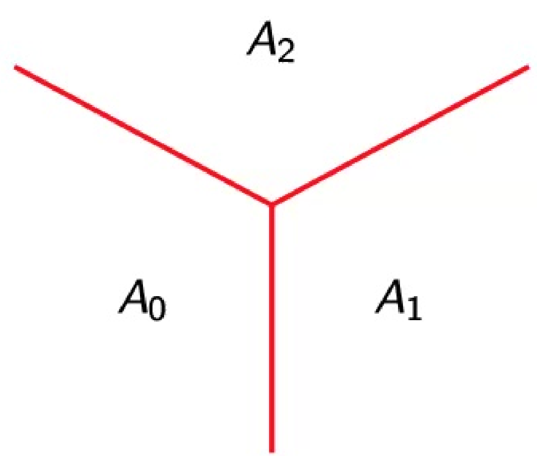
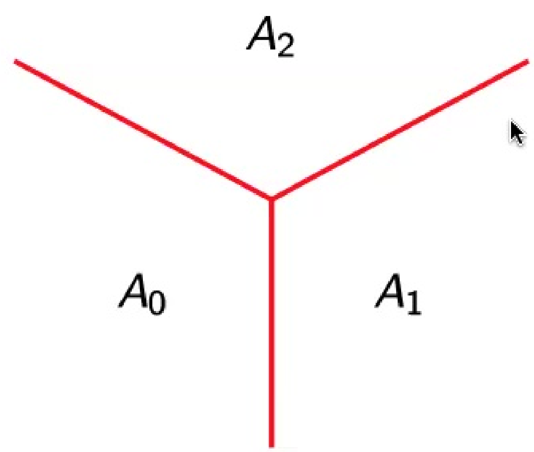


Any invertible or gapped state satisfies this condition (Main tool of the proof: Lieb-Robinson bound and the integral transform used in the quasi-adiabatic continuation).

The key property

More precise formulation is the exactness of the following complex:

$$\ldots \xrightarrow{\partial} \bigoplus_{j,k} \mathfrak{D}^{\psi}_{A_j A_k} \xrightarrow{\partial} \bigoplus_{j} \mathfrak{D}^{\psi}_{A_j} \xrightarrow{\partial} \mathfrak{D}^{\psi} \to 0$$

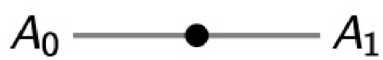


Berry classes for 1d systems

Let M be a smooth manifold. We define a smooth family of states $\psi : M \to \mathcal{X}_1$ by requiring that there exists $\mathsf{G} \in \Omega^1(M, \mathfrak{D})$ such that

$$d\langle \mathcal{A} \rangle_\psi = \langle D\mathcal{A} \rangle_\psi := \langle d\mathcal{A} + [\mathsf{G}, \mathcal{A}] \rangle_\psi .$$

We have $\mathsf{F} = d\mathsf{G} + \frac{1}{2}[\mathsf{G}, \mathsf{G}]$ satisfying $D\mathsf{F} = 0$. Note that $\langle [\mathsf{F}, \mathcal{A}] \rangle_\psi = \langle DD\mathcal{A} \rangle_\psi = dd\langle \mathcal{A} \rangle_\psi = 0$. Therefore F preserves ψ and there is a splitting $\mathsf{F} = \mathsf{F}_{A_0} + \mathsf{F}_{A_1}$ such that $\mathsf{F}_{A_i} \in \Omega^2(M, \mathfrak{D}^{\psi}_{A_i})$.



$D\mathsf{F}_{A_0}$ is a 3-form taking values in local observables localized on the interface $A_0 A_1$. The Berry class is given by

$$[\langle D\mathsf{F}_{A_0} \rangle_\psi] \in H^3(M, i\mathbb{R}).$$

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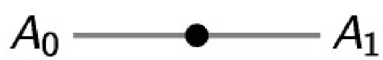

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General construction

There is a general construction for any dimension d that produces a class in $H^{d+2}(M, i\mathbb{R})$.

All the information can be packaged into a single equation

$$dG^{\bullet} + \frac{1}{2}\{G^{\bullet}, G^{\bullet}\} = -\partial G^{\bullet}$$

where $G^{\bullet} = G + g^{(0)} + g^{(1)} + \ldots$ for $g^{(k)} \in \Omega^{k+2}(M, C_k(\mathfrak{D}^{\psi}))$ and $\{\cdot, \cdot\}$ is a natural bracket on the extended complex.

The Berry class is given by

$$[\langle g^{(d)}_{A_0 \ldots A_d} \rangle_{\psi}] \in H^{d+2}(M, i\mathbb{R})$$

and does not depend on the choice of the solution of the equation or regions $A_0, \ldots, A_d$.

Equivariant classes

Suppose we have an on-site action of a Lie group G and consider G-invariant states up to a G-invariant equivalence.

Intuition: the partition function of a (d+1)-dimensional QFT describing the lattice system at low energy may depend on the background gauge fields, e.g.

$$Z_{(2+1)d} \sim \exp(i\frac{k}{4\pi}\int_{\Sigma} AdA)$$

where A is a $U(1)$ gauge field. We should be able to define an invariant taking values in $H^4(BU(1),\mathbb{R}) \cong \mathbb{R}$ that corresponds to k.

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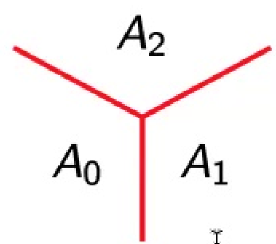
Example: 2d system with $U(1)$ symmetry

Let Q be the total charge for the $U(1)$ action that is a sum of on-site terms. We can split

$$Q = \tilde{Q}_{A_0} + \tilde{Q}_{A_1} + \tilde{Q}_{A_2}$$

such that $\tilde{Q}_{A_i} \in \mathfrak{D}^{\psi}_{A_i}$. We can modify $\tilde{Q}_{A_i}$ so that $[Q, \tilde{Q}_{A_i}] = 0$.



Note that $[\tilde{Q}_{A_0}, \tilde{Q}_{A_1}]$ is an almost local observable. The equivariant Berry class can be defined by

$$\sigma := 4\pi i \langle [\tilde{Q}_{A_0}, \tilde{Q}_{A_1}] \rangle_{\psi}.$$

Example: 2d system with $U(1)$ symmetry

Remarks:

- When the state is a ground state of some gapped Hamiltonian, σ coincides with the Hall conductance.
- For invertible states we have $\sigma \in 2\mathbb{Z}$ for bosonic systems and $\sigma \in \mathbb{Z}$ for fermionic.
- Defined for any state that has no local spontaneous symmetry breaking.

Equivariant classes

Remarks:

- One can similarly define invariants for any term in the effective action that consists of gauge fields corresponding to internal symmetries described by a Lie group G and variations of the parameters of the family.
- Example: $\int \phi \wedge F$ corresponds to the Thouless pump invariant of a family of 1d $U(1)$-invariant states.
- Example: $\frac{1}{(2\pi)^2} \int AdAdA$ corresponds to the invariant of a 4d $U(1)$-invariant state.
- For finite group symmetry G, one can define invariants for invertible states in low dimensions using similar methods.
- For the construction of invariants coming from a spatial structure (such as "gravitational Chern-Simons coupling") we need a better understanding of the class of topologically ordered states.

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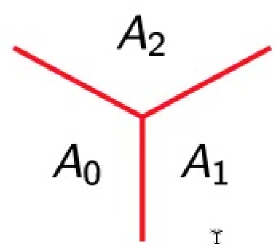

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It does not depend on the splitting or the choice of regions A_0, A_1, A_2. This class in fact takes values in the equivariant cohomology of a point $H^4_{U(1)}(\mathrm{pt}, \mathbb{R}) \cong \mathbb{R}$.

Chiral states in 2d

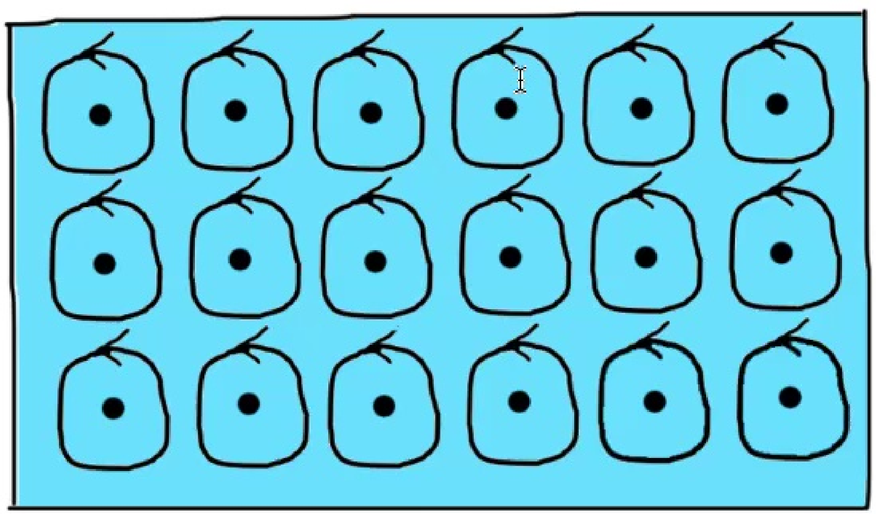
- Non-chiral states can be efficiently described by commuting projectors or tensor networks (Levin-Wen model).
- One of the main features of chiral states is the non-existence of boundaries with short-range correlations.
- Various candidates for chiral states of lattice spin systems have been proposed inspired by Laughlin wave function.
- While some expected properties of such states can be checked numerically, it is difficult to verify that they are representatives of the correct topological phase.
- To the best of our knowledge, no general construction exists for an arbitrary unitary rational vertex algebra.
- A representative of a non-trivial invertible 2d state has not been constructed. ($(E_8)_1$ state conjectured by Kitaev.)

Chiral states in 2d: general construction

Let V be a good unitary rational vertex operator algebra, and let $\mathcal{V}$ be the Hilbert space of the vacuum module. We define a model on a finite lattice Γ with on-site Hilbert spaces $\mathcal{V}_j \cong \mathcal{V}$ for $j \in \Gamma$.

For a collection of disjoint holomorphic embeddings of unit disks we define

$$\langle \Psi_\Gamma | : \bigotimes_{j \in \Gamma} \mathcal{V}_j \to \mathbb{C}$$



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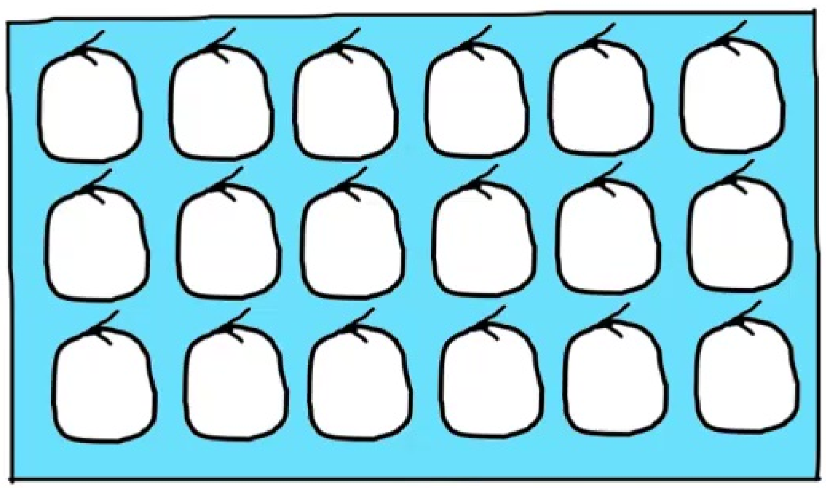

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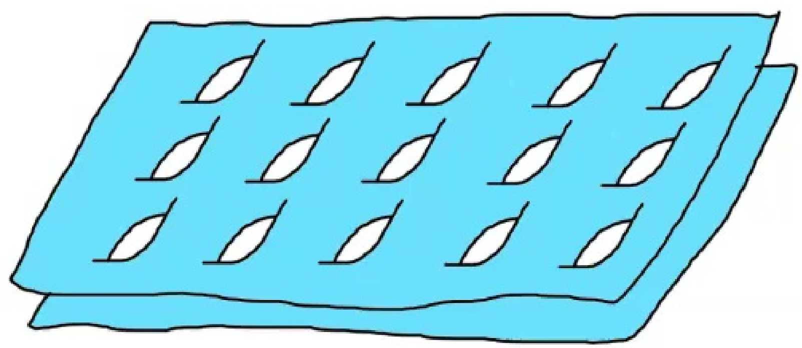
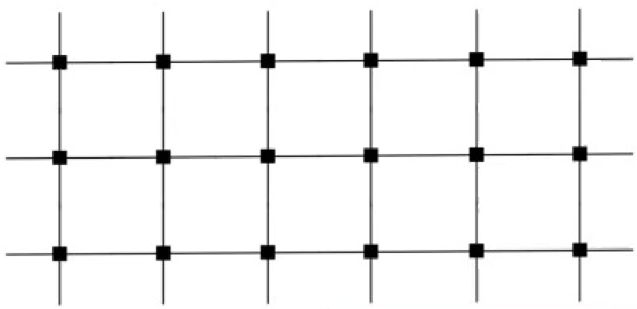
The norm $\langle \Psi_\Gamma | \Psi_\Gamma \rangle$ is given by the partition function on the following surface



It can be computed by cutting the surface into elementary blocks, so that we get a tensor network



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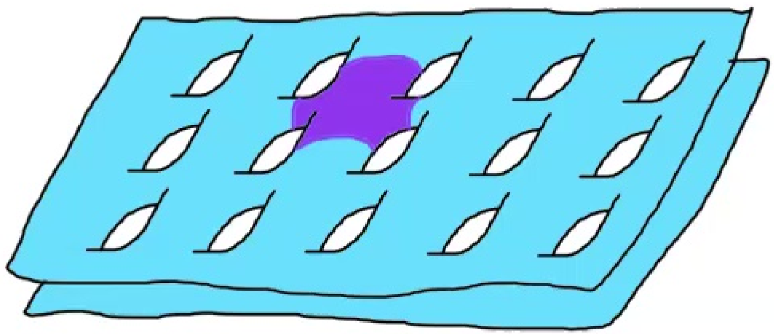

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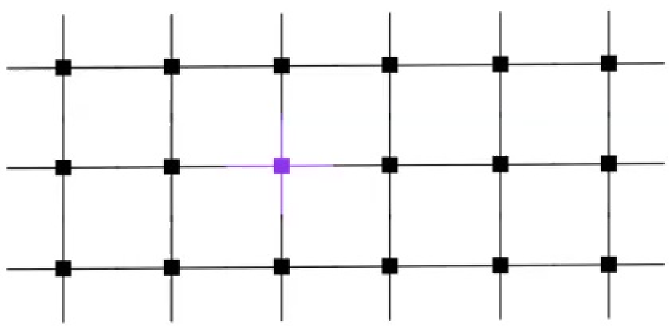
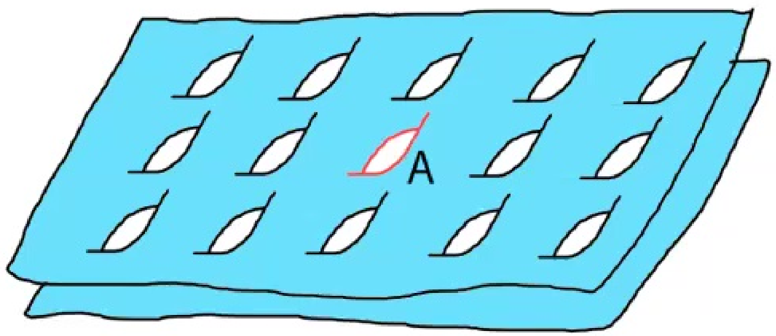


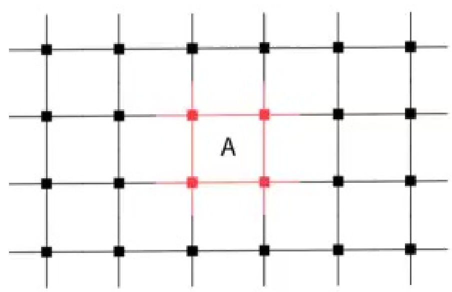
Chiral states in 2d: general construction

The average

$$\langle \mathcal{O}_A \rangle_{\Psi_\Gamma} := \frac{\langle \Psi_\Gamma | \mathcal{O}_A | \Psi_\Gamma \rangle}{\langle \Psi_\Gamma | \Psi_\Gamma \rangle}$$

is given by the partition function on the surface with insertions





Chiral states in 2d: general construction

Claims:

- The states are well-defined in the thermodynamic limit and have rapid decay of correlations in the bulk at least for large enough holes.
- The holes effectively "screen" the correlations.
- The correlation length depends on the size of the holes.
- Insertions of vertex operators and variations of the moduli of the surface can be implemented by a local Hamiltonian evolution of the lattice system, and therefore define a state in the same topological phase.
- The state is not a tensor network state. However, it can be obtained as a limit of a family of tensor network states together with a decoupled complex conjugated copy.

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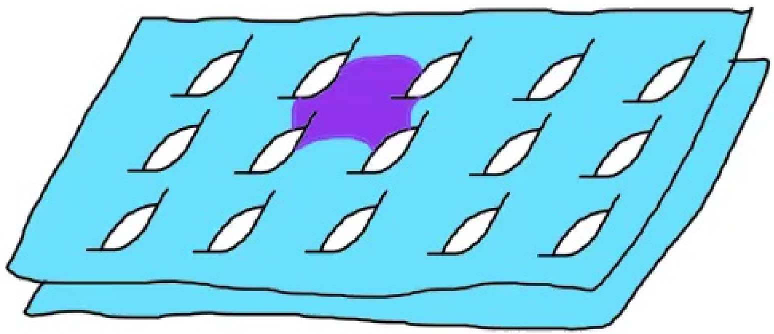

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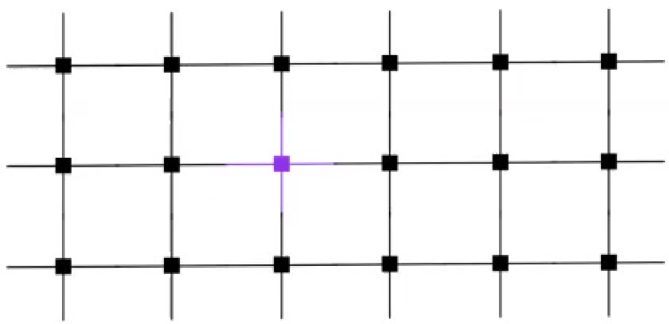

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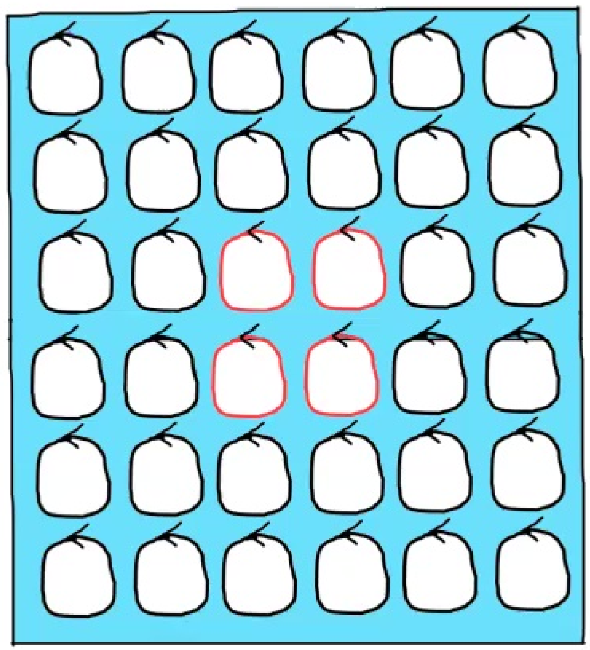
Chiral states in 2d: $U(1)$ symmetry

If V has a $U(1)$ symmetry, the construction naturally defines $U(1)$ invariant states with the on-site action being the action on the vacuum module of the $U(1)$ charge. The action of the charge Q_A on a region A corresponds to the insertion of $\int_{\partial A} J(z) \frac{dz}{2\pi i}$ into the defining correlator.



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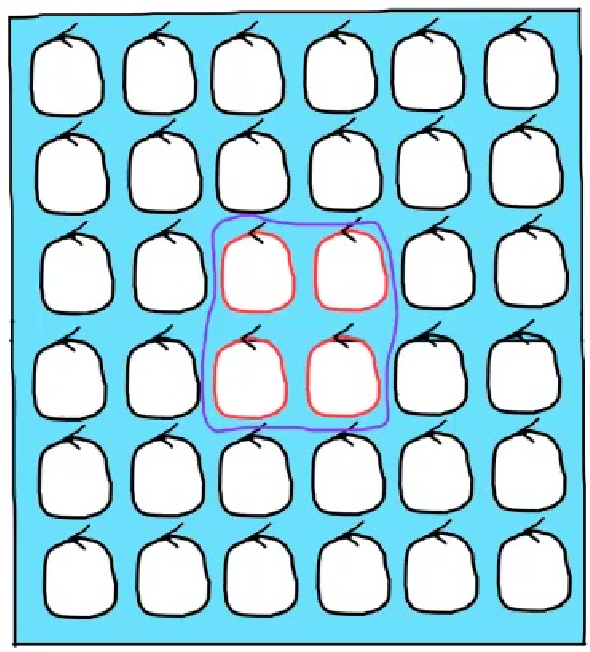

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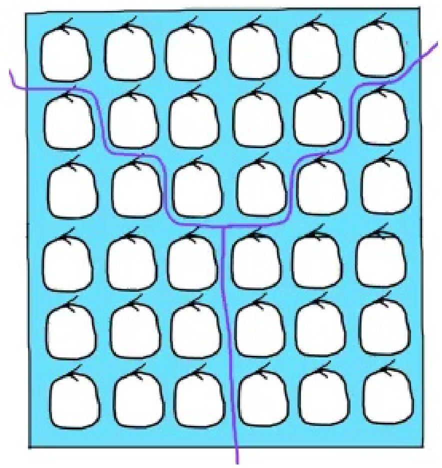

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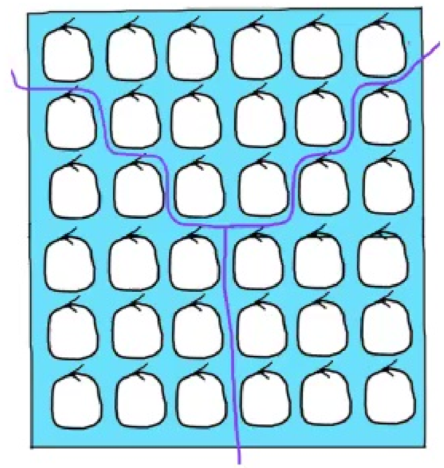

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Conclusion

- We have defined the topological invariants generalizing the Berry classes and their equivariant analogs for a certain class of states that includes gapped states.
- The construction of chiral 2d states from a unitary rational vertex operator algebra has been proposed that generates examples of 2d states with non-trivial equivariant Berry classes.

General construction

There is a general construction for any dimension d that produces a class in $H^{d+2}(M, i\mathbb{R})$.

All the information can be packaged into a single equation

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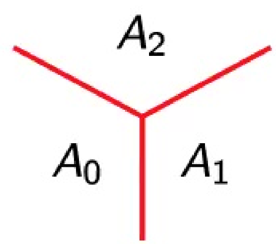
Example: 2d system with $U(1)$ symmetry

Let Q be the total charge for the $U(1)$ action that is a sum of on-site terms. We can split

$$\mathrm{Q} = \tilde{\mathrm{Q}}_{A_0} + \tilde{\mathrm{Q}}_{A_1} + \tilde{\mathrm{Q}}_{A_2}$$

such that $\tilde{\mathrm{Q}}_{A_i} \in \mathfrak{D}^{\psi}_{A_i}$. We can modify $\tilde{\mathrm{Q}}_{A_i}$ so that $[\mathrm{Q}, \tilde{\mathrm{Q}}_{A_i}] = 0$.



Note that $[\tilde{\mathrm{Q}}_{A_0}, \tilde{\mathrm{Q}}_{A_1}]$ is an almost local observable. The equivariant Berry class can be defined by

$$\sigma := 4\pi i \langle [\tilde{\mathrm{Q}}_{A_0}, \tilde{\mathrm{Q}}_{A_1}] \rangle_{\psi}.$$

It does not depend on the splitting or the choice of regions A_0, A_1, A_2. This class in fact takes values in the equivariant cohomology of a point $H^4_{U(1)}(\mathrm{pt}, \mathbb{R}) \cong \mathbb{R}$.