Title: Bootstrapping the lattice Yang-Mills theory

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Series: Quantum Fields and Strings

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Abstract: I will speak about my recent work with Vladimir Kazakov where we study the SU(Nc) lattice Yang-Mills theory in the planar limit, at dimensions D=2,3,4, via the numerical bootstrap method. It combines the Makeenko-Migdal loop equations, with the cut-off L on the maximal length of loops, and the positivity conditions on certain correlation matrices. Our algorithm is inspired by the pioneering paper of P. Anderson and M. Kruczenski but it is significantly more efficient, as it takes into account the symmetries of the lattice theory and uses the relaxation procedure in the line with our previous work on matrix bootstrap. We thus obtain the rigorous upper and lower bounds on the plaquette average at various couplings and dimensions. The results are quickly improving with the increase of cutoff L. For D=4 and L=16, the lower bound data appear to be close to the Monte Carlo data in the strong coupling phase and the upper bound data in the weak coupling phase reproduce well the 3-loop perturbation theory. We attempt to extract the information about the gluon condensate from this data. Our results suggest that this bootstrap approach can provide a tangible alternative to, so far uncontested, the Monte Carlo approach.

Zoom link: https://pitp.zoom.us/j/96101268796?pwd=QkdJbm9GUzQ2YnIrM1NlcUt2Z3Nvdz09

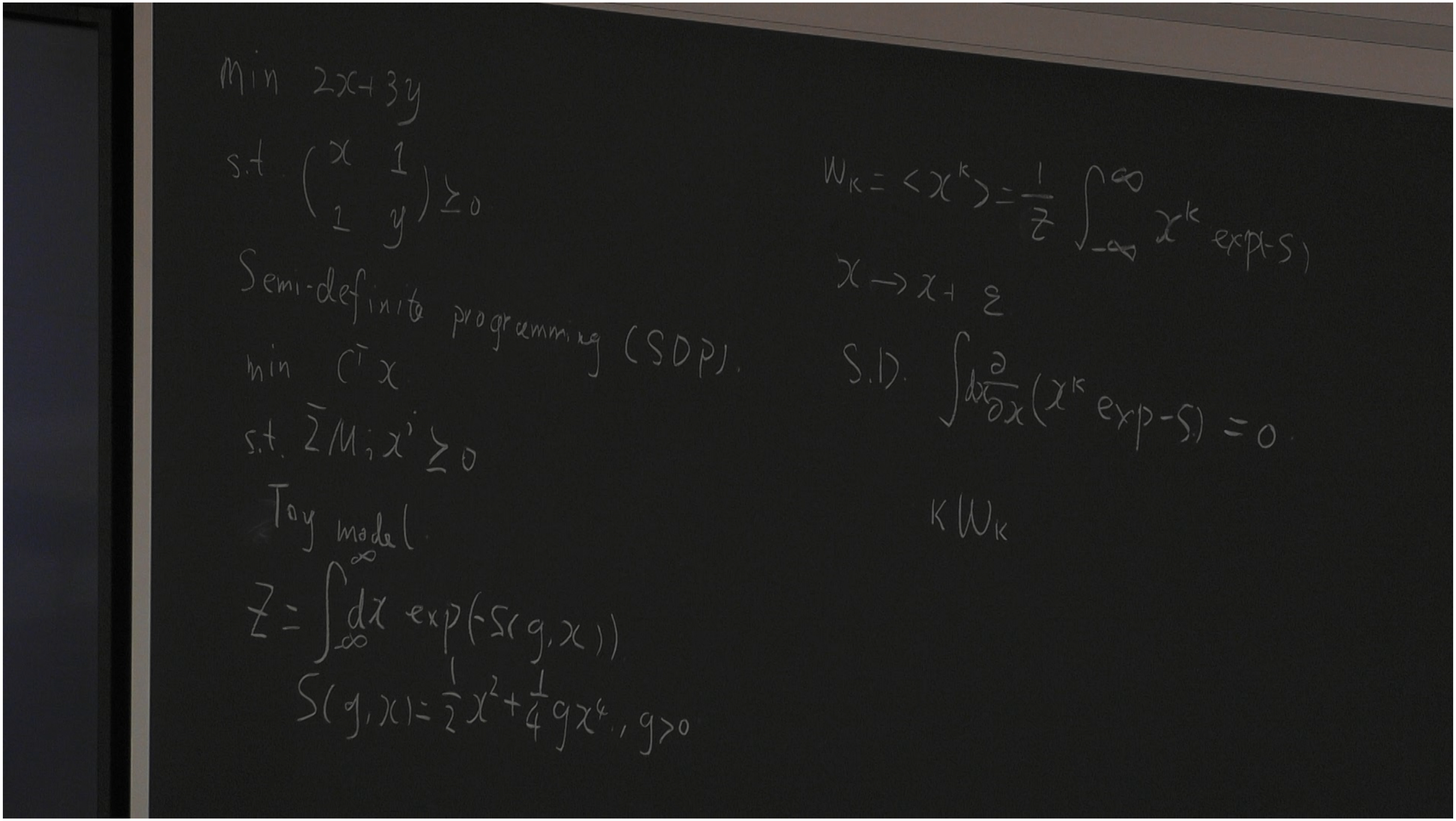
BOOTSTRAP THE LATTICE YANG-MILLS THEORY

work with Vladimir Kazakov

Zechuan Zheng

December 9, 2022

École Normale Supérieure


min 2x+3y
s.t. (x 1 / 1 y) ≥ 0
Semi-definite programming (SDP).
min c^T x
s.t. Σ M_i x^i ≥ 0
Toy model
Z = ∫_{-∞}^{∞} dx exp(-S(g,x))
S(g,x) = 1/2 x^2 + 1/4 g x^4, g>0
W_k = <x^k> = 1/Z ∫_{-∞}^{∞} x^k exp(-S)
x → x + ε
S.D. ∫dx ∂/∂x (x^k exp-S) = 0
k W_k

$x \to x + \varepsilon$

...mming (SDP).

$(g, x))$

$\frac{1}{4} g x^4 , \; g>0$

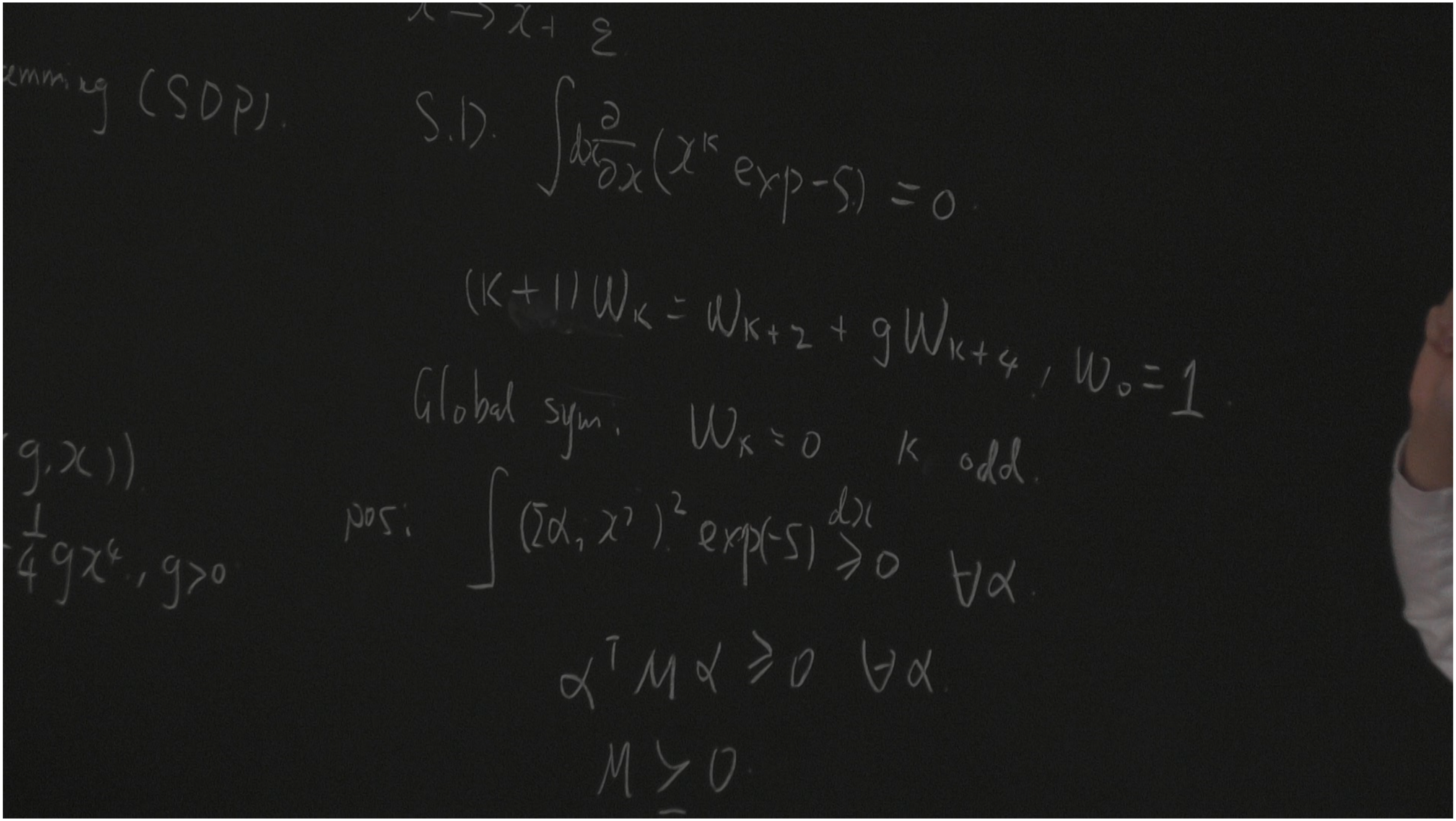
S.D. $\int dx \frac{\partial}{\partial x}(x^k \exp -S) = 0$

$$(k+1) W_k = W_{k+2} + g W_{k+4}, \quad W_0 = 1$$

Global sym: $W_k = 0$ k odd

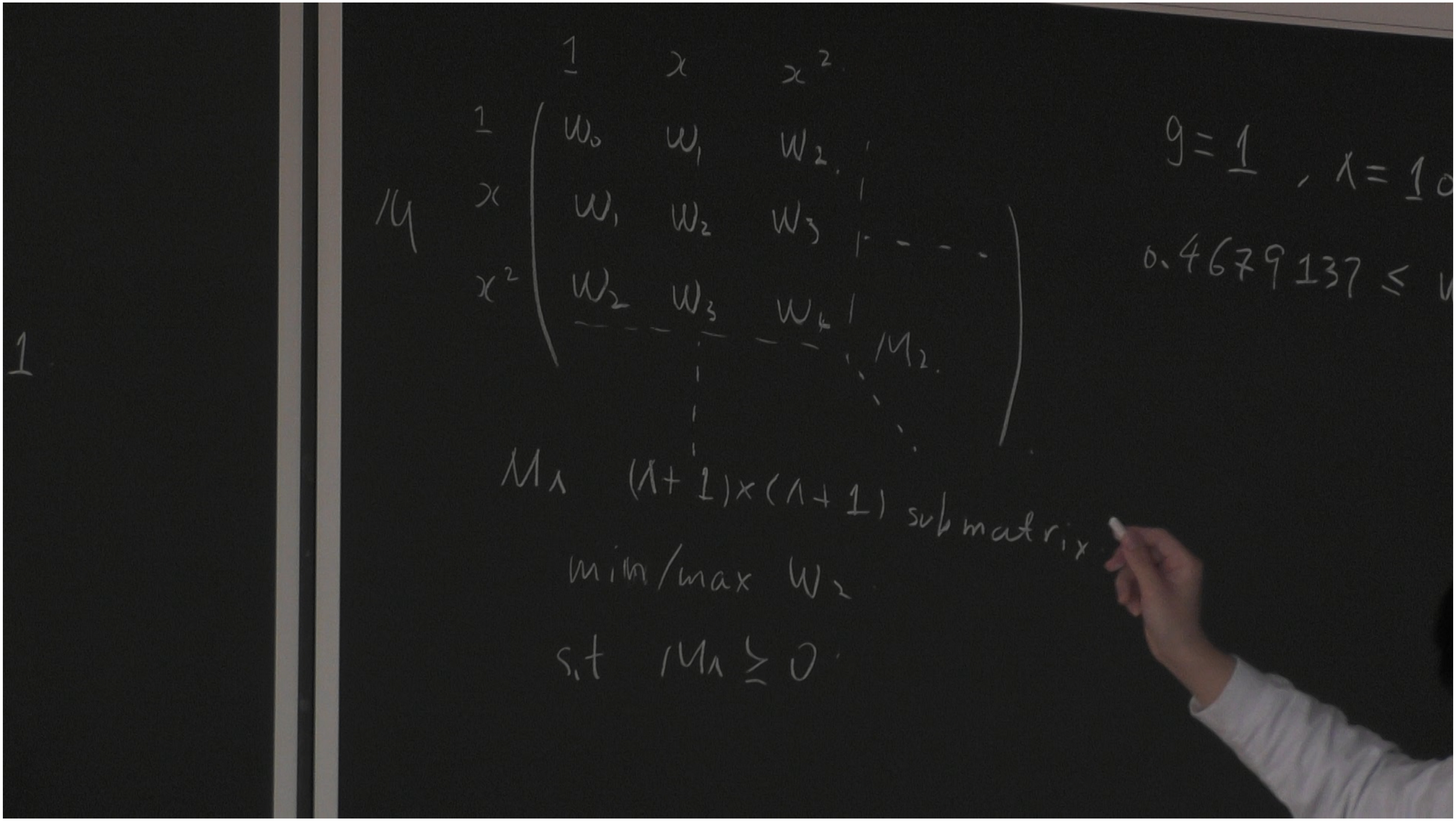
pos: $\int (\sum \alpha_i x^i)^2 \exp(-S)\, dx \geq 0 \quad \forall \alpha$

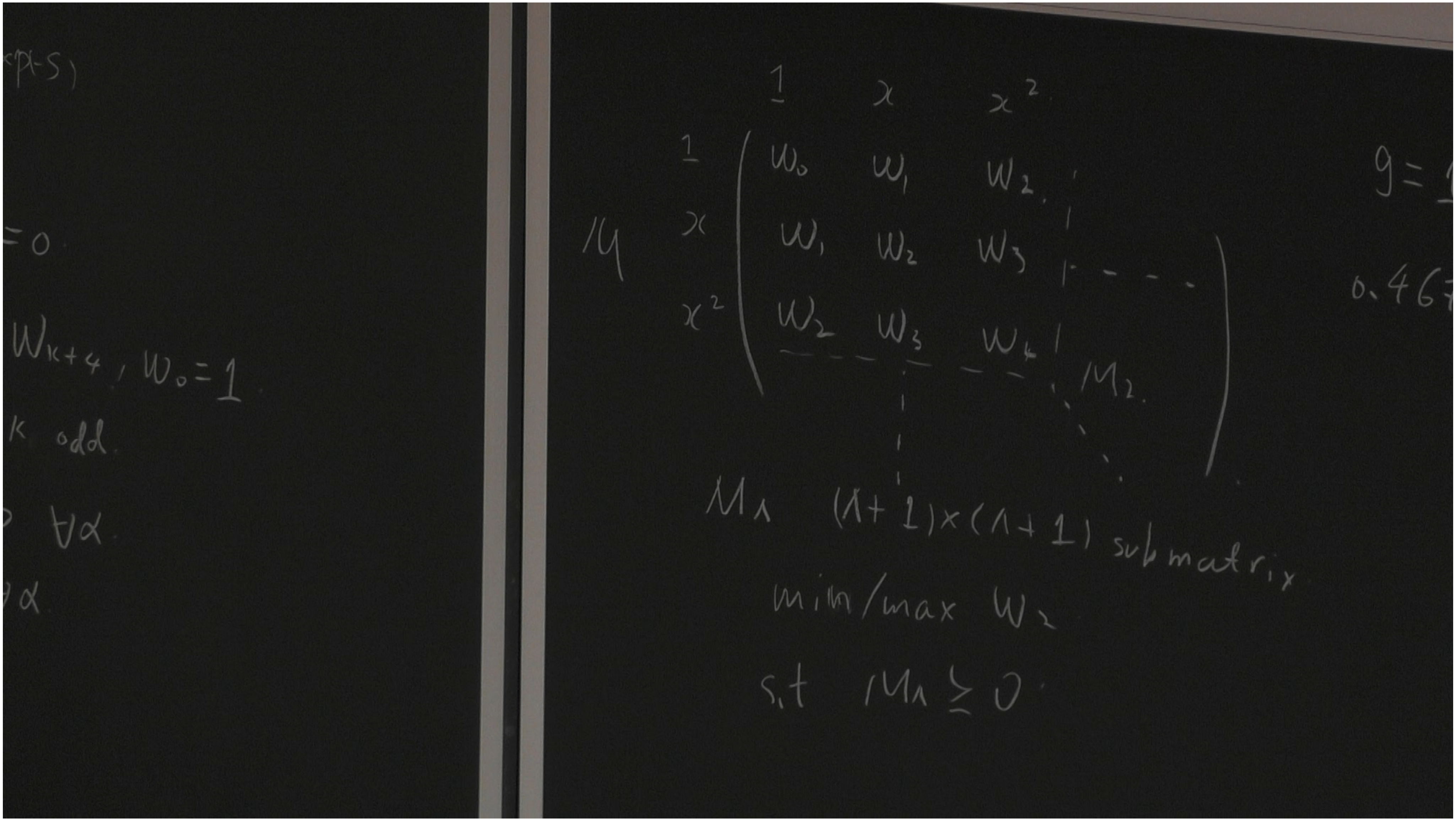
$$\alpha^T M \alpha \geq 0 \quad \forall \alpha$$

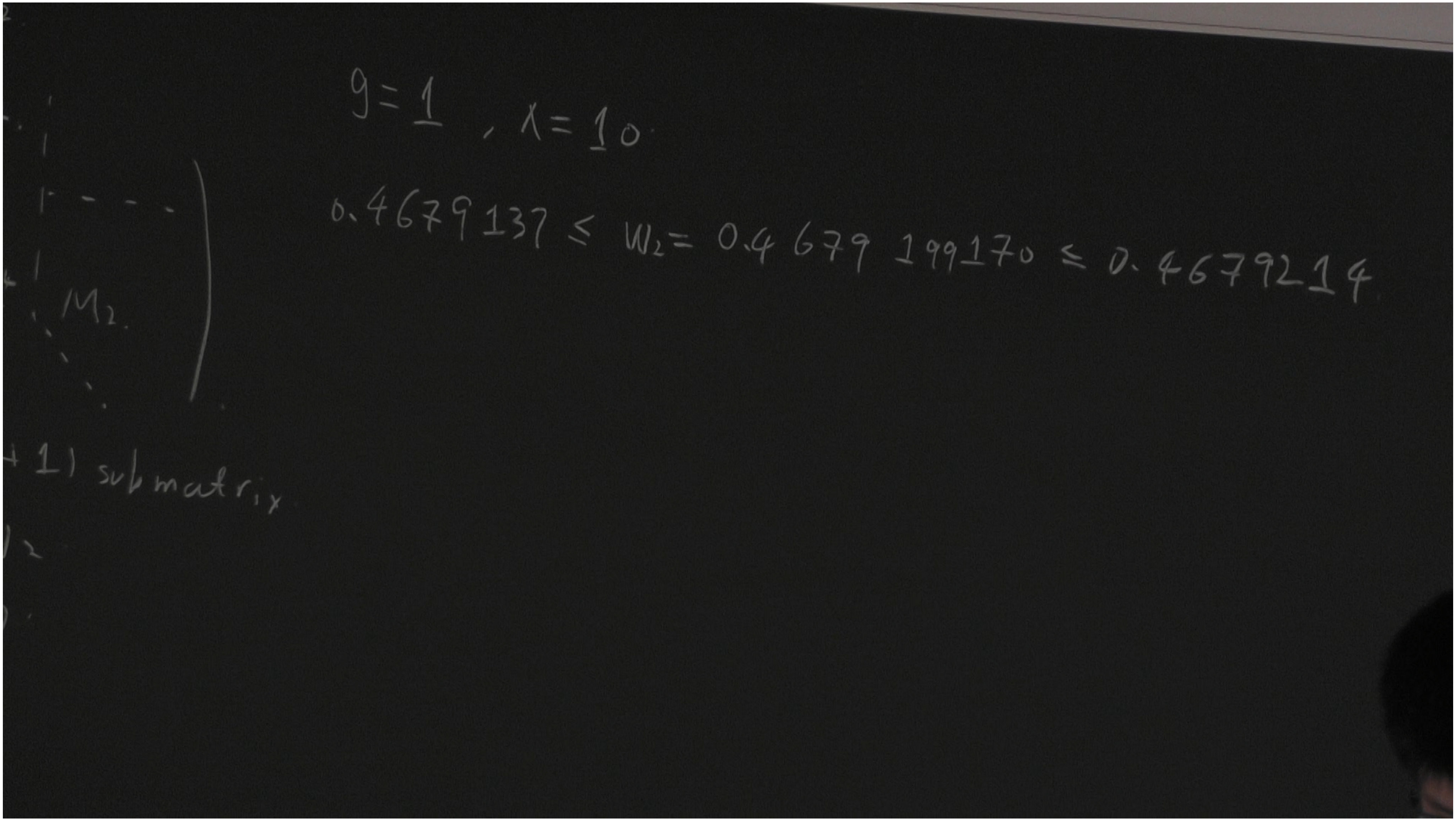


$x \to x + \varepsilon$
(SDP)
S.D. $\int dx \frac{\partial}{\partial x}(x^k \exp - S) = 0$
$(k+1) W_k = W_{k+2} + g W_{k+4}$, $W_0 = 1$
Global sym: $W_k = 0$ k odd
pos: $\int (\sum \alpha_i x^i)^2 \exp(-S)\, dx \geq 0 \;\; \forall \alpha$
$\alpha^T M \alpha \geq 0 \;\; \forall \alpha$
$M \succeq 0$

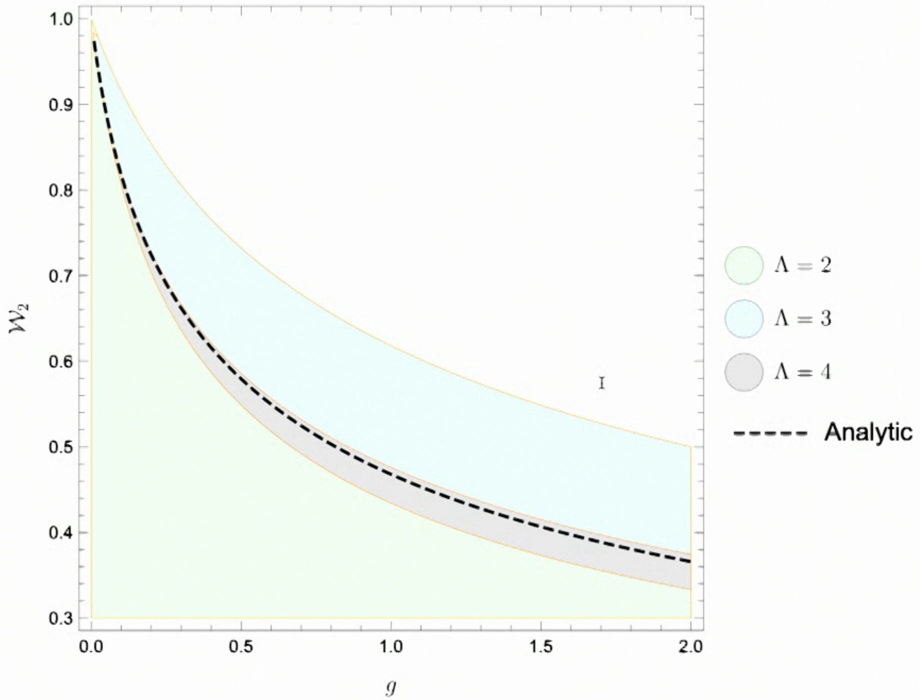
$$\mathcal{M} = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ \\ \end{array} \begin{array}{c} 1 \quad\quad x \quad\quad x^2 \\ \begin{pmatrix} w_0 & w_1 & w_2 & \\ w_1 & w_2 & w_3 & \cdots \\ w_2 & w_3 & w_4 & \\ & \vdots & & \ddots \end{pmatrix} \end{array}$$


M
1 x x²
w₀ w₁ w₂
w₁ w₂ w₃
w₂ w₃ w₄
M₂
g = 1
0.4679137 ≤
Mλ (λ+1)×(λ+1) submatrix
min/max w₂
s.t Mλ ≥ 0


1 x x²
1 (w₀ w₁ w₂
x w₁ w₂ w₃
x² w₂ w₃ w₄
M M₂
M_Λ (Λ+1)×(Λ+1) submatrix
min/max w₂
s.t M_Λ ≥ 0
W_{k+4}, w₀=1
k odd
∀α
g=
0.46
=0


g = 1 , λ = 10
0.4679137 ≤ W2 = 0.4679199170 ≤ 0.4679214
M2
1) submatrix

Result



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
$\mathcal{W}_2$
0.0
0.5
1.0
1.5
2.0
g
$\Lambda = 2$
$\Lambda = 3$
$\Lambda = 4$
Analytic

$g = \underline{1}$, $\lambda = \underline{1}0$.

$0.4679137 \leq W_2 = 0.4679199\underline{1}70 \leq 0.46792\underline{1}4$.

Rmk: 1. justification.

2. generalization.

matrix.

BACKGROUND

It is characterized by using the Equations of Motion (EOM).

- Matrix model: 2002.08387 (Lin), 2108.04830 (Kazakov & Zheng) and 2108.08803 (Koch et al.)...
- Quantum systems: 2004.10212 (Han & Hartnoll & Kruthoff) and follow-up works. See also: 2110.10701 and 2205.12325 (Hastings et al.). There are earlier attempts in this direction.
- Classical dynamical systems: 1610.05335, 1705.07096, 1807.09814 (Goluskin et al.)...
- Lattice models: 1612.08140 (Anderson & Kruczenski) and 2203.11360 (Kazakov & Zheng) for lattice gauge theory. Recent results on Lattice Ising model: 2206.12538 (Cho et al.)

Multi-matrix bootstrap: an example

Here we propose to study the following two-matrix model:

$$Z = \lim_{N\to\infty} \int d^{N^2}A\, d^{N^2}B\, \mathrm{e}^{-N\mathrm{tr}\left(-h[A,B]^2/2+A^2/2+gA^4/4+B^2/2+gB^4/4\right)} \quad (4)$$

The integration is over Hermitian matrix. To the best of our knowledge, this model with general g and h value, is not solvable!

Matrix bootstrap

$$\mathrm{Tr}A^2,\ \mathrm{Tr}A^4,\ \mathrm{Tr}A^2B^2,\ \mathrm{Tr}ABAB,\ \mathrm{Tr}A^6,\ \mathrm{Tr}A^4B^2,\ \mathrm{Tr}A^3BAB,\ \mathrm{Tr}A^2BA^2B,\ \mathrm{Tr}A^8,$$
$$\mathrm{Tr}A^6B^2,\ \mathrm{Tr}A^5BAB,\ \mathrm{Tr}A^4BA^2B,\ \mathrm{Tr}A^4B^4,\ \mathrm{Tr}A^3BA^3B,\ \mathrm{Tr}A^3BAB^3,\ \mathrm{Tr}A^3B^2AB^2,$$
$$\mathrm{Tr}A^2BABAB^2,\ \mathrm{Tr}A^2BAB^2AB,\ \mathrm{Tr}A^2B^2A^2B^2,\ \mathrm{Tr}ABABABAB\ \ldots \tag{5}$$

	Toy model	two-matrix model
Observable	$\mathcal{W}_k$	$\mathrm{Tr(Words)}$
Action	Quartic action	Quartic+$\mathrm{Commutator}^2$
EOM	Linear recursion relations	loop equations
Symmetry	$\mathbb{Z}_2$	Dihedral group D_4
positivity	Hankel matrix	Hermitian
Convex?	Yes	No (relaxation)

Multi-matrix bootstrap: an example

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The integration is over Hermitian matrix. To the best of our knowledge, this model with general g and h value, is not solvable!

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CUTOFF=4: LOOP EQUATIONS

$$
\begin{aligned}
1 &= \mathrm{Tr}A^2 + g\mathrm{Tr}A^4 - h(-2\mathrm{Tr}A^2B^2 + 2\mathrm{Tr}ABAB)\\
0 &= -2\mathrm{Tr}A^2 + \mathrm{Tr}A^4 - h(2\mathrm{Tr}A^3BAB - 2\mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^6\\
0 &= -\mathrm{Tr}A^2 + \mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2BA^2B + 2\mathrm{Tr}A^3BAB - \mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^4B^2\\
0 &= -h(2\mathrm{Tr}A^2BA^2B - 2\mathrm{Tr}A^3BAB) + g\mathrm{Tr}A^3BAB + \mathrm{Tr}ABAB\\
\beta &= -2\mathrm{Tr}A^4 + \mathrm{Tr}A^6 - h(2\mathrm{Tr}A^5BAB - 2\mathrm{Tr}A^6B^2) + g\mathrm{Tr}A^8\\
\beta &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^4B^2 - h(-\mathrm{Tr}A^3B^2AB^2 + 2\mathrm{Tr}A^3BAB^3 - \mathrm{Tr}A^4B^4) + g\mathrm{Tr}A^6B^2\\
0 &= -2\mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2B^2A^2B^2 + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^6B^2\\
0 &= -\mathrm{Tr}A^4 + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^4B^4 - h(-\mathrm{Tr}A^4BA^2B + 2\mathrm{Tr}A^5BAB - \mathrm{Tr}A^6B^2)\\
0 &= \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^2BAB^2AB - \mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3BAB^3) + g\mathrm{Tr}A^5BAB - \mathrm{Tr}ABAB\\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^5BAB - 2\mathrm{Tr}ABAB - h(-2\mathrm{Tr}A^2BABAB^2 + 2\mathrm{Tr}ABABABAB)\\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^3BAB^3 - h(-\mathrm{Tr}A^3BA^3B + 2\mathrm{Tr}A^4BA^2B - \mathrm{Tr}A^5BAB)\\
0 &= g\mathrm{Tr}A^3BA^3B + \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^3B^2AB^2 - 2\mathrm{Tr}A^3BAB^3)\\
0 &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^2BA^2B - h(-\mathrm{Tr}A^2BAB^2AB + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + g\mathrm{Tr}A^4BA^2B\\
\beta &= \mathrm{Tr}A^2BA^2B + g\mathrm{Tr}A^3B^2AB^2 - h(2\mathrm{Tr}A^3BA^3B - 2\mathrm{Tr}A^4BA^2B)\\
\beta &= (\mathrm{Tr}A^2)^2.
\end{aligned}
\tag{6}
$$

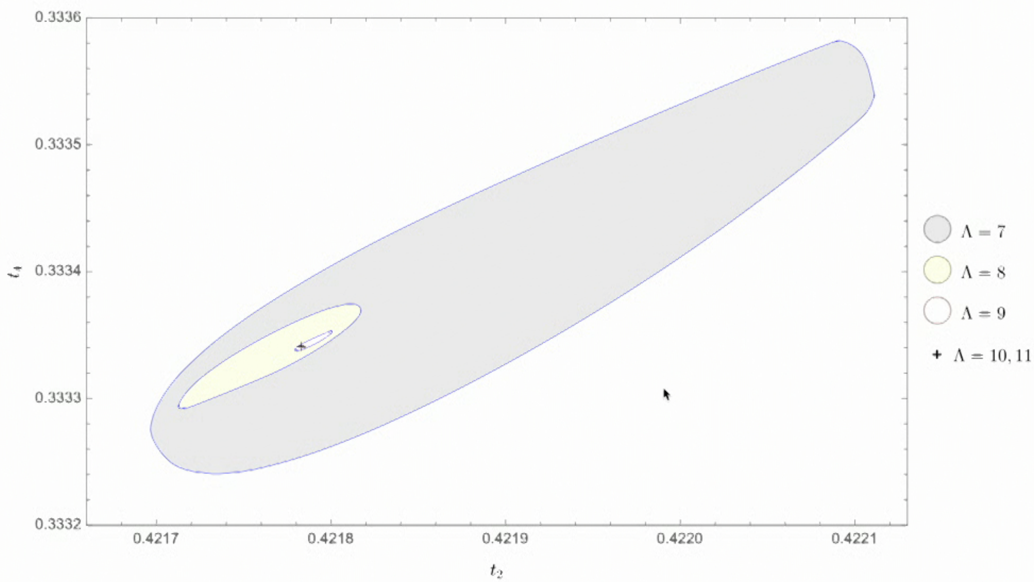
Example: correlation matrix

$$
\begin{pmatrix}
\mathrm{Tr}A^2 & \mathrm{Tr}A^4 & \mathrm{Tr}A^2B^2 & \mathrm{Tr}ABAB & \mathrm{Tr}A^2B^2 \\
\mathrm{Tr}A^4 & \mathrm{Tr}A^6 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^4B^2 \\
\mathrm{Tr}A^2B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^2BA^2B \\
\mathrm{Tr}ABAB & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^2BA^2B & \mathrm{Tr}A^3BAB \\
\mathrm{Tr}A^2B^2 & \mathrm{Tr}A^4B^2 & \mathrm{Tr}A^2BA^2B & \mathrm{Tr}A^3BAB & \mathrm{Tr}A^4B^2
\end{pmatrix} \tag{7}
$$

CUTOFF=4: LOOP EQUATIONS

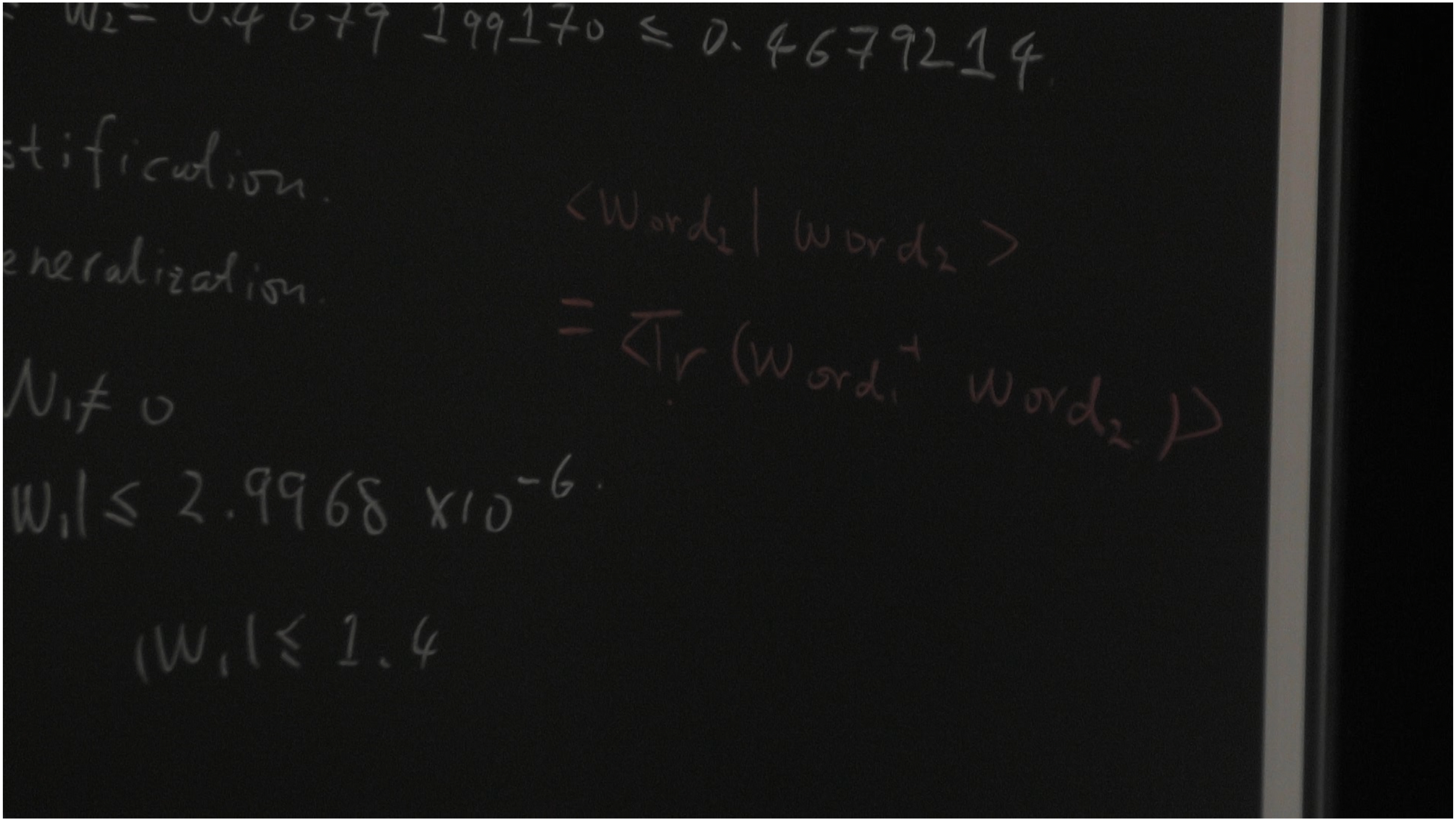
$$
\begin{aligned}
1 &= \mathrm{Tr}A^2 + g\mathrm{Tr}A^4 - h(-2\mathrm{Tr}A^2B^2 + 2\mathrm{Tr}ABAB) \\
0 &= -2\mathrm{Tr}A^2 + \mathrm{Tr}A^4 - h(2\mathrm{Tr}A^3BAB - 2\mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^6 \\
0 &= -\mathrm{Tr}A^2 + \mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2BA^2B + 2\mathrm{Tr}A^3BAB - \mathrm{Tr}A^4B^2) + g\mathrm{Tr}A^4B^2 \\
0 &= -h(2\mathrm{Tr}A^2BA^2B - 2\mathrm{Tr}A^3BAB) + g\mathrm{Tr}A^3BAB + \mathrm{Tr}ABAB \\
\beta &= -2\mathrm{Tr}A^4 + \mathrm{Tr}A^6 - h(2\mathrm{Tr}A^5BAB - 2\mathrm{Tr}A^6B^2) + g\mathrm{Tr}A^8 \\
\beta &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^4B^2 - h(-\mathrm{Tr}A^3B^2AB^2 + 2\mathrm{Tr}A^3BAB^3 - \mathrm{Tr}A^4B^4) + g\mathrm{Tr}A^6B^2 \\
0 &= -2\mathrm{Tr}A^2B^2 - h(-\mathrm{Tr}A^2B^2A^2B^2 + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^6B^2 \\
0 &= -\mathrm{Tr}A^4 + \mathrm{Tr}A^4B^2 + g\mathrm{Tr}A^4B^4 - h(-\mathrm{Tr}A^4BA^2B + 2\mathrm{Tr}A^5BAB - \mathrm{Tr}A^6B^2) \\
0 &= \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^2BAB^2AB - \mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3BAB^3) + g\mathrm{Tr}A^5BAB - \mathrm{Tr}ABAB \\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^5BAB - 2\mathrm{Tr}ABAB - h(-2\mathrm{Tr}A^2BABAB^2 + 2\mathrm{Tr}ABABABAB) \\
0 &= \mathrm{Tr}A^3BAB + g\mathrm{Tr}A^3BAB^3 - h(-\mathrm{Tr}A^3BA^3B + 2\mathrm{Tr}A^4BA^2B - \mathrm{Tr}A^5BAB) \\
0 &= g\mathrm{Tr}A^3BA^3B + \mathrm{Tr}A^3BAB - h(2\mathrm{Tr}A^3B^2AB^2 - 2\mathrm{Tr}A^3BAB^3) \\
0 &= -\mathrm{Tr}A^2B^2 + \mathrm{Tr}A^2BA^2B - h(-\mathrm{Tr}A^2BAB^2AB + 2\mathrm{Tr}A^2BABAB^2 - \mathrm{Tr}A^3B^2AB^2) + g\mathrm{Tr}A^4BA^2B \\
\beta &= \mathrm{Tr}A^2BA^2B + g\mathrm{Tr}A^3B^2AB^2 - h(2\mathrm{Tr}A^3BA^3B - 2\mathrm{Tr}A^4BA^2B) \\
\beta &= (\mathrm{Tr}A^2)^2.
\end{aligned}
\tag{6}
$$

Result



$$\Lambda = 11,\ g = h = 1: \begin{cases} 0.421783612 \leq \langle \mathrm{Tr} A^2 \rangle \leq 0.421784687 \\ 0.333341358 \leq \langle \mathrm{Tr} A^4 \rangle \leq 0.333342131 \end{cases} \tag{8}$$

8


W2= 0.4679 199170 ≤ 0.46792214.
stification.
eneralization.
W1≠ 0
|W1| ≤ 2.9968 x10^-6.
|W1| ≤ 1.4
<Word1 | Word2>
= Tr (Word1^† Word2.)

Compare with MC

Compared to the MC study of the same model 2111.02410 (Jha), we are convinced that for this model bootstrap is at least two order of magnitude more efficient than MC.

- MC: 80-85 hours for N=800 simulation to get 4.5 digits.
- Bootstrap: $\sim$ 40 hours to get 6 digits. (These are old results and can be greatly improved by at least one order of magnitude.)

$x \to x + \varepsilon$

PI.

S.D. $\int dx \frac{\partial}{\partial x}\left(x^k \exp - S\right) = 0.$

$W_2(g) = \sum_{i=0}^{\infty} (\#)\, g^i$

$(k+1) W_k = W_{k+2} + g W_{k+4}, \quad W_0 = 1.$

Global sym.: $W_k = 0$ k odd.

pos: $\int \left(\sum \alpha_i x^i\right)^2 \exp(-S)\, dx \geq 0 \quad \forall \alpha.$

$\alpha^T M \alpha \geq 0 \quad \forall \alpha.$

$M \succ 0.$

Lattice Gauge theory

We are going to bootstrap the large N_c limit of the following theory:

$$Z = \int \prod_{x,\,\mu} \mathrm{d}U_\mu(x) \exp(-S) \tag{9}$$

$$S = -\frac{N_c}{2\lambda} \sum_P \mathrm{tr}(U_P + U_P^\dagger) \tag{10}$$

where U_P is the product of four unitary link variables around the plaquette P and we sum up over all plaquettes P, including both orientations. In our last work we bootstrap the one plaquette average:

$$u_P = \frac{1}{N_c} \langle \mathrm{tr} U_P \rangle \tag{11}$$

2. generalization.

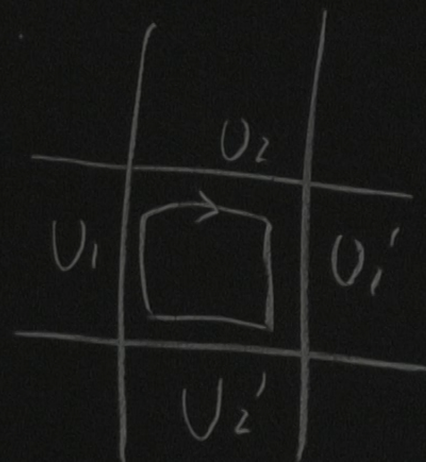
3. $W_1 \neq 0$

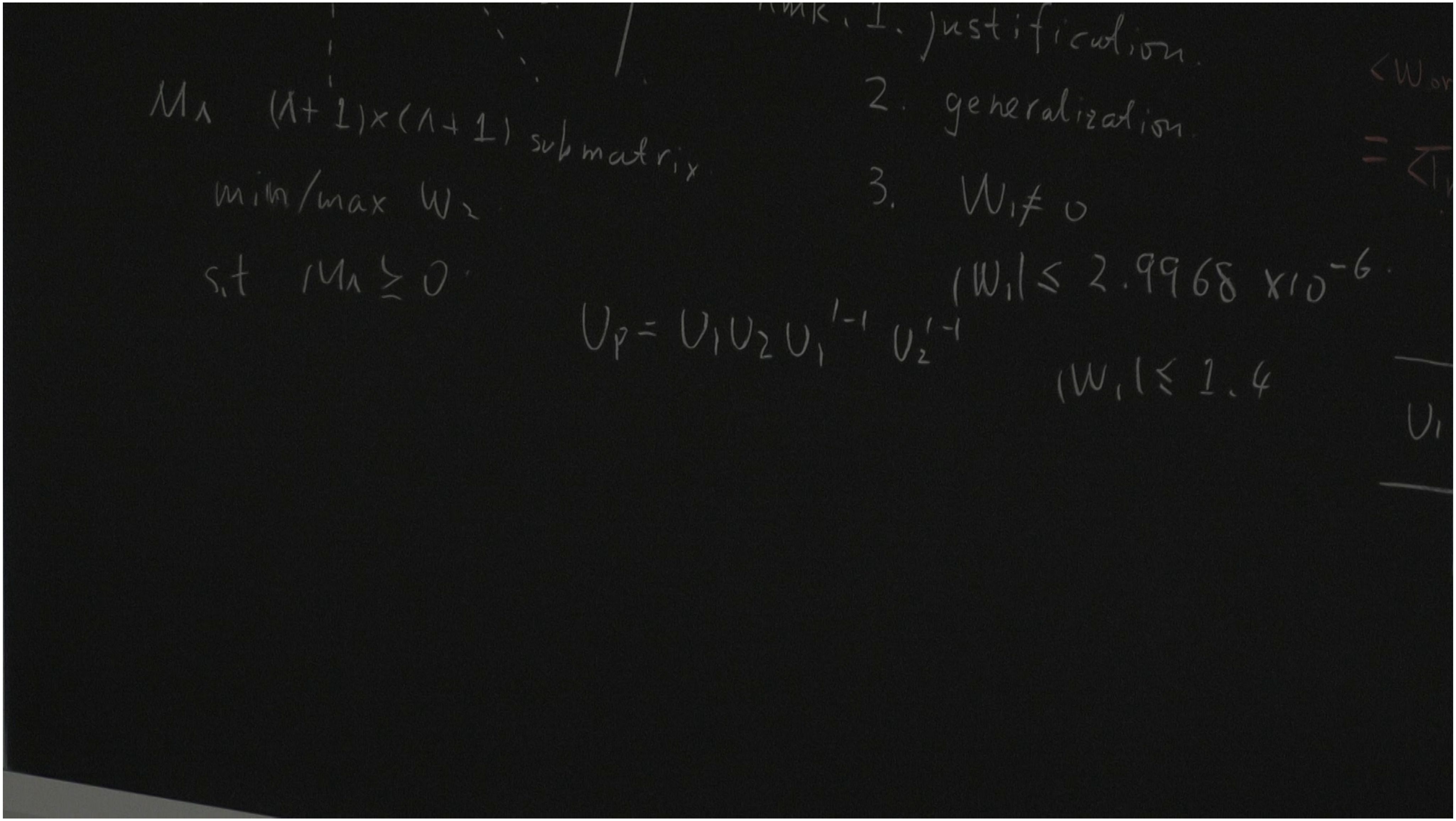
$|W_1| \leq 2.9968 \times 10^{-6}$.

$|W_1| \leq 1.4$

$\langle word_1 | word_2 \rangle$

$= \langle Tr(word_1^{\dagger} \; word_2) \rangle$




Rmk. 1. justification.
2. generalization.
3. W1 ≠ 0
|W1| ≤ 2.9968 ×10^-6.
|W1| ≤ 1.4
MΛ (Λ+1)×(Λ+1) submatrix.
min/max W2.
s.t MΛ ≥ 0.
Up = U1U2U1'^-1 U2'^-1

COMPARE

	Plaquette model	Lattice YM $SU(\infty)$
Observable	$\mathcal{W}_k$	Wilson loops
Action	Quartic action	Wilson Action
EOM	Linear recursion relations	MM loop equations
Symmetry	$\mathbb{Z}_2$	lattice symmetry+C
positivity	Hankel matrix	Hermitian+Reflection
Convex?	Yes	No (relaxation)

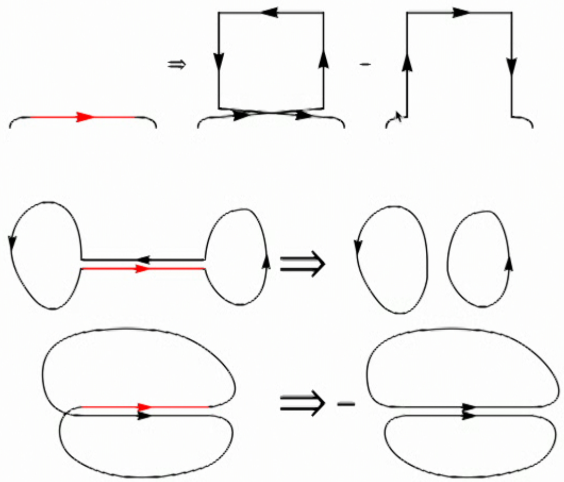
We improved the method of 1612.08140 (Anderson & Kruczenski) by the following points:

- Symmetry reduction and Reflection positivity
- Large N relaxation
- Back-track loop equations
- Some other technical improvements...

Makeenko-Migdal loop equations

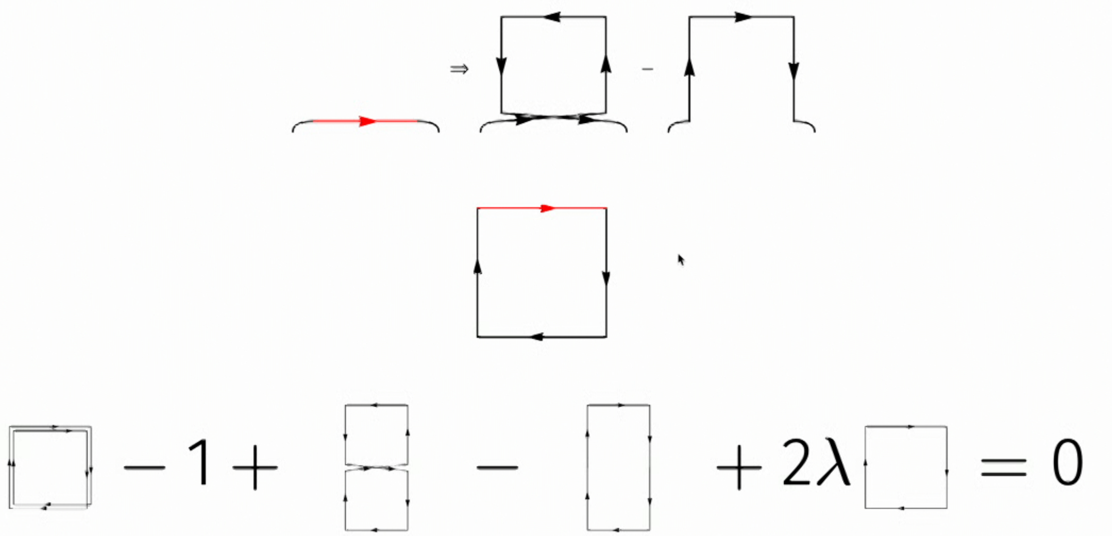
Doing the following infinitesimal transformation $U_\mu(x) \to U_\mu(x)(1 + i\epsilon)$ to the Wilson loop $\mathcal{W}[C]$, we can get the following loop equations schematically:

$$(\text{linear}) + 2\lambda\mathcal{W}[C] = 2\lambda(\text{nonlinear}) \quad (12)$$



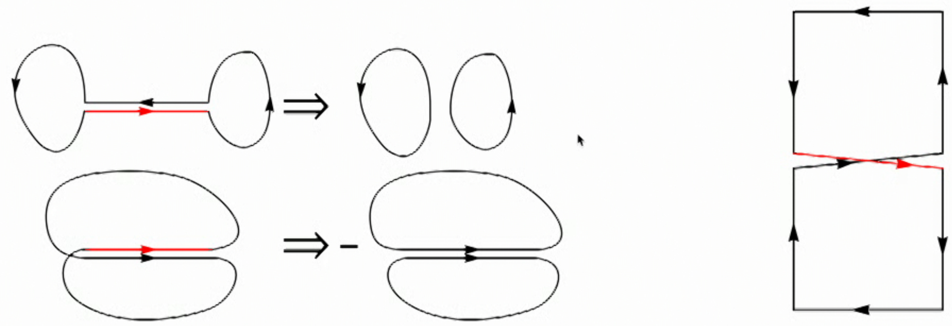
Makeenko-Migdal loop equations

$$(\text{linear}) + 2\lambda\mathcal{W}[C] = 2\lambda(\text{nonlinear}) \qquad (13)$$



Makeenko-Migdal loop equations

$$(\text{linear}) + 2\lambda\mathcal{W}[C] = 2\lambda(\text{nonlinear}) \qquad (15)$$



Positivity by inner product

Generalization: Any inner products defined on the vector space of operators or its subspace could leads to positivity condition:

$$\langle \mathcal{O}|\mathcal{O}\rangle = \langle \mathcal{O}^\dagger \mathcal{O}\rangle = \alpha^{*\mathrm{T}} \mathcal{M} \alpha \geq 0 \Leftrightarrow \mathcal{M} \succeq 0. \quad (16)$$

In the above case of single-variable integration and Hermitian matrix integration, we were taking adjoint to be Hermitian conjugation:

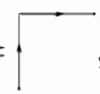
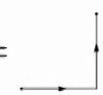
$$\mathcal{O}^\dagger = \mathcal{O}^{*\mathrm{T}} \quad (17)$$

Positivity by Hermitian Conjugation

In parallel to the bootstrap for Hermitian matrix model, we have:

$$\mathrm{Path}^{*\mathrm{T}} = \mathrm{Reverse} \circ \mathrm{Path} \tag{18}$$

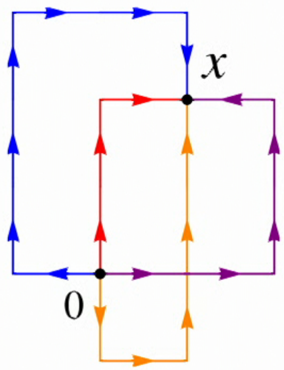
For a simplest example:

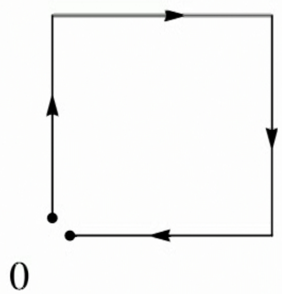
$$\mathrm{Path}_1 = $$  $$, \quad \mathrm{Path}_2 = $$  (19)

$$\begin{array}{c} \\ \mathrm{Path}_1^\dagger \\ \mathrm{Path}_2^\dagger \end{array} \begin{array}{c} \begin{array}{cc} \mathrm{Path}_1 & \mathrm{Path}_2 \end{array} \\ \begin{pmatrix} 1 & u_P \\ u_P & 1 \end{pmatrix} \end{array} \succeq 0. \tag{20}$$

Positivity by Hermitian Conjugation

Of course the matrix can be arbitrarily big when we consider multiple Wilson paths:





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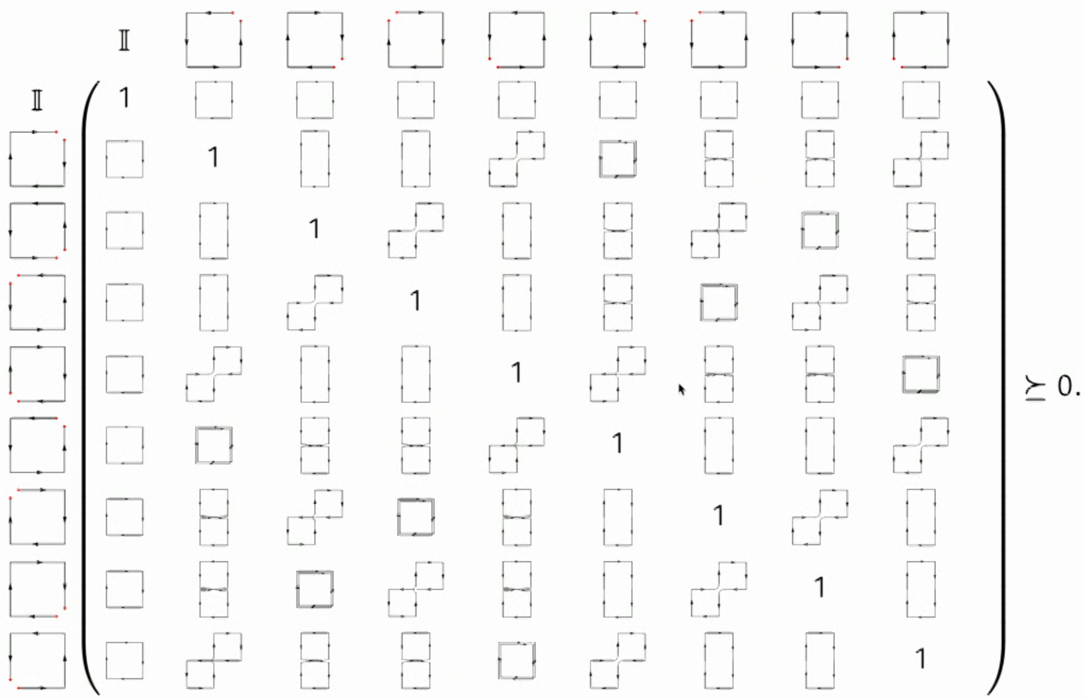
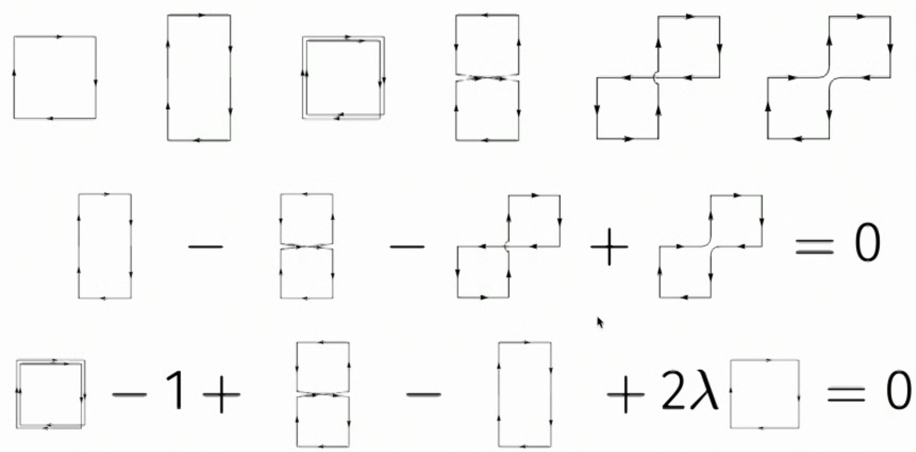
Positivity by Hermitian Conjugation



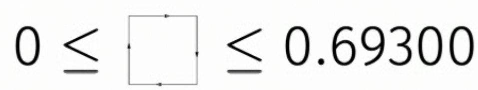
Bootstrap

There are actually 6 Wilson loops in the matrix:



(21)

After the optimization, we get ($\lambda = 1$):



Compare

	Toy model	Lattice YM $\mathrm{SU}(\infty)$
Observable	$\mathcal{W}_k$	Wilson loops
Action	Quartic action	Wilson Action
EOM	Linear recursion relations	MM loop equations
Symmetry	$\mathbb{Z}_2$	lattice symmetry+C
positivity	Hankel matrix	Correlation Matrix+?
Convex?	Yes	No (relaxation)

Reflection Positivity

We can also define the inner product by reflection positivity:

$$\mathcal{O}^\dagger = \Theta\mathcal{O} \tag{22}$$

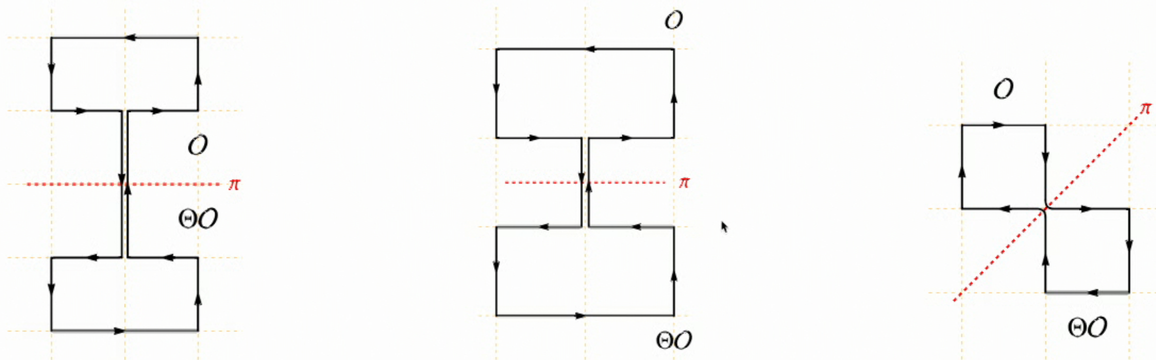
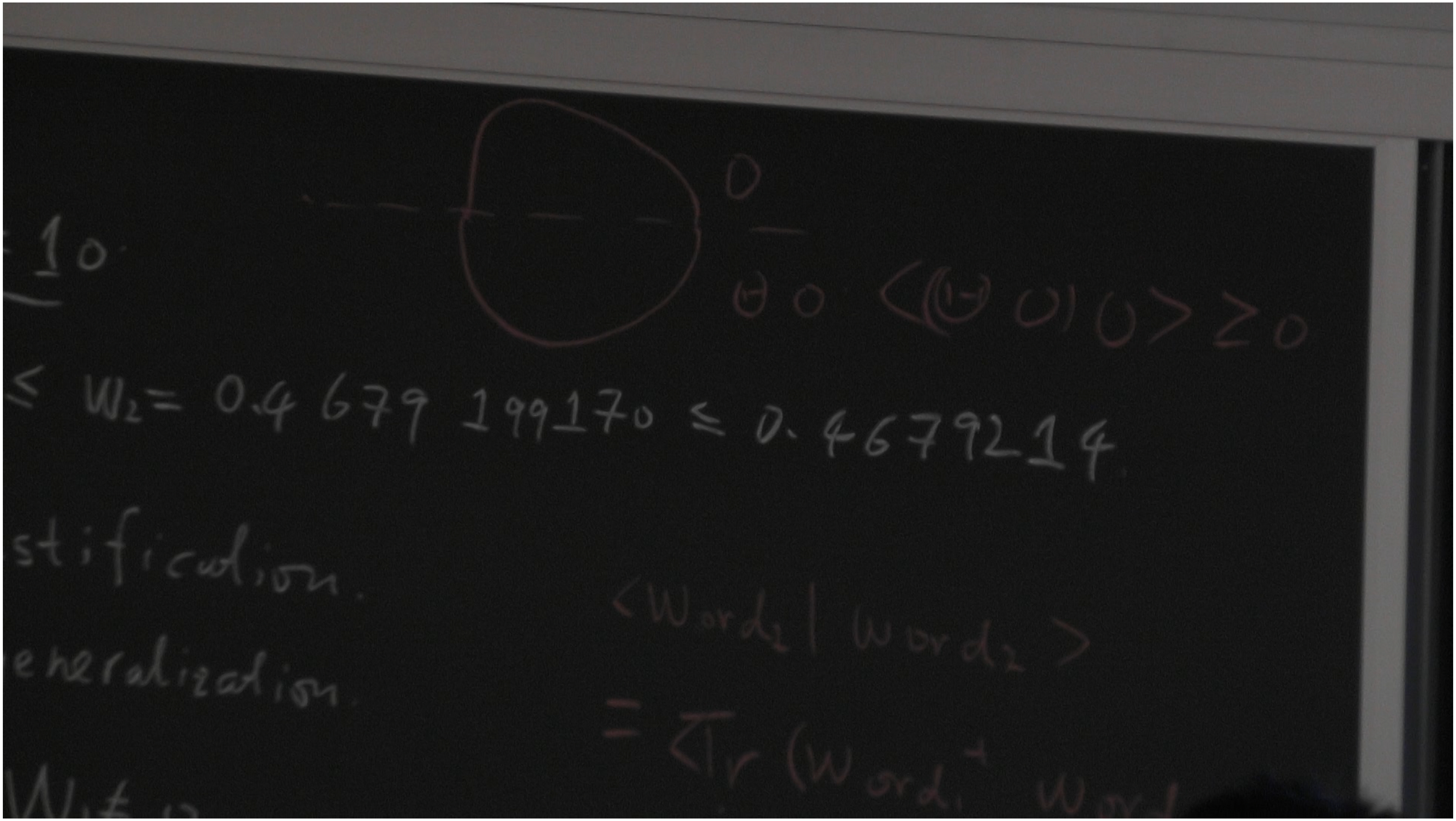
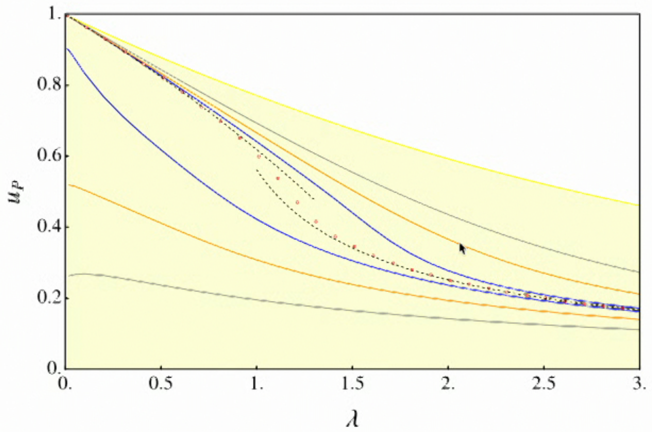


Figure: Three reflection symmetries on the lattice allowing new positivity conditions on Wilson loops combining the original and reflected Wilson lines.


≤ W2= 0.4679 199170 ≤ 0.46792214
stification.
eneralization.
<Word1 | Word2>

Reflection Positivity

Reflection Positivity is a new independent positivity condition (Gray curve and Orange curve).



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Convex?	Yes	No (relaxation)

Symmetry reduction

Coming from the matrix of inner products, our positivity conditions is formally similar to S-matrix. It is a well-known fact that we can decompose the S-matrix w.r.t the spin channel.

We notice that our inner product defined above is invariant under some symmetry group.

$$\langle (g \circ \mathcal{O}_1) | (g \circ \mathcal{O}_2) \rangle = \langle \mathcal{O}_1 | \mathcal{O}_2 \rangle, \; \forall g \in G \tag{23}$$

We can decompose the positivity conditions w.r.t the irreducible representation of the symmetry group. (This is mathematically guaranteed)

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Symmetry Group

For the correlation matrix with $0 \to 0$, the invariant group G is $B_d \times \mathbb{Z}_2$. Here B_d is the Hyperoctahedral group in D spacetime dimensions. It acts on the Wilson path by doing the corresponding spacetime rotation and reflection on the lattice. $\mathbb{Z}_2$ is the group reversing the path.

Dimension	Hermitian Conjugation	site&link reflection	diagonal reflection
2	$B_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
3	$B_3 \times \mathbb{Z}_2$	$B_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$
4	$B_4 \times \mathbb{Z}_2$	$B_3 \times \mathbb{Z}_2$	$B_2 \times \mathbb{Z}_2^2$

Table: Invariant groups of correlation and reflection matrices $0 \to 0$

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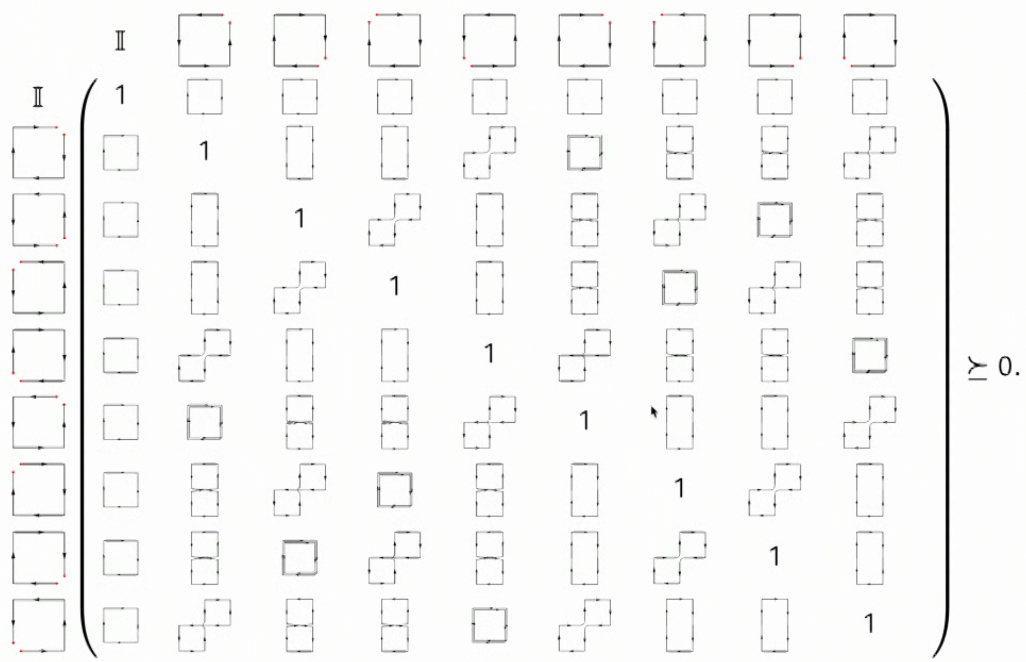
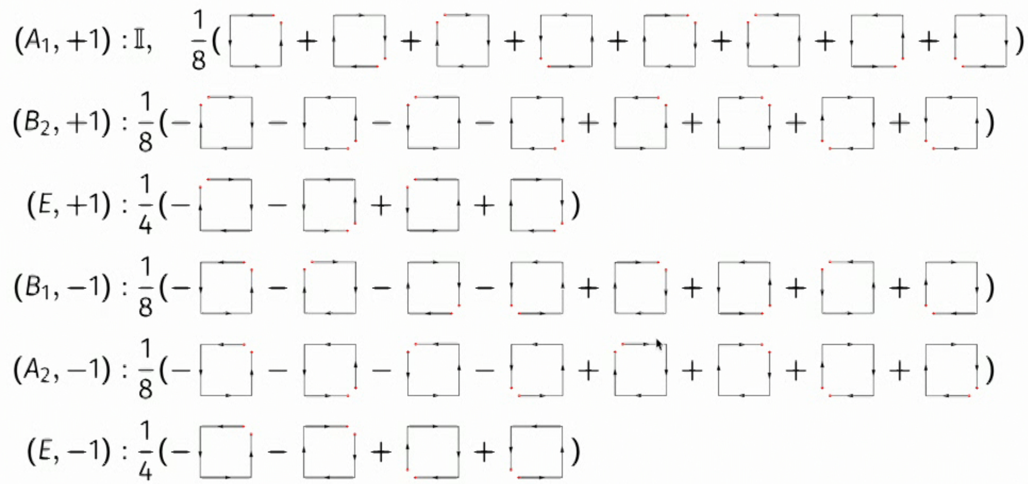
Positivity by Hermitian Conjugation



Symmetry Group



(24)

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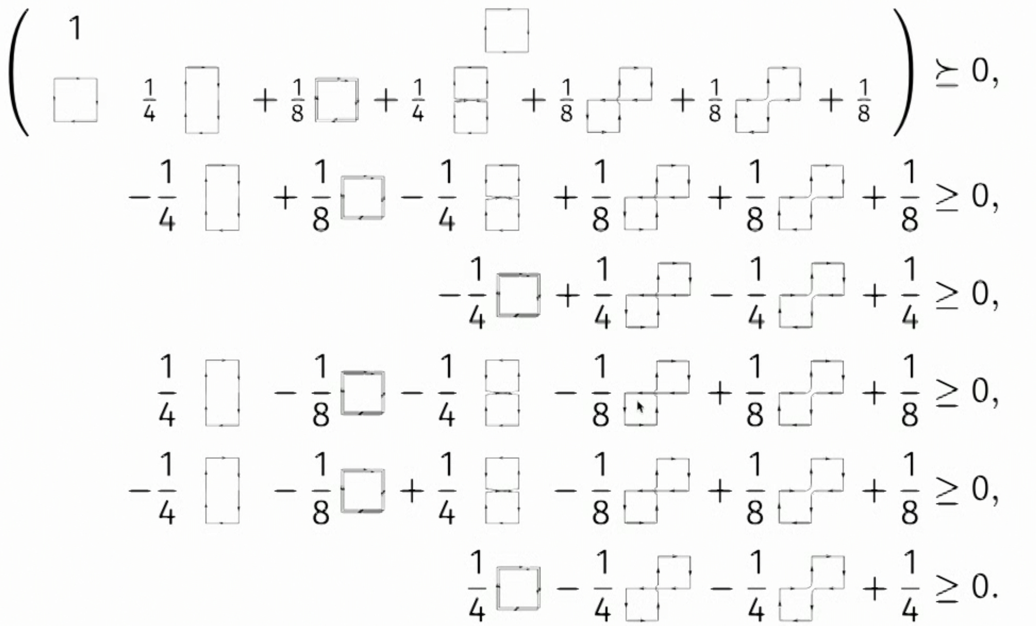
Symmetry Group



Selection of multiplets of Wilson path

We discard positivity constraints from less important Wilson Paths.

To better illustrate the efficiency of these reduction and selection techniques, take an example of the correlation matrix for the paths $0 \to 0$, at $3D$ and $L_{\mathrm{max}} = 16$: it has a huge size 6505×6505. After the symmetry reduction and truncation of the multiplets, the positivity of the correlation matrix becomes the positivity conditions of 20 smaller matrices [1], each with size:

$$\begin{aligned} &38, 15, 25, 18, 62, 33, 68, 75, 56, 78, \\ &22, 18, 34, 15, 56, 33, 57, 76, 69, 73 \end{aligned} \tag{26}$$

So the SDP gets greatly simplified.

[1] The invariant group $B_3 \times \mathbb{Z}_2$ has 20 irreducible representations.

Compare

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Convex?	Yes	No (relaxation)

Relaxation

We can relax them to make them convex by replacing $x^2 = T_1$ with $x^2 \leq T_1$ or, in the positive semi-definite matrix form,

$$\begin{pmatrix} 1 & x \\ x & T_1 \end{pmatrix} \succeq 0\,. \tag{27}$$

Relaxation

Our general strategy: we treat the quadratic terms in the loop equations as independent variables, and replace the algebraic equality by the convex inequality:

$$Q = xx^{\mathrm{T}} \tag{28}$$

to:

$$\mathcal{R} = \begin{pmatrix} 1 & x^{\mathrm{T}} \\ x & Q \end{pmatrix} \succeq 0. \tag{29}$$

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Relaxation

In our example, at $L_{\max} = 12$, the relaxation matrix is:

$$\begin{pmatrix} 1 & \square \\ \square & q \end{pmatrix} \succeq 0. \tag{30}$$

Here q is the variable in place of square of $\square$.

Final scheme

$$
\begin{aligned}
\text{min / max} \quad & \square, \\
\text{subject to} \quad & \text{MM loop equations} \\
& \mathsf{CM}^{\text{irrep}} \succeq 0, \\
& \mathsf{RefM}^{\text{irrep}} \succeq 0, \times 3 \\
& \mathcal{R} \succeq 0
\end{aligned}
\tag{31}
$$

Result

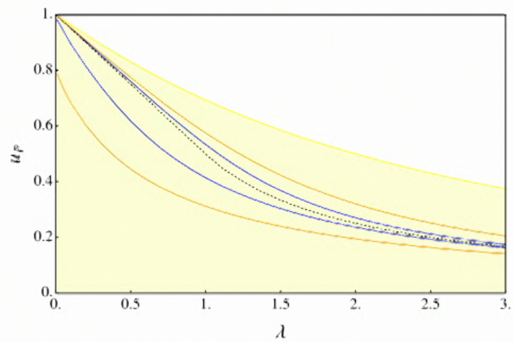
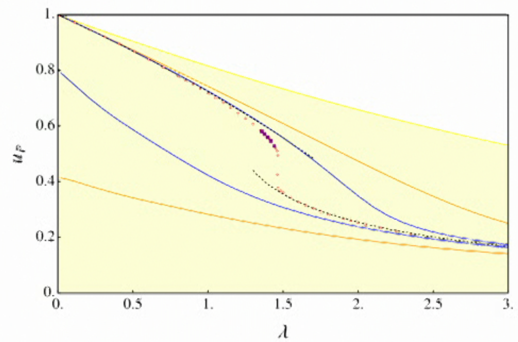


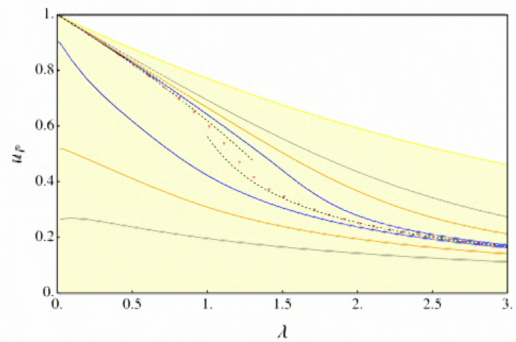
Figure: 2D: the upper and lower bounds from our bootstrap at $L_{\max} = 8$ (yellow region), $L_{\max} = 12$ (orange curves) and $L_{\max} = 16$ (blue curves). The dashed line is the exact solution.

$$u_P = \begin{cases} 1 - \frac{\lambda}{2}, & \text{for} \lambda \leq 1 \\ \frac{1}{2\lambda}, & \text{for} \lambda \geq 1 \end{cases} \tag{32}$$

Result



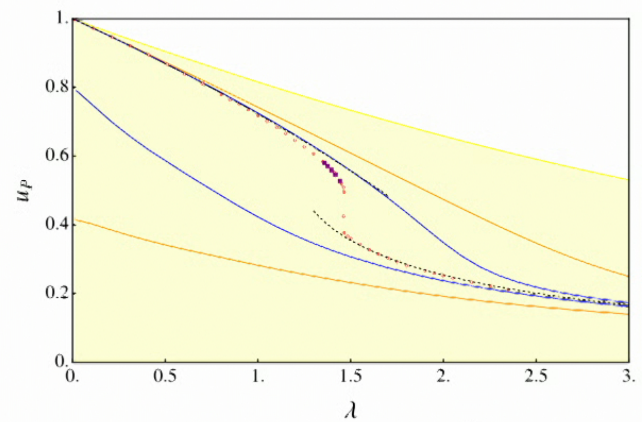
(a) 4D



(b) 3D

Figure: Our bootstrap results for plaquette average in $4D$ and $3D$ LGT: upper bounds at $L_{\mathbf{max}} = 8$ (yellow domain) at $L_{\mathbf{max}} = 12$ (orange curves) and $L_{\mathbf{max}} = 16$ (blue curves). The red circles represent the MC data for $SU(10)$ LGT (with 5 purple squares for $SU(12)$). Dashed upper and lower lines represent the 3-loop PT and strong coupling expansion, respectively.

Future direction: technical improvement



This is a general framework, any technical breakthrough will benefit other project in this category: Hierarchy in operators, solver of SD equation, parallelization, effective action... After this, we expect to reach $L_{\max} = 24$ in the near future.

Future direction: possible applications

- Finite N bootstrap(soon)
- Glueball spectrum
- Confinement
- ...

They are all related to observables in the asymptotic region

QUESTIONS?