Title: Discrete shift and quantized charge polarization: New invariants in crystalline topological states

Speakers: Naren Manjunath

Series: Quantum Matter

Date: December 14, 2022 - 3:30 PM

URL: https://pirsa.org/22120024

Abstract: In this talk I will describe a topological response theory that predicts the physical manifestation of a class of topological invariants in systems with crystalline symmetry. I focus on two such invariants, the 'discrete shift' and a quantized charge polarization. Guided by theory, I discuss how these invariants can be extracted from lattice models by measuring the fractional charge at lattice disclinations and dislocations, as well as from the angular and linear momentum of magnetic flux. These methods are illustrated using the Hofstadter model of spinless fermions in a background magnetic field; they give new topological invariants in this model for the first time since the quantized Hall conductance was computed by TKNN in 1982.

Zoom link: https://pitp.zoom.us/j/93633131128?pwd=d2h4U1l0ZVU5aE1ORURkdFNSanB4dz09

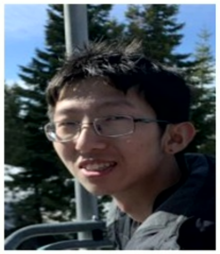
New invariants in topological crystalline states

Naren Manjunath
Perimeter Institute

Dec 14, 2022

2204.05320 and 2211.09127

Collaborators



Yuxuan Zhang

Gautam Nambiar

Maissam Barkeshli

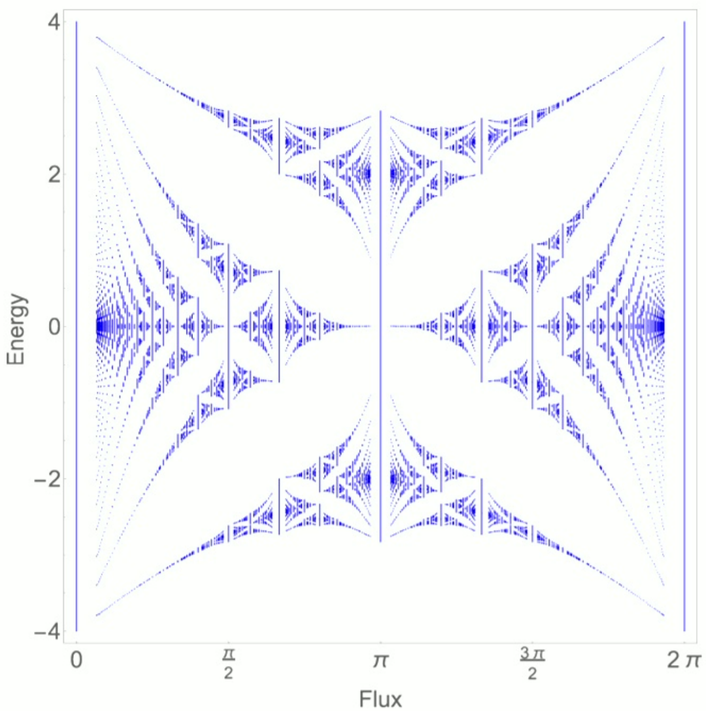
Yu-An Chen (UMD)

Po-Shen Hsin (UCLA)

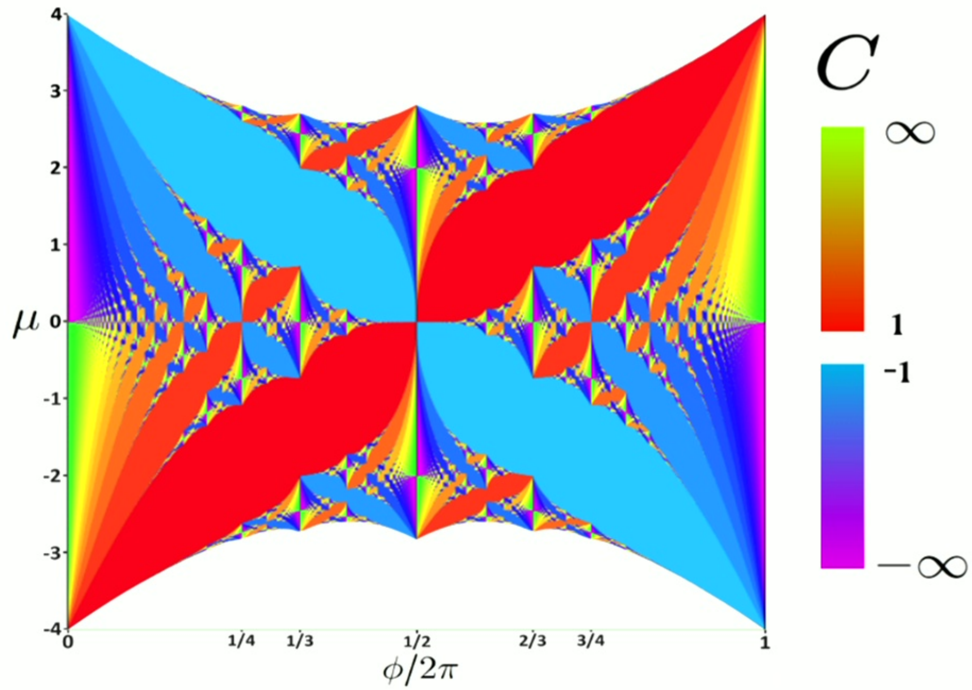
Vladimir Calvera (Stanford)

Topological invariants in condensed matter
(very briefly)

$$H = -\mu \sum_i c_i^\dagger c_i + \sum_{<ij>} e^{-iA_{ij}} c_i^\dagger c_j + \text{h.c.}$$



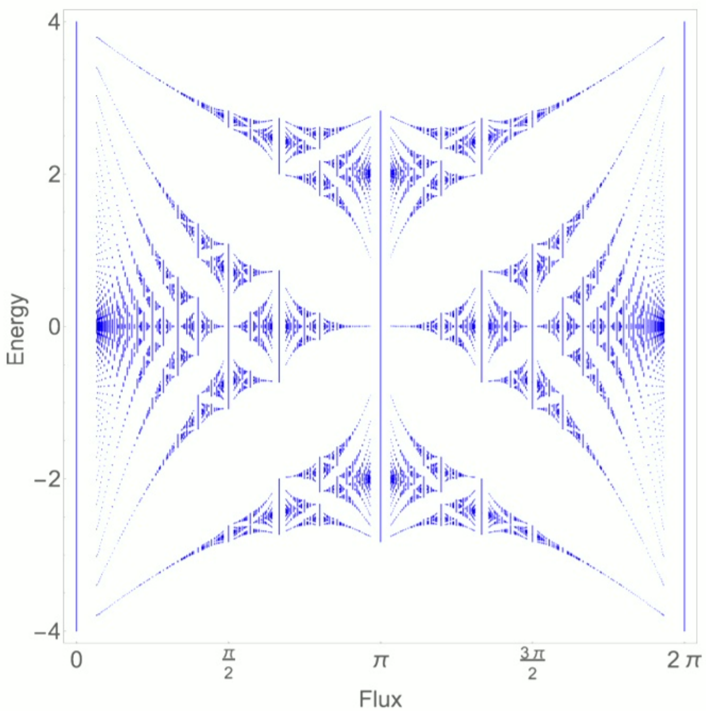
Hofstadter (1976); Harper (1955)



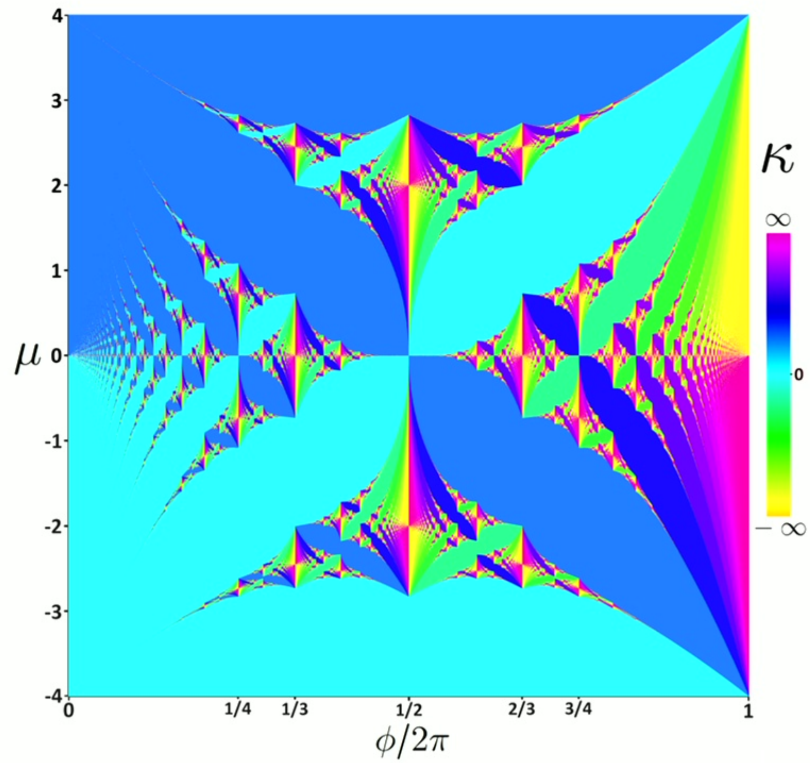
TKNN (1982); Osadchy, Avron (2001)

Topological invariants in condensed matter
(very briefly)

$$H = -\mu \sum_i c_i^\dagger c_i + \sum_{<ij>} e^{-iA_{ij}} c_i^\dagger c_j + \text{h.c.}$$



Hofstadter (1976); Harper (1955)



$$\nu = C\frac{\phi}{2\pi} + \kappa, \quad \kappa \in \mathbb{Z}$$

AZ	G/H	$\bar{d}=0$	$\bar{d}=1$	$\bar{d}=2$	$\bar{d}=3$	$\bar{d}=4$	$\bar{d}=5$	$\bar{d}=6$	$\bar{d}=7$
Complex case:									
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}\longleftarrow$
AIII	$\mathrm{U}(N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$
Real case:									
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}\longleftarrow$
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}\longleftarrow$
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0
C	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0
CI	$\mathrm{Sp}(2N)$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$

Ryu, Schnyder, Furusaki, Ludwig, NJP (2010); Kitaev (2009)

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
Z_2^T	Z_1	Z_2	Z_1	Z_2
$Z_2^T\times\mathrm{trn}$	Z_1	Z_2	Z_2^2	Z_2^4
Z_n	Z_n	Z_1	Z_n	Z_1
$Z_n\times\mathrm{trn}$	Z_n	Z_n	Z_n^2	Z_n^4
$U(1)$	Z	Z_1	Z	Z_1
$U(1)\times\mathrm{trn}$	Z	Z	Z^2	Z^4
$U(1)\rtimes Z_2^T$	Z	Z_2	Z_2	Z_2^2
$U(1)\rtimes Z_2^T\times\mathrm{trn}$	Z	$Z\times Z_2$	$Z\times Z_2^3$	$Z\times Z_2^8$
$U(1)\times Z_2^T$	Z_1	Z_2^2	Z_1	Z_2^3
$U(1)\times Z_2^T\times\mathrm{trn}$	Z_1	Z_2^2	Z_2^4	Z_2^9
$U(1)\rtimes Z_2$	Z_2	Z_2	$Z\times Z_2$	Z_2
$U(1)\times Z_2$	$Z\times Z_2$	Z_1	$Z\times Z_2^2$	Z_1
$Z_n\rtimes Z_2^T$	Z_n	$Z_2\times Z_{(2,n)}$	$Z_{(2,n)}^2$	$Z_2\times Z_{(2,n)}^2$
$Z_n\times Z_2^T$	$Z_{(2,n)}$	$Z_2\times Z_{(2,n)}$	$Z_{(2,n)}^2$	$Z_2\times Z_{(2,n)}^2$
$Z_n\rtimes Z_2$	$Z_2\times Z_{(2,n)}$	$Z_{(2,n)}$	$Z_n\times Z_2\times Z_{(2,n)}$	$Z_{(2,n)}^2$
$Z_m\times Z_n$	$Z_m\times Z_n$	$Z_{(m,n)}$	$Z_m\times Z_n\times Z_{(m,n)}$	$Z_{(m,n)}^2$
$D_2\times Z_2^T=D_{2h}$	Z_2^2	Z_2^4	Z_2^6	Z_2^9
$Z_m\times Z_n\times Z_2^T$	$Z_{(2,m)}\times Z_{(2,n)}$	$Z_2\times Z_{(2,m)}\times Z_{(2,n)}\times Z_{(m,n)}$	$Z_{(2,m,n)}^2\times Z_{(2,m)}^2\times Z_{(2,n)}^2$	$Z_2\times Z_{(2,m,n)}^4\times Z_{(2,m)}^2\times Z_{(2,n)}^2$
$SU(2)$	Z_1	Z_1	Z	Z_1
$SO(3)$	Z_1	Z_2	Z	Z_1
$SO(3)\times\mathrm{trn}$	Z_1	Z_2	$Z\times Z_2^2$	$Z^3\times Z_2^3$
$SO(3)\times Z_2^T$	Z_1	Z_2^2	Z_2	Z_2^3
$SO(3)\times Z_2^T\times\mathrm{trn}$	Z_1	Z_2^2	Z_2^5	Z_2^{12}

Chen, Gu, Liu, Wen, PRB (2013)

AZ	G/H	$\bar{d}=0$	$\bar{d}=1$	$\bar{d}=2$	$\bar{d}=3$	$\bar{d}=4$	$\bar{d}=5$	$\bar{d}=6$	$\bar{d}=7$
Complex case:									
A	$\mathrm{U}(N+M)/\mathrm{U}(N)\times\mathrm{U}(M)$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}\longleftarrow$
AIII	$\mathrm{U}(N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$
Real case:									
AI	$\mathrm{Sp}(N+M)/\mathrm{Sp}(N)\times\mathrm{Sp}(M)$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}\longleftarrow$
BDI	$\mathrm{U}(2N)/\mathrm{Sp}(2N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	$\mathrm{O}(2N)/\mathrm{U}(N)$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$
DIII	$\mathrm{O}(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$
AII	$\mathrm{O}(N+M)/\mathrm{O}(N)\times\mathrm{O}(M)$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0	$\mathbf{0}\longleftarrow$
CII	$\mathrm{U}(N)/\mathrm{O}(N)$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0	0
C	$\mathrm{Sp}(2N)/\mathrm{U}(N)$	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	0
CI	$\mathrm{Sp}(2N)$	0	0	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbf{0}$	$\longleftarrow\mathbb{Z}$

Ryu, Schnyder, Furusaki, Ludwig, NJP (2010); Kitaev (2009)

Symm. group	$d=0$	$d=1$	$d=2$	$d=3$
Z_2^T	Z_1	Z_2	Z_1	Z_2
$Z_2^T \times \mathrm{trn}$	Z_1	Z_2	Z_2^2	Z_2^4
Z_n	Z_n	Z_1	Z_n	Z_1
$Z_n \times \mathrm{trn}$	Z_n	Z_n	Z_n^2	Z_n^4
$U(1)$	Z	Z_1	Z	Z_1
$U(1) \times \mathrm{trn}$	Z	Z	Z^2	Z^4
$U(1) \rtimes Z_2^T$	Z	Z_2	Z_2	Z_2^2
$U(1) \rtimes Z_2^T \times \mathrm{trn}$	Z	$Z \times Z_2$	$Z \times Z_2^3$	$Z \times Z_2^8$
$U(1) \times Z_2^T$	Z_1	Z_2^2	Z_1	Z_2^3
$U(1) \times Z_2^T \times \mathrm{trn}$	Z_1	Z_2^2	Z_2^4	Z_2^9
$U(1) \rtimes Z_2$	Z_2	Z_2	$Z \times Z_2$	Z_2
$U(1) \times Z_2$	$Z \times Z_2$	Z_1	$Z \times Z_2^2$	Z_1
$Z_n \rtimes Z_2^T$	Z_n	$Z_2 \times Z_{(2,n)}$	$Z_{(2,n)}^2$	$Z_2 \times Z_{(2,n)}^2$
$Z_n \times Z_2^T$	$Z_{(2,n)}$	$Z_2 \times Z_{(2,n)}$	$Z_{(2,n)}^2$	$Z_2 \times Z_{(2,n)}^2$
$Z_n \rtimes Z_2$	$Z_2 \times Z_{(2,n)}$	$Z_{(2,n)}$	$Z_n \times Z_2 \times Z_{(2,n)}$	$Z_{(2,n)}^2$
$Z_m \times Z_n$	$Z_m \times Z_n$	$Z_{(m,n)}$	$Z_m \times Z_n \times Z_{(m,n)}$	$Z_{(m,n)}^2$
$D_2 \times Z_2^T = D_{2h}$	Z_2^2	Z_2^4	Z_2^6	Z_2^9
$Z_m \times Z_n \times Z_2^T$	$Z_{(2,m)} \times Z_{(2,n)}$	$Z_2 \times Z_{(2,m)} \times Z_{(2,n)} \times Z_{(m,n)}$	$Z_{(2,m,n)}^2 \times Z_{(2,m)}^2 \times Z_{(2,n)}^2$	$Z_2 \times Z_{(2,m,n)}^4 \times Z_{(2,m)}^2 \times Z_{(2,n)}^2$
$SU(2)$	Z_1	Z_1	Z	Z_1
$SO(3)$	Z_1	Z_2	Z	Z_1
$SO(3) \times \mathrm{trn}$	Z_1	Z_2	$Z \times Z_2^2$	$Z^3 \times Z_2^3$
$SO(3) \times Z_2^T$	Z_1	Z_2^2	Z_2	Z_2^3
$SO(3) \times Z_2^T \times \mathrm{trn}$	Z_1	Z_2^2	Z_2^5	Z_2^{12}

Chen, Gu, Liu, Wen, PRB (2013)

Wallpaper group #	$\mathcal{C}(G)=\mathcal{C}_0(G)$	pgSPT invariants	Weak pgSPT invariants
1	–	–	–
2	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	–
3	$\mathbb{Z}_2^2$	–	$\mathbb{Z}_2^2$
4	–	–	–
5	$\mathbb{Z}_2$	–	$\mathbb{Z}_2$
6	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8$	–
7	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$
8	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	–
9	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$	–
10	$\mathbb{Z}_4^2 \times \mathbb{Z}_2$	$\mathbb{Z}_4^2 \times \mathbb{Z}_2$	–
11	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^6$	–
12	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	–
13	$\mathbb{Z}_3^3$	$\mathbb{Z}_3^3$	–
14	$\mathbb{Z}_2$	–	$\mathbb{Z}_2$
15	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_2$
16	$\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathbb{Z}_6 \times \mathbb{Z}_3 \times \mathbb{Z}_2$	–
17	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	–

Huang, Song, Huang, Hermele, PRB (2017)

G_b	E_0^{1D}	E_0^{0D}	$\mathcal{G}_0$
$p1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
$p2$	0	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pm	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^5 \times \mathbb{Z}_2$
pg	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_4$
cm	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2$
pmm	0	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^4$
pmg	$\mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
pgg	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
cmm	0	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$
$p4$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4 \times \mathbb{Z}_2$
$p4m$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^3$	$\mathbb{Z}_2^4 \times \mathbb{Z}_2^3$
$p4g$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$
$p3$	0	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$	$\mathbb{Z}_2 \times \mathbb{Z}_3^3$
$p3m1$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2$
$p31m$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$
$p6$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$	$\mathbb{Z}_2^2 \times \mathbb{Z}_3^2$
$p6m$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^3 \times \mathbb{Z}_2^2$

Zhang, Yang, Qi, Gu, PRR (2022)

There is a very rich set of topological invariants with lattice symmetries

Questions

- What response properties do these lattice invariants correspond to?
- How do we measure these invariants in lattice models?
- Are there relationships between different invariants?

1. Crystalline gauge theory

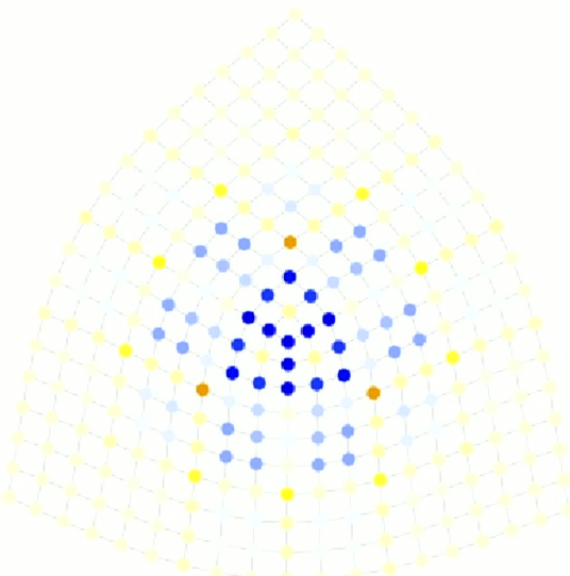
- An effective response theory for charge conservation, discrete translation and rotation symmetries based on group cohomology
- Non-perturbative (e.g. does not appeal to band theory)
- Predicts lattice versions of known continuum invariants + invariants with no continuum analog
- Valid for general gapped topological phases, but numerically we focus on 'invertible' fermionic phases (i.e. without fractionalized excitations)

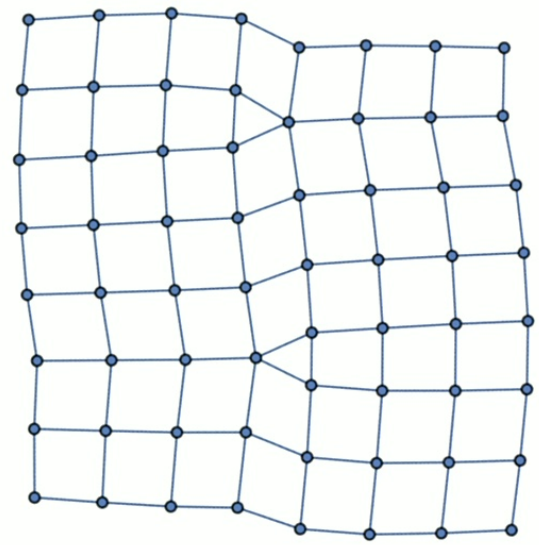
$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_o}{2\pi} A \wedge d\omega + \frac{\vec{\mathscr{P}}_o}{2\pi} \cdot A \wedge \vec{T} + \frac{\kappa}{2\pi} A \wedge A_{XY} + \ldots$$

NM, Barkeshli, 2012.11603
NM, Barkeshli, PRR (2021)

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_{\text{o}}}{2\pi} A \wedge d\omega + \frac{\vec{\mathscr{P}}_{\text{o}}}{2\pi} \cdot A \wedge \vec{T} + \frac{\kappa}{2\pi} A \wedge A_{XY} + \ldots$$

$$H = -\mu \sum_i c_i^\dagger c_i + \sum_{<ij>} e^{iA_{ij}} c_i^\dagger c_j + \text{h.c.}$$

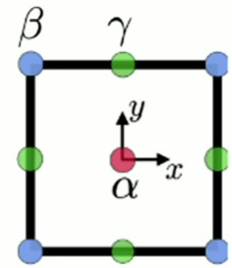




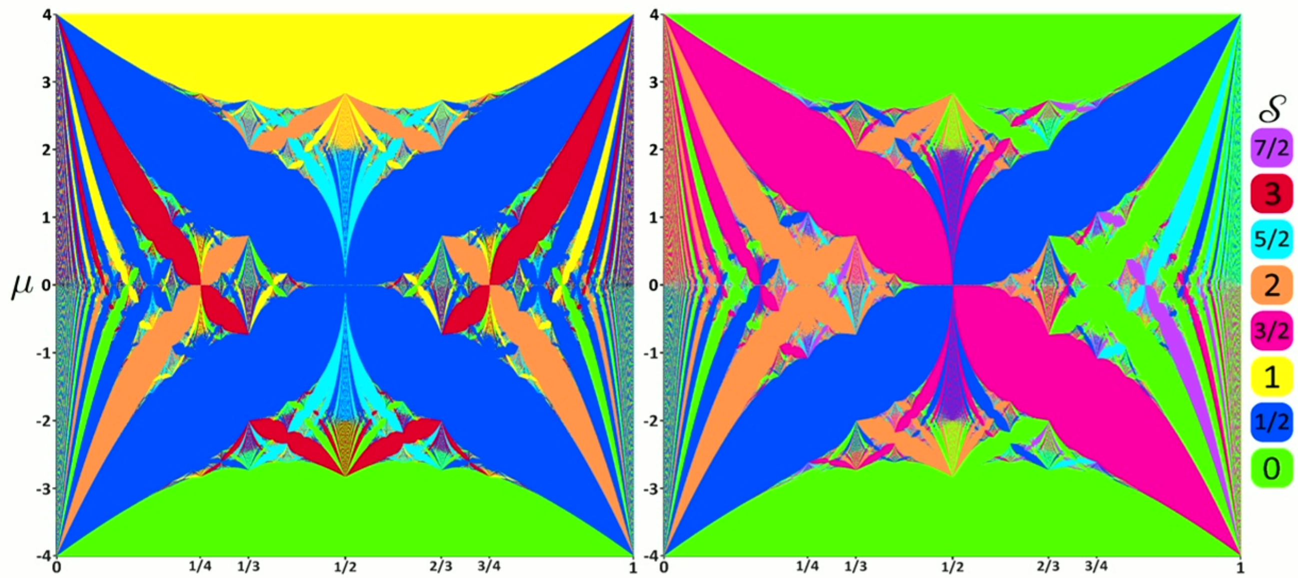
3. Quantized charge polarization
(defined for Chern insulators)

2. Discrete shift

A $\mathbb{Z}_M$ invariant defined for each high symmetry point



$$\mathscr{S}_{\rm o} = \frac{C}{2} \mod 1 \qquad \begin{array}{rl} C \text{ odd} & : \mathscr{S} \in \{1/2, 3/2, \ldots, M-1/2\} \\ C \text{ even} & : \mathscr{S} \in \{0, 1, \ldots, M-1\} \end{array}$$



(First new invariants in Hofstadter model since 1982!)

Zhang, NM, Nambiar, Barkeshli (2022)

Effective response theories

- IQHE ($U(1)$ symmetry) is defined by

$$J_x = -\sigma_H E_y, \delta N_e = \sigma_H \delta\Phi; \quad \sigma_H = C\frac{e^2}{h}$$

- Response captured by effective action with $U(1)$ gauge field A:

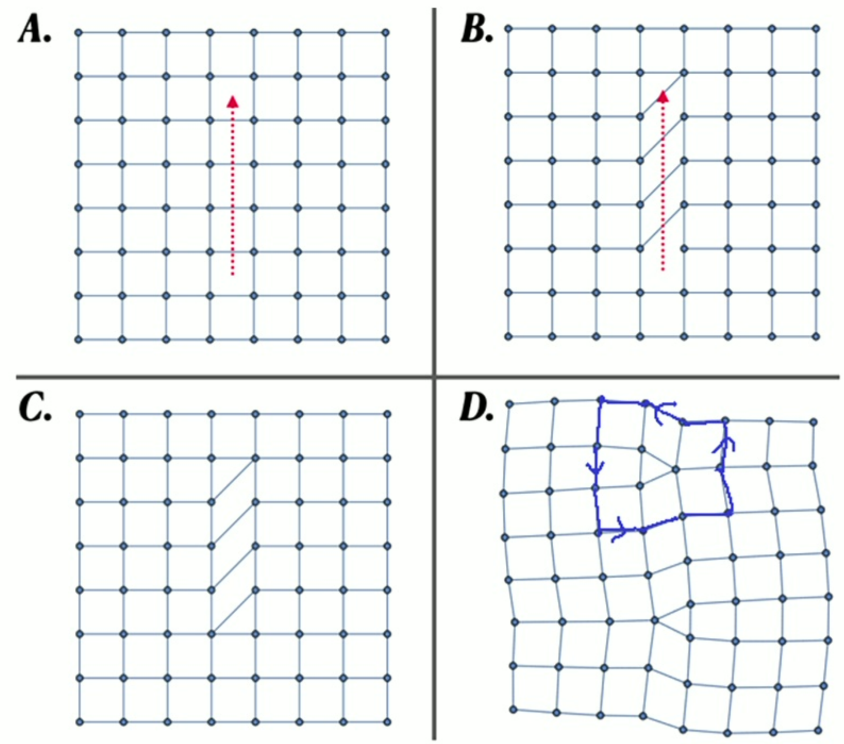
$$\mathcal{L} = \frac{C}{4\pi}(A_t B - A_x E_y + A_y E_x) =: \frac{C}{4\pi} A \wedge dA$$

- Classification = integers = $\mathbb{Z}$

- Can we emulate this for crystalline symmetries?

Lattice defects

Dislocation is defined by its Burgers vector $\vec{b}$
(independent of choice of loop used to measure it)



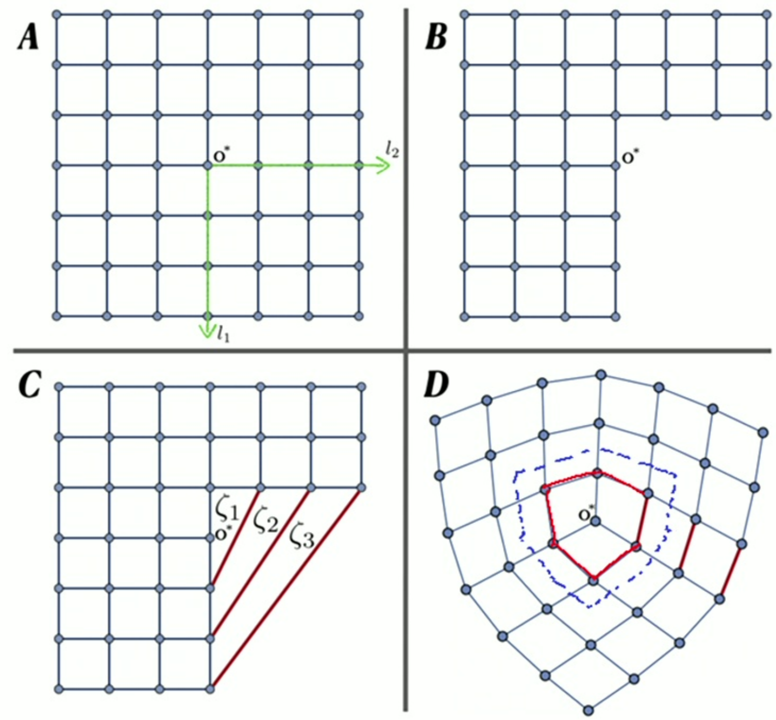
Disclinations are defined by a pair $(\vec{b}_o, \Omega)$

($\vec{b}_o$ depends on the choice of loop and *not* on the origin o* used to create the disclination from the clean lattice)

$$\vec{b}_{\mathrm{o}+\vec{v}} = \vec{b}_{\mathrm{o}} + (1 - U(\Omega))\vec{v}$$

Hence for a fixed Ω, the different Burgers vectors fall into distinct equivalence classes

$$\vec{b}_{\mathrm{o}} \simeq \vec{b}_{\mathrm{o}} + (1 - U(\Omega))\vec{\Lambda}, \quad \Lambda \in \mathbb{Z}^2$$



$$b_{\beta,x} + b_{\beta,y} = 0 \mod 2$$
$$b_{\alpha,x} + b_{\alpha,y} = 1 \mod 2$$

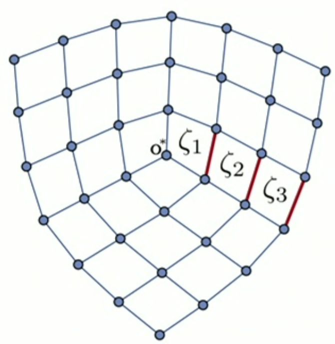
Crystalline gauge fields
(encode lattice defects)

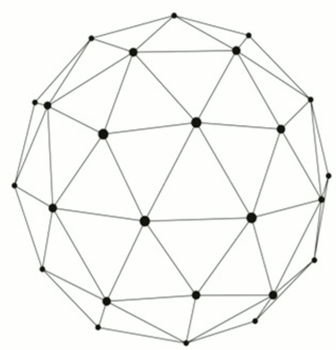
Two options: define them as real valued differential 1-forms in the continuum with quantized holonomies

Or as discrete gauge fields on the simplices of a triangulation of the manifold

$$A \in \mathbb{R}; \quad A \sim A + 2\pi$$
$$\omega \in \tfrac{2\pi}{M}\mathbb{Z}; \quad \omega \sim \omega + 2\pi$$
$$\vec{R} = (X, Y); \quad X, Y \in 2\pi\mathbb{Z}$$

$$G = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$$
$$\begin{aligned}(\vec{R}_{ij}, \omega_{ij}) &\to (\vec{r}_i, h_i)^{-1}(\vec{R}_{ij}, \omega_{ij})(\vec{r}_j, h_j) \\ &= ((U(-h_i)(\vec{R}_{ij} + U(\omega_{ij})\vec{r}_j - \vec{r}_i), \omega_{ij} + h_j - h_i)\end{aligned}$$





- Gauge invariant quantities: $\prod_{ij\in\gamma_o}(\vec{R}_{ij},\omega_{ij}) \simeq (\sum_{ij\in\gamma_o}\vec{R}_{ij}, \sum_{ij\in\gamma_o}\omega_{ij}) \simeq (2\pi\vec{b}_o, \Omega)$

(holonomies depend on the origin o of the loop)

$\vec{T} = d\vec{R}, d\omega$ give dislocation and disclination density (up to equivalences among Burgers vectors)

Also define an area element $A_{XY} : \int_W \frac{A_{XY}}{2\pi} = N_W; \quad A_{XY} = \frac{1}{2\pi} X \wedge Y \text{ if } \omega = 0$

- Identify $(\vec{R}, \omega)$ with coframe fields and spin connection associated to the metric

Kleinert (1980s, elasticity theory)
Thorngren, Else, PRX (2019)
Song, He, Vishwanath, Wang, PRR (2021)
NM, Barkeshli, PRR (2021)
...

Topological terms from group cohomology
(for internal symmetries)

- (2+1)D Bosonic SPTs with U(1) symmetry: $\mathcal{L} = \frac{n}{2\pi} A \wedge dA; \quad n \in \mathbb{Z}; n \times \frac{2e^2}{h} = \sigma_H$
- Classified by $H^3\big(U(1), U(1)\big) = \mathbb{Z}$ (for internal symmetries) Chen, Gu, Liu, Wen, PRB (2013)

This group also classifies functions

$$f(\mathbf{g},\mathbf{h},\mathbf{k}) \mod 2\pi, \quad f : U(1)^3 \to U(1)$$
$$\mathbf{g},\mathbf{h},\mathbf{k} \in U(1), \text{ i.e. } \mathbf{g} \sim \mathbf{g} + 2\pi$$
$$df = 0$$
$$f \simeq f + db$$

Representative: $f(\mathbf{g},\mathbf{h},\mathbf{k}) = \frac{n}{2\pi}\mathbf{g}([\mathbf{h}]_{2\pi} + [\mathbf{k}]_{2\pi} - [\mathbf{h}+\mathbf{k}]_{2\pi}) \mod 2\pi; \quad n \in \mathbb{Z}$

Knowing f, consider a triangulated manifold with A a flat U(1) gauge field. Define $\mathcal{L} = A^* f$:

$$\mathcal{L}[0123] = f(A_{01}, A_{12}, A_{23})$$

Extension to fermions and crystalline symmetries

Group cohomology classification can be generalized to invertible fermionic phases
$\mathcal{L} = A^* f$, but f is not a cocycle:

$$df = \mathcal{O}_4[c_-, n_1, n_2, \omega_2] \mod 1$$

Topological terms have different quantization, e.g. $\frac{C}{4\pi} A \wedge dA$

Gu, Wen, PRB (2014)
Wang, Gu, PRX (2020)
Barkeshli, Chen, Hsin, NM, PRB (2022)
Aasen, Bonderson, Knapp (2021)

Extension to crystalline symmetries:

Important claim: classification of G invertible phases is isomorphic to that of invertible phases with some effective **internal** symmetry G^{eff} (conjecture with evidence)

For bosonic systems without reflections, $G^{eff} \cong G$
For fermionic systems, there is an explicit algebraic formula to find G^{eff}

Thorngren, Else, PRX (2018)
Debray (2021)
NM, Calvera, Barkeshli (2022)
...

The full response theory

$$G_f = U(1)^f \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$$

Gauge field for $G_f^{eff} \cong G_f$: $\mathrm{B} = \left(\delta A, \vec{R}, \omega\right)$
$\delta A =$ deviation of vector potential from some background
But total magnetic flux can be written in terms of full vector potential A

$$T_{\mathbf{y}}^{-1} T_{\mathbf{x}}^{-1} T_{\mathbf{y}} T_{\mathbf{x}} = e^{i\phi \hat{N}}$$

$$F := d\delta A + \frac{\phi}{2\pi} A_{XY}$$
$$= dA$$

$$\mathcal{L} = \boxed{\frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_{\mathrm{o}}}{2\pi} A \wedge d\omega + \frac{\vec{\mathscr{P}}_{\mathrm{o}}}{2\pi} A \wedge \vec{T} + \frac{\kappa}{2\pi} A \wedge A_{XY}}$$
$$+ \frac{\ell}{4\pi} \omega \wedge d\omega + \frac{\vec{\mathscr{P}}_{s,\mathrm{o}}}{2\pi} \omega \wedge \vec{T} + \frac{\kappa_s}{2\pi} \omega \wedge A_{XY} + \ldots$$

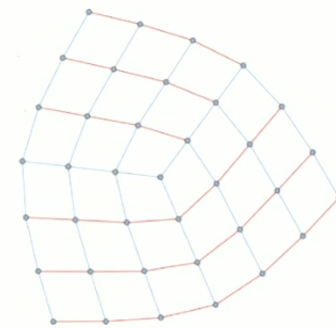
NM, Barkeshli, PRR (2021); Zhang, NM, Nambiar, Barkeshli (2022)

Facts from continuum QHE

- Filling relation on torus: $N_e = \sigma_H N_\phi$
- On a sphere: $N_e = \sigma_H N_\phi + 2\mathcal{S}$
- On a genus g surface: $N_e = \sigma_H N_\phi + 2(1-g)\mathcal{S}$ Wen, Zee, PRL (1992)
- S is an intrinsic universal property of the quantum Hall phase
- $2S \in Z$
- For C filled Landau levels, $S = \frac{C^2}{2}$

Discrete shift $\frac{\mathcal{S}_{\rm o}}{2\pi} A \wedge d\omega$

Biswas, Son, PNAS (2016)
Liu, Vishwanath, Khalaf, PRX (2019)
Han, Wang, Ye, PRB (2019)
Li, Zhu, Benalcazar, Hughes, PRB (2020)
NM, Barkeshli, PRR (2021)
...



- Fractional charge assigned to elementary disclination of angle $\frac{2\pi}{M}$ is

$$Q = \frac{\mathcal{S}_{\rm o}}{M} \mod 1$$
$$\mathcal{S}_{\rm o} \simeq \mathcal{S}_{\rm o} + M$$

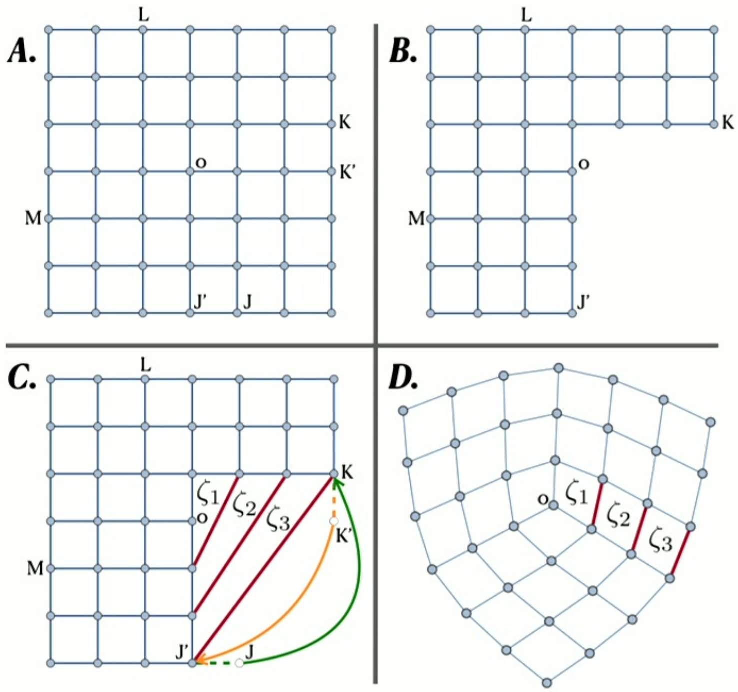
- $2M$ possible values in principle: $0, \frac{1}{2}, 1, \ldots, M - \frac{1}{2}$
- But

$$\tilde{C}^M_{M,\rm o} = +1 \implies \mathcal{S}_{\rm o} = \frac{C}{2} \mod 1$$

- $C = 0$: related to fractional corner charges, HOTIs

Constructing a disclination

Disclination Hamiltonian is fully determined by clean Hamiltonian
(up to local operators at the disclination core)



$$\tilde{C}_{4,\lambda} = \hat{C}_4 e^{i\sum_j \lambda_j \hat{n}_j}; \quad \tilde{C}^4_{4,\lambda} = +1$$

$$c^\dagger_{J'} c_J e^{-iA_{J'J}} \to c^\dagger_{J'}\left(\tilde{C}_{4,\lambda} c_J \tilde{C}^\dagger_{4,\lambda}\right) e^{-iA_{J'J}}$$
$$= e^{-i(A_{J'J}+\lambda_J)} c^\dagger_{J'} c_K$$

$$\implies A^{\text{defect}}_{J'K} = A_{J'J} + \lambda_J$$

There is still ambiguity in rotation operator
(can be fixed if we demand equal flux in all plaquettes)

Zhang, NM, Nambiar, Barkeshli (2022)

Fractional disclination charge

- Charge in region W:

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_o}{2\pi} A \wedge d\omega + \cancel{\frac{\vec{\mathscr{P}}_o}{2\pi} \cdot A \wedge \vec{T}} + \frac{\kappa}{2\pi} A \wedge A_{XY} + \dots$$

$$Q_W = \int_W \frac{\delta \mathcal{L}}{\delta A_0} = C(\frac{\phi}{2\pi} N_{W,o} + \frac{\delta \Phi_{W,o}}{2\pi}) + \mathscr{S}_o \frac{\Omega_W}{2\pi} + \kappa N_{W,o}$$

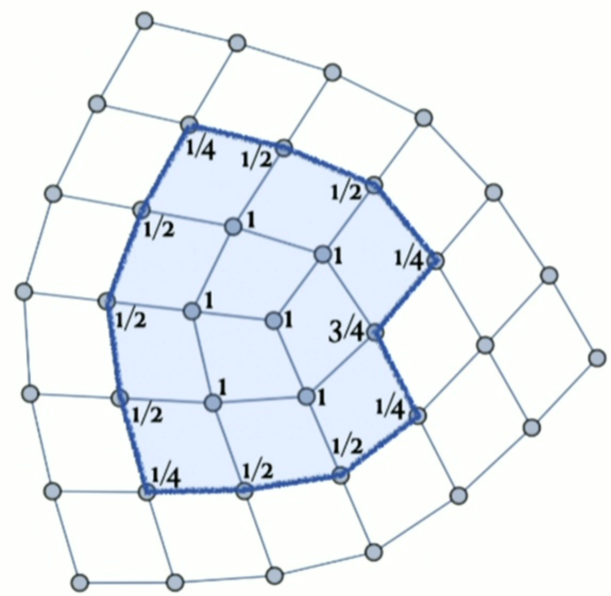
$$= \cancel{C \frac{\delta \Phi_W}{2\pi}} + \mathscr{S}_o \frac{\Omega_W}{2\pi} + \nu N_{W,o}$$

$$= \frac{\mathscr{S}_o}{4} + \nu N_{W,o}$$

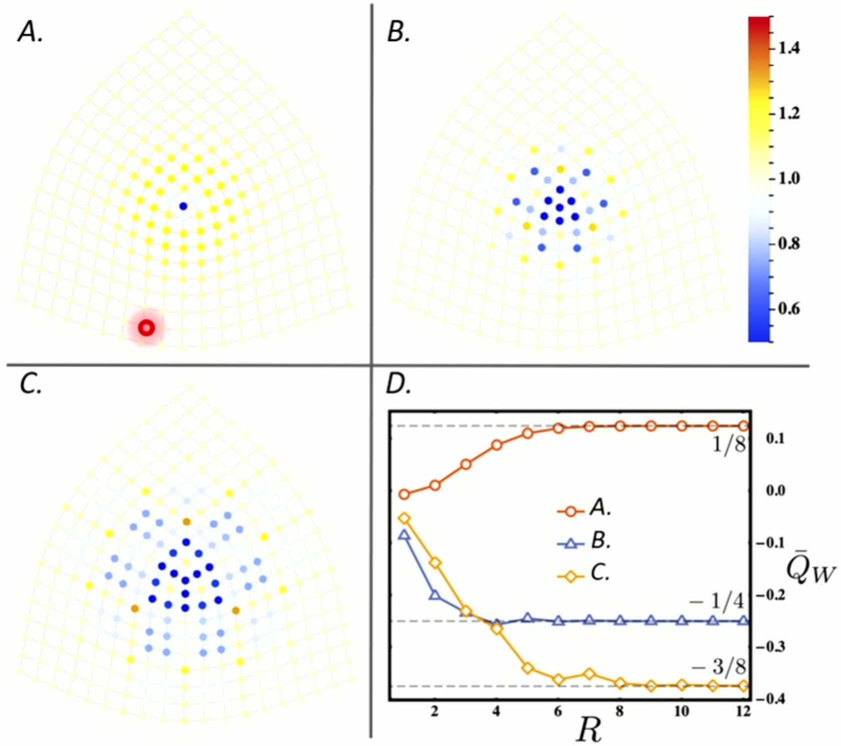
$$Q_W \equiv \sum_{i \in W} \mathrm{wt}(i) Q_i$$

- $N_{W,o}$ chosen to be an integer

- Weight satisfies $Q_W + Q_{W'} = Q_{W \cup W'}$



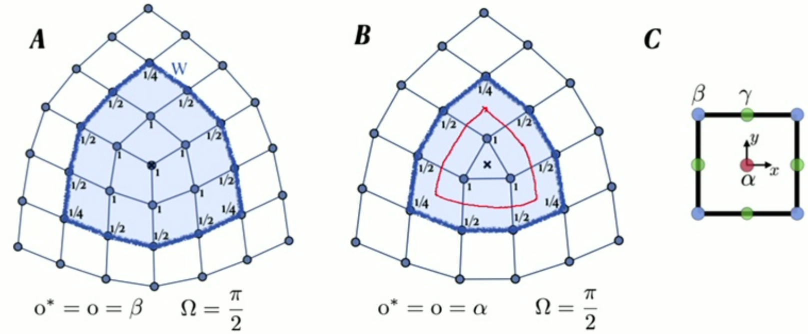
Zhang, NM, Nambiar, Barkeshli (2022)



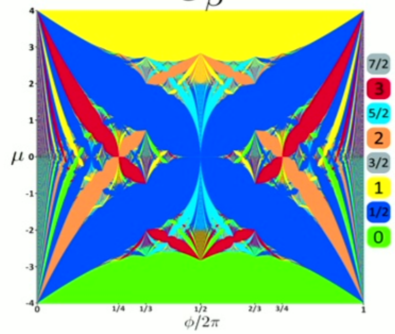
Zhang, NM, Nambiar, Barkeshli (2022)

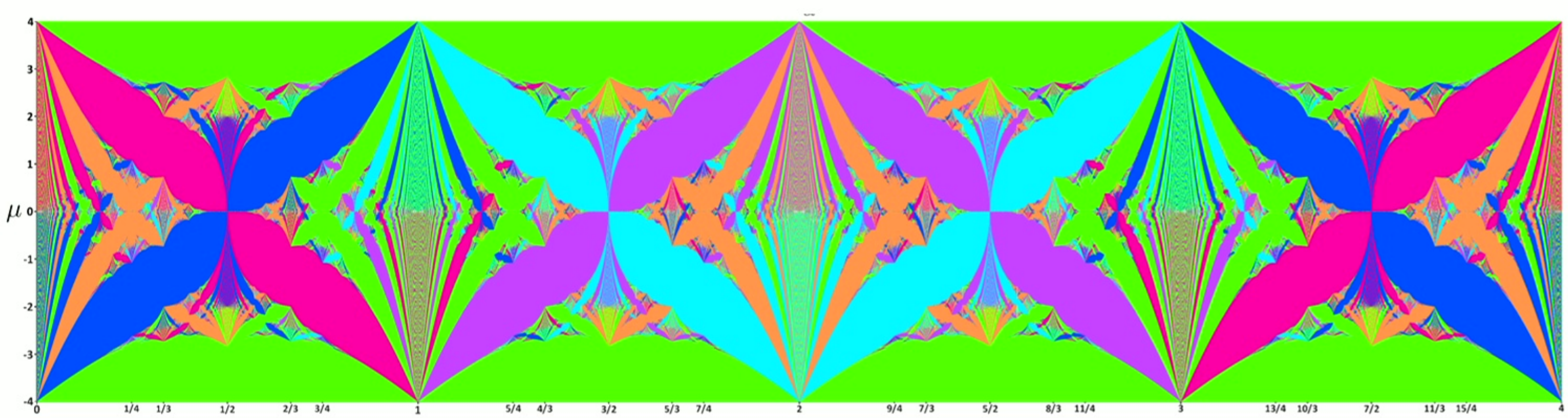
Origin dependence

A disclination created with $\tilde{C}_{4,\alpha}, \tilde{C}_{4,\alpha}^{4} = +1$, has flux $3\phi/4$ in the irregular plaquette
Therefore the defect Hamiltonian is only periodic under $\phi \to \phi + 8\pi$



$\mathcal{S}_\beta$



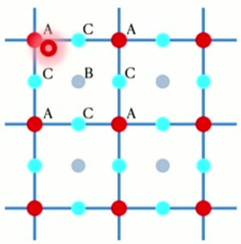


$\mathcal{S}_\alpha$

$$\tilde{C}^{M}_{M,\mathrm{o}} = +1 \implies \mathscr{S}_{\mathrm{o}} = \frac{C}{2} \mod 1$$

- Can always find one state with this property, for each C

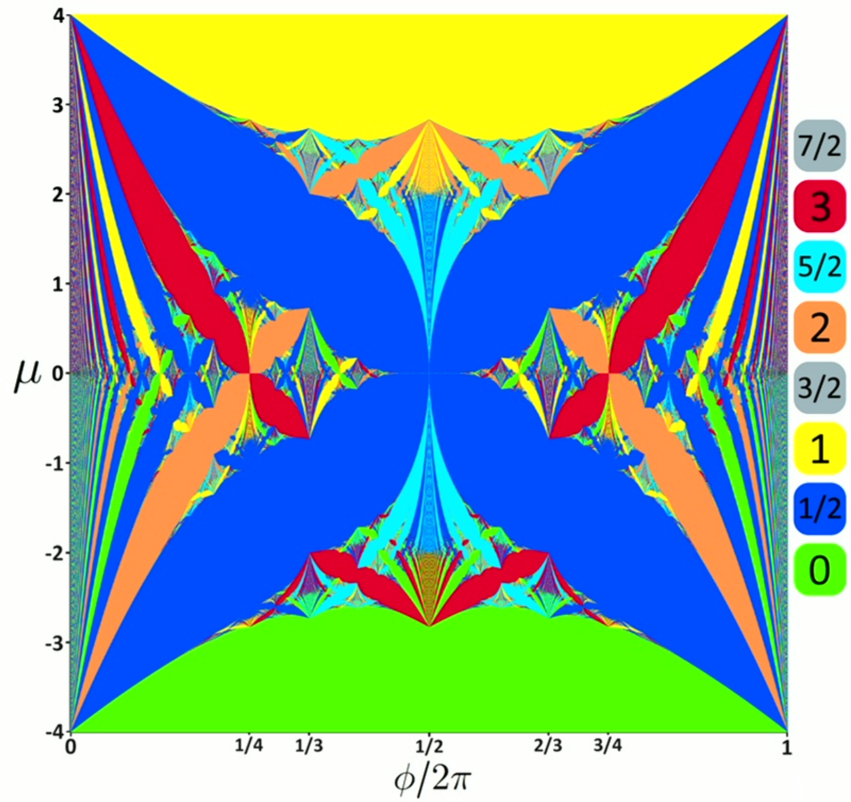
- When $C = 0$:



$$C = 0 \implies \mathscr{S}_{\mathrm{o}} = n_{\mathrm{o}}$$

- Now use stacking arguments

- Nonperturbative argument also available, using CEP



Zhang, NM, Nambiar, Barkeshli (2022)

Dual response: $\frac{\mathcal{S}_o}{2\pi} A \wedge d\omega = \frac{\mathcal{S}_o}{2\pi} \omega \wedge dA + \frac{\mathcal{S}_o}{2\pi} d(\omega \wedge A)$

Liu, Vishwanath, Khalaf, PRX (2019)
You, Bibo, Pollmann, PRR (2020)

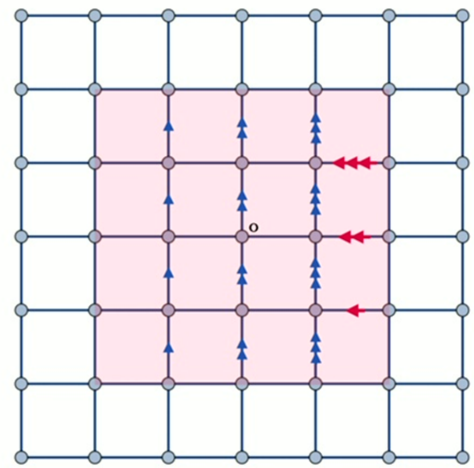
- Insert m total flux quanta uniformly on torus, and measure

$$\tilde{C}_{4,o} |\Psi_{\text{flux}}\rangle = e^{il_o \pi/2} |\Psi_{\text{flux}}\rangle$$

- Use same rotation operator as for disclinations

$$l_o(m) = \mathcal{S}_o m + C\frac{m^2}{2} + K(C, L) \mod 4$$

- This verifies a very non-trivial duality of TQFT



Zhang, NM, Nambiar, Barkeshli (2022)

Quantization of $\frac{\vec{\mathscr{P}}_{\rm o}}{2\pi} \cdot A \wedge \vec{T}$

Fang, Gilbert, Bernevig, PRB (2012)
Jadaun, Xiao, Niu, Banerjee, PRB (2013)
NM, Barkeshli, PRR (2021)

- A defect with dislocation Burgers vector $\vec{b}_o$ is assigned charge $\vec{\mathscr{P}}_{\rm o} \cdot \vec{b}_{\rm o} \mod 1$

- Rotating coordinate system by an angle $\frac{2\pi}{M}$ should not change the prediction

$$\implies \quad \vec{\mathscr{P}}_{\rm o} \cdot \vec{b}_{\rm o} = \vec{\mathscr{P}}_{\rm o} \cdot U(\frac{2\pi}{M})\vec{b}_{\rm o} \mod 1, \quad \forall \vec{b}_{\rm o}$$

- E.g. square lattice: $\vec{\mathscr{P}}_{\rm o} \cdot \hat{x} = \vec{\mathscr{P}}_{\rm o} \cdot \hat{y} = -\vec{\mathscr{P}}_{\rm o} \cdot \hat{x} \mod 1$

$$\vec{\mathscr{P}}_{\rm o} = (0,0) \text{ or } (1/2,1/2) \mod \mathbb{Z}^2$$

- Classification :

$M = 2 : \{(0,0), (\frac{1}{2},0), (0,\frac{1}{2}), (\frac{1}{2},\frac{1}{2})\}$ $\qquad \mathbb{Z}_2 \times \mathbb{Z}_2$

$M = 3 : \{(0,0), (\frac{1}{3},\frac{2}{3}), (\frac{2}{3},\frac{1}{3})\}$ $\qquad \mathbb{Z}_3$

$M = 4 : \{(0,0), (\frac{1}{2},\frac{1}{2})\}$ $\qquad \mathbb{Z}_2$

$M = 6 : \{(0,0)\}$ $\qquad \mathbb{Z}_1$

Polarization from dislocation charge

van Miert, Ortix, PRB (2018)
Li, Zhu, Benalcazar, Hughes, PRB (2020)
Zhang, NM, Nambiar, Barkeshli (2022)

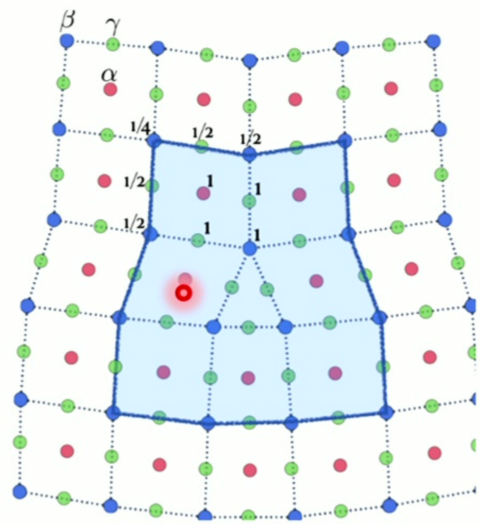
- Theory: $$Q_W = \frac{C}{2\pi}\delta\Phi_{W,\mathrm{o}} + \nu N_{W,\mathrm{o}} + \vec{\mathscr{P}}_{\mathrm{o}} \cdot \vec{b}$$

$$N_{W,\mathrm{o}} = n_{\mathrm{irreg,o}} + \text{known integer}$$
$$\delta\Phi_{W,\mathrm{o}} := \phi_{\mathrm{irreg}} - \phi n_{\mathrm{irreg,o}}$$

- Calculation reduces to computing $n_{\mathrm{irreg,o}}$

- Fix by requiring that in the C=0 limit,

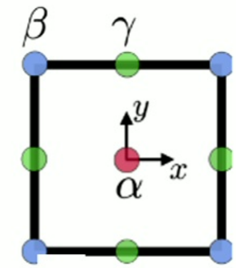
polarization agrees with a dipole moment calculation

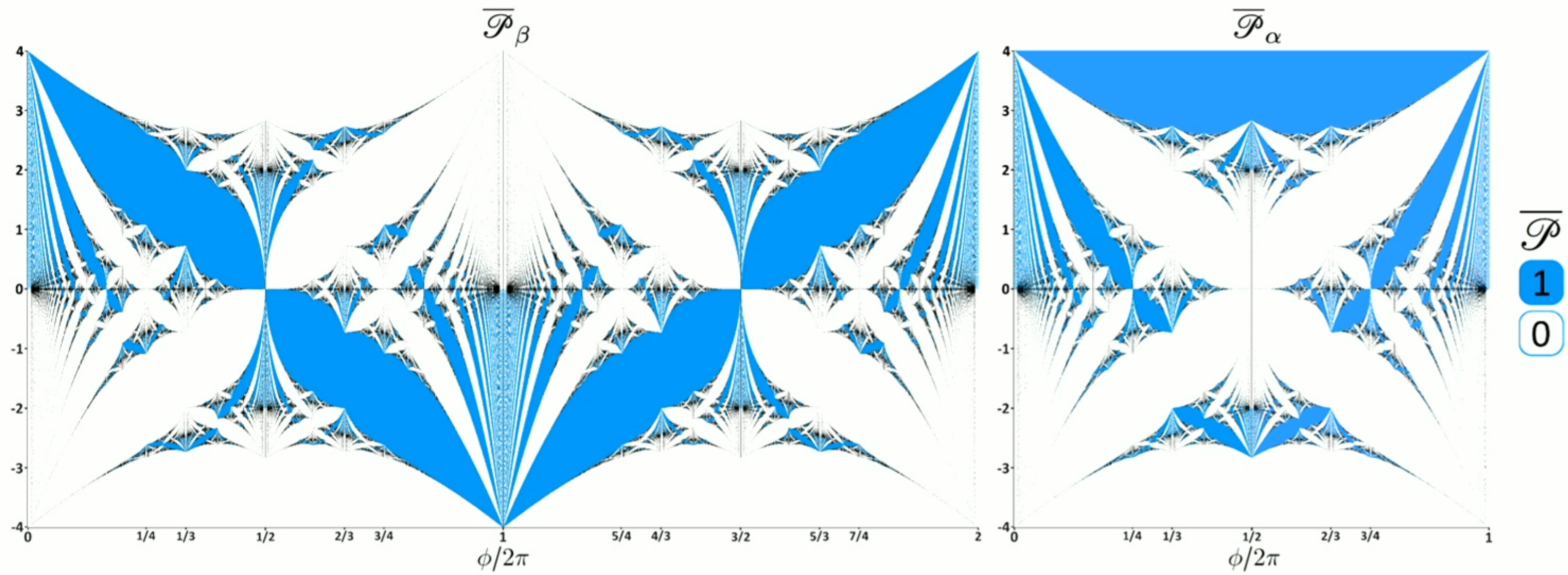


$n_{\mathrm{irreg},\beta} = 1/2; n_{\mathrm{irreg},\alpha} = 0$

- (Origin dependence of polarization is consistent with all known definitions)

Polarization in Hofstadter model





$$(\mathcal{S}_\beta, \overline{\mathscr{P}}_\beta, \kappa) = (\mathcal{S}_\alpha + 2\overline{\mathscr{P}}_\alpha - \kappa, \overline{\mathscr{P}}_\alpha + \kappa, \kappa)$$

Polarization ≈ difference of shifts

Why is this interesting?

- Developing an intrinsically many-body definition of charge polarization valid for d>1 is a very old problem

  King-Smith, Vanderbilt, PRB (1993)
  Resta, PRL (1998); Resta, Vanderbilt (2007)
  Watanabe, Oshikawa, PRX (2018)

- Definition via Berry phase in momentum space gives an answer independent of origin for BZ when $C = 0$

- For $C \neq 0$ the definition depends on the origin for BZ and was generally assumed to be physically not meaningful

- Polarization was previously shown to be defined in a Chern insulator if we fix an origin in the BZ, but this is not conventional wisdom

  Coh, Vanderbilt, PRL (2009)

- Dislocation charge defines a quantized polarization in 2d, in a many-body setting, for Chern insulators (given rotational symmetries)

Dual response: linear momentum of flux

Song, Wang, Vishwanath, He, Nat. Comm. (2019)
Song, He, Vishwanath, Wang, PRR (2021)
Zhang, NM, Nambiar, Barkeshli (2022)

- Define approximate translation operator satisfying

$$A_{i+\hat{y},j+\hat{y}} = A_{ij} - \lambda_i + \lambda_j$$

- Insert m total flux quanta and take expectation value

$$\frac{\vec{\mathscr{P}}_{\mathrm{o}}}{2\pi} \cdot \vec{R} \wedge dA$$

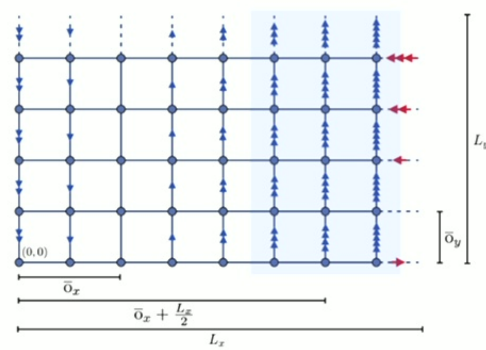
$$\tilde{T}_y := \hat{T}_y e^{i\sum_j \lambda_j \hat{n}_j}$$

$$\langle \Psi | \tilde{T}_y | \Psi \rangle = e^{-\gamma + i p_{\lambda,y}}$$

$$p_{\lambda,y} = -\mathscr{P}_{\mathrm{o},y} m + K_y$$

Freedom in λ:

- $\lambda \to \lambda + \chi$: Fix partially by constraining the flux in a dislocation created using $\tilde{T}_y$
- Fix location of strip which breaks translation symmetry



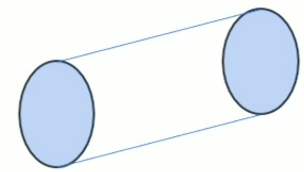
Empirically: only one choice of strip which gives quantized results throughout Hofstadter model

(resulting polarization agrees with dislocation charge!)

Location of strip can be changed when $C = 0$

Further comments

- Additional response: Dimensional reduction to 1d gives a length dependent 1d polarization,



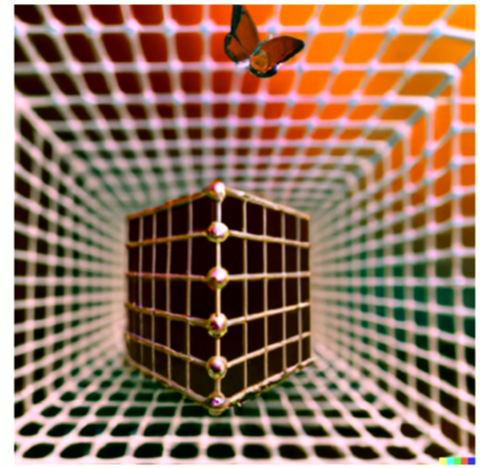
$$-\mathcal{P}_{\mathcal{O},x} = \frac{Ck\phi}{2\pi} L_y + L_y \mathscr{P}_{\mathrm{o},y} + K \mod 1$$

- Story generalizes almost completely to M=2,3,6

- Polarization ≈ difference of shifts

M	$\mathscr{S}_{\mathrm{o}+\vec{v}}$	$\vec{\mathscr{P}}_{\mathrm{o}+\vec{v}}$
2	$\mathscr{S}_{\mathrm{o}} - 4\vec{v}\cdot\vec{\mathscr{P}}_{\mathrm{o}} + 4\kappa(v_x^2+v_y^2+v_xv_y)$	$\vec{\mathscr{P}}_{\mathrm{o}} + (-v_y\kappa, v_x\kappa)$
4	$\mathscr{S}_{\mathrm{o}} + 2\overline{\mathscr{P}}_{\mathrm{o}} - \kappa$	$\vec{\mathscr{P}}_{\mathrm{o}} + (-\frac{1}{2}\kappa, \frac{1}{2}\kappa)$
3	$\mathscr{S}_{\mathrm{o}} - 3v_y\overline{\mathscr{P}}_{\mathrm{o}} - 3\kappa(v_x^2+v_y^2+v_xv_y)$	$\vec{\mathscr{P}}_{\mathrm{o}} + (-v_y\kappa, v_x\kappa)$
6	$\mathscr{S}_{\mathrm{o}}$	0

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_{\text{o}}}{2\pi} A \wedge d\omega + \frac{\vec{\mathscr{P}}_{\text{o}}}{2\pi} A \wedge \vec{T} + \frac{\nu}{2\pi} A \wedge A_{XY}$$
$$+ \frac{\ell}{4\pi} \omega \wedge d\omega + \frac{\vec{\mathscr{P}}_{s,\text{o}}}{2\pi} \omega \wedge \vec{T} + \frac{\nu_s}{2\pi} \omega \wedge A_{XY} + \dots$$



Classification summary