

Title: Topology of the Fermi sea: ordinary metals as topological materials

Speakers: Pok Man Tam

Series: Quantum Matter

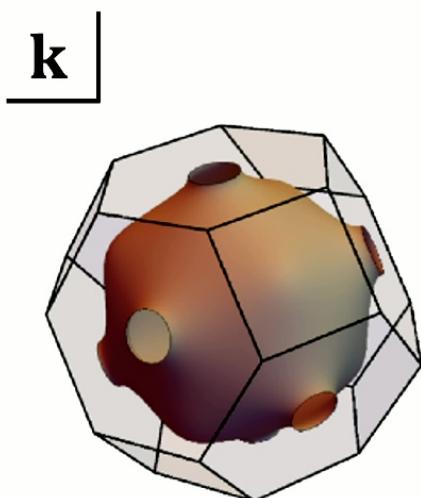
Date: December 08, 2022 - 3:00 PM

URL: <https://pirsa.org/22120022>

Abstract: It has long been known that the quantum ground state of a metal is characterized by an abstract manifold in the momentum space called the Fermi sea. Fermi sea can be distinguished topologically in much the same way that a ball can be distinguished from a donut by counting the number of holes. The associated topological invariant, i.e. the Euler characteristic (χ_F), serves to classify metals. Here I will survey two recent proposals relating χ_F to experimental observables, namely: (i) equal-time density/number correlations, and (ii) Andreev state transport along a planar Josephson junction. Moreover, from the perspective of quantum information, I will explain how multipartite entanglement in real space probes the Fermi sea topology in momentum space. Our works not only suggest a new connection between topology and entanglement in gapless quantum matters, but also suggest accessible experimental platforms to extract the topology in metals.

Zoom link: <https://pitp.zoom.us/j/98944473905?pwd=ak5nVmd4N0pSdXpjOFM0YnFJdnJ4dz09>

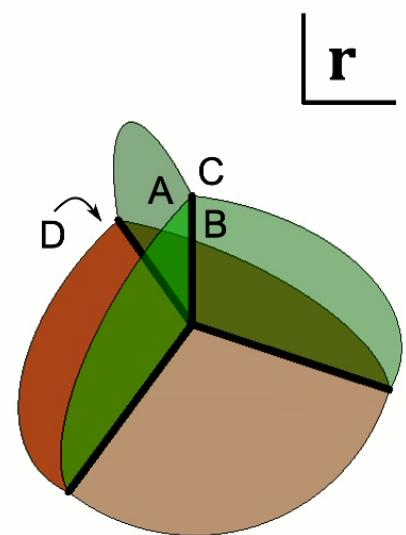
Topology of the Fermi sea: Ordinary metals as topological materials



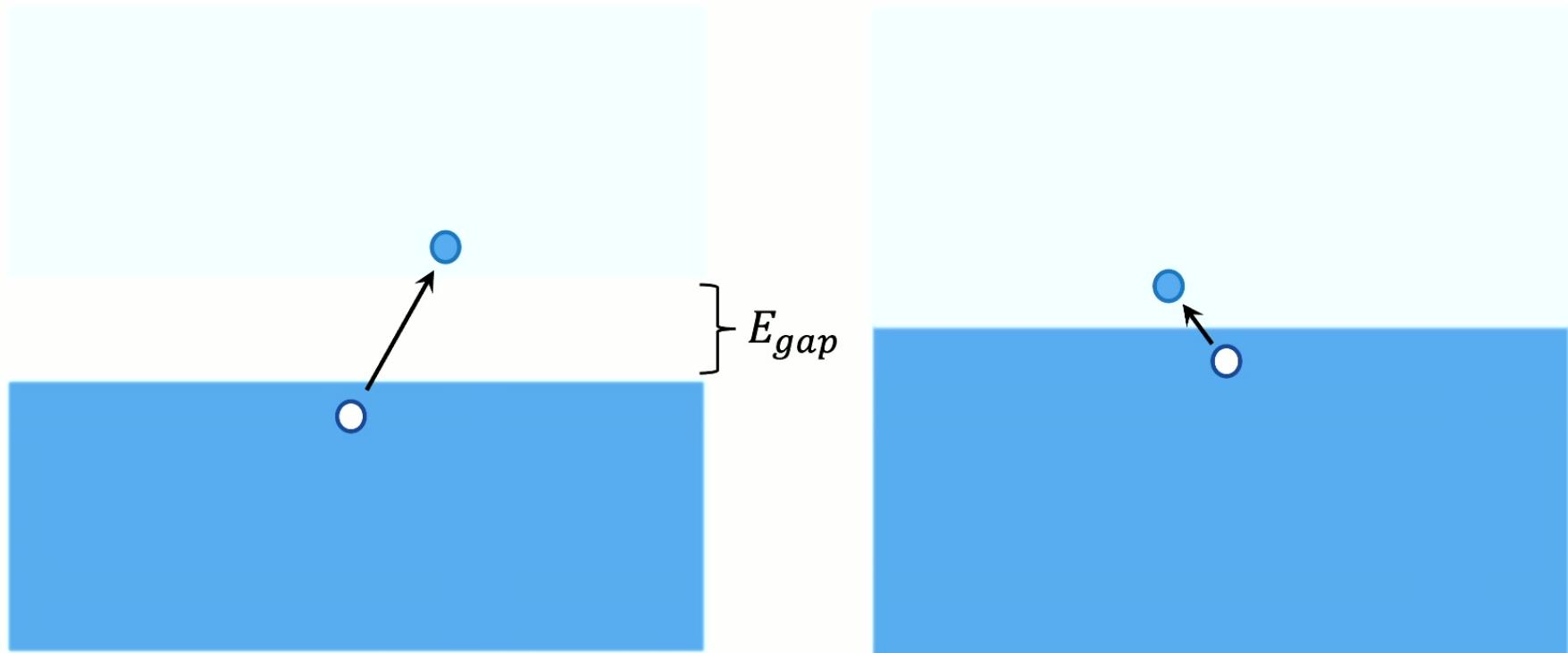
Pok Man Tam
University of Pennsylvania

Tam, Claassen, and Kane, Phys. Rev. X 12, 031022
Tam and Kane, arXiv:2210.08048

Perimeter Institute
12/08/2022



Gapped vs Gapless



Topology in condensed matter

- Topological band insulator

- Su-Schrieffer-Heeger chain (1D), Chern insulator/IQHE (2D), TRS topological insulators (2D, 3D), etc.

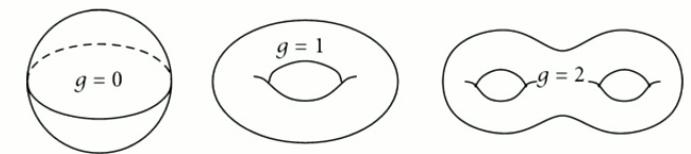
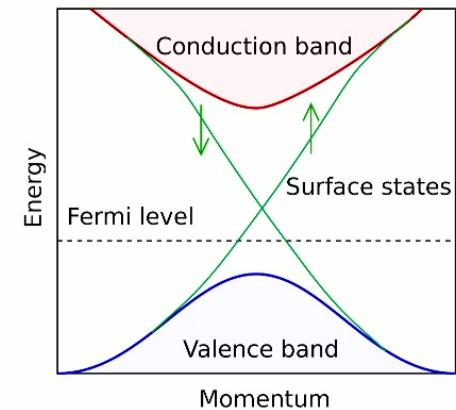
$$\nu_m = \frac{i}{2\pi} \int_{BZ} d^2\mathbf{k} \nabla \times \langle u_m | \nabla_k | u_m \rangle \Rightarrow \sigma_{xy} = \frac{e^2}{h} \cdot \nu$$

TKNN, PRL (1982)

- Gapless boundary modes \Leftrightarrow quantized responses
- Topological order

- Fractional quantum Hall states (e.g. Laughlin, Moore-Read states, etc.), and quantum spin liquids (e.g. Kitaev's toric code)
- Bulk hosts gapped *anyons* \Leftrightarrow Edge hosts gapless excitations

Moore and Read, Nucl. Phys B (1991)



Topology in condensed matter

- Topological band insulator

- Su-Schrieffer-Heeger chain (1D), Chern insulator/IQHE (2D), TRS topological insulators (2D, 3D), etc.

$$\nu_m = \frac{i}{2\pi} \int_{BZ} d^2\mathbf{k} \nabla \times \langle u_m | \nabla_k | u_m \rangle \Rightarrow \sigma_{xy} = \frac{e^2}{h} \cdot \nu$$

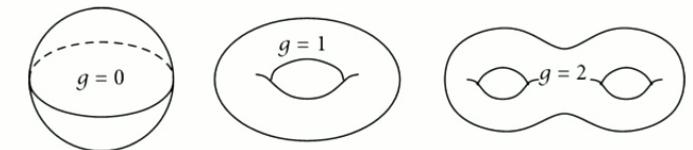
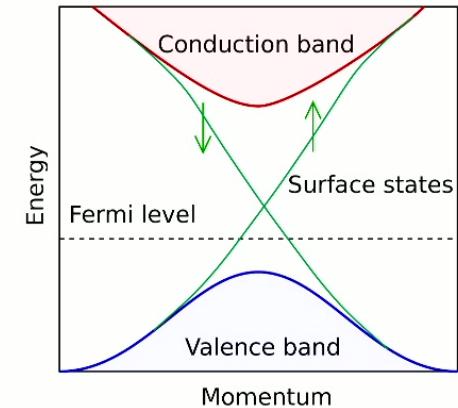
TKNN, PRL (1982)

- Gapless boundary modes \Leftrightarrow quantized responses

- Topological order

- Fractional quantum Hall states (e.g. Laughlin, Moore-Read states, etc.), and quantum spin liquids (e.g. Kitaev's toric code)
- Bulk hosts gapped *anyons* \Leftrightarrow Edge hosts gapless excitations

Moore and Read, Nucl. Phys B (1991)



Topology of wavefunction

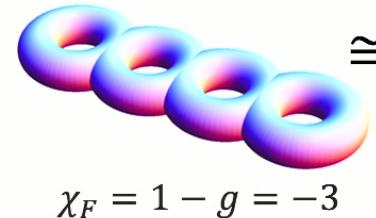
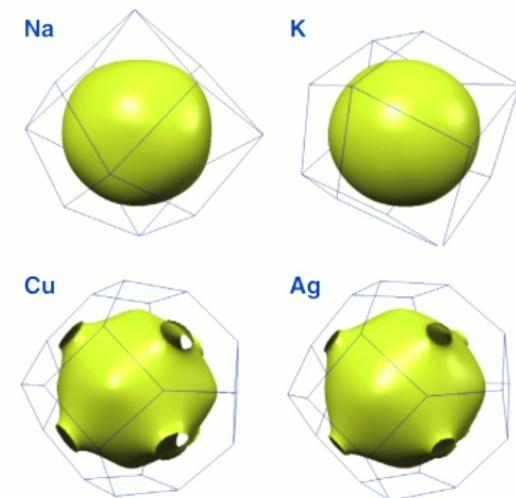
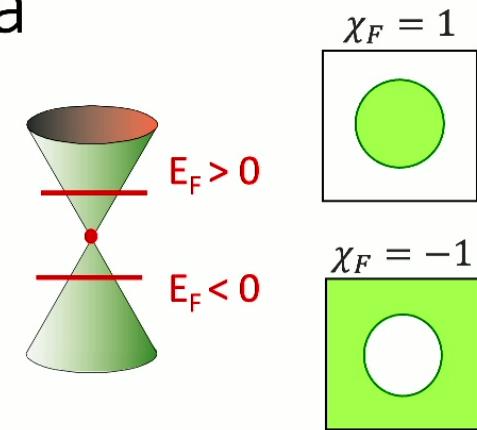
Topology of Fermi sea

Metal exhibits an **intrinsic topology** associated to the shape of its **Fermi sea**

$$\Rightarrow \chi_F$$

What are the consequences of this type of topology?

Any **quantized** physical quantities (response, correlation, etc.) distinguish metals with topologically distinct Fermi sea?



Topology of Fermi sea

Metal exhibits an **intrinsic topology** associated to the shape of its **Fermi sea**

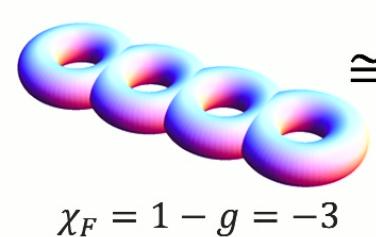
$$\Rightarrow \chi_F$$

What are the consequences of this type of topology?

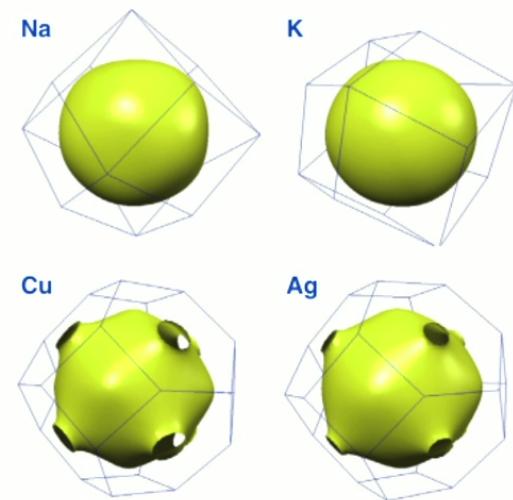
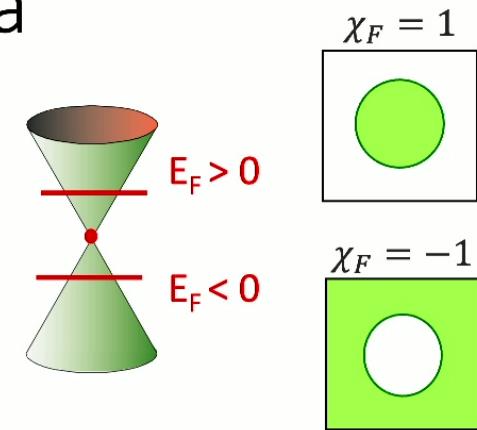
Any **quantized** physical quantities (response, correlation, etc.) distinguish metals with topologically distinct Fermi sea?



How to count holes in the Fermi sea without diving in?



$$\chi_F = 1 - g = -3$$

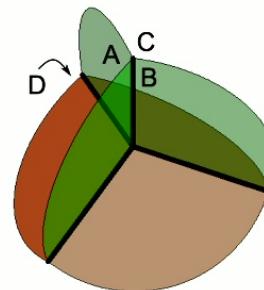


Probing the Fermi Sea Topology

1. For general D-dimension:

- (D+1)-point density correlation
- Multipartite number correlation
- Multipartite entanglement: mutual information

Tam, Claassen, and Kane, Phys. Rev. X 12, 031022

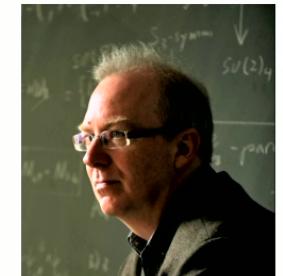
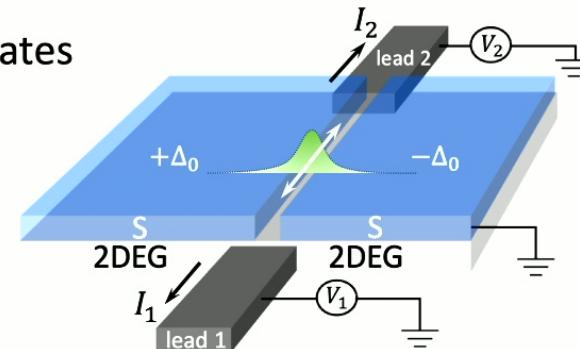


Martin Claassen

2. For D=2:

- Quantized transport by Andreev states
- Topological rectification effect

Tam and Kane, arXiv:2210.08048



Charles Kane

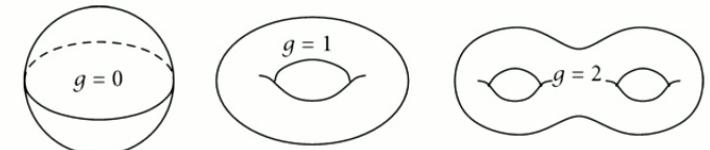
Euler characteristic

- For surfaces of convex polyhedron (Euler, 1758):

Tetrahedron V=4, E=6, F=4	Cube or hexahedron V=8, E=12, F=6	Octahedron V=6, E=12, F=8	Dodecahedron V=12, E=30, F=20	Icosahedron V=20, E=30, F=12
				

$$V - E + F = 2$$

generally, $(2 - 2g)$



- Modern definition for topological space in arbitrary dimension, in terms of *Betti numbers* (b_ℓ):

$$\chi = \sum_{\ell=0}^D (-1)^\ell b_\ell$$

b_0 : number of connected components
 $b_{\ell>0}$: number of ℓ -dimensional holes

Topological space of interest here is the Fermi sea $\Rightarrow \chi_F$

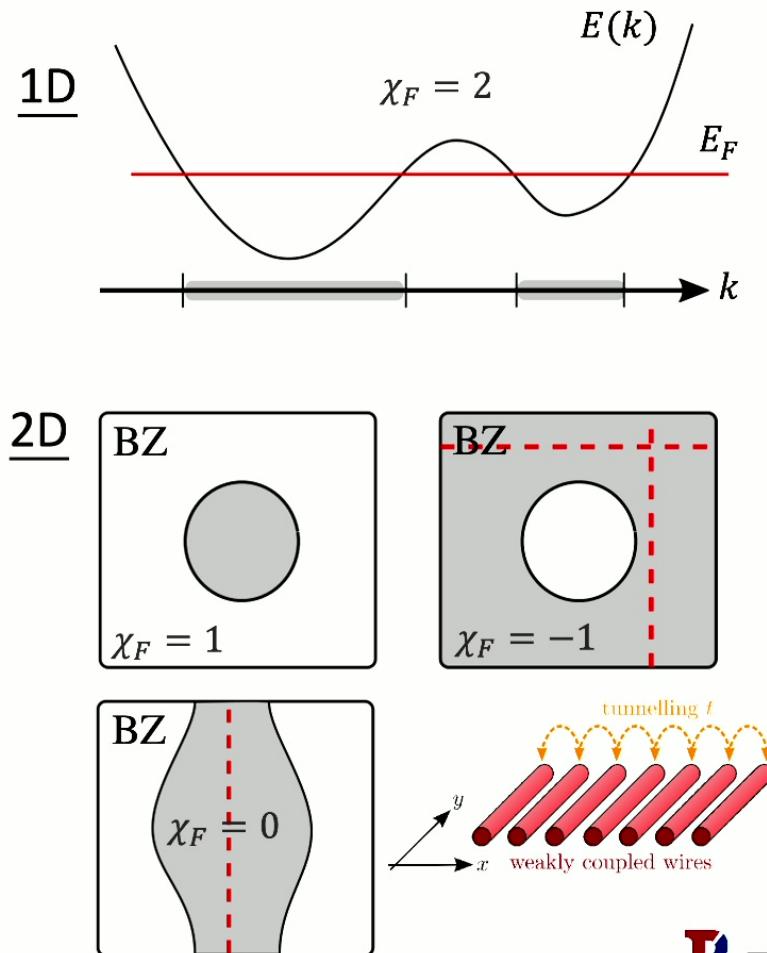
Examples

- 1D metal: $\chi_F = b_0$ counts the number of connected components of Fermi sea
- 2D metal: $\chi_F = b_0 - b_1$
 - Electron-pocket contributes $\chi_F = 1 - 0 = +1$
 - Hole-pocket contributes $\chi_F = 1 - 2 = -1$
 - $\chi_F = \# \text{ of e-like pockets} - \# \text{ of h-like pockets}$
- Odd vs even dimensions:

$$\chi_{\partial F} = \begin{cases} 2\chi_F & D \text{ odd} \\ 0 & D \text{ even.} \end{cases}$$

- 3D metal with Fermi surface of genus g :

$$\chi_{\partial F} = 2 - 2g \Rightarrow \chi_F = 1 - g$$



How to measure χ_F ?

- Obvious way: map out the geometry of Fermi sea using quantum oscillation experiments, e.g. de Haas-van Alphen effect. The topology is then fixed by the geometry.
Shoenberg (1962); Aschroft and Mermin (1976)
- But can one “*count holes in the Fermi sea without diving in*”? Can one probe the topology of Fermi sea in **momentum**-space through physical quantities constructed in the **real**-space?
- **Entanglement entropy** has proved useful in classifying phases of matter
 - Calabrese-Cardy formula for D=1 gapless systems
 - Levin-Wen/Kitaev-Preskill formula for D=2 gapped topological orders

Bipartite Entanglement Entropy

- von Neumann entropy:

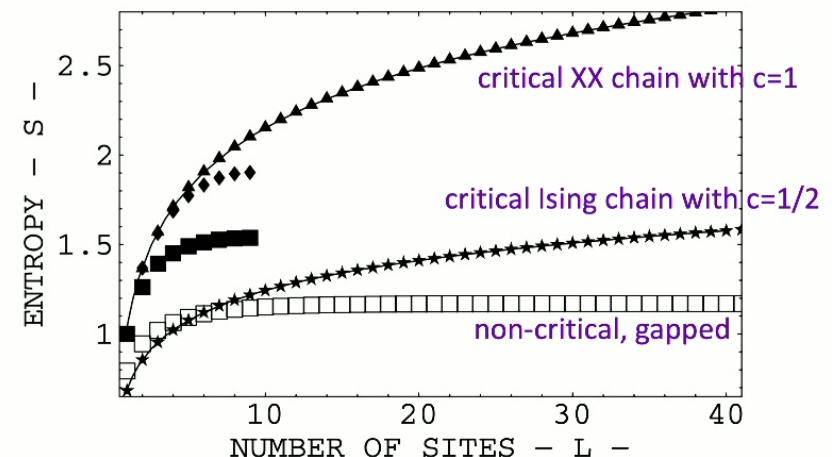
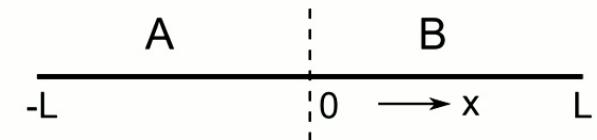
$$\rho_A = \text{Tr}_B[\rho_{AB}] \Rightarrow S_A = -\text{Tr}_A[\rho_A \log \rho_A]$$

- Universal finite-size scaling for a 1+1D critical phase:

$$S_A(L) = \frac{c}{6} \log \frac{L}{a} + \mathcal{O}(L^0)$$

Holzhey, Larsen, and Wilczek, Nucl. Phys. B (1994);
Calabrese and Cardy, J. Stat. Mech. (2004)

- Central charge c appears as a universal coefficient of the log-divergence (i.e. independent of cut-off “ a ”)



Vidal, Latorre, Rico and Kitaev, PRL (2003)

Bipartite Entanglement Entropy

- von Neumann entropy:

$$\rho_A = \text{Tr}_B[\rho_{AB}] \Rightarrow S_A = -\text{Tr}_A[\rho_A \log \rho_A]$$

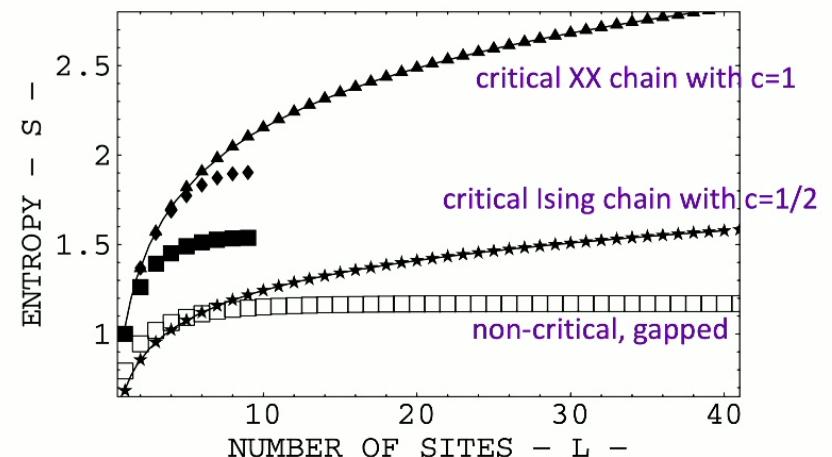
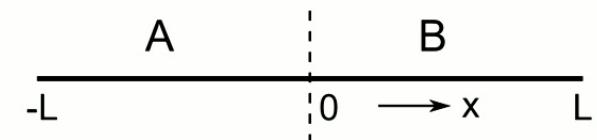
- Universal finite-size scaling for a 1+1D critical phase:

$$S_A(L) = \frac{c}{6} \log \frac{L}{a} + \mathcal{O}(L^0)$$

Holzhey, Larsen, and Wilczek, Nucl. Phys. B (1994);
Calabrese and Cardy, J. Stat. Mech. (2004)

- Central charge c appears as a universal coefficient of the log-divergence (i.e. independent of cut-off “ a ”)

For 1D fermions: $c = \chi_F$



Vidal, Latorre, Rico and Kitaev, PRL (2003)

Topological Entanglement Entropy for D=2

- For a 2D topological order:

$$S_A = \underbrace{\alpha L_A}_{\text{boundary-law}} - \log \mathcal{D} + \dots$$

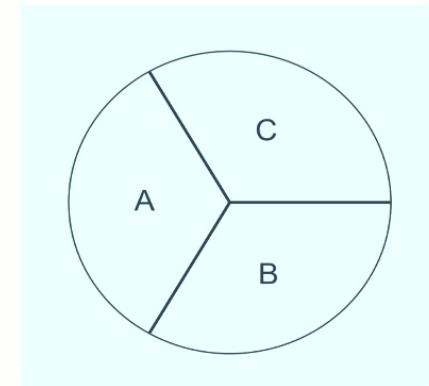
topological
 (total quantum dimension)

Kitaev and Preskill, PRL (2006)
 Levin and Wen, PRL (2006)

- The *tri-partite mutual information* isolates the topological piece:

$$\begin{aligned} I_3 &= S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} \\ &= -\log \mathcal{D} \end{aligned}$$

- Non-universal boundary terms are completely canceled.



Topology of Fermi sea can be extracted in similar manner

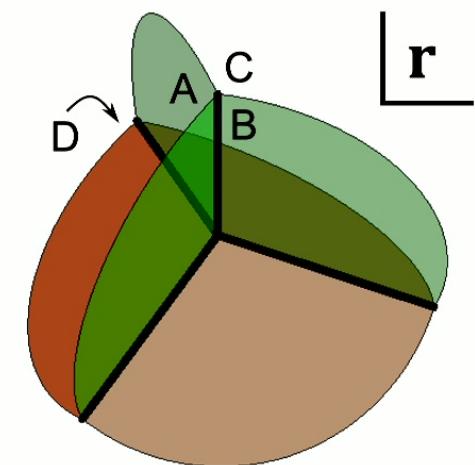
Topological Mutual Information

- The topology of **D-dimensional Fermi sea** (χ_F) is encoded in the leading logarithmic divergence of the **($D + 1$)-partite mutual information**
- For $D = 3$, using 4-partite mutual information:

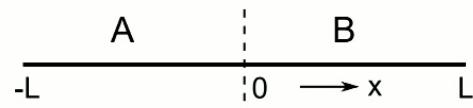
$$\begin{aligned} I_4 &= S_A + S_B + S_C + S_D \\ &\quad - S_{AB} - S_{AC} - S_{AD} - S_{BC} - S_{BD} - S_{CD} \\ &\quad + S_{ABC} + S_{ABD} + S_{ACD} + S_{BCD} - S_{ABCD} \end{aligned}$$

$$I_4 = \frac{\chi_F}{5\pi^2} \log^3 \frac{L}{a}$$

(3D Fermi sea)



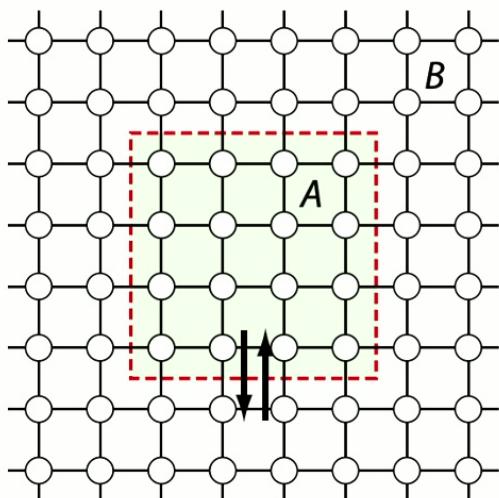
(insensitive to geometry*)



$$I_2 = S_A + S_B - S_{AB} = \frac{\chi_F}{3} \log \frac{L}{a}$$

(1D Fermi sea)

Entanglement and fluctuation



Song *et al.*, PRB (2012)
Klich and Levitov, PRL (2009)

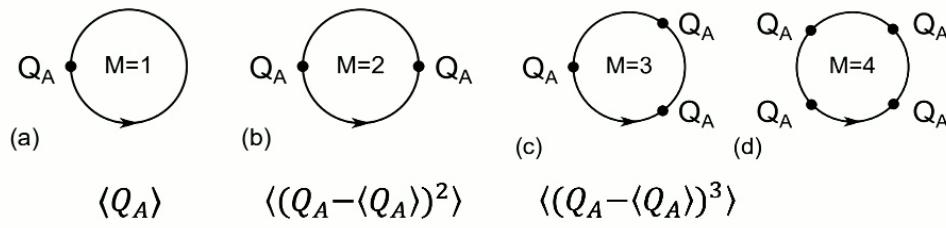
- For non-interacting fermions, bipartite EE is completely determined by the fluctuation of particle number (charge) and its higher-order cumulants:

$$S_{A,n} = \text{Tr}_A[\rho_A^n] \\ = \prod_p \langle e^{\frac{2\pi i p}{n} Q_{A,p}} \rangle$$

$$S_A = \sum_{l=1}^{\infty} 2\zeta_{2l} \langle Q_A^{2l} \rangle_c.$$

$$\zeta_2 = \pi^2/6 \\ \zeta_4 = \pi^4/90$$

- Cumulants (connected correlation function) of charge $\langle Q_A^M \rangle_c$:



Universal Density Correlations

$$\langle Q^M \rangle_c \xleftarrow{\int_r} \langle \rho(\mathbf{r}_1) \rho(\mathbf{r}_2) \dots \rho(\mathbf{r}_M) \rangle_c \xleftarrow{F.T.} \langle \rho(\mathbf{q}_1) \rho(\mathbf{q}_2) \dots \rho(\mathbf{q}_M) \rangle_c$$

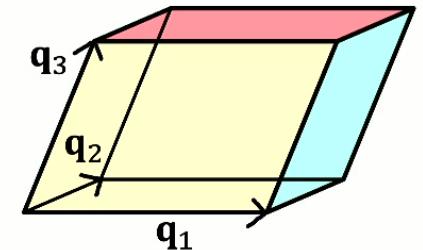
Translation invariance requires $\mathbf{q}_1 + \mathbf{q}_2 + \dots + \mathbf{q}_M = 0$

$$\rho(\mathbf{q}) = \int d^D \mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \rho(\mathbf{r}) = \int \frac{d^D \mathbf{k}}{(2\pi)^D} c_{\mathbf{k}}^\dagger c_{\mathbf{k}+\mathbf{q}}$$

In spatial dimension D , for small \mathbf{q} (long wavelength limit), we find:

$$\int_{\mathbf{q}_{D+1}} \langle \rho(\mathbf{q}_1) \rho(\mathbf{q}_2) \dots \rho(\mathbf{q}_{D+1}) \rangle_c = \chi_F \cdot |\det \mathcal{Q}|$$

where $\mathcal{Q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_D)$



$$D = 3: |(\mathbf{q}_1 \times \mathbf{q}_2) \cdot \mathbf{q}_3|$$

Universal Density Correlations

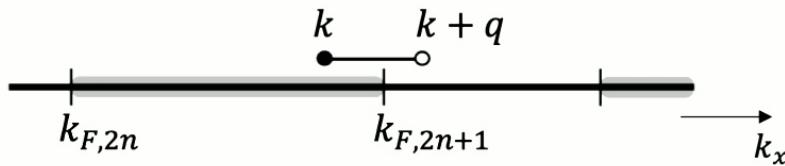
$$\int_{\mathbf{q}_{D+1}} \langle \rho(\mathbf{q}_1) \rho(\mathbf{q}_2) \dots \rho(\mathbf{q}_{D+1}) \rangle_c = \chi_F \cdot |\det \mathcal{Q}|$$

D = 1

$$\int dk f_k \bar{f}_{k+q} = \chi_F |q|$$

$$f_k = \theta(E_F - E_k)$$

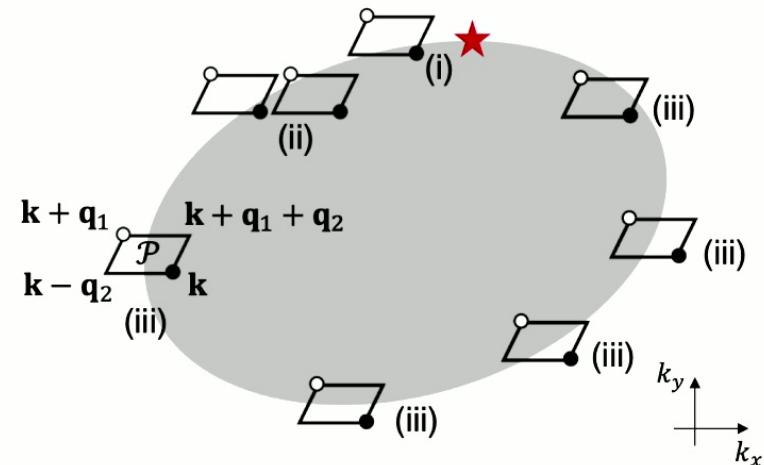
$$\bar{f}_k \equiv 1 - f_k$$



$$\chi_F = b_0 \text{ (number of components of Fermi sea)}$$

D = 2

$$\int d^2\mathbf{k} f_{\mathbf{k}} \bar{f}_{\mathbf{k}+\mathbf{q}_1} (\bar{f}_{\mathbf{k}+\mathbf{q}_1+\mathbf{q}_2} - f_{\mathbf{k}-\mathbf{q}_2}) = \chi_F |\mathbf{q}_1 \times \mathbf{q}_2|$$

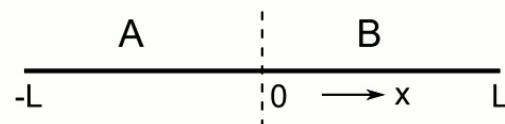


$$\chi_F = \# \text{ of e-like pockets} - \# \text{ of h-like pockets}$$

Multipartite Number Correlations

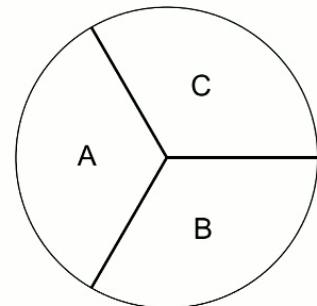
1D metal:

$$\langle Q_A Q_B \rangle_c = -\frac{\chi_F}{2\pi^2} \log \Lambda$$



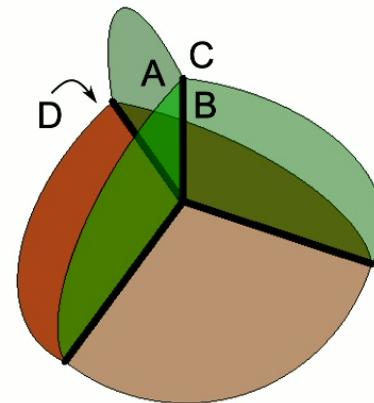
2D metal:

$$\langle Q_A Q_B Q_C \rangle_c = \frac{3\chi_F}{8\pi^4} \log^2 \Lambda$$



3D metal:

$$\langle Q_A Q_B Q_C Q_D \rangle_c = -\frac{3\chi_F}{8\pi^6} \log^3 \Lambda$$

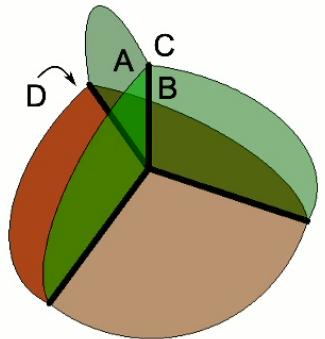


$$(\Lambda \equiv \frac{L}{a})$$

General D:

$$\langle Q_{A_1} \dots Q_{A_{D+1}} \rangle_c = \chi_F \cdot (-1)^D \frac{(D+1)!}{(2\pi)^{2D}} \log^D \Lambda$$

Mutual Information in D=3



$$\begin{aligned} \mathcal{I}_4 = & S_A + S_B + S_C + S_D - S_{AB} - S_{AC} - S_{AD} \\ & - S_{BC} - S_{BD} - S_{CD} + S_{ABC} + S_{ABD} \\ & + S_{ACD} + S_{BCD} - S_{ABCD}. \end{aligned}$$

$$S_A = \sum_{l=1}^{\infty} 2\zeta_{2l} \langle Q_A^{2l} \rangle_c.$$



$$\mathcal{I}_4 = -48\zeta_4 \langle Q_A Q_B Q_C Q_D \rangle_c = \frac{\chi_F}{5\pi^2} \log^3 \Lambda$$

$$(\Lambda \equiv \frac{L}{a})$$

Topology of 3D Fermi sea is encoded in 4-partite entanglement

Charge-Weighted Entanglement

$$S_A = -\text{Tr}_A[\rho_A \log \rho_A] \rightarrow S_A^Q = -\text{Tr}[(Q_A - \langle Q_A \rangle) \rho_A \log \rho_A]$$

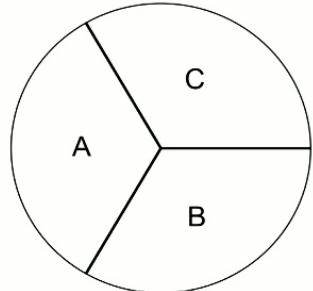
[Related to *charged EE*, c.f. Matsuura, Wen, Hung and Ryu, PRB (2016)]

$$S_A^Q = \sum_{l=1}^{\infty} 2\zeta_{2l} \langle Q_A^{2l+1} \rangle_c$$

Under particle-hole transformation: $S_A^Q \rightarrow -S_A^Q$

Tri-partite *charge-weighted* mutual information:

$$\mathcal{I}_3^Q = S_A^Q + S_B^Q + S_C^Q - S_{AB}^Q - S_{BC}^Q - S_{CA}^Q + S_{ABC}^Q = 12\zeta_2 \langle Q_A Q_B Q_C \rangle_c$$



$$\mathcal{I}_3^Q = \frac{3\chi_F}{4\pi^2} \log^2 \Lambda$$

Topology of 2D Fermi sea is encoded in 3-partite entanglement

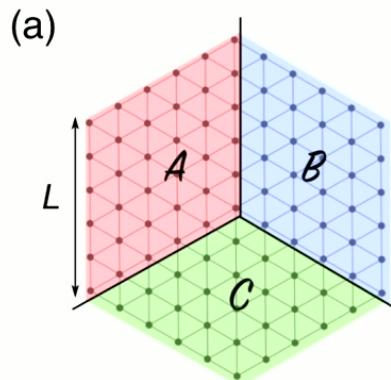
Numerics for D=2

- For free fermions, the correlation matrix $(C_A)_{ij} = \langle c_i^\dagger c_j \rangle$ determines the entanglement entropy:

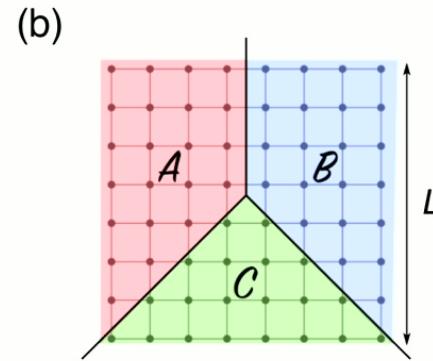
Chung and Peschel, PRB (2001)

$$S_A^Q = \text{Tr}[(1 - C_A)C_A \log(C_A^{-1} - 1)] \rightarrow \mathcal{I}_3^Q = S_A^Q + S_B^Q + S_C^Q - S_{AB}^Q - S_{BC}^Q - S_{CA}^Q + S_{ABC}^Q$$

- Two types of tight-binding models, two tri-partition geometries:

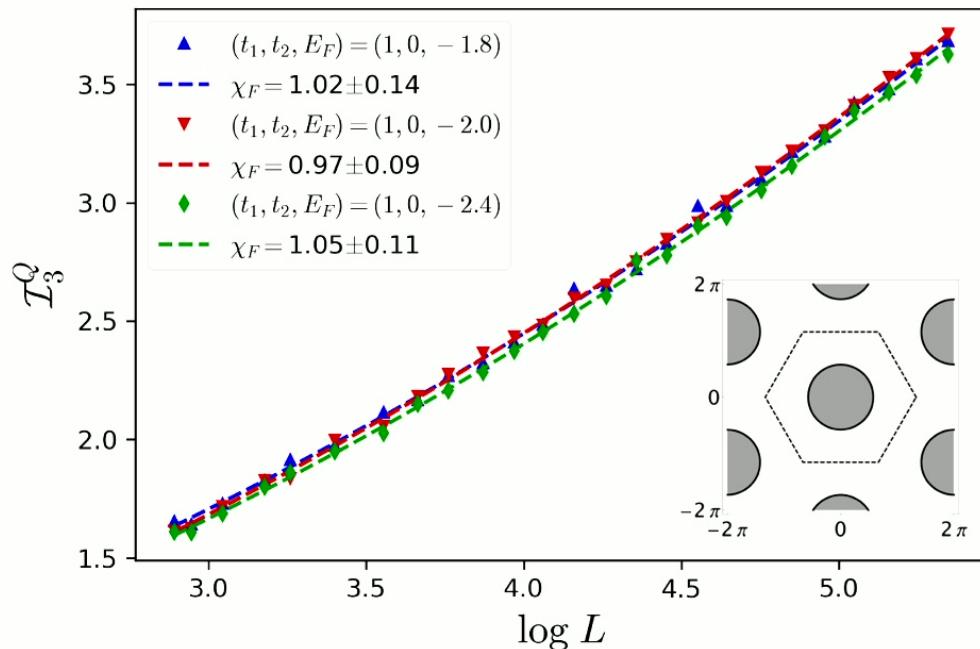
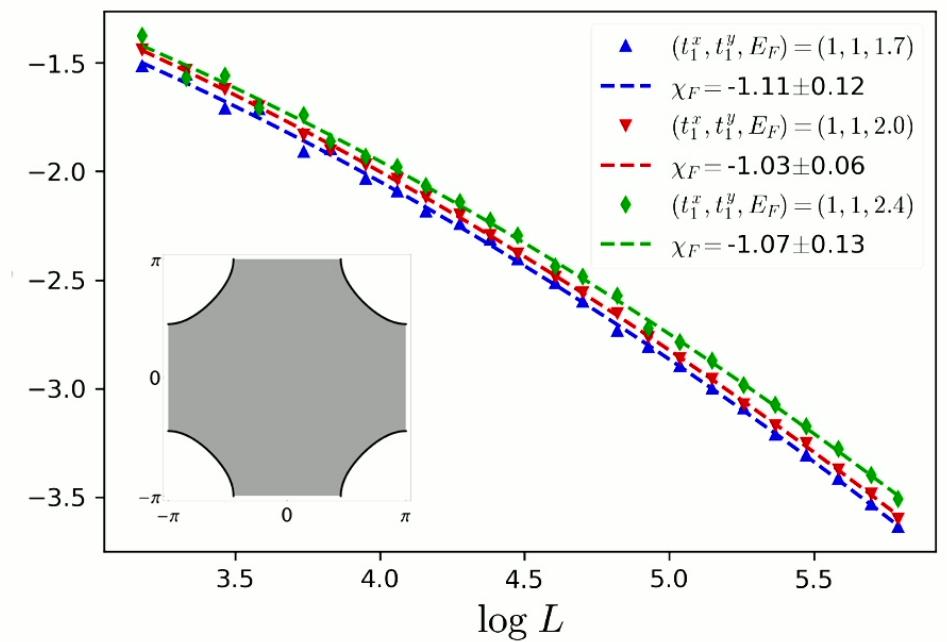


$$H_{\text{tri}} = -t_1 \sum_{\langle i,j \rangle} c_i^\dagger c_j - t_2 \sum_{\langle\langle i,j \rangle\rangle} c_i^\dagger c_j.$$



$$H_{\text{sq}} = - \sum_{\ell=1}^3 \sum_i (t_\ell^x c_i^\dagger c_{i+\ell\hat{x}} + t_\ell^y c_i^\dagger c_{i+\ell\hat{y}} + \text{H.c.}).$$

$$\mathcal{I}_3^Q = \frac{3\chi_F}{4\pi^2} \log^2\left(\frac{L}{a}\right) + \mathcal{O}(L^0)$$

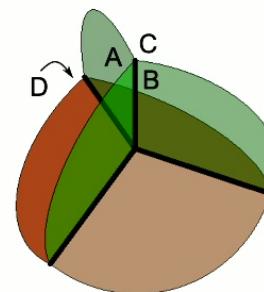
Triangular lattice with $\chi_F = 1$ Square lattice with $\chi_F = -1$ 

Probing the Fermi Sea Topology

1. For general D-dimension:

- (D+1)-point density correlation
- Multipartite number correlation
- Multipartite entanglement: mutual information

Tam, Claassen, and Kane, Phys. Rev. X 12, 031022

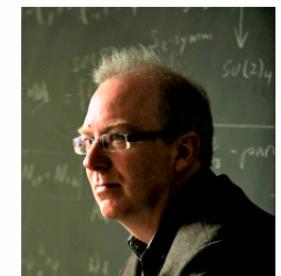
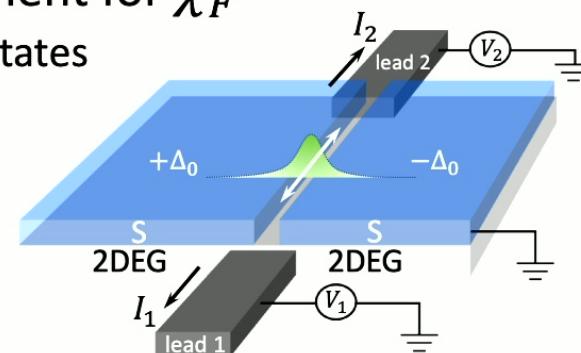


Martin Claassen

2. For D=2: accessible measurement for χ_F

- Quantized transport by Andreev states
- Topological rectification effect

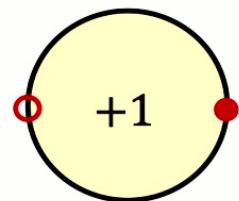
Tam and Kane, arXiv:2210.08048



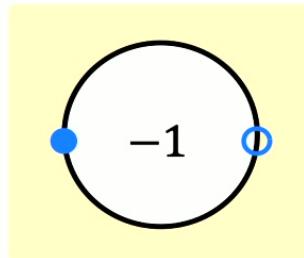
Charles Kane

Euler characteristic (revisited)

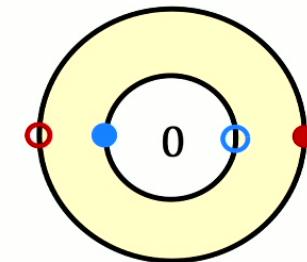
χ_F :



$$c_e = 1, \quad c_h = 0$$

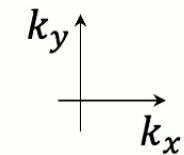


$$c_e = 0, \quad c_h = 1$$



$$c_e = 1, \quad c_h = 1$$

- : $v_x > 0$
- : $v_x < 0$

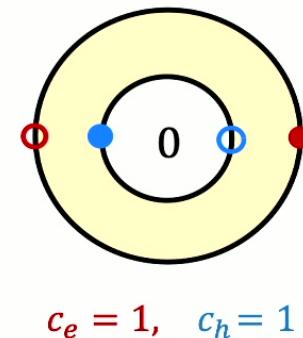
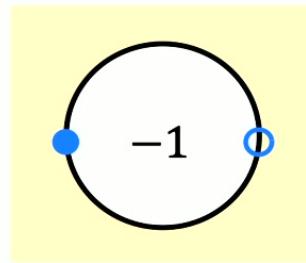
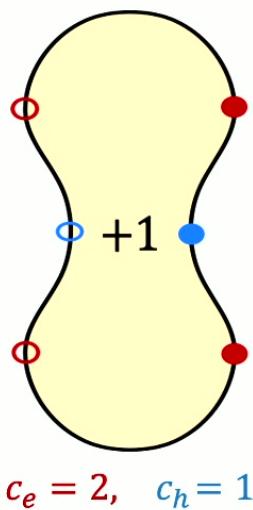


c_e : # of **convex** critical points, with $v_x > 0$
 c_h : # of **concave** critical points, with $v_x > 0$

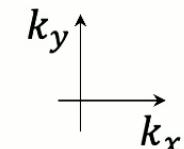
$$\chi_F = c_e - c_h$$

Euler characteristic (revisited)

χ_F :



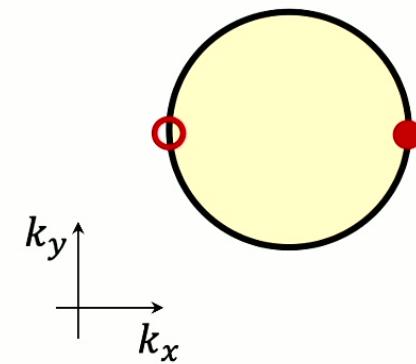
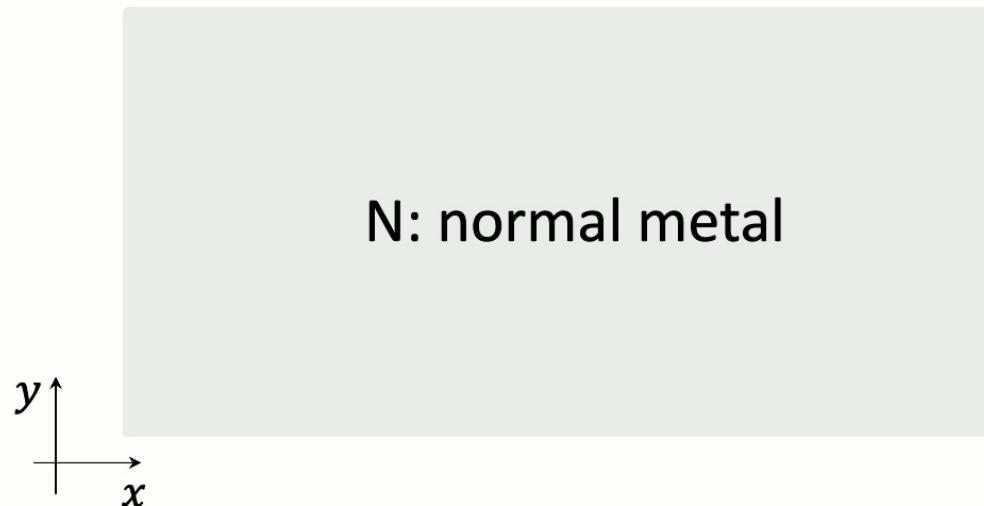
- : $v_x > 0$
- : $v_x < 0$



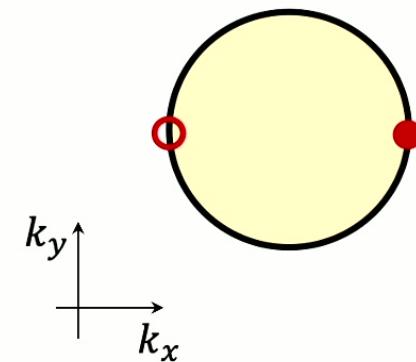
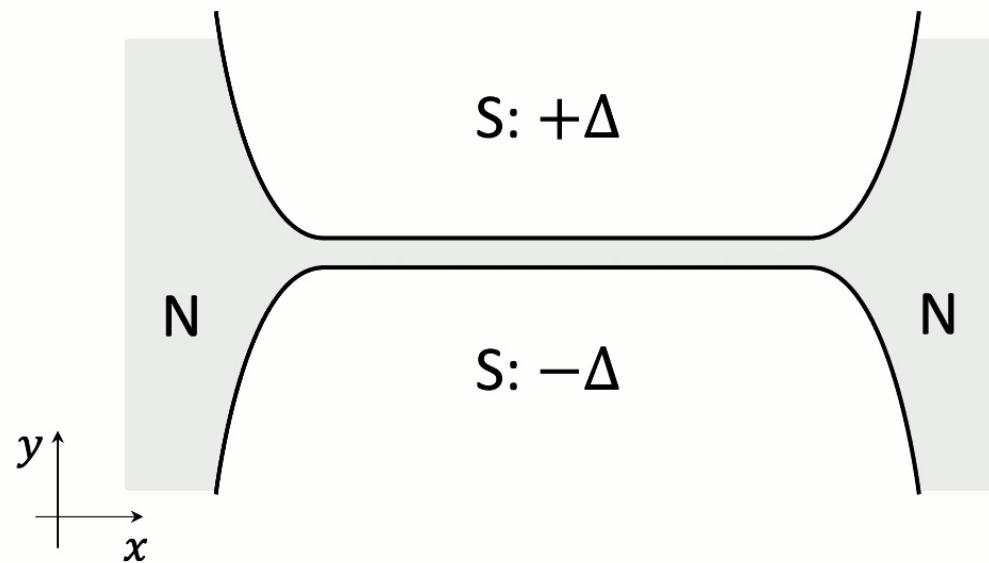
c_e : # of **convex** critical points, with $v_x > 0$
 c_h : # of **concave** critical points, with $v_x > 0$

$$\chi_F = c_e - c_h$$

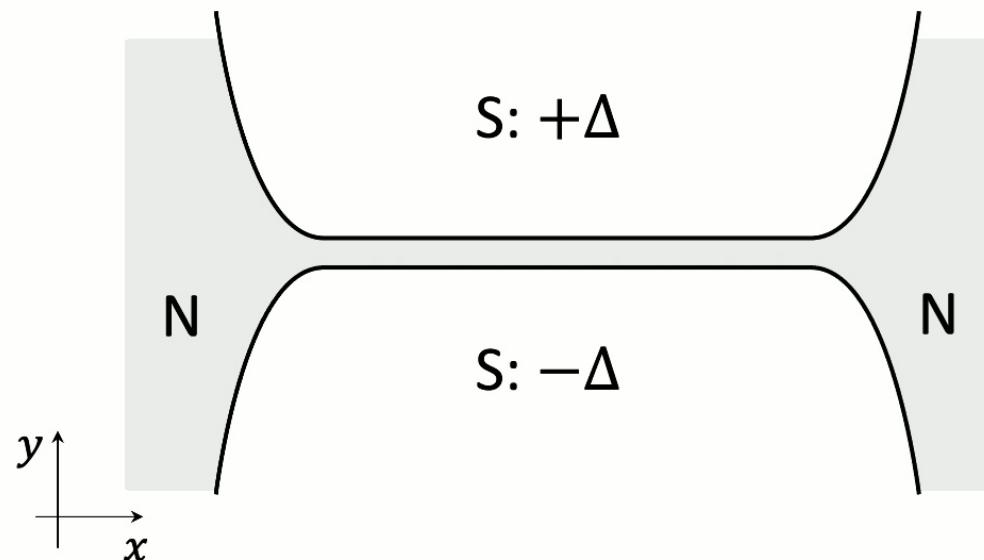
Andreev state in π -junction



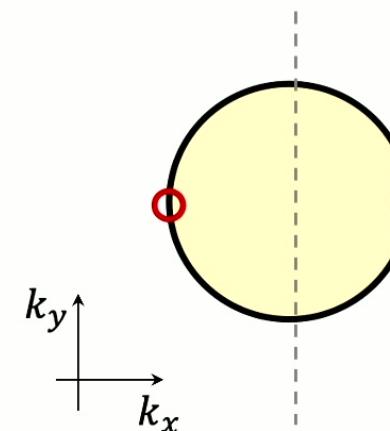
Andreev state in π -junction



Andreev state in π -junction

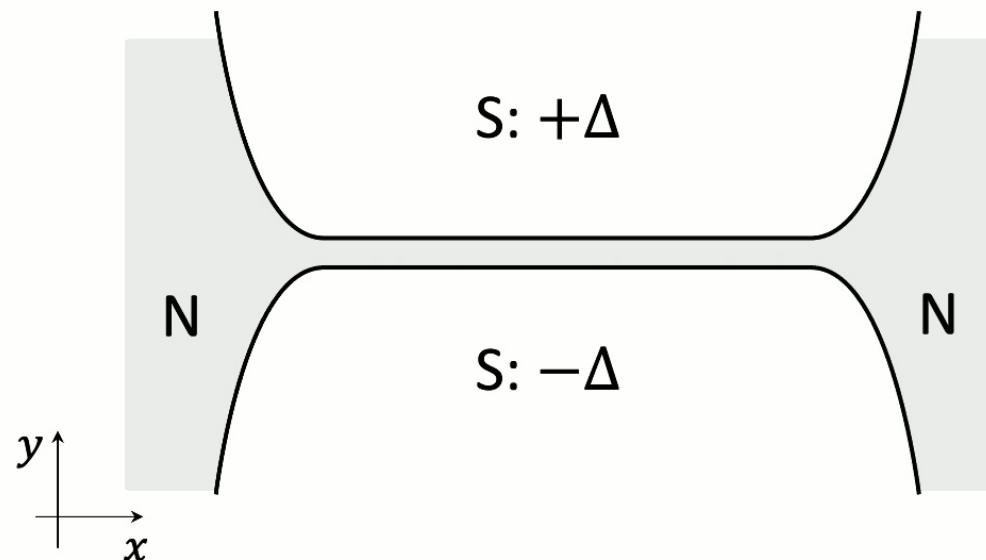


$$H_{\text{BdG}}(k_x) = v_y(k_x) \cdot (-i\partial_y) \cdot \tau_z \\ + \text{sgn}(y) |\Delta| \cdot \tau_y$$

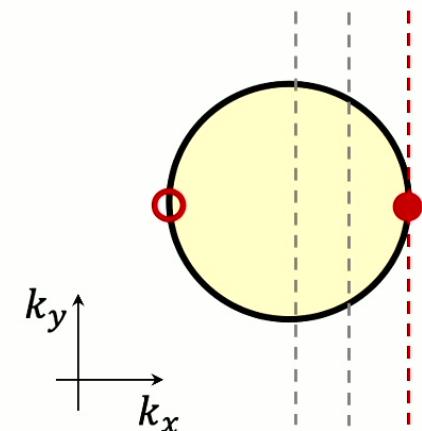


Andreev states are *zero-modes*

Andreev state in π -junction



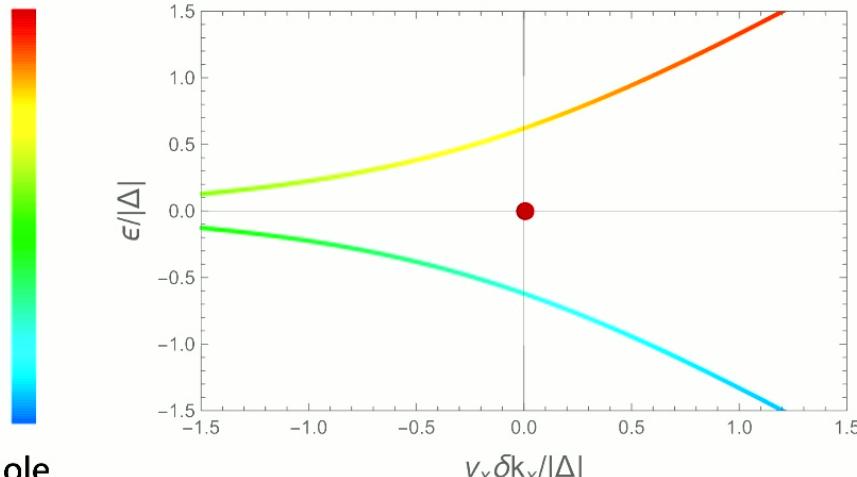
$$H_{\text{BdG}}(k_x) = \cancel{v_y(k_x) \cdot (-i\partial_y) \cdot \tau_z} + \text{sgn}(y) |\Delta| \cdot \tau_y$$



Andreev states are zero-modes (*away from critical points*)

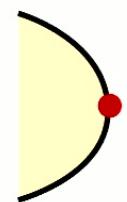
Dispersive Andreev states

particle

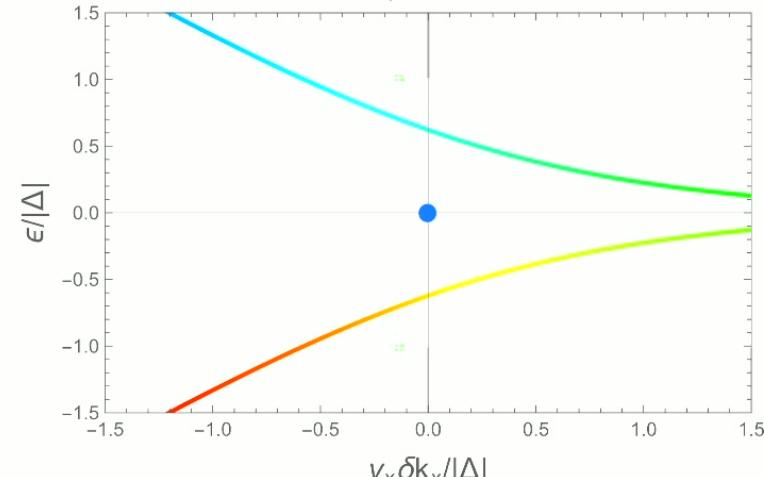


hole

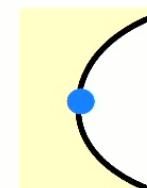
Electron-like:



$k_F W = 10.$

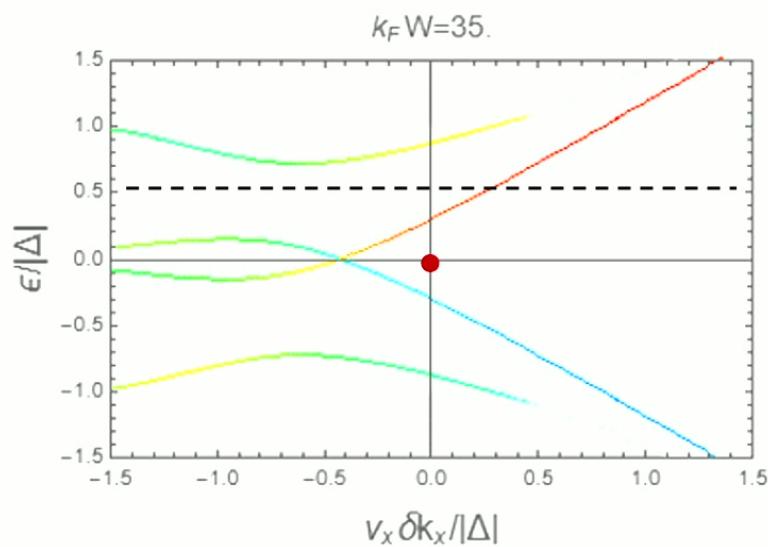


Hole-like:



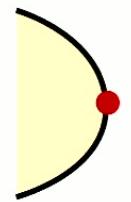
Dispersive Andreev states

particle

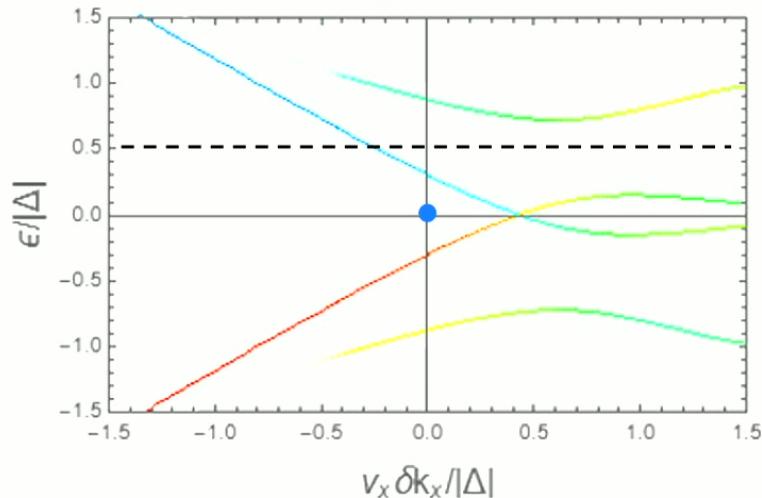


hole

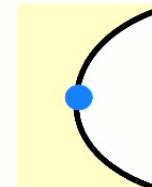
Electron-like:



$k_F W = 35.$

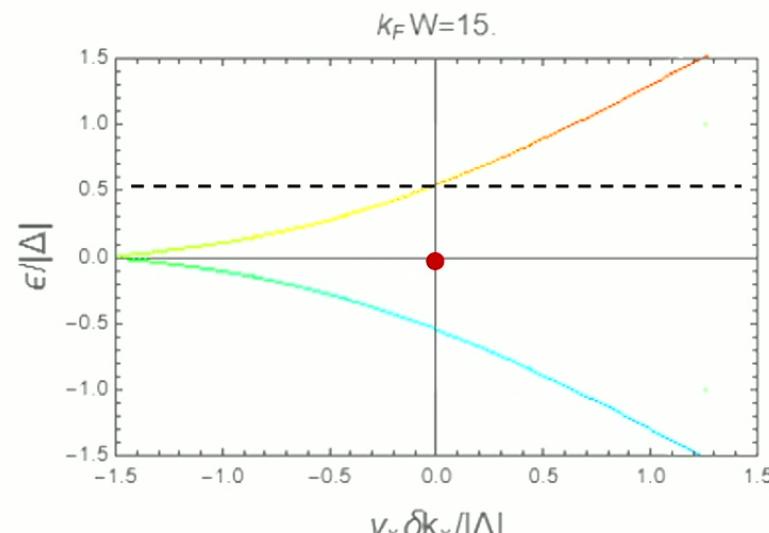
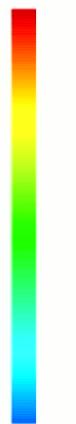


Hole-like:



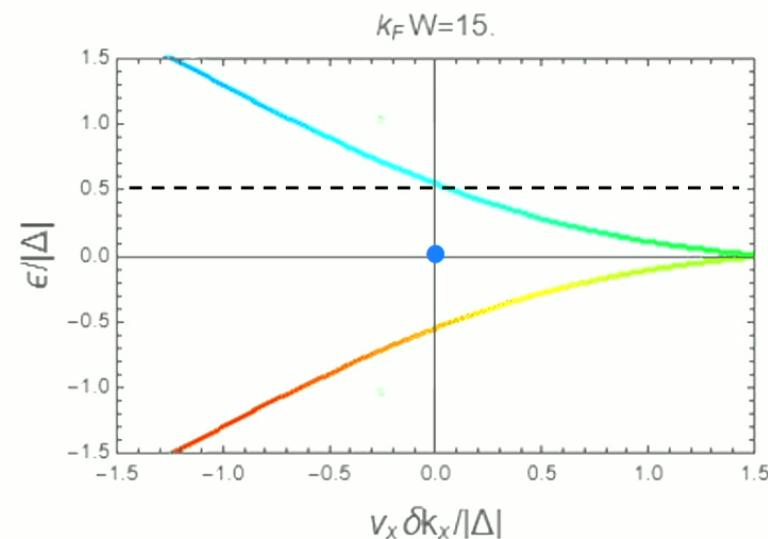
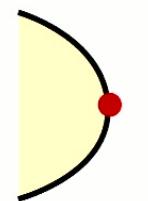
Dispersive Andreev states

particle

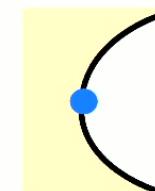


hole

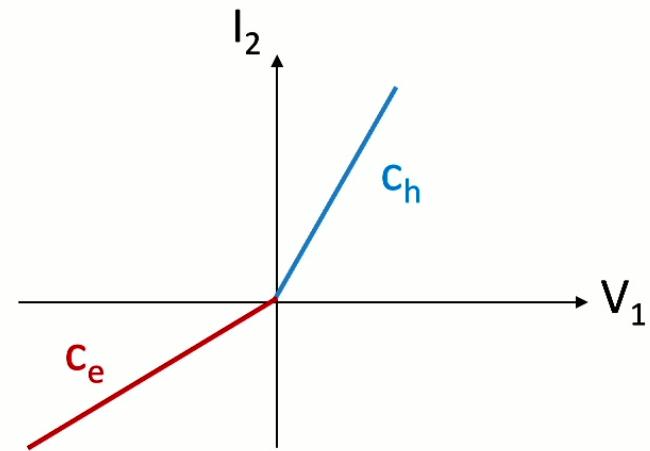
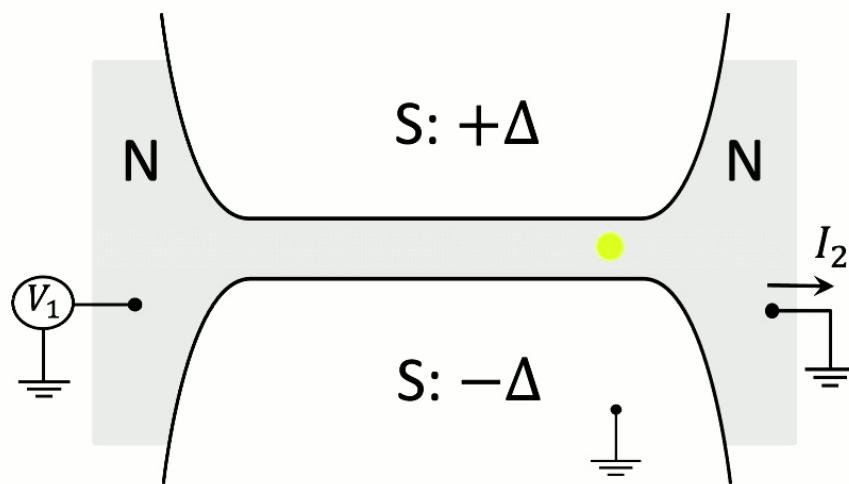
Electron-like:



Hole-like:



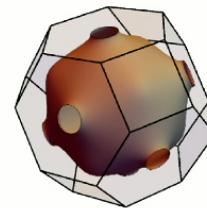
Quantized transport *along* junction



$$G_{21} \equiv \frac{dI_2}{dV_1} = \frac{2e^2}{h} [\textcolor{red}{c}_e \cdot \theta(-V_1) + \textcolor{blue}{c}_h \cdot \theta(V_1)]$$

$$\chi_F = \textcolor{red}{c}_e - \textcolor{blue}{c}_h$$

Conclusion

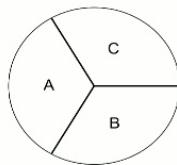


- Universal density/number correlations

$$\int_{\mathbf{q}_{D+1}} \langle \rho(\mathbf{q}_1)\rho(\mathbf{q}_2) \dots \rho(\mathbf{q}_{D+1}) \rangle_c = \chi_F \cdot |\det \mathcal{Q}|$$

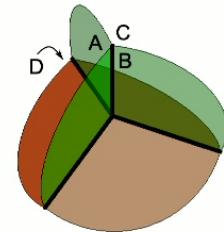
$$\langle Q_{A_1} \dots Q_{A_{D+1}} \rangle_c = \chi_F \cdot (-1)^D \frac{(D+1)!}{(2\pi)^{2D}} \log^D \Lambda$$

- Multipartite entanglement

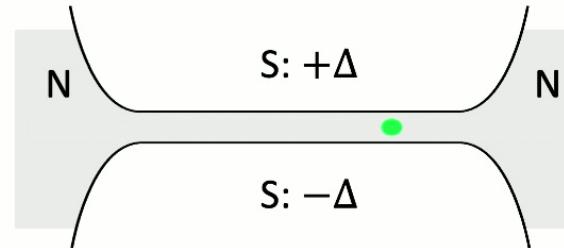


2D: $\mathcal{I}_3^Q = \frac{3\chi_F}{4\pi^2} \log^2 \Lambda$

3D: $\mathcal{I}_4 = \frac{\chi_F}{5\pi^2} \log^3 \Lambda$



- Quantized transport by Andreev state



$$\langle I \rangle_{d.c.} = \chi_F \cdot \frac{2e^2 V_{a.c.}}{\pi h}$$

- Other probes for χ_F ?

- Generalized Landauer conductance (nonlinear, 3-terminal for 2D), see Kane, PRL (2022).
- How about 3D ??
- Strong-correlation effects? Topological classification for *non-Fermi liquids*