Title: Statistical Physics - Lecture 221213

Speakers:

Collection: Statistical Physics (2022/2023)

Date: December 13, 2022 - 9:00 AM

URL: https://pirsa.org/22120011

Lorenz Model

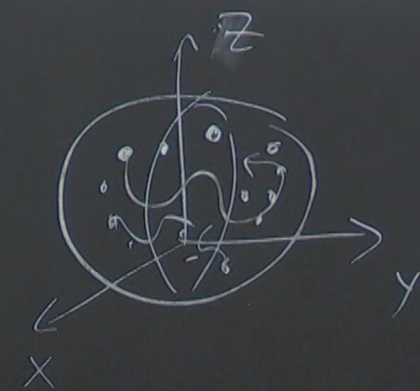
$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$

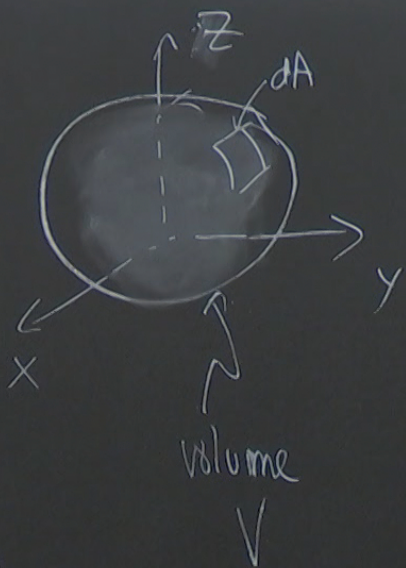
Basic properties

- coupled
- non-linear
- 3d
- dissipative



ies

, volume shrinking



flow into dA in time dt

$$\vec{u}\cdot\hat{n}\,dA\,dt$$

$$V(t+\Delta t) = V(t) + \int_{\substack{\text{area}\\ \text{of surface}}} \vec{u}\cdot\hat{n}\,dA\,\Delta t$$

$$\frac{dV}{dt} = \int_{\text{surface}} \vec{u}\cdot\hat{n}\,dA = \int_{\text{Vol}} \vec{\nabla}\cdot\vec{u}\,dV$$

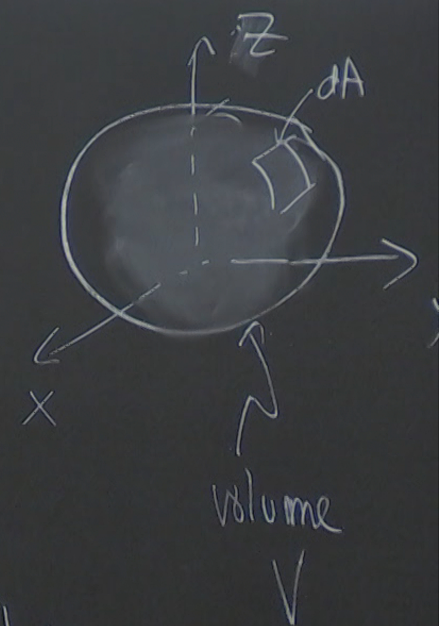
Lorenz Model

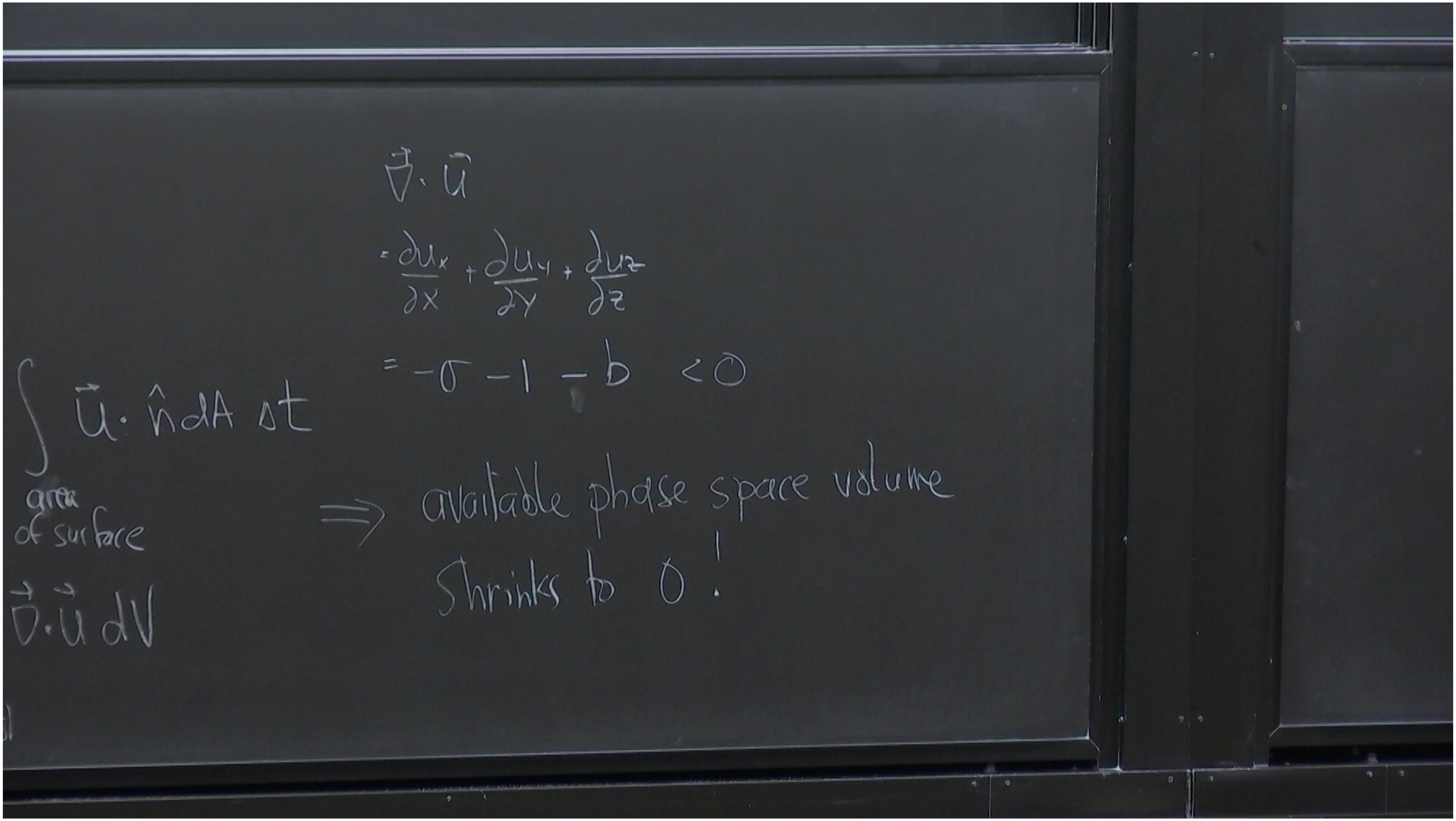
$$\begin{cases}\dot{x} = \sigma(y-x)\\ \dot{y} = rx - y - xz\\ \dot{z} = xy - bz\end{cases}$$

$$\equiv \dot{\vec{u}}$$

Basic properties

- coupled
- non-linear
- 3d
- dissipative, volume shrinking




$\vec{\nabla}\cdot\vec{u}$
$= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$
$= -\sigma - 1 - b < 0$
$\int \vec{u}\cdot\hat{n}\,dA\,\Delta t$
area of surface
$\Rightarrow$ available phase space volume
Shrinks to 0!
$\vec{\nabla}\cdot\vec{u}\,dV$

Fixed pts

(a) $x = y = z = 0$

(b) $C^{\pm}$: $x^* = y^*$

$\rightarrow \dot{y} = 0 \Rightarrow r x^* - x^* - x^* z^* = 0$

$\rightarrow z^* = (r-1)$

$\dot{z} = 0 \quad x^{*2} = b(r-1)$

$$\Rightarrow \boxed{\begin{array}{l} x^* = \pm\sqrt{b(r-1)} \\ z^* = r-1 \end{array}}$$

Stability at (0,0,0)

$$\begin{aligned} x &= \varepsilon_x \\ y &= \varepsilon_y \\ z &= \varepsilon_z \end{aligned} \quad \rightarrow \quad \begin{aligned} \dot{\varepsilon}_x &= \sigma(\varepsilon_y - \varepsilon_x) \\ \dot{\varepsilon}_y &= r\varepsilon_x - \varepsilon_y \\ \dot{\varepsilon}_z &= -b\varepsilon_z \end{aligned}$$

$$\dot{\vec{\varepsilon}} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \vec{\varepsilon}$$

$$\text{eigenvalues} \quad \overset{\lambda_1}{-b}, \quad \overset{\lambda_{2,3}}{\tfrac{1}{2}\left[-(1-\sigma) \pm \sqrt{(1+\sigma)^2 - 4\sigma(r-1)}\right]}$$

$\underline{r<1}$: all $\lambda_i < 0$ locally stable

in fact $(0,0,0)$ globally stable

Global Stability

$H \equiv$ Lyapunov function $= \dfrac{x^2}{\sigma} + y^2 + z^2$

show $\dot{H} < 0$

$$\frac{\dot{H}}{2} = \frac{x\dot{x}}{\sigma} + y\dot{y} + z\dot{z}$$

$$= xy - x^2 + rxy - y^2 - \cancel{xyz}$$

$$+ \cancel{xyz} - bz^2$$

$z\dot{z}$

$-y^2 - \cancel{xyz}$

$-bz^2$

$-x^2 - y^2 + xy + rxy - bz^2$

$-(x-y)^2 + xy + rxy - bz^2$

$-(x-y)^2 - bz^2 + xy(r-1) < 0$

$r < 1$

$\Rightarrow$ all points in state space flow to $(0,0,0)$

$r > 1$

$- bz^2$

bz^2

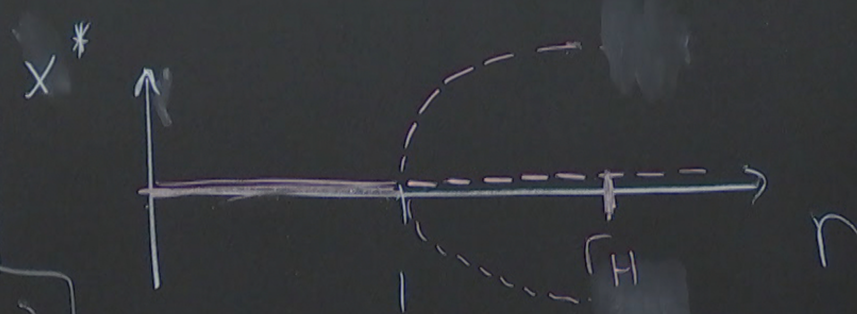
$r-1) < 0$

$r<1$

ate space flow to $(0,0,0)$

$\underline{r>1}$ 2 eigenvalues < 0

1 > 0



(b) $C^{\pm}$

linear stability

$$x = x^* + \varepsilon_x$$

$$y = y^* + \varepsilon_y$$

$$z = z^* + \varepsilon_z$$

$$\vec{\varepsilon} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z^* & -1 & x^* \\ x^* & x^* & -b \end{pmatrix} \vec{\varepsilon}$$

3 eigenvalues:

$r > r_H$ $\quad C^{\pm}$ unstable in all directions

all $\lambda_i > 0$

bility

nov function $= \frac{x^2}{\sigma} + y^2 + z^2$

$\dot{H} < 0$

$$\frac{\dot{H}}{2} = \frac{x\dot{x}}{\sigma} + y\dot{y} + z\dot{z}$$

$$= xy - x^2 + rxy - y^2 - \cancel{xyz}$$

$$+ \cancel{xyz} - bz^2$$

$-(x-y)^2 - bz^2$

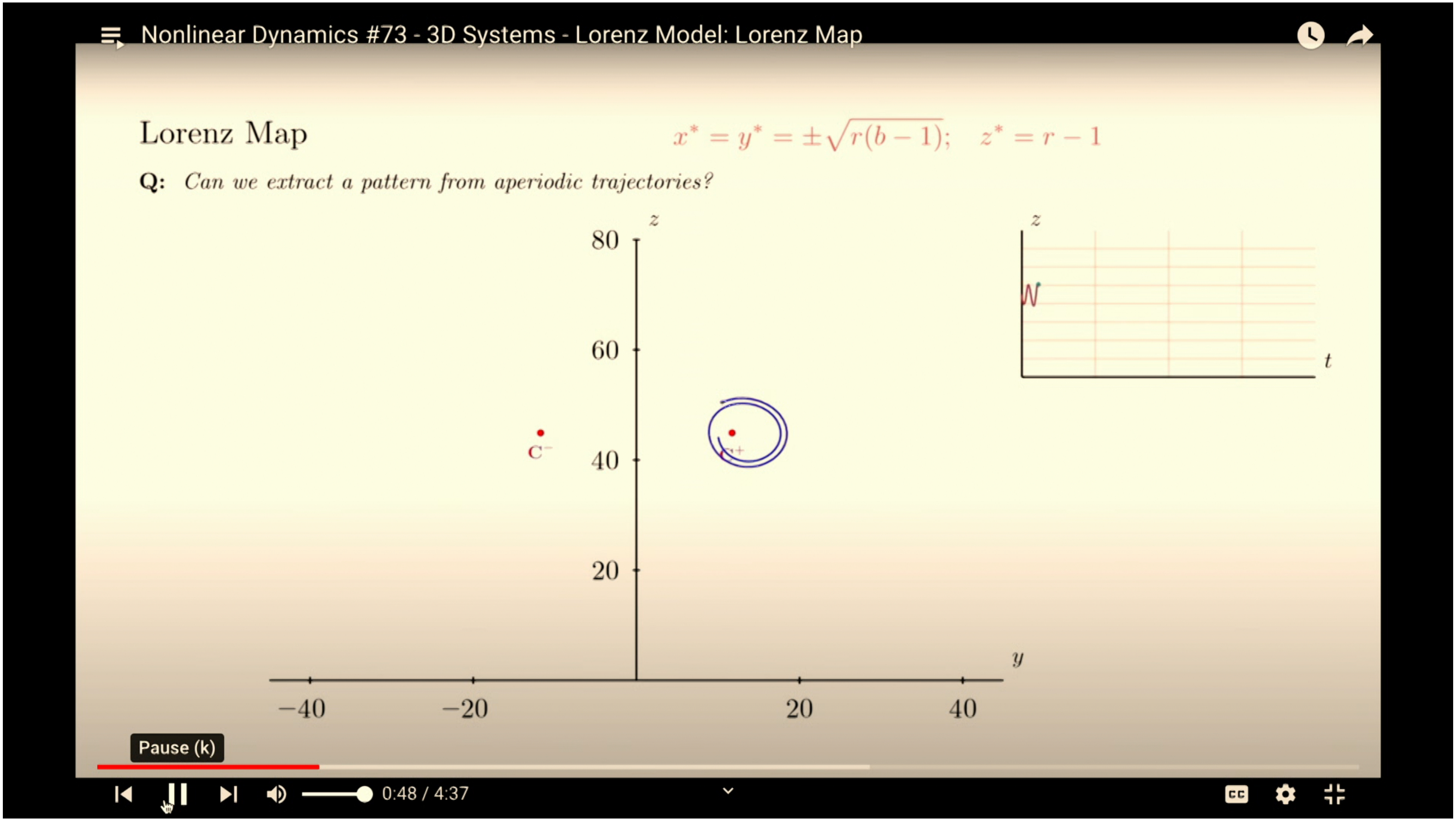
$- xy(r-1)$

$- x^2 - y^2 + xy +$

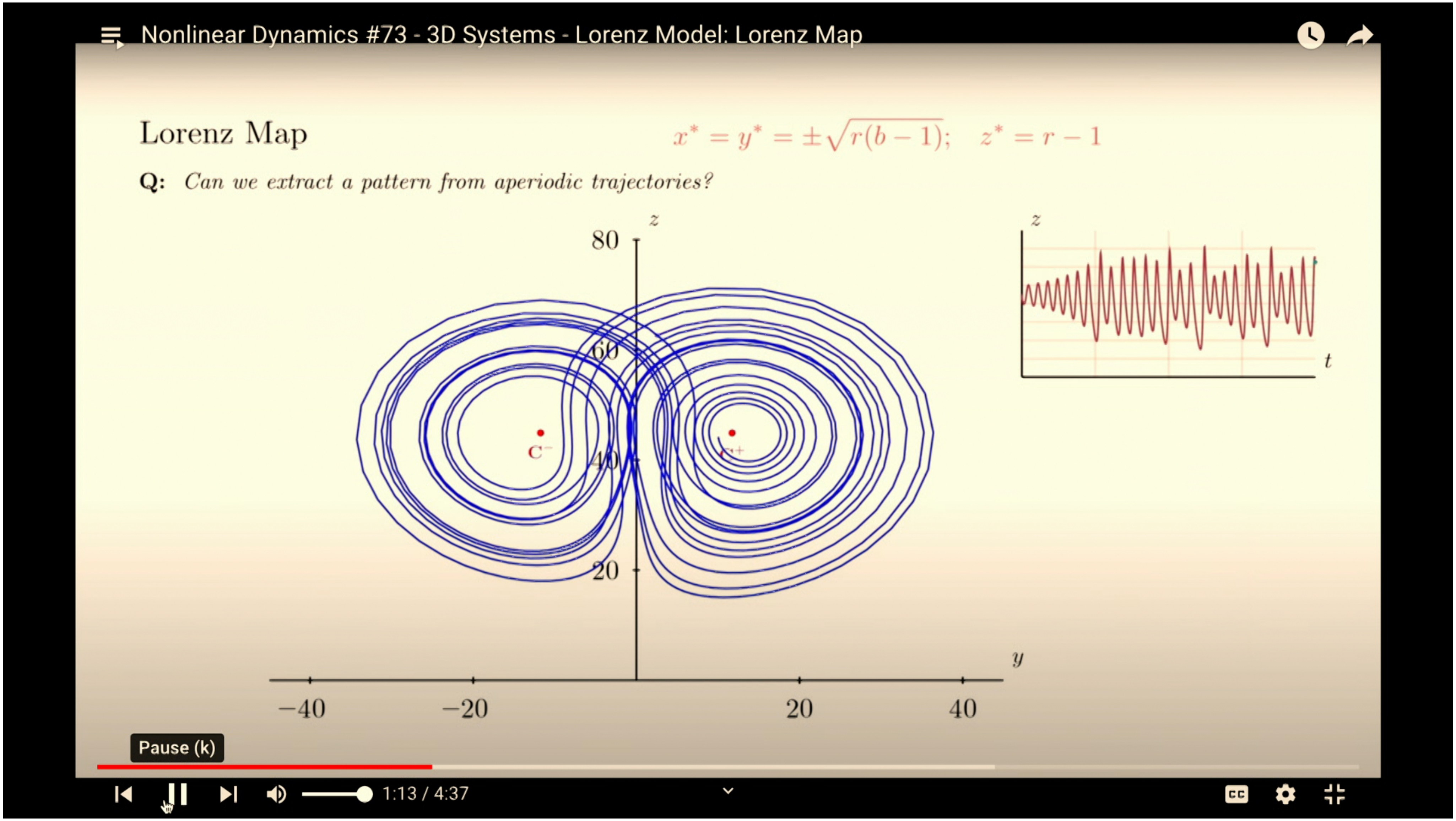
$-(x-y)^2 - xy$

$-(x-y)^2 - bz^2$

$\Rightarrow$ all point



Nonlinear Dynamics #73 - 3D Systems - Lorenz Model: Lorenz Map
Lorenz Map
$x^* = y^* = \pm\sqrt{r(b-1)}; \quad z^* = r - 1$
Q: Can we extract a pattern from aperiodic trajectories?
z
80
60
40
20
C⁻
y
−40
−20
20
40
z
t
Pause (k)
0:48 / 4:37



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t
Pause (k)
1:13 / 4:37

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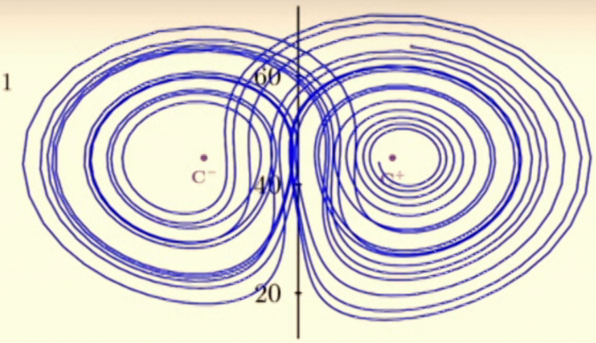


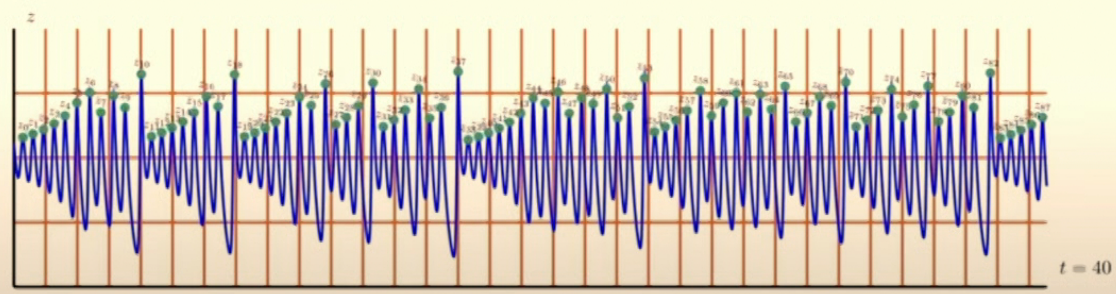
Q: *Can we extract a pattern from aperiodic trajectories?*

Observations: depart one spiral when $z_{\max} > z_T$

$z - z_T$ determines: entry point on next spiral

\# cycles before next switch

Idea: z_n determines z_{n+1}



Play (k)

2:54 / 4:37

Nonlinear Dynamics #73 - 3D Systems - Lorenz Model: Lorenz Map

Lorenz Map

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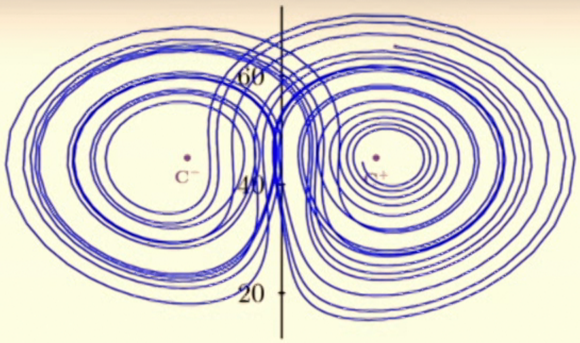

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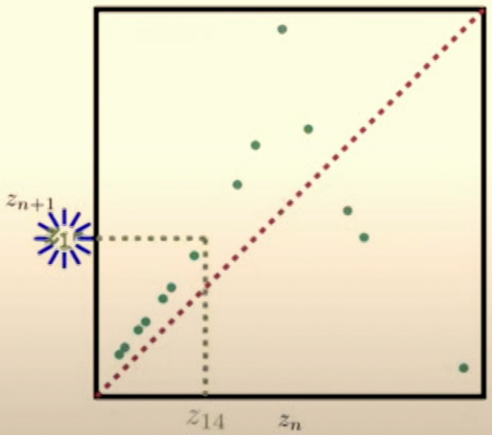
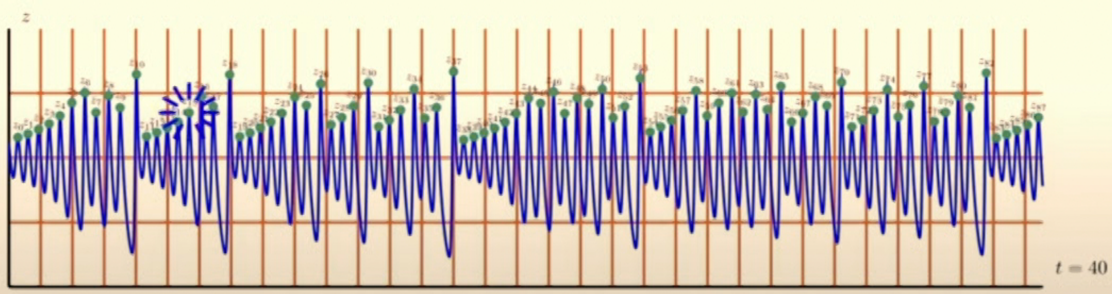




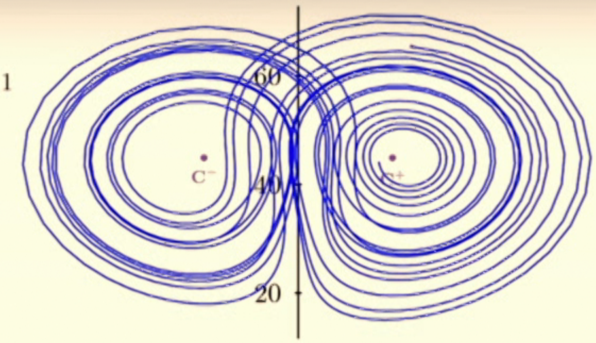
Pause (k)

3:31 / 4:37

Nonlinear Dynamics #73 - 3D Systems - Lorenz Model: Lorenz Map

Lorenz Map

$x^* = y^* = \pm\sqrt{r(b-1)};\ z^* = r - 1$



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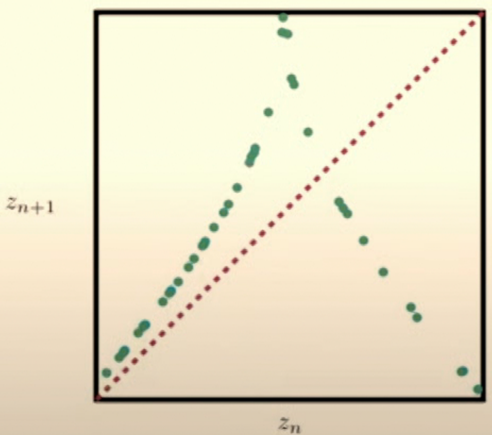
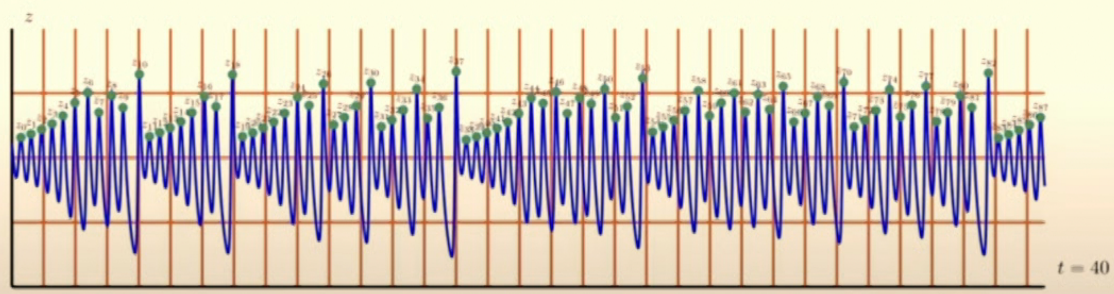
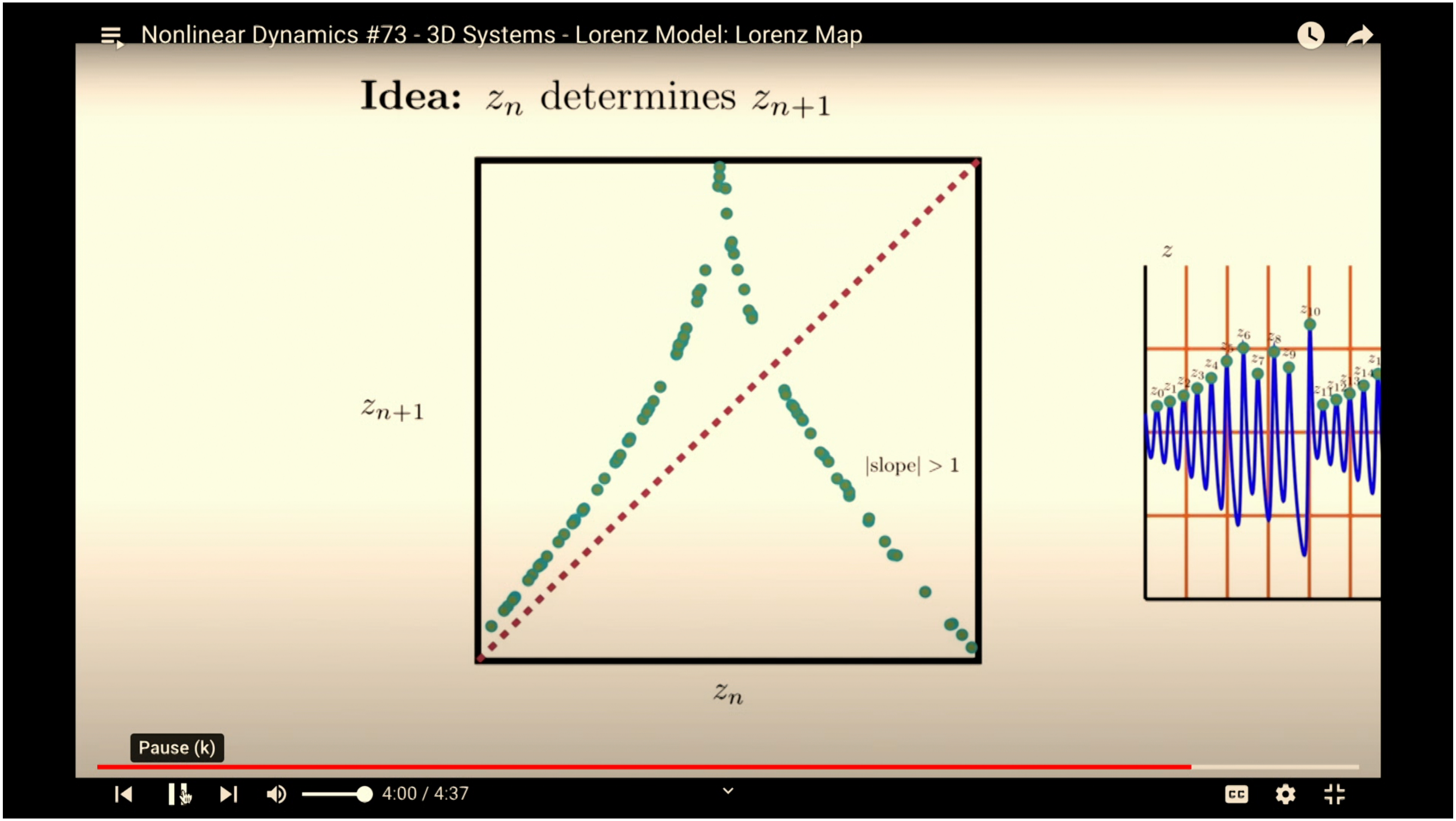




Pause (k)

3:40 / 4:37

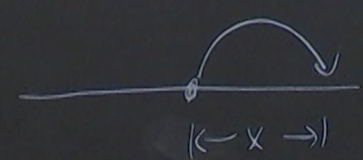

Nonlinear Dynamics #73 - 3D Systems - Lorenz Model: Lorenz Map
Idea: z_n determines z_{n+1}
z_{n+1}
$|\text{slope}| > 1$
z_n
z
Pause (k)
4:00 / 4:37

Central Limit Theorem

then $P_N(x)$

given single-step hopping prob.

$p(x)$



with $\underline{p(x)}$ iid variable

independent identically distributed

$$\langle x \rangle = 0 \; ; \; \langle x^2 \rangle \equiv \sigma^2 < \infty$$

then $P_N(x) \to$ Gaussian

$e^{ikx'} e^{ik(x-x')}$

Chapman Kolmogorov eqn

$$\int \left[P_N(x) = \int P_{N-1}(x')\, p(x-x')\, dx' \right] e^{ikx}\, dx \Rightarrow P_N(k) =$$

introduce Fourier transforms

$$p(k) = \int dx\, p(x)\, e^{ikx}$$

$$P_N(k) = \int dx\, P_N(x)\, e^{ikx}$$

y distributed

$^2 < \infty$

ssian

$$e^{ikx'}\, e^{ik(x-x')}$$
$$\uparrow$$

$$P_{N-1}(x')\; p(x-x')\; dx' \Big]\, e^{ikx}\, dx \;\Rightarrow\; P_N(k) = P_{N-1}(k)\, p(k)$$

$$= P_0(k)\,\big[p(k)\big]^N$$

$$p(k) = \int dx\; p(x)\, e^{ikx} = \int dx\, p(x)\Big[1 + ikx - \frac{k^2x^2}{2} - \frac{ik^3x^3}{3!} + \dots\Big] \;\to\; p(k)^N$$

$$P_N(k) = \int dx\; P_N(x)\, e^{ikx} = 1 - \frac{k^2}{2}\sigma^2 + O(k^3)$$

$$P_N(k) = [p(k)]^N \approx \left(1 - \frac{k^2\sigma^2}{2} + \cdots\right)^N$$

$$\approx e^{-Nk^2\sigma^2/2}$$

$$P_N(x) = \frac{1}{2\pi}\int dk\, e^{-ikx - N^2 k^2\sigma^2/2}$$

$P_N(k) = \int dx\, P_N(x)\, e \quad \cdot\; 1 - \frac{k}{2}\sigma + O(k^3) \quad 3!$

$$\frac{1}{2\pi}\int dk\; e^{-\left(ik\sqrt{\frac{N\sigma^2}{2}} - \frac{x}{\sqrt{2N\sigma^2}}\right)^2 - x^2/2N\sigma^2}$$

$$\sigma^2 = \langle x^2\rangle - \langle x\rangle^2$$

$$\frac{1}{2\pi}\, e^{-x^2/2N\sigma^2}\int dk\; e^{-k^2a^2 - b}$$

$$\boxed{P_N(x) = \frac{1}{\sqrt{2\pi N\sigma^2}}\, e^{-x^2/2N\sigma^2}}$$

$\rightarrow$ if $\langle x\rangle \neq 0$

$$P_N(x) = \frac{e^{-(x - N\langle x\rangle)^2/2N\sigma^2}}{\sqrt{2\pi N\sigma^2}}$$