

Title: Statistical Physics - Lecture 221212

Speakers:

Collection: Statistical Physics (2022/2023)

Date: December 12, 2022 - 9:00 AM

URL: <https://pirsa.org/22120010>

Complex Networks

A basic observable: degree distribution

$N_k \equiv \# \text{ nodes of degree } k$

A basic observable: degree distribution

$N_k \equiv \# \text{ nodes of degree } k$

$$\frac{dN_k}{dN} = p_{k-1} N_{k-1} - p_k N_k + \delta_{k,1}$$

↑
prob of attaching
to node of degree $k-1$



Complex Networks

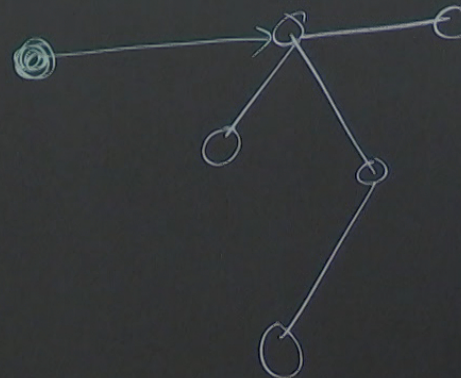
A basic observable: degree distribution

$N_k \equiv \# \text{ nodes of degree } k$

$$\frac{dN_k}{dN} = p_{k-1} N_{k-1} - p_k N_k + \delta_{k,1}$$

↑
prob of attaching
to node of degree $k-1$

growth process



Random Rec

Random Recursive Tree

$P_k = 1/N$ (Uniform Attachment)

$$\frac{dN_k}{dN} = \frac{N_{k-1}}{N} - \frac{N_k}{N} + \delta_{k,1}$$

$$\frac{dN_1}{dN} = -\frac{N_1}{N} + 1$$

Random Recursive Tree

$P_k = 1/N$ (Uniform Attachment)

$$\frac{dN_k}{dN} = \frac{N_{k-1}}{N} - \frac{N_k}{N} + \delta_{k,1}$$

$$\dot{N}_1 = -\frac{N_1}{N} + 1$$
$$(N\dot{N}_1 + N_1 = 1)$$

$$\frac{dN_1}{dN} = -\frac{N_1}{N} + 1$$

Random Recursive Tree

$$P_k = \frac{1}{N} \quad (\text{Uniform Attachment})$$

$$\frac{dN_k}{dN} = \frac{N_{k-1}}{N} - \frac{N_k}{N} + \delta_{k,1}$$

$$\frac{dN_1}{dN} = -\frac{N_1}{N} + 1$$

$$\dot{N}_2 = \frac{N_1}{N} - \frac{N_2}{N}$$

$$(N\dot{N}_2 + N_2) = N_1 = \frac{N}{2}$$

$$\rightarrow N_2 = N_1/4$$

Random Recursive Tree

$P_k = 1/N$ (Uniform Attachment)

$$\frac{dN_k}{dN} = \frac{N_{k-1}}{N} - \frac{N_k}{N} + \delta_k$$

$$\frac{dN_1}{dN} = -\frac{N_1}{N} + 1$$

$$\dot{N}_1 = -\frac{N_1}{N} + 1$$

$$N \dot{N}_1 + N_1 = N$$

$$(N \dot{N}_1) = N - N_1$$

$$N \dot{N}_1 = \frac{N^2}{2}$$

$$N_1 = N/2$$

$$\dot{N}_2 = \frac{N_1}{N} - \frac{N_2}{N}$$

$$(N \dot{N}_2 + N_2) = N_1 = \frac{N}{2}$$

$$\rightarrow N_2 = N_1/4$$

⋮

$$N_k = N/2^k$$

Recursive Tree

$$P_k = 1/N \quad (\text{Uniform Attachment})$$

$$\frac{1}{N} + \delta_{k,1} \leftarrow$$

$$\dot{N}_1 = -\frac{N_1}{N} + 1$$

$$N\dot{N}_1 + N_1 = N$$

$$(N\dot{N}_1) = N$$

$$NN_1 = \frac{N^2}{2}$$

$$N_1 = N/2$$

$$\dot{N}_2 = \frac{N_1}{N} - \frac{N_2}{N}$$

$$(N\dot{N}_2 + N_2) = N_1 = \frac{N}{2}$$

$$\rightarrow N_2 = N_1/4$$

⋮

$$N_k = N/2^k$$

$$\text{easier: } N_k = N n_k$$

$$\Rightarrow n_k = n_{k-1} - n_k + \delta_{k,1}$$

$$2n_1 = 1 \quad n_1 = 1/2$$

$$n_2 = n_1 - n_2 \rightarrow n_2 = 1/4$$

⋮

$$n_k = 2^{-k}$$

Preferential Attachment

$$P_k = \frac{A_k}{\sum_j A_j N_j} \quad \text{with } A_k = k^\gamma$$

Preferential Attachment

$$P_k = \frac{A_k}{\sum_j A_j N_j} \quad \text{with } A_k = k^\gamma$$

preferential attachment $0 < \gamma \leq 1$

$$\frac{dN_k}{dN} = \frac{A_{k-1} N_{k-1} - A_k N_k}{\sum_j A_j N_j} + \delta_{k,1}$$

Moments

$$M_0 = \sum_{k=1}^{\infty} N_k = N$$

$$M_1 = \sum_{k=1}^{\infty} k N_k =$$

Moments

$$M_0 = \sum_{k=1}^{\infty} N_k = N$$

$$M_1 = \sum_{k=1}^{\infty} k N_k = 2N$$

$$M_0 < M_\gamma = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$
$$= \mu N$$

Preferential Attachment

$$P_k = \frac{A_k}{\sum_j A_j N_j} \quad \text{with } A_k = k^\gamma$$

preferential attachment $0 < \gamma \leq 1$

$$\frac{dN_k}{dN} = \frac{A_{k-1} N_{k-1} - A_k N_k}{\underbrace{\sum_j A_j N_j}_{\mu N}} + \delta_{k,1}$$

Moments

$$M_0 = \sum_{k=1}^{\infty} N_k = N$$

$$M_1 = \sum_{k=1}^{\infty} k N_k = 2N$$

$$M_0 < M_\gamma = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$
$$= \mu N$$

Moments

$$M_0 = \sum_{k=1}^{\infty} N_k = N$$

$$M_1 = \sum_{k=1}^{\infty} k N_k = 2N$$

$$M_0 < M_\gamma = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$
$$= \mu N$$

also: $N_k = N n_k$

$$\Rightarrow n_k = \frac{A_{k-1} n_{k-1} - A_k n_k + \delta_{k,1}}{\mu}$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right) = 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right)$$

Moments

$$M_0 = \sum_{k=1}^{\infty} N_k = N$$

$$M_1 = \sum_{k=1}^{\infty} k N_k = 2N$$

$$M_0 < M_1 = \sum_{k=1}^{\infty} k^2 N_k < M_1$$
$$= \mu N$$

also: $N_k = N n_k$

$$\Rightarrow \frac{A_{k-1} n_{k-1} - A_k n_k}{\mu} + \delta_{k,1}$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right)^{-1}$$
$$= \frac{\mu}{A_1} \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$N N_1 = \frac{N}{2}$$

$$N_1 = N/2$$

$$N_k = N/2^k$$

$$1/k = 2^k$$

Moments

$$\text{also: } N_k = N n_k$$

$$M_0 = \sum_{k=1}^{\infty} N_k = N$$

$$M_1 = \sum_{k=1}^{\infty} k N_k = 2N$$

$$M_0 < M_1 = \sum_{k=1}^{\infty} k^2 N_k < M_1$$

$$= \mu N$$

$$\Rightarrow n_k = \frac{A_{k-1} n_{k-1} - A_k n_k + \delta_{k,1}}{\mu}$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right)^{-1}$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right)^{-1}$$

$$= \frac{\mu}{A_1} \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$n_2 = \left(1 + \frac{\mu}{A_1}\right)^{-1} \left(1 + \frac{\mu}{A_2}\right)^{-1} \frac{\mu}{A_2}$$

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \left(1 + \frac{\mu}{A_j}\right)^{-1}$$

$$n_2 = \left(1 + \frac{\mu}{A_1}\right)^{-1} \left(1 + \frac{\mu}{A_2}\right)^{-1} \frac{\mu}{A_2}$$

$$n_k = \frac{\mu}{A_k} \prod_{j=1}^k \left(1 + \frac{\mu}{A_j}\right)^{-1}$$

5. generic cases:

$\gamma = 0$	RRT
$0 < \gamma < 1$	"complex" networks
$\gamma = 1$	
$2 > \gamma > 1$	singular
$\gamma \geq 2$	singular

$$k=1$$

$$= \mu N$$

$$0 < \gamma < 1$$

$$n_k = \frac{\mu}{k^\gamma} \prod_{j=1}^k \left(1 + \frac{\mu}{j^\gamma}\right)^{-1}$$

$$= \frac{\mu}{k^\gamma} \exp\left(-\sum_{j=1}^k \ln\left(1 + \frac{\mu}{j^\gamma}\right)\right)$$

$$\sim \frac{\mu}{k^\gamma} \exp\left(-\int_1^k dx \frac{\mu}{x^\gamma}\right)$$

$$n_k \sim \frac{\mu}{k^\gamma} \exp\left(-\frac{\mu}{1-\gamma} k^{1-\gamma}\right)$$

stretched exponential

$$k=1$$

$$= \mu N$$

$$0 < \gamma < 1$$

$$n_k = \frac{\mu}{k^\gamma} \prod_{j=1}^k \left(1 + \frac{\mu}{j^\gamma}\right)^{-1}$$

$$= \frac{\mu}{k^\gamma} \exp\left(-\sum_{j=1}^k \ln\left(1 + \frac{\mu}{j^\gamma}\right)\right)$$

$$\sim \frac{\mu}{k^\gamma} \exp\left(-\int_1^k dj \frac{\mu}{j^\gamma}\right)$$

$$n_k \sim \frac{\mu}{k^\gamma} \exp\left(-\frac{\mu k^{1-\gamma}}{1-\gamma}\right)$$

stretched exponential

$$M_1 = \sum_{k=1}^{\infty} k N_k = \sum N$$

$$M_0 < M_1 = \sum_{k=1}^{\infty} k^{\gamma} N_k < M_1$$

$$= \mu N$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right)^{-1}$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$0 < \gamma < 1$$

$$n_k = \frac{\mu}{k^{\gamma}} \prod_{j=1}^k \left(1 + \frac{\mu}{j^{\gamma}}\right)^{-1}$$

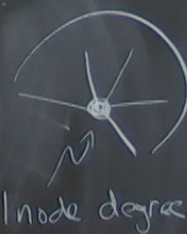
$$= \frac{\mu}{k^{\gamma}} \exp\left(-\sum_{j=1}^k \frac{1}{j^{\gamma}}\right)$$

$$\sim \frac{\mu}{k^{\gamma}}$$

$$n_k \sim \frac{\mu}{k^{\gamma}} \exp\left(-\frac{\mu k^{1-\gamma}}{1-\gamma}\right)$$

stretched exponential

$\gamma > 2$ "Bible" probability



$$M_1 = \sum_{k=1}^{\infty} k N_k = 2N$$

$$M_0 < M_\gamma = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$

$$= \mu N$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right)^{-1}$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$0 < \gamma < 1$$

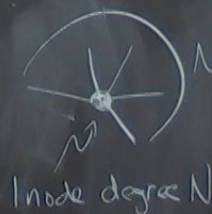
$$n_k = \frac{\mu}{k^\gamma} \prod_{j=1}^k \left(1 + \frac{\mu}{j^\gamma}\right)^{-1}$$

$$= \frac{\mu}{k^\gamma} \exp\left(-\sum_{j=1}^k \ln\left(1 + \frac{\mu}{j^\gamma}\right)\right)$$

$$\sim \frac{\mu}{k^\gamma} \exp\left(-\int_1^k dj \frac{\mu}{j^\gamma}\right)$$

$$n_k \sim \frac{\mu}{k^\gamma} \exp\left(-\frac{\mu k^{1-\gamma}}{1-\gamma}\right) \quad \text{stretched exponential}$$

$\gamma > 2$ "Bible" probability



$$P_N = \frac{N^\gamma}{\sum k^\gamma N_k} = \frac{N^\gamma}{N + N^\gamma}$$

Bible probability

$$\prod_{N=1}^{\infty} \frac{1}{1 + N^{1-\gamma}}$$

$$\text{BP: } \prod_{N=1}^{\infty} (1 + N^{1-\gamma})^{-1}$$

$$= \exp \left(- \sum_{N=1}^{\infty} \ln(1 + N^{1-\gamma}) \right)$$

$$\sim \exp \left(- \int dN N^{1-\gamma} \right)$$

$$\rightarrow \begin{matrix} > 0 & \gamma > 2 \\ & & \end{matrix}$$

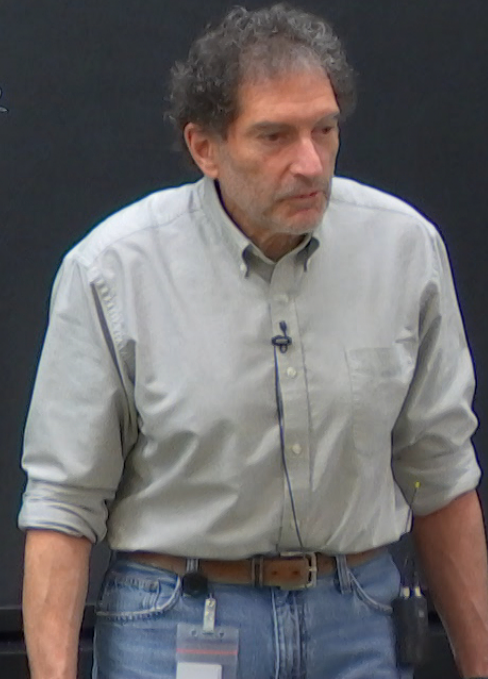
$$= 0 \quad \gamma \leq 2$$

$$\begin{aligned} \rightarrow & \gamma > 0 & \gamma > 2 \\ & = 0 & \gamma \leq 2 \end{aligned}$$

Linear Preferential Attachment ($\gamma=1$) $A_k = k$

$$\begin{aligned} n_k &= \frac{2}{k} \prod_{j=1}^k \left(1 + \frac{2}{j}\right)^{-1} \\ &= \frac{2}{k} \prod_{j=1}^k \left(\frac{j}{j+2}\right) \end{aligned}$$

$\rightarrow \frac{2}{k} \quad \frac{1}{3} \frac{2}{4} \frac{3}{5} \frac{4}{6} \dots \frac{(k-1)}{k} \frac{k}{k+1} \frac{k}{k+2}$
 $= \frac{4}{k(k+1)(k+2)}$



$$\begin{aligned} &\rightarrow >0 & \gamma > 2 \\ & & =0 & \gamma \leq 2 \end{aligned}$$

Linear Preferential Attachment ($\gamma=1$) $A_k = k$

$$\begin{aligned} n_k &= \frac{2}{k} \prod_{j=1}^k \left(1 + \frac{2}{j}\right)^{-1} \\ &= \frac{2}{k} \prod_{j=1}^k \left(\frac{j}{j+2}\right) \end{aligned}$$

$$\begin{aligned} &\rightarrow \frac{2}{k} \quad \frac{1}{3} \quad \frac{2}{4} \quad \frac{3}{5} \quad \frac{4}{6} \quad \dots \quad \frac{\cancel{k-1}}{k} \quad \frac{\cancel{k}}{\cancel{k+1}} \quad \frac{k}{k+2} \\ &= \frac{4}{k(k+1)(k+2)} \sim k^{-3} ! \end{aligned}$$

$$\begin{aligned} &\rightarrow >0 & \gamma > 2 \\ & & =0 & \gamma \leq 2 \end{aligned}$$

Linear Preferential Attachment ($\gamma=1$) $A_k = k$

$$\begin{aligned} n_k &= \frac{2}{k} \prod_{j=1}^k \left(1 + \frac{2}{j}\right)^{-1} \\ &= \frac{2}{k} \prod_{j=1}^k \left(\frac{j}{j+2}\right) \end{aligned}$$

$\rightarrow \frac{2}{k} \quad \frac{1}{3} \quad \frac{2}{4} \quad \frac{3}{5} \quad \frac{4}{6} \quad \dots \quad \frac{(k-1)}{k} \quad \frac{k}{k+1} \quad \frac{k}{k+2}$

$$= \frac{4}{k(k+1)(k+2)} \sim k^{-3} !$$

$$= 4 \frac{\Gamma(k)}{\Gamma(k+3)}$$

$$= 4 \frac{\Gamma(k)}{\Gamma(k+3)}$$

Shifted Linear Preferential Attachment $A_k = k + \lambda$

$$\mu = \frac{\sum A_k N_k}{N} = \frac{\sum (k + \lambda) N_k}{N} = 2 + \lambda$$

$$n_k = \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(1 + \frac{2 + \lambda}{j + \lambda} \right)^{-1}$$

$$= \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(\frac{j + \lambda}{j + 2\lambda + 2} \right)$$

$$= 4 \frac{\Gamma(k)}{\Gamma(k+3)}$$

Shifted Linear Preferential Attachment $A_k = k + \lambda$

$$\mu = \frac{\sum A_k N_k}{N} = \frac{\sum (k + \lambda) N_k}{N} = 2 + \lambda$$

$$2 + \lambda$$

$$n_k = \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(1 + \frac{2 + \lambda}{j + \lambda} \right)^{-1}$$

$$= \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(\frac{j + \lambda}{j + 2\lambda + 2} \right)$$

$$= 4 \frac{\Gamma(k)}{\Gamma(k+3)}$$

Shifted Linear Preferential Attachment $A_k = k + \lambda$

$$\mu = \frac{\sum A_k N_k}{N} = \frac{\sum (k + \lambda) N_k}{N} = 2 + \lambda$$

$$n_k = \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(1 + \frac{2 + \lambda}{j + \lambda} \right)^{-1}$$

$$= \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(\frac{j + \lambda}{j + 2\lambda + 2} \right)$$

$$\frac{2 + \lambda}{k + \lambda}$$

$$\frac{(1 + \lambda)(2 + \lambda)(3 + \lambda) \dots (k + \lambda)}{(3 + 2\lambda)(4 + 2\lambda) \dots (k + 2 + 2\lambda)}$$

$$= 4 \frac{\Gamma(k)}{\Gamma(k+3)}$$

Shifted Linear Preferential Attachment $A_k = k + \lambda$

$$\mu = \frac{\sum A_k N_k}{N} = \frac{\sum (k + \lambda) N_k}{N} = 2 + \lambda$$

$$n_k = \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(1 + \frac{2 + \lambda}{j + \lambda} \right)^{-1}$$

$$= \frac{2 + \lambda}{k + \lambda} \prod_{j=1}^k \left(\frac{j + \lambda}{j + 2\lambda + 2} \right)$$

$$\frac{2 + \lambda}{k + \lambda}$$

$$\frac{(1 + \lambda)(2 + \lambda)(3 + \lambda) \dots (k + \lambda)}{(3 + 2\lambda)(4 + 2\lambda) \dots (k + 2 + 2\lambda)}$$

$$= \frac{2 + \lambda}{k + \lambda}$$

$$\frac{\Gamma(k + \lambda + 1)}{\Gamma(1 + \lambda)} \frac{\Gamma(3 + 2\lambda)}{\Gamma(k + 3 + 2\lambda)}$$

$$A_k = k + \lambda$$

$$\frac{2+\lambda}{k+\lambda} \frac{(1+\lambda)(2+\lambda)(3+\lambda)\dots(k+\lambda)}{(3+2\lambda)(4+2\lambda)\dots(k+2+2\lambda)}$$

$$= \frac{2+\lambda}{k+\lambda} \frac{\Gamma(k+\lambda+1)}{\Gamma(1+\lambda)} \frac{\Gamma(3+2\lambda)}{\Gamma(k+3+2\lambda)}$$

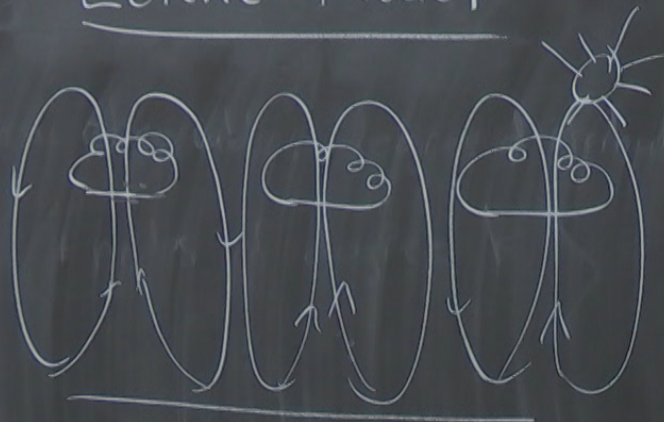
$$\lim_{k \rightarrow \infty} \frac{\Gamma(k+a)}{\Gamma(k+b)} \approx k^{a-b}$$

$$= (2+\lambda) \frac{\Gamma(k+\lambda)}{\Gamma(1+\lambda)} \frac{\Gamma(3+2\lambda)}{\Gamma(k+3+2\lambda)}$$

$$\sim \frac{(2+\lambda)}{\Gamma(1+\lambda)} \frac{\Gamma(3+2\lambda)}{k}$$

✓
- (3+λ)

Lorenz Model



$$\dot{X} = \sigma(y - x)$$

$$\dot{Y} = -rx - y - xz$$

$$\dot{Z} = xy - bz$$

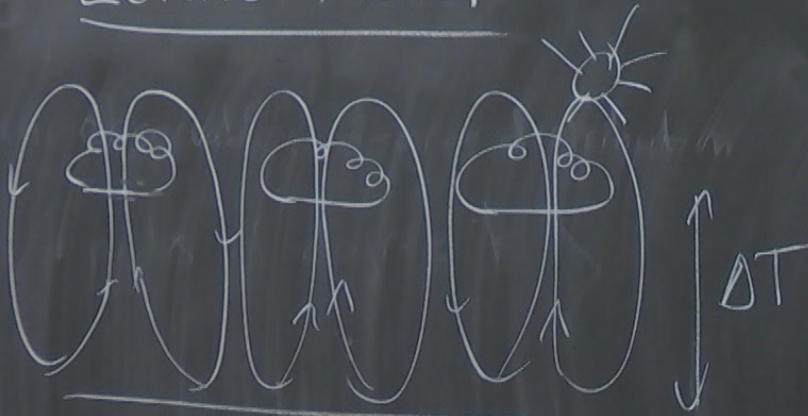
$x \sim$ convective intensity

$$\frac{dN}{dt} = \frac{\sum A_j N_j}{\sum N_j} + \sigma_{k,l}$$

$$M_0 < M_\gamma = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$

$$n_1 = \left(1 + \frac{A}{M}\right)$$

Lorenz Model



$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = -rx - y - xz$$

$$\dot{z} = xy - bz$$

$$\sigma \sim$$

$$r$$

$$b$$

$x \sim$ convective intensity

$y \sim$ Temp diff.

$z \sim$ deviation of T from linearity

$$M_0 < M_\infty = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right) = 1$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{A_1}{\mu}\right) \frac{A_1}{\mu}$$

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = -rX - Y - XZ$$

$$\dot{Z} = XY - bZ$$

$X \sim$ convective intensity

$Y \sim$ Temp diff.

$Z \sim$ deviation of T from linearity

$$\sigma \sim \text{Prandtl \#} = \frac{\text{viscosity}}{\text{heat conduction coeff}}$$

r

b

$$M_0 < M_\gamma = \sum_{k=1}^{\infty} k^\gamma N_k < M_1$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

$$n_1 = \left(1 + \frac{A_1}{\mu}\right) = 1$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{A_1}{\mu}\right)$$

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = -rX - Y - XZ$$

$$\dot{Z} = XY - bZ$$

$X \sim$ convective intensity

$Y \sim$ Temp diff.

$Z \sim$ deviation of T from linearity

$$\sigma \sim \text{Prandtl \#} = \frac{\text{viscosity}}{\text{heat conduction coeff}}$$

$$r \sim \text{Rayleigh \#} \propto \Delta T h^3$$

$$b \sim \text{aspect ratio of convection rolls}$$

$$n_1 = -\frac{A_1 n_1}{\mu} + 1$$

M_1

$$n_1 = \left(1 + \frac{A_1}{\mu}\right) = 1$$

$$n_2 = \frac{A_1 n_1 - A_2 n_2}{\mu}$$

$$n_2 \left(1 + \frac{A_2}{\mu}\right) = \frac{n_1 A_1}{\mu} = \left(1 + \frac{\mu}{A_1}\right)^{-1}$$

$$\sigma \sim \text{Prandtl \#} = \frac{\text{viscosity}}{\text{heat conduction coeff}}$$

$$r \sim \text{Rayleigh \#} \propto \Delta T h^3$$

$$b \sim \text{aspect ratio of convection rolls}$$

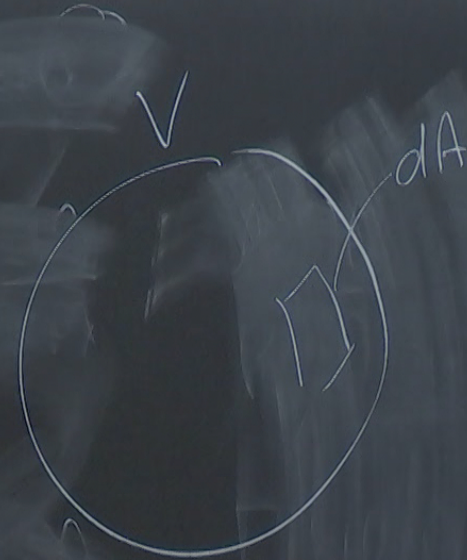
Facts of LM

- nonlinear
- 3d

Eqs are dissipative

$V(t) \equiv$ volume in state space

$$V(t+\Delta t) = V(t) + \int dA \vec{v} \cdot \hat{n} \Delta t$$



$$\sigma \sim \text{Prandtl \#} = \frac{\text{viscosity}}{\text{heat conduction coeff}}$$

$$r \sim \text{Rayleigh \#} \propto \Delta T h^3$$

$$b \sim \text{aspect ratio of convection rolls}$$

$$\vec{r} = L \vec{r}$$

from linearity

Eqs are dissipative

$V(t) \equiv$ volume in state space

$$V(t + \Delta t) = V(t) + \int dA \vec{v} \cdot \hat{n} \Delta t$$

$$= V(t) + \int dV \vec{v} \cdot \vec{\nabla} dt$$

Eqs are dissipative

$V(t) \equiv$ volume in state space

$$\begin{aligned} V(t + \Delta t) &= V(t) + \int dA \vec{v} \cdot \hat{n} \Delta t \\ &= V(t) + \int dV \vec{\nabla} \cdot \vec{v} \Delta t \end{aligned}$$

$$\frac{dV}{dt} = \int dV \vec{\nabla} \cdot \vec{v}$$

$$= (2 +$$

$$\lim_{k \rightarrow \infty} \frac{\Gamma(k+1)}{\Gamma(k+1)} \approx k$$

$$\sigma(y-x)$$

$$-rx - y - xz$$

$$xy - bz$$

convective intensity

emp diff.

variation of T from linearity

$$\sigma \sim \text{Prandtl \#} = \frac{\text{viscosity}}{\text{heat conduction coeff}}$$

$$r \sim \text{Rayleigh \#} \propto \Delta T h^3$$

$$b \sim \text{aspect ratio of convection rolls}$$

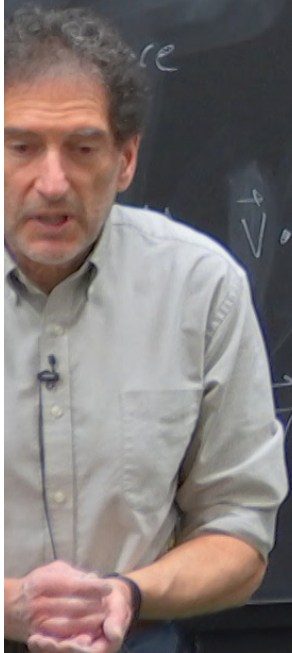
$$\vec{r} = L \vec{r} = V \vec{r}$$

$$\frac{\mu}{k\delta} \exp\left(-\int_0^k dj \frac{\mu}{j\delta}\right)$$

Bible probability

$$\prod_{j=1}^{\infty} \frac{1}{1+N^{1-j}}$$

Inode degree N



$$\frac{dV}{dt} = \int dV \vec{\nabla} \cdot \vec{V}$$

$$< 0$$

$$= (2+\lambda) \frac{\Gamma(k+\lambda)}{\Gamma(1+\lambda)} \frac{\Gamma(3+2\lambda)}{\Gamma(k+3+2\lambda)}$$

$$\sim \frac{(2+\lambda)}{\Gamma(1+\lambda)} \frac{\Gamma(3+2\lambda)}{\Gamma(k+3+2\lambda)}$$

$$\lim_{k \rightarrow \infty} \frac{\Gamma(k+a)}{\Gamma(k+b)} \approx k^{a-b}$$