Title: Statistical Physics - Lecture 221206

Speakers:
Date: December 06, 2022 - 9:00 AM

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Scaling Solution for Aggregation

$$\frac{\partial c}{\partial t}(x,t) = \frac{1}{2}\int_0^x dy\; K(y, x-y)\, c(x-y,t)\, c(y,t) - \int_0^\infty dy\; K(x,y)\, c(x,t)\, c(y,t)$$

homogeneity : $K(ax, ay) = a^\lambda K(x,y)$

Scaling Ansatz:

$$c(x,t) = \frac{1}{s(t)^2} f\left(x/s(t)\right)$$ normalization

$S(t)$ = characteristic cluster size

normalization $\int c(x,t)\,dx = 1$ - total mass

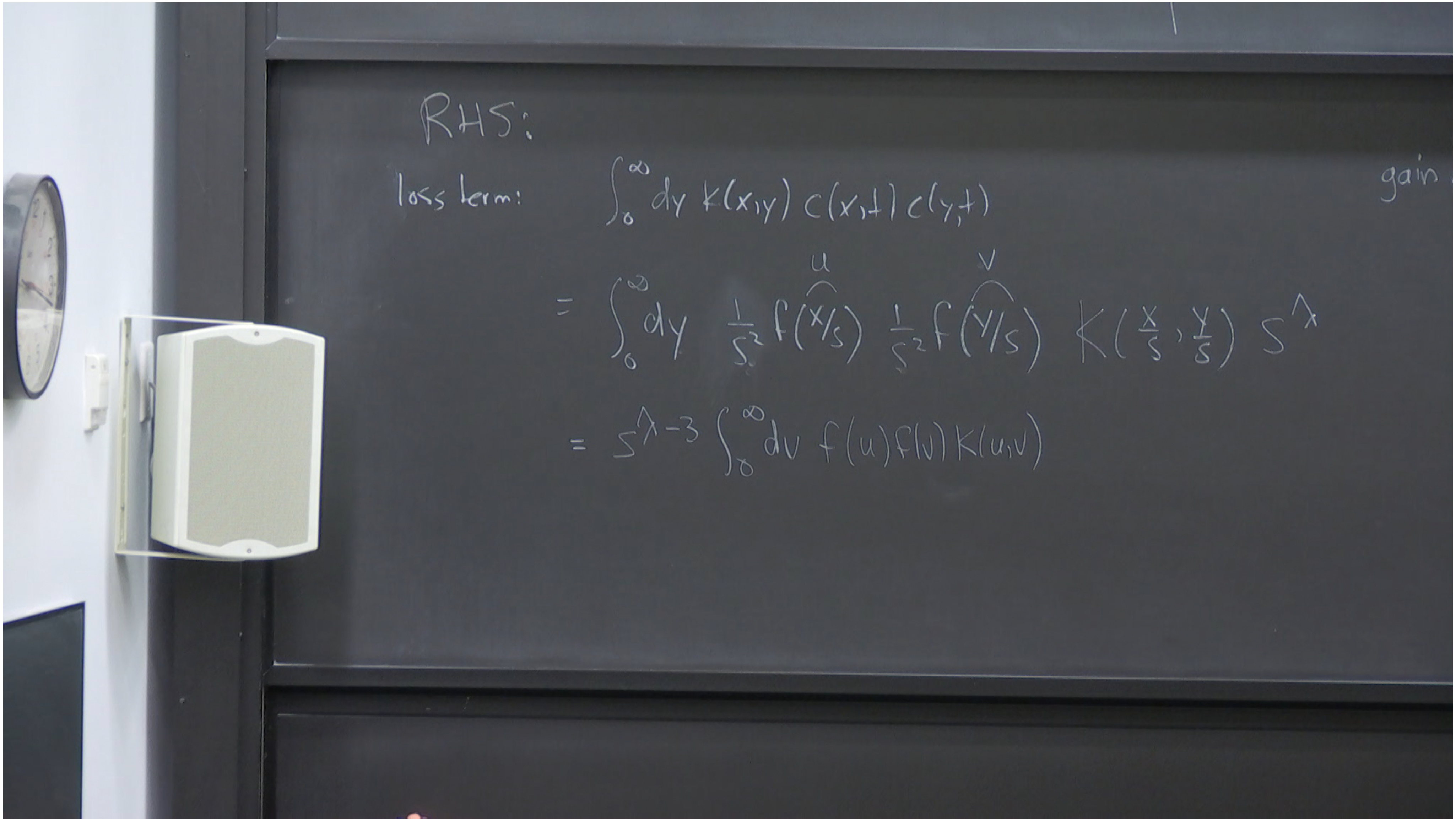
$$\int x \, \frac{1}{s^2} f(x/s)\, dx$$

$$= \underbrace{\int z\, f(z)\, dz}_{O(1)} \qquad z = x/s$$

LHS:

$$\frac{\partial c}{\partial t} = -\frac{1}{s^3} 2\dot{s}\, f(z) + \frac{1}{s^2} f'(z)\left(-\frac{x}{s^2}\right)\dot{s}$$

$$= -\frac{\dot{s}}{s^3}\left(2f + zf'\right)$$


RHS:
loss term: $\int_0^\infty dy\, K(x,y)\, c(x,t)\, c(y,t)$
$= \int_0^\infty dy\, \frac{1}{s^2} f(\overbrace{x/s}^{u})\, \frac{1}{s^2} f(\overbrace{y/s}^{v})\, K(\frac{x}{s}, \frac{y}{s})\, s^\lambda$
$= s^{\lambda-3} \int_0^\infty dv\, f(u) f(v) K(u,v)$
gain

$,t)\, c(y,t)$

$\frac{1}{s^2} f(\overset{v}{y/s})\; K\left(\frac{x}{s}, \frac{y}{s}\right) s^\lambda$

$(u) f(v) K(u,v)$

gain term $\int_0^x dy\; c(y,t)\, c(x-y,t)\, K(y,x-y)$

$$= \int_0^x dy\; \frac{1}{s^2} f\left(\frac{y}{s}\right) \frac{1}{s^2} f\left(\frac{x-y}{s}\right) K\left(\frac{y}{s}, \frac{x-y}{s}\right) s^\lambda$$

$$= s^{\lambda-3} \int_0^u f(v) f(u-v) K(v,u-v)\, dv$$

gregation

$c(x-y,t)\,c(y,t))$

$c(x,t)\,c(y,t))$

$q^{\lambda} K(x,y))$

Scaling Ansatz:

$$c(x,t) = \frac{1}{S(t)^2} f(x/S(t)) \longrightarrow \text{normalization}$$

$S(t)$ = characteristic cluster size

$$\int x\, c(x,t)\, dx = 1 \text{ - total mass}$$

$$\int x \frac{1}{s^2} f(x/s)\, dx$$

$$= \underbrace{\int u\, f(u)\, du}_{O(1)} \qquad u = x/s$$

$= 1 -$ total mass

dx

$u \qquad u = x/s$

LHS:

$$\frac{\partial c}{\partial t} = -\frac{1}{s^3} 2\dot{s} f(u) + \frac{1}{s^2} f'(u) \left(-\frac{x}{s^2}\right) \dot{s}$$

$$= -\frac{\dot{s}}{s^3}\left(2f + uf'\right)$$

$$\Rightarrow \quad -\frac{\dot{S}}{S^3}\left(2f+uf'\right) = S^{\lambda-3}\left[\overbrace{\int_0^u dv\, f(v)f(u-v)K(v,u-v) - \int_0^\infty dv\, f(u)f(v)K(u,v)}^{H(u)}\right]$$

time

$$\underbrace{\dot{S}\,S^{-\lambda}}_{\text{time only}} = \left[-\frac{H(u)}{2f+uf'}\right] \equiv \underbrace{\mathcal{H}(u)}_{u \text{ only}} \equiv \Lambda$$

e dependance

homogeneity index

$$\dot{S}\, S^{-\lambda} = \Lambda$$

$$S(t) \simeq \begin{cases} t^{\frac{1}{1-\lambda}} & \lambda < 1 \\ e^{\Lambda t} & \lambda = 1 \\ (t - t_g)^{-\frac{1}{\lambda - 1}} & \lambda > 1 \end{cases}$$

u dependence

$\mathcal{H}(u) = \Lambda$ insoluble

Some general insights

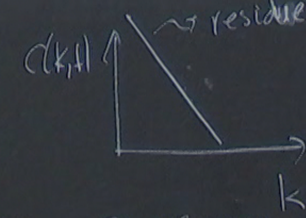
$$K(i,j) = \begin{pmatrix} SS & SL \\ LS & LL \end{pmatrix}$$

$$K(ai,aj) = a^{\lambda} K(i,j)$$

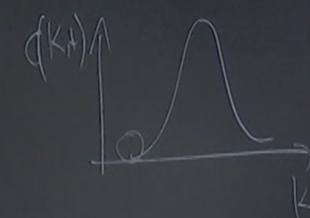
$$K(1,j) \sim j^{\nu}$$

Class I $\quad LL \gg SS, LS \qquad \lambda > \nu$

$\rightsquigarrow$ residue left behind



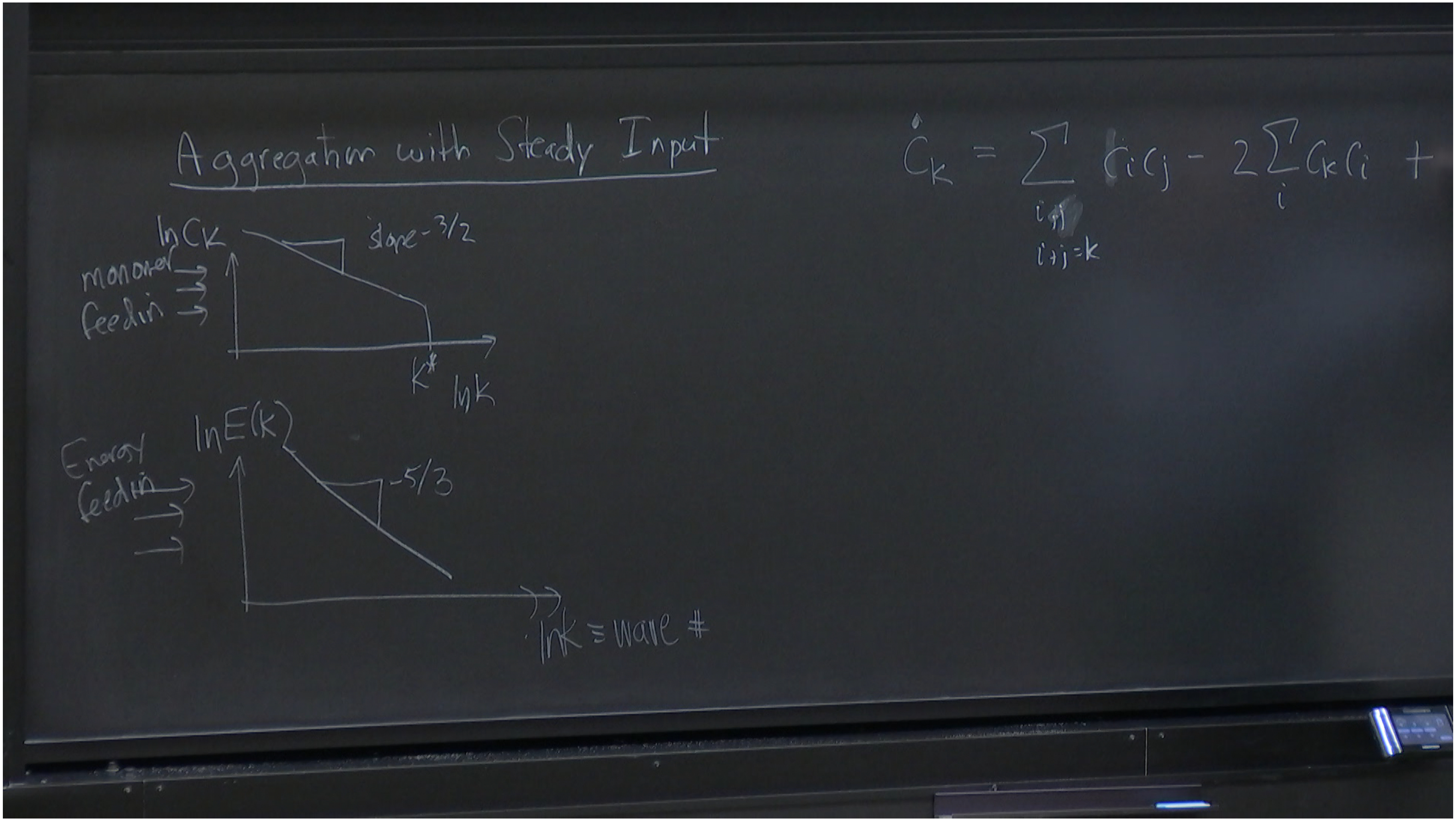
Class II $\quad SL, LS \gg LL, SS \qquad \lambda < \nu$



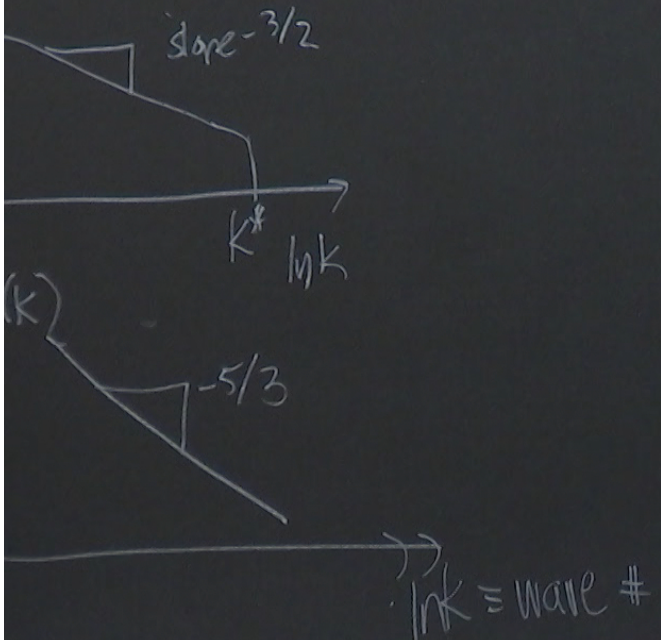
class III $\quad LL \sim SS \sim SL, LS$

$\lambda = \nu$

marginal


Aggregation with Steady Input
$\dot{C}_k = \sum_{\substack{i,j \\ i+j=k}} C_i C_j - 2\sum_i C_k C_i +$
ln Ck
slope -3/2
monomer feedin
k*
ln k
ln E(k)
-5/3
Energy feedin
ln k ≡ wave #

...regation with Steady Input



$$\sum_{k=1}^{\infty}\left[\dot{C}_k = \sum_{\substack{i,j\\ i+j=k}} C_i C_j - 2\sum_i C_k C_i + \delta_{k,1}\right] z^k$$

Steady state

$$0 = \frac{\partial g(z,t)}{\partial t} = g(z,t)^2 - 2g(z,t)\,g(1,t) + z$$

$z \qquad g(z)^2 - 2g(z)\,g(1) + z = 0$

$z=1 \qquad -g(1)^2 + 1 = 0 \quad \Rightarrow g(1) = 1$

$$\left[\dot{c}_k = \sum_{\substack{i,j \\ i+j=k}} c_i c_j - 2\sum_i c_k c_i + \delta_{k,1}\right] z^k$$

$$g^2 - 2g + z = 0$$

$$\boxed{g = 1 - \sqrt{1 - z^2}}$$

$$\frac{\partial g(z,t)}{\partial t} = g(z,t)^2 - 2g(z,t)g(1,t) + z$$

$$z \quad g(z)^2 - 2g(z)g(1) + z = 0$$

$$z=1 \quad -g(1)^2 + 1 = 0 \quad \Rightarrow g(1) = 1$$

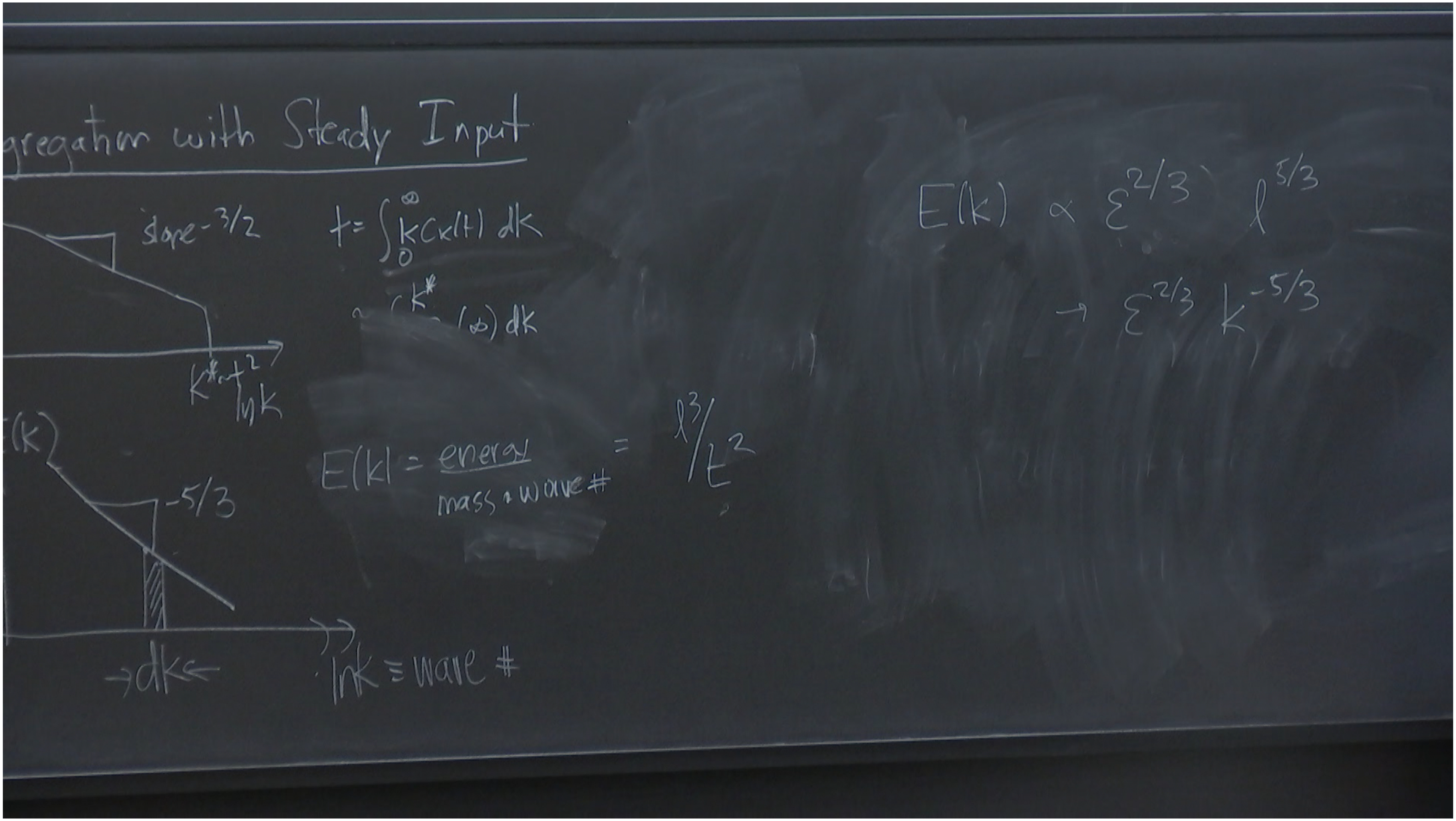
$$g = \sum_k c_k z^k = 1 - \sqrt{1-z}$$

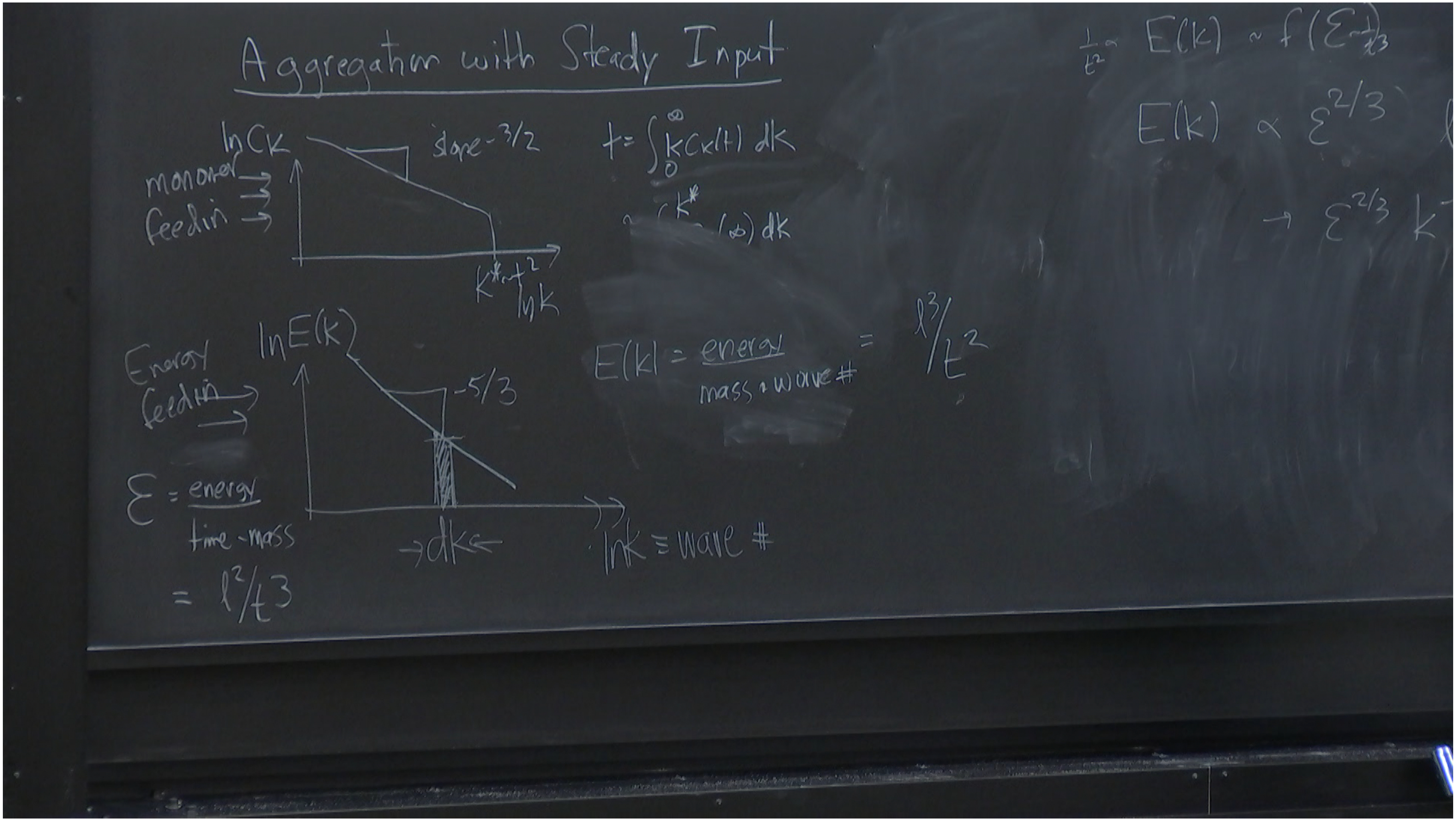
$$\sqrt{1-z} = \sqrt{1+(-z)} = 1 + \frac{1}{2}(-z) + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{(-z)^2}{2!} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-z^3)}{3!} + \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\frac{(-z^4)}{4!} \cdots$$

$$= (1+x)^{1/2}$$

$$(k+a)(k+a-1)\cdots a = \frac{\Gamma(k+a-1)}{\Gamma(a)}$$

$$\left.\begin{array}{l} \Gamma(k) = (k-1)! \\ k\Gamma(k) = \Gamma(k+1) \end{array}\right\}$$


gregation with Steady Input
slope -3/2
t = ∫₀^∞ k C_k(t) dk
k* ~ t²
ln k
E(k)
-5/3
→dk←
ln k ≡ wave #
E(k) = energy / mass * wave #
= l³/t²
E(k) ∝ ε^{2/3} l^{5/3}
→ ε^{2/3} k^{-5/3}


Aggregation with Steady Input
ln Ck
slope -3/2
monomer feedin
t = ∫₀^∞ k Ck(t) dk
k* ~ t²
ln k
Energy feedin
ln E(k)
-5/3
ε = energy / time·mass
= l²/t³
→dk←
ln k ≡ wave #
E(k) = energy / mass·wave # = l³/t²
E(k) ∝ ε^2/3
→ ε^2/3 k

Product Kernel Aggregation $\quad K_{ij} = ij$

$$\dot{C}_k = \frac{1}{2} \sum_{\substack{i,j \\ i+j=k}} i j\, c_i c_j - \sum_{i=1}^{\infty} i k\, c_i c_k$$

$$\boxed{\dot{C}_k = \frac{1}{2} \sum_{i+j=k} i j\, c_i c_j - k c_k}$$

$K_{ij} = ij$

$\sum_{i=1}^{\infty} ik\, c_i c_k$

$\boxed{k c_k}$

Moments

$M_n = \sum k^n c_k$

$\dot{M}_0 = \frac{1}{2} M_1^2 - M_1 \to -\frac{1}{2}$ for $M_1 = 1$ $\quad M_0(t) = 1 - t/2$!

$\dot{M}_1 = \frac{1}{2} \sum_{i,j} (i+j)\, ij\, c_i c_j - M_2$

$= M_1 M_2 - M_2 = 0$ if $M_1 = 1$

$\dot{M}_2 = \frac{1}{2}$

ck

$M_0(t) = 1 - t/2$!

M_2

$M_1 = 1$

$$\dot{M}_2 = \frac{1}{2}\sum_{i,j} (i+j)^2\, ij\, c_i c_j - M_3$$

$$= \cancel{M_3 M_1} + M_2^2 - \cancel{M_3}$$

$$= M_2^2 \qquad M_2(t) = \frac{1}{1-t}$$