Title: Quantum Constraint Problems can be complete for BQP, QCMA, and BPP

Speakers: Alexander Meiburg

Series: Perimeter Institute Quantum Discussions

Date: November 30, 2022 - 2:45 PM

URL: https://pirsa.org/22110119

Abstract: Constraint satisfaction problems are known to always be "easy" or "hard", in the sense of being either solvable in P or being NP-complete, with no intermediate difficulty levels. The quantum analog of constraint problems, frustration-free Hamiltonians, are known to exhibit at least two more levels of complexity: QMA (for arbitrary local Hamiltonians) and MA (for stoquastic Hamiltonians). Wondering if other complexity classes can occur, we answer in the affirmative: there are interactions which can be freely arranged on qubits in any arrangement, such that the resulting frustration problem is BQP-complete, and captures exactly the difficulty of quantum computation. Simple modifications of this construction show that quantum constraint problems can be complete for QCMA and BPP as well. Based on https://arxiv.org/abs/2101.08381

Quantum Constraint Problems can be complete for BQP, QCMA, and BPP

[arXiv:2101.08381]

Alexander Meiburg
University of California, Santa Barbara

Outline

- Classical constraint problems, CSP dichotomy theorem
- The quantum setting: frustration-free Hamiltonians
- Four known cases
- What about BQP?
  - Adapting BQP to QCMA and BPP (well, coRP)
- Construction details
- Future work

Constraint satisfaction problems (classical)

- Variables chosen from some finite set (True/False; Red/Green/Blue)
- Relationships between fixed number of variables
  - v1 is true *or* v2 is false
  - Exactly three of (v3, v5, v10, v12, v19) are red
- Problem: is there an allowed assignment of variables?
- Sometimes the problem is very easy
  - Can follow a chain of implications and deduce an answer if there is one, or prove there isn't. Class P.

CSPs (classical)

- Can be viewed as minimizing an energy functional:
  - E(v1, v2) = 1 *if (v1 is false and v2 is true), else* 0.
  - E(v3, v5, v10) = 1 *if all of (v3,v5,v10) are not red, else* 0.
- Overall Hamiltonian is a sum of these interactions
- Question: Is there an E_{Tot}=0 state?

Dichotomy

- Was observed in the 80's that almost every problem was either in P (easy) or maximally hard (NP-complete).
- Classic "intermediate" problems like Factoring or Graph Isomorphism don't have an obvious form as a CSP
  - A given instance of factoring can be written as an instance of 3SAT. But there's no type of "factoring clauses" that includes factoring but doesn't let you also build much harder problems.
- CSP Dichotomy Conjecture: *No CSP is NP-intermediate.*
- Independently proved by both Zhuk and Bulatov in 2017, became the CSP Dichotomy Theorem

Constraint problems (quantum)

- Variables are now qubits (or, generally, qudits)
- Form a Hamiltonian from a sum of local projectors
  - $H(v1,v2) = (1 + \sigma_{1,x}\sigma_{2,y})/2$
  - $H(v4,v5,v8) = 1 - |002\rangle\langle 002| - |12+\rangle\langle 12+|$
- Does this Hamiltonian have a zero-energy ground state?
  - *i.e.* Is this Hamiltonian frustration-free, *or* is the ground state energy larger than zero?
    - Technically asking "is it larger than 1/n^100"
- Hard to find the answer. But given the ground state, easy to check.
  - Measure the provided ground state on each local projector. Positive chance to find violated term.

Constraint problems (quantum)

- Problems checkable given a quantum state: QMA
- Kitaev (2002) showed 5-local Hamiltonians on qubits ("Quantum 5-SAT") is universal for QMA, that is, QMA-complete.
  - Improved to Quantum 3-SAT (Gosset and Nagaj, 2013)
  - Quantum 2-SAT has an efficient algorithm for determining frustration (Bravyi, 2006)

Classifications - in the quantum setting

- Classical problems can still be realized as quantum constraint problems, so there are “P” and “NP” quantum problems.
- Kitaev showed that there are QMA (quantum NP) complete problems.
- In 2008, Bravyi & Terhal show that “stoquastic” frustration-free Hamiltonians are MA-complete.
  - Stoquastic: the off-diagonal elements of the operators are real and non-positive. These are Hamiltonians “with no sign problem”, and permit efficient Monte-Carlo methods in many settings.
  - MA-complete: the same as NP, but verification is allowed be probabilistic.
    - “Give me your proof, I’ll run many checks, and >80% of my checks should pass.”

Completeness

Fact:

There is a classical constraint problem (*HornSAT*) that is **P-complete** - any polynomial time program can be turned into *HornSAT* with very little effort (logspace).

Question:

Is there a class of Quantum CSPs that captures exactly the power of *quantum* computers? **BQP-complete**

Completeness

Question:

Is there a class of Quantum CSPs that captures exactly the power of quantum computers? **BQP-complete**

- **BQP**: Problems with a quantum circuit that to solve them,
  - $>2/3$ correctness, polynomial time
- **BQP-complete**: Problems that are sufficiently flexible to capture all of BQP.

Completeness

Question:

Is there a class of Quantum CSPs that captures exactly the power of quantum computers? **BQP-complete**

- **BQP**: Problems with a quantum circuit that to solve them,
  - $>2/3$ correctness, polynomial time
- **BQP-complete**: Problems that are sufficiently flexible to capture all of BQP.
  - Simple example: "What is the output of this quantum circuit"

Completeness

Question:

Is there a class of Hamiltonians whose ground states capture exactly the power of quantum computers? (BQP-complete)

- Have to be flexible enough to simulate a full quantum computer
- Have to be constrained enough that a quantum computer can systematically proceed through and check for frustration.
- BQP is most “naturally” about quantum circuits. Nothing about ground states of Hamiltonians!

New results

- Yes! There is a BQP-complete class of Hamiltonian problems.
- Precise statement: there is a fixed list of interactions $\{H_1, H_2, H_3, H_4, H_5\}$ such that applying these to any qubits in any configuration gives a total Hamiltonian **H** that...
  - ...can be used to simulate an arbitrary quantum computer C: **H**(C) is frustration free iff C returns "1"
  - ...can be solved on a quantum computer: linear-time algorithm to determine if **H** is frustration free or not.

New results

- Bonus: once this set of interactions was designed, offered straightforward modifications to get two more new classes.
  - QCMA: “Quantum Classical Merlin-Arthur”. Problems checkable by a quantum computer given a classical solution string of bits
    - Harder than BQP (needs a solution) but easier than QMA (it isn’t a *quantum* solution)
  - coRP: “Randomized polynomial”. Problems checkable by a classical computer with a source of randomness. No proof string needed.
    - A very “classic” complexity class. Very few complete problems, surprisingly!
    - Some technical reasons here about one-sided error: QMA $\rightarrow$ QMA1, BPP $\rightarrow$ coRP

Classical difficulty levels

Of constraint problems

- P
- NP (Cook, 1971)

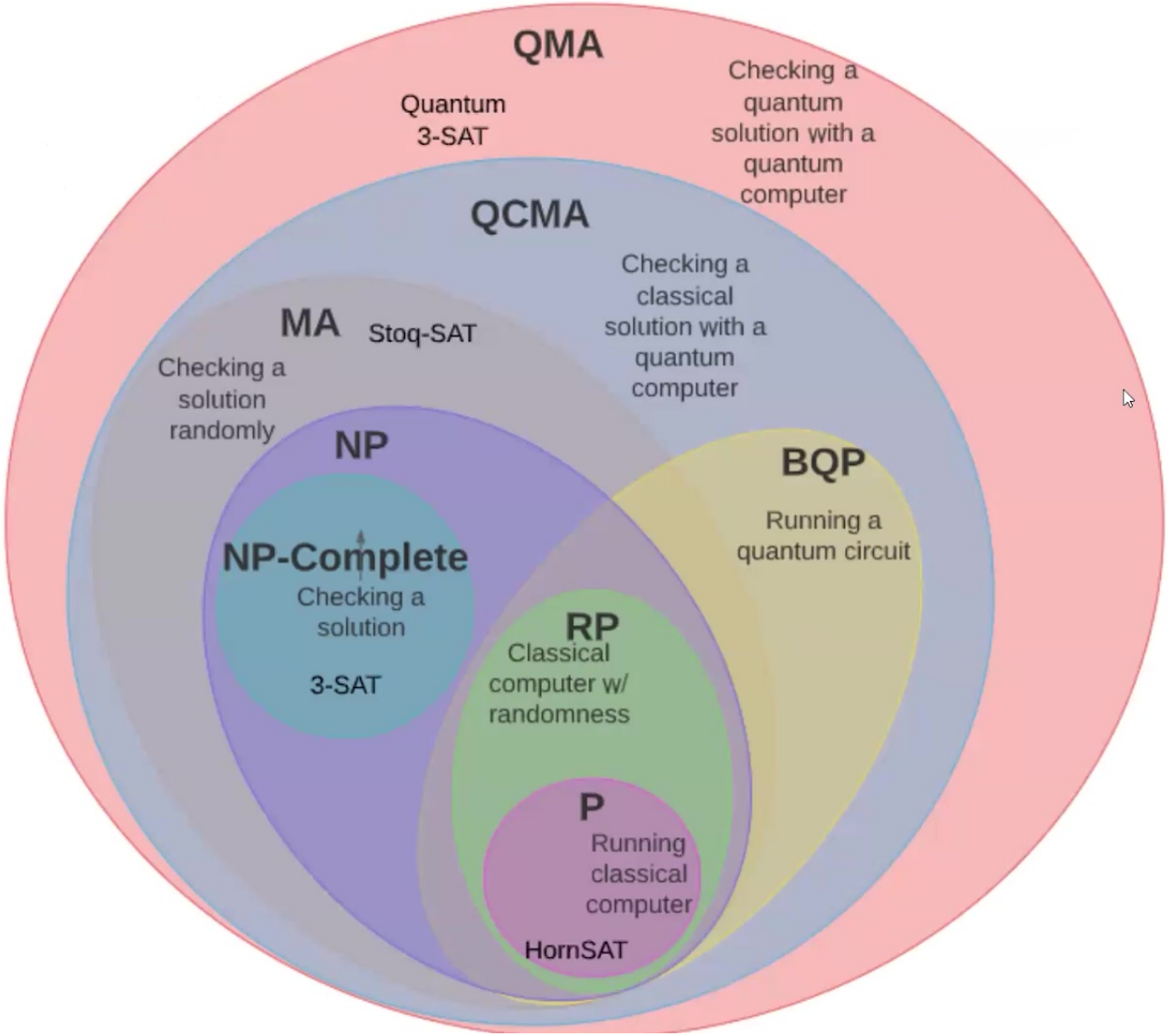
Complete list ✔

Quantum difficulty levels

Of constraint problems

- P
- coRP (new)
- NP
- MA (Bravyi, 2008)
- BQP (new)
- QCMA (new)
- QMA (Kitaev, 2002)

More to find ?


QMA
Quantum
3-SAT
Checking a
quantum
solution with a
quantum
computer
QCMA
Checking a
classical
solution with a
quantum
computer
MA
Stoq-SAT
Checking a
solution
randomly
NP
BQP
Running a
quantum circuit
NP-Complete
Checking a
solution
3-SAT
RP
Classical
computer w/
randomness
P
Running
classical
computer
HornSAT

Construction of a BQP-complete Hamiltonian

- Going to build a “dictionary” from circuits to Hamiltonians
- Circuit ➔ Hamiltonian:
  - Every circuit can be embedded in a Hamiltonian
  - Hamiltonian has low-energy state iff circuit outputs “1”
  - One such embedding was done with Kitaev’s clock construction.
- Hamiltonian ➔ Circuit:
  - Every Hamiltonian can be analyzed as a circuit
  - … or, if not a circuit exactly, then fragments of circuits, that can each be processed.

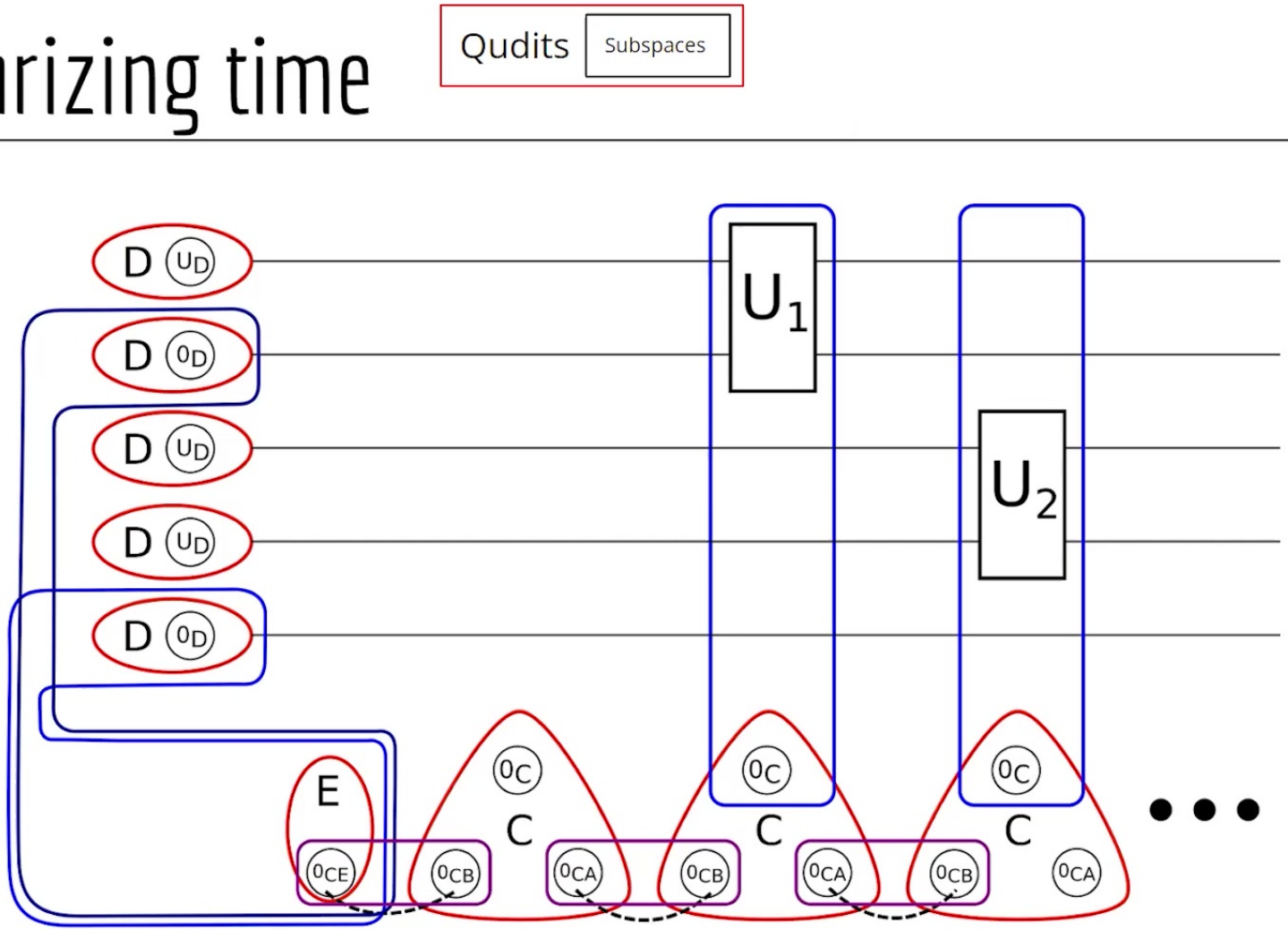
Construction of a BQP-complete Hamiltonian

- Idea: start with Kitaev's QMA-complete construction.
- Some qubits are "data", some are "time", overall the ground state is a "history" state (superposition of full computational history of circuit)
- Gives the ability to build any quantum circuit!
  - *But* that circuit can take any input: could be the "solution" quantum state. Can't have that!
  - *Also*, allows many configurations that are *not* quantum circuits:
    - Could use a "time" bit as "data" (what does this mean?), or have multiple "time" lines
    - Could couple leave "time" bits uncoupled
    - Input could be left blank or unusually constrained

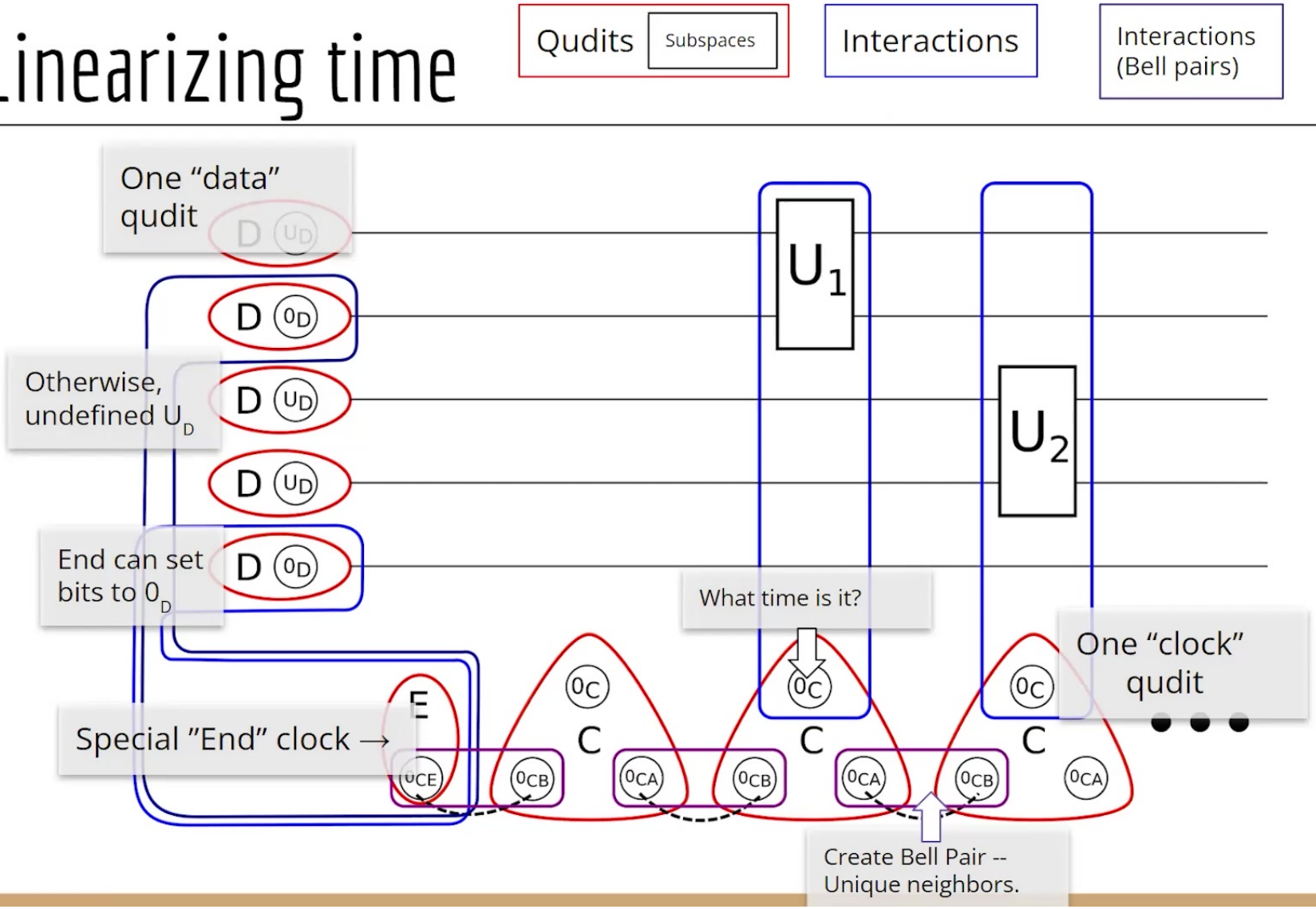
Construction of a BQP-complete Hamiltonian

- Modify Kitaev's clock-circuits to be easily solvable.
- First, separate "data" and "clock" into separate states.
  - Qubits become qudits, with d=4: "data-0", "data-1", "clock-0", "clock-1".
- Penalize interactions that "don't look like circuits".
  - Local frustration appears.
  - Checker can quickly find these local problems and return "FRUSTRATED".
- Any absent constraints will make circuit trivially satisfiable
  - Failed to initialize the circuit correctly? Okay, we can put everything in an extra "dumb" state $|U\rangle$ that will satisfy everything else.

Linearizing time



Linearizing time



Construction of a BQP-complete Hamiltonian

- Some “problems” include:
  - Time without the endpoints
    - All qubits end up in “undefined” state and work out trivially
    - Also applies to “circular time”
  - Multiple “timelines” acting on the same set of bits.
    - Choose gate set such that clock-data entanglement is guaranteed.
    - Bound the entanglement between data with each clock line.
    - Conclude that there is frustration.

Construction of a BQP-complete Hamiltonian

- When all the “fixes” are accounted for, interactions are:
  - 5-local
  - on 13-qudits
- Some reductions show this could be reduced to
  - 80-local interactions
  - On qubits

Freebies

- QCMA (Quantum Classical Merlin-Arthur)
  - Like BQP or QMA, now instead of starting with |0⟩ or |psi⟩ we start with Z bitstring |k⟩
  - Start with BQP circuit
  - Alternate “End” clause now allows input to be 0 or 1
  - But makes a copy (in the Z basis) to an extra qubit so entanglement is undetectable
- RP (Randomized Polynomial-time)
  - Like BQP, but all of the operations are classical, and we have randomness
  - Initial states are can be set to |0⟩ or |+⟩
  - Only allowed gates are classical gates

But as a physicist...

... why should I care about classifying the *algorithms* that can solve different Hamiltonians?

- Really a statement about the types of entanglement present in different Hamiltonians
  - P or NP: “essentially classical” entanglement
  - BQP or QCMA: “efficiently preparable” entanglement
  - QMA: entanglement that is likely *not* efficiently preparable

Future questions...

- Two other interesting complexity classes: StoqMA and TIM
  - No longer restricted to frustration-free (instead, “What is the ground-state energy?”)
  - StoqMA: stoquastic interactions. Should be “easier” in some senses
  - TIM: Transverse-field Ising Model. Relevant for DWave machines. Not obviously easier, but not shown to be universal either
- Looking for ways towards a classification
  - Dimitry Zhuk’s proof for classical problems centered on ‘polymorphisms’. Unlikely to carry over to quantum case nicely. (Many nice notions of ‘quantum polymorphism’ don’t behave)
- More physically reasonable constructions for BQP, QCMA, RP.

Thank you!