Title: First-Passage Processes in Physics and Beyond

Speakers: Sidney Redner

Series: Colloquium

Date: November 30, 2022 - 2:00 PM

URL: https://pirsa.org/22110115

Abstract: A fundamental aspect of a random walk is determining when it reaches a specified threshold position for the first time. This first-passage time, and more generally, the distribution of first passage times underlies many non-equilibrium phenomena, such as the triggering of integrate and fire neurons, the statistics of cell division, and the execution of stock options. The computation of the first-passage time and its distribution is both simple and beautiful, with profound connections to electrostatic potential theory. I will present some aspects of these fundamentals and then discuss applications of first-passage ideas to diverse phenomena, including stochastic search processes and a toy model of wealth sharing.

Zoom link: https://pitp.zoom.us/j/98293478936?pwd=NTR3dWZoNElWRmd2NVJ1bzk5aC9ZQT09

First-Passage Processes in Physics and Beyond

Perimeter Institute, Nov 30, 2022

collaborators: D. ben Avraham, P. L. Krapivsky, B. De Bruyne, J. Random-Furling & NSF support

Sid Redner, Santa Fe Institute, santafe.edu/~redner

First-Passage Processes in Physics and Beyond

Perimeter Institute, Nov 30, 2022

collaborators: D. ben Avraham, P. L. Krapivsky, B. De Bruyne, J. Random-Furling & support

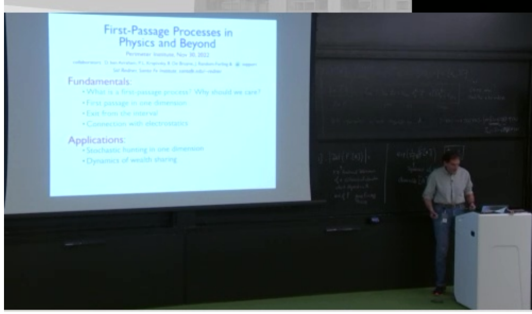
Sid Redner, Santa Fe Institute, santafe.edu/~redner

Fundamentals:

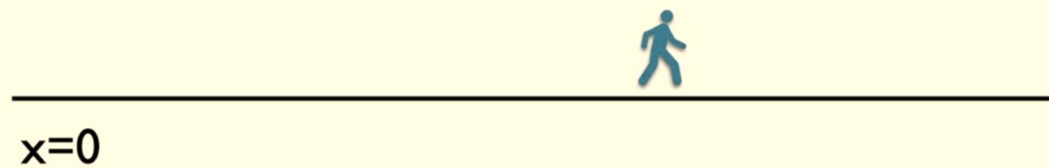
- What is a first-passage process? Why should we care?
- First passage in one dimension
- Exit from the interval
- Connection with electrostatics

Applications:

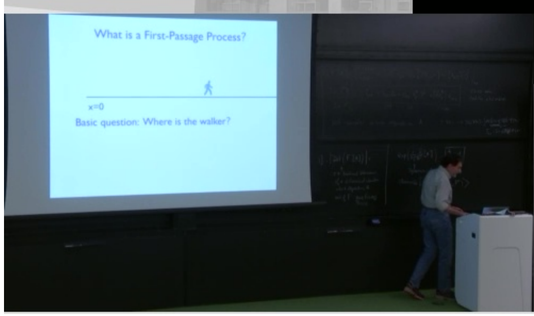
- Stochastic hunting in one dimension
- Dynamics of wealth sharing

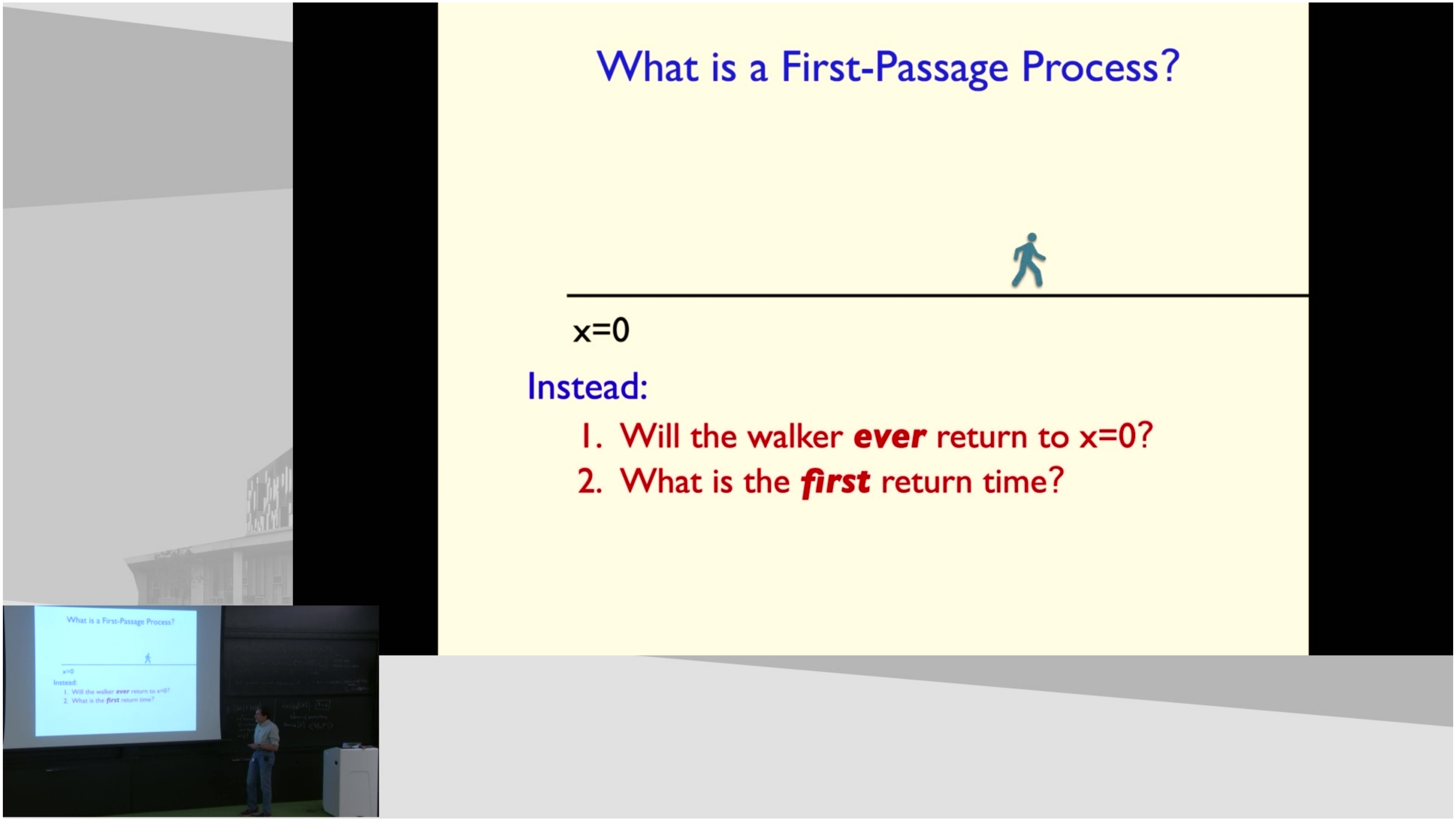


What is a First-Passage Process?



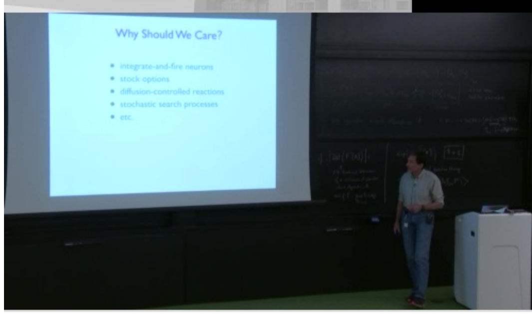
Basic question: Where is the walker?




What is a First-Passage Process?
x=0
Instead:
1. Will the walker ever return to x=0?
2. What is the first return time?

Why Should We Care?

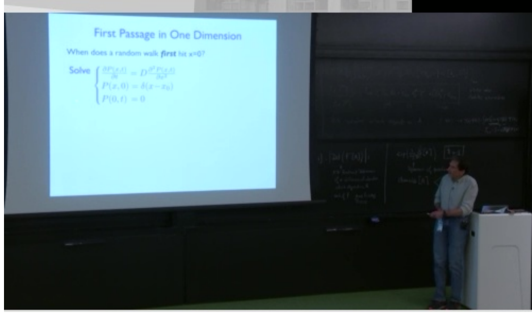
- integrate-and-fire neurons
- stock options
- diffusion-controlled reactions
- stochastic search processes
- etc.



First Passage in One Dimension

When does a random walk ***first*** hit x=0?

Solve
$$\begin{cases} \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} \\ P(x,0) = \delta(x-x_0) \\ P(0,t) \ = 0 \end{cases}$$

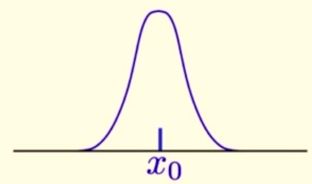


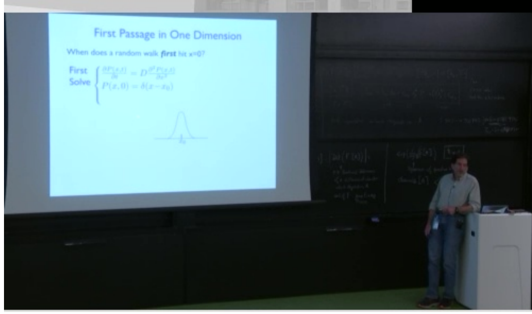
First Passage in One Dimension

When does a random walk ***first*** hit x=0?

First Solve

$$\begin{cases} \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} \\ P(x,0) = \delta(x-x_0) \\ \end{cases}$$



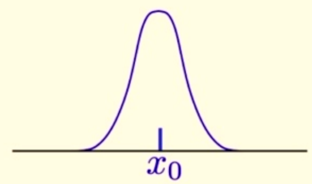


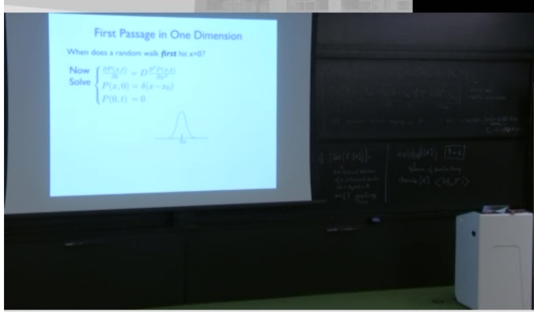
First Passage in One Dimension

When does a random walk ***first*** hit x=0?

Now Solve

$$\begin{cases} \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} \\ P(x,0) = \delta(x-x_0) \\ P(0,t) = 0 \end{cases}$$

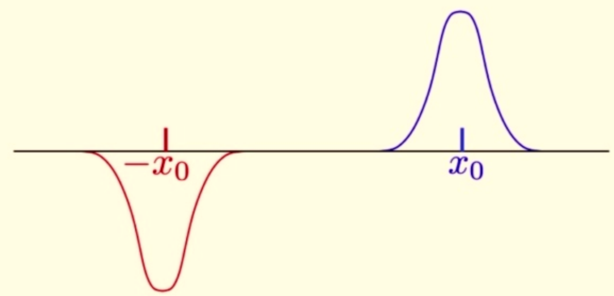


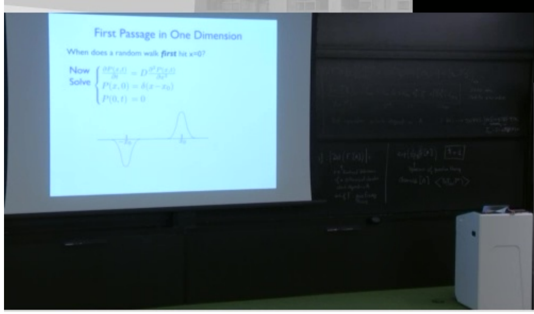


First Passage in One Dimension

When does a random walk ***first*** hit x=0?

Now Solve
$$\begin{cases} \frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} \\ P(x,0) = \delta(x-x_0) \\ P(0,t) = 0 \end{cases}$$

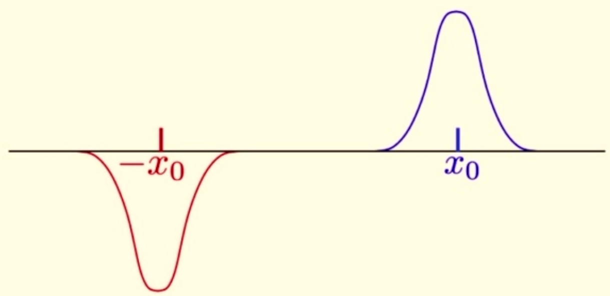


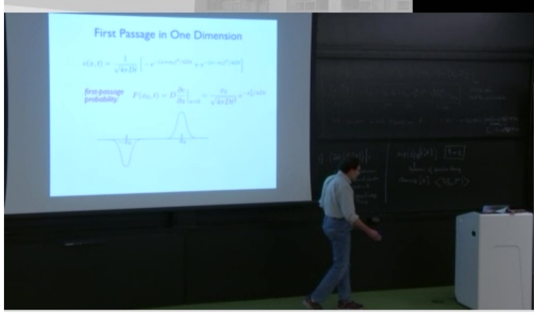


First Passage in One Dimension

$$c(x,t) = \frac{1}{\sqrt{4\pi Dt}}\left[-\,e^{-(x+x_0)^2/4Dt} + e^{-(x-x_0)^2/4Dt}\right]$$

first-passage probability: $$F(x_0,t) = D\frac{\partial c}{\partial x}\bigg|_{x=0} = \frac{x_0}{\sqrt{4\pi Dt^3}}\,e^{-x_0^2/4Dt}$$



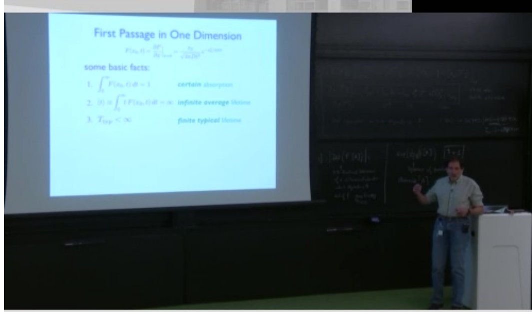


First Passage in One Dimension

$$F(x_0,t) = \frac{\partial P}{\partial x}\Big|_{x=0} = \frac{x_0}{\sqrt{4\pi D t^3}}\, e^{-x_0^2/4Dt}$$

some basic facts:

1. $\int_0^\infty F(x_0,t)\,dt = 1$ — ***certain*** absorption
2. $\langle t\rangle \equiv \int_0^\infty t\,F(x_0,t)\,dt = \infty$ — ***infinite average*** lifetime
3. $T_{\rm typ} < \infty$ — ***finite typical*** lifetime



First Passage in One Dimension

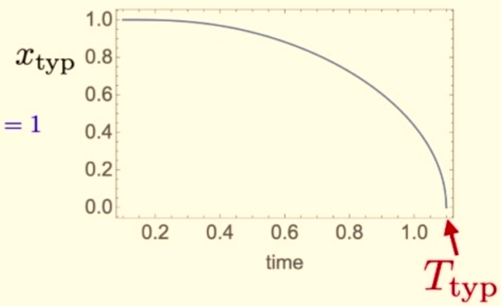
$$F(x_0,t) = \frac{\partial P}{\partial x}\Big|_{x=0} = \frac{x_0}{\sqrt{4\pi D t^3}}\, e^{-x_0^2/4Dt}$$

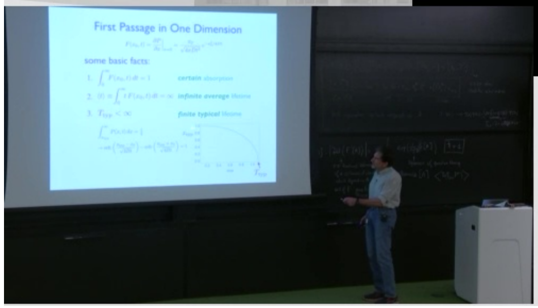
some basic facts:

1. $\int_0^\infty F(x_0,t)\,dt = 1$ ***certain*** absorption
2. $\langle t\rangle \equiv \int_0^\infty t\,F(x_0,t)\,dt = \infty$ ***infinite average*** lifetime
3. $T_{\rm typ} < \infty$ ***finite typical*** lifetime

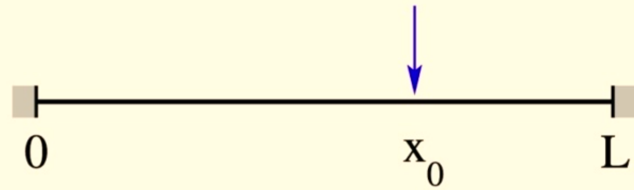
$$\int_{x_{\rm typ}}^\infty P(x,t)\,dx = \tfrac{1}{2}$$

$$\to \operatorname{erfc}\left(\frac{x_{\rm typ}-x_0}{\sqrt{4Dt}}\right) - \operatorname{erfc}\left(\frac{x_{\rm typ}+x_0}{\sqrt{4Dt}}\right) = 1$$

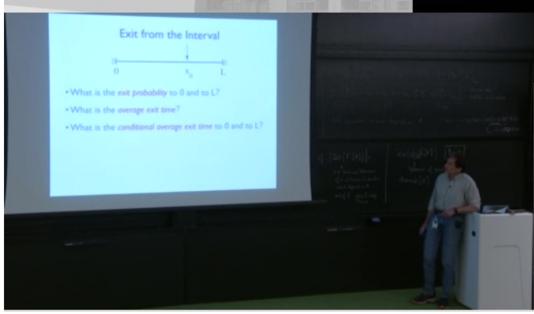




Exit from the Interval



- What is the *exit probability* to 0 and to L?
- What is the *average exit time*?
- What is the *conditional average exit time* to 0 and to L?



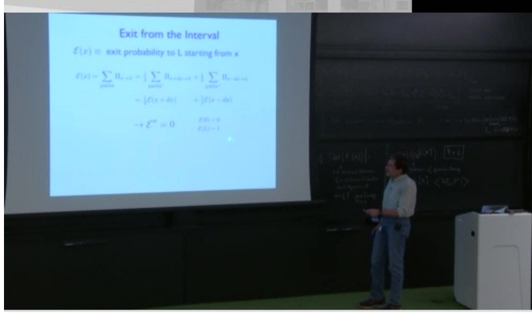
Exit from the Interval

$\mathcal{E}(x) \equiv$ exit probability to L starting from x

$$\mathcal{E}(x) = \sum_{\text{paths}} \Pi_{x\to L} = \tfrac{1}{2} \sum_{\text{paths}'} \Pi_{x+dx\to L} + \tfrac{1}{2} \sum_{\text{paths}''} \Pi_{x-dx\to L}$$

$$= \tfrac{1}{2}\mathcal{E}(x+dx) \qquad + \tfrac{1}{2}\mathcal{E}(x-dx)$$

$$\to \mathcal{E}'' = 0 \qquad \begin{array}{l} \mathcal{E}(0) = 0 \\ \mathcal{E}(L) = 1 \end{array}$$

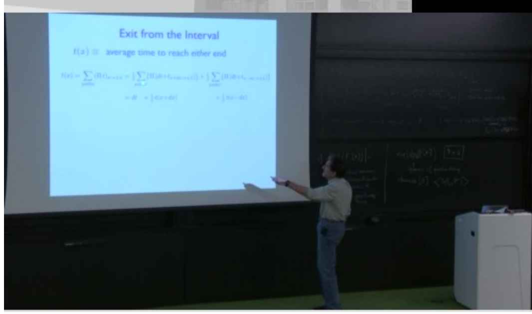


Exit from the Interval

$t(x) \equiv$ average time to reach *either* end

$$t(x) = \sum_{\text{paths}} (\Pi\, t)_{x\to\pm L} = \tfrac{1}{2}\sum_{\text{paths}'} \{\Pi\,[dt + t_{x+dx\to\pm L}]\} + \tfrac{1}{2}\sum_{\text{paths}''} \{\Pi\,[dt + t_{x-dx\to\pm L}]\}$$

$$= dt \quad + \tfrac{1}{2}\, t(x+dx) \qquad\qquad + \tfrac{1}{2}\, t(x-dx)$$



Exit from the Interval

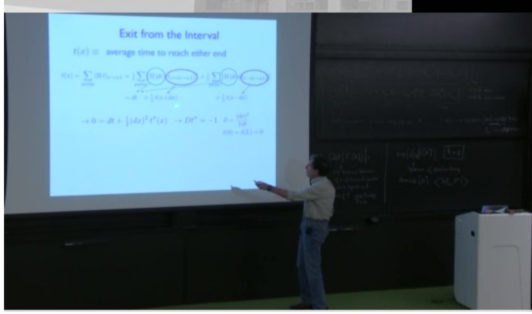
$t(x) \equiv$ average time to reach *either* end

$$t(x) = \sum_{\text{paths}} (\Pi\, t)_{x\to\pm L} = \tfrac{1}{2}\sum_{\text{paths}'} \left\{\Pi\left[dt + t_{x+dx\to\pm L}\right]\right\} + \tfrac{1}{2}\sum_{\text{paths}''} \left\{\Pi\left[dt + t_{x-dx\to\pm L}\right]\right\}$$

$$= dt \quad + \tfrac{1}{2}\, t(x+dx) \qquad + \tfrac{1}{2}\, t(x-dx)$$

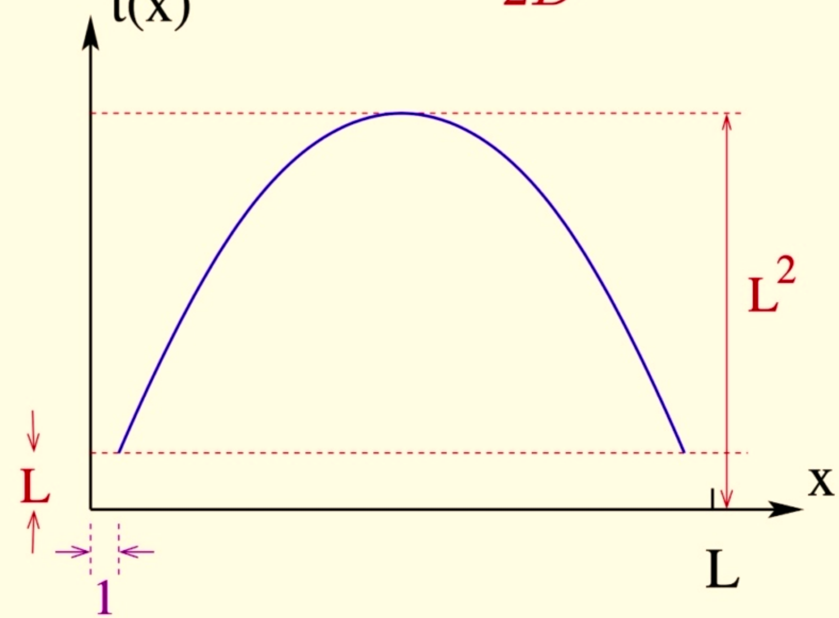
$$\to 0 = dt + \tfrac{1}{2}(dx)^2\, t''(x) \quad \to Dt'' = -1 \qquad D = \frac{(dx)^2}{2\,dt}$$

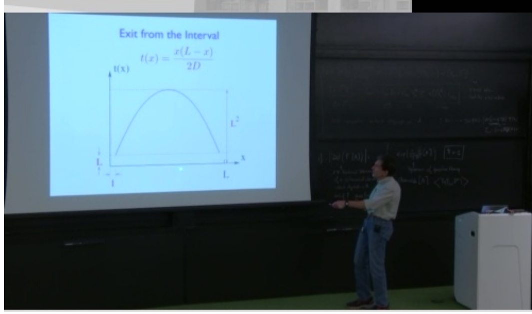
$$t(0) = t(L) = 0$$



Exit from the Interval

$$t(x) = \frac{x(L-x)}{2D}$$





Exit from the Interval

$t(x) \equiv$ average time to reach *either* end

$$t(x) = \sum_{\text{paths}} (\Pi\, t)_{x\to\pm L} = \tfrac{1}{2}\sum_{\text{paths}'}\{\Pi\,[dt + t_{x+dx\to\pm L}]\} + \tfrac{1}{2}\sum_{\text{paths}''}\{\Pi\,[dt + t_{x-dx\to\pm L}]\}$$

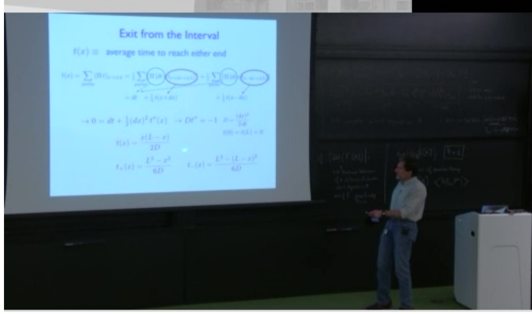
$$= dt \quad + \tfrac{1}{2}\, t(x+dx) \qquad\qquad + \tfrac{1}{2}\, t(x-dx)$$

$$\to 0 = dt + \tfrac{1}{2}(dx)^2\, t''(x) \quad \to Dt'' = -1 \qquad D = \frac{(dx)^2}{2\,dt}$$

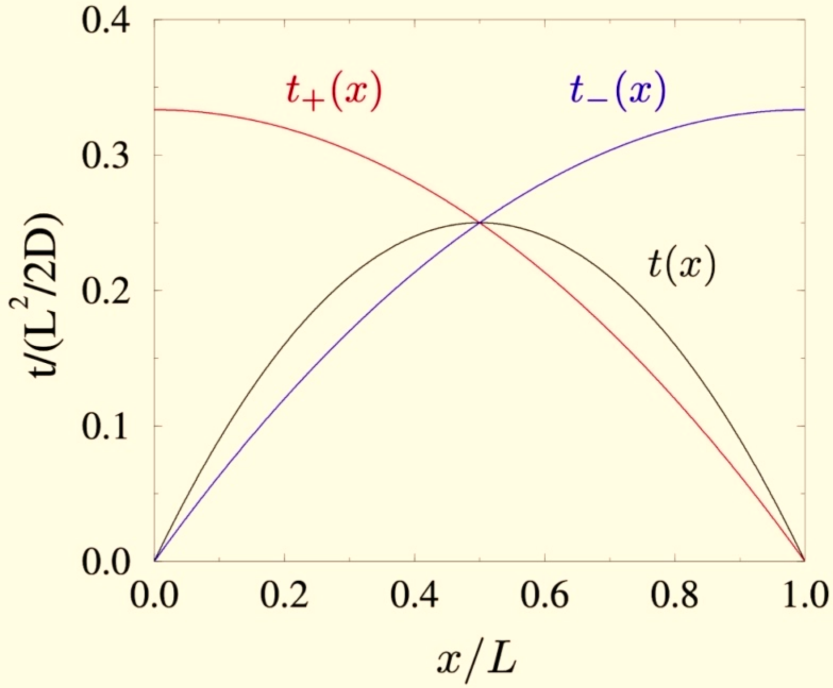
$$t(0) = t(L) = 0$$

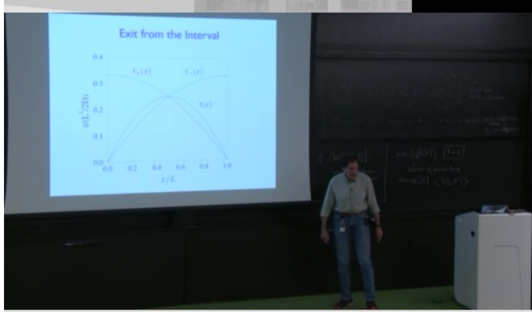
$$t(x) = \frac{x(L-x)}{2D}$$

$$t_+(x) = \frac{L^2 - x^2}{6D} \qquad t_-(x) = \frac{L^2 - (L-x)^2}{6D}$$

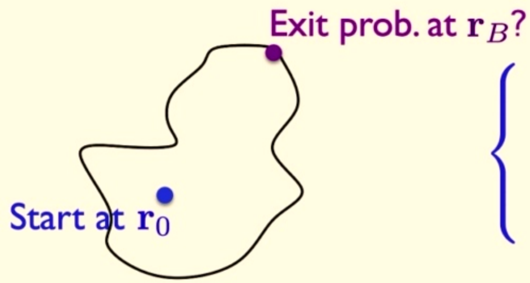


Exit from the Interval

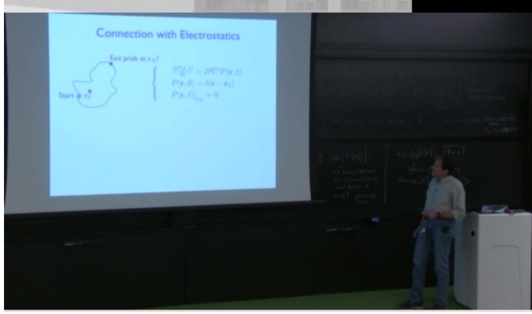




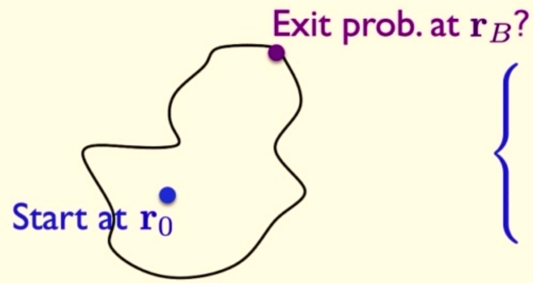
Connection with Electrostatics



$$
\begin{cases}
\frac{\partial P(\mathbf{r},t)}{\partial t} = D\nabla^2 P(\mathbf{r},t) \\
P(\mathbf{r},0) = \delta(\mathbf{r}-\mathbf{r}_0) \\
P(\mathbf{r},t)\big|_{\mathbf{r}_B} = 0
\end{cases}
$$



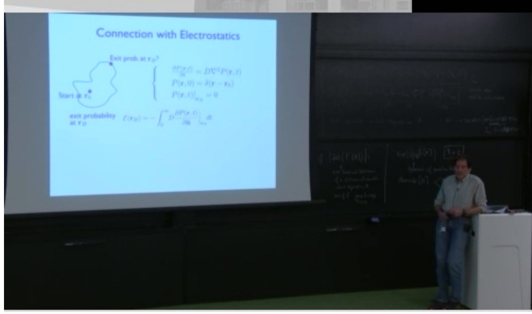
Connection with Electrostatics



$$\begin{cases} \frac{\partial P(\mathbf{r},t)}{\partial t} = D\nabla^2 P(\mathbf{r},t) \\ P(\mathbf{r},0) = \delta(\mathbf{r}-\mathbf{r}_0) \\ P(\mathbf{r},t)\big|_{\mathbf{r}_B} = 0 \end{cases}$$

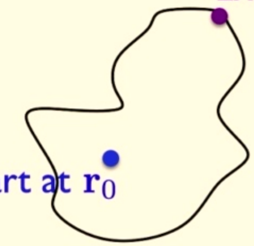
exit probability at $\mathbf{r}_B$

$$\mathcal{E}(\mathbf{r}_B) = -\int_0^\infty D\frac{\partial P(\mathbf{r},t)}{\partial \hat{\mathbf{n}}}\bigg|_{\mathbf{r}_B} dt$$



Connection with Electrostatics

Exit prob. at $\mathbf{r}_B$?



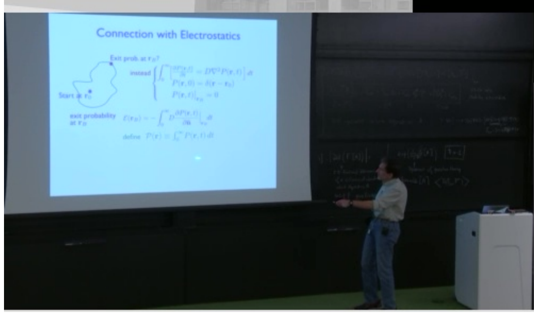
Start at $\mathbf{r}_0$

instead $\begin{cases} \int_0^\infty \left[\frac{\partial P(\mathbf{r},t)}{\partial t} = D\nabla^2 P(\mathbf{r},t) \right] dt \\ P(\mathbf{r},0) = \delta(\mathbf{r}-\mathbf{r}_0) \\ P(\mathbf{r},t)\big|_{\mathbf{r}_B} = 0 \end{cases}$

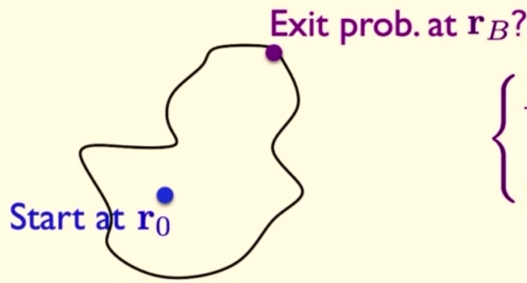
exit probability at $\mathbf{r}_B$

$$\mathcal{E}(\mathbf{r}_B) = -\int_0^\infty D \frac{\partial P(\mathbf{r},t)}{\partial \hat{\mathbf{n}}}\Big|_{\mathbf{r}_B} dt$$

define $\mathcal{P}(\mathbf{r}) \equiv \int_0^\infty P(\mathbf{r},t)\, dt$



Connection with Electrostatics

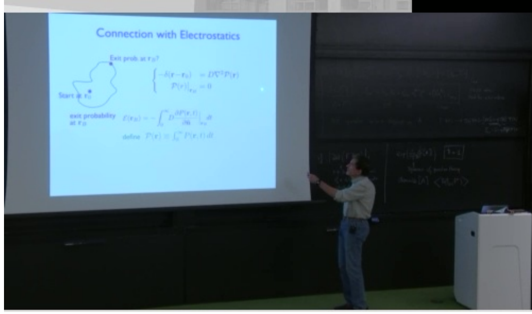


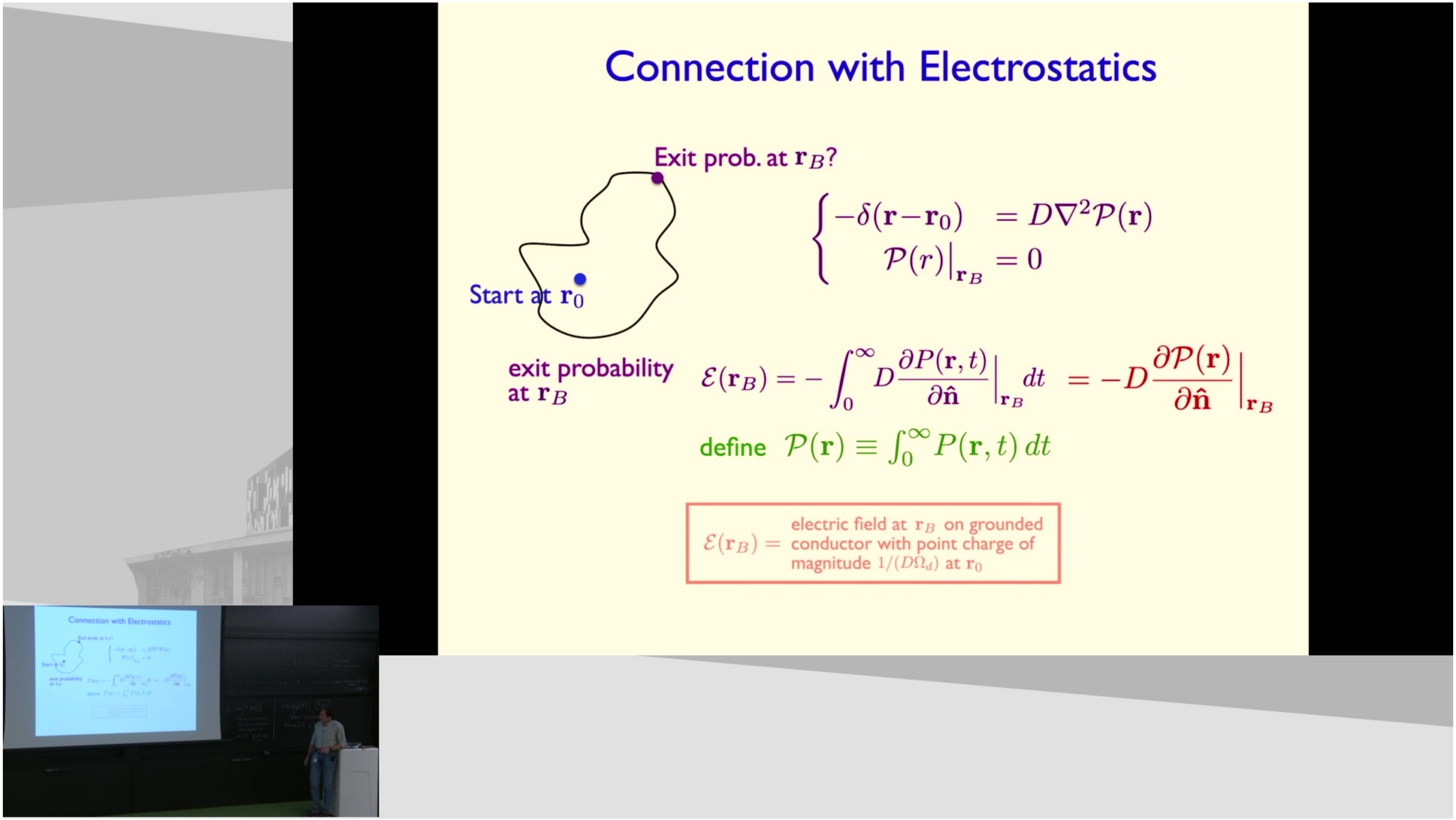
$$\begin{cases} -\delta(\mathbf{r}-\mathbf{r}_0) & = D\nabla^2\mathcal{P}(\mathbf{r}) \\ \quad \mathcal{P}(r)\big|_{\mathbf{r}_B} & = 0 \end{cases}$$

exit probability at $\mathbf{r}_B$

$$\mathcal{E}(\mathbf{r}_B) = -\int_0^\infty D\frac{\partial P(\mathbf{r},t)}{\partial \hat{\mathbf{n}}}\Big|_{\mathbf{r}_B} dt$$

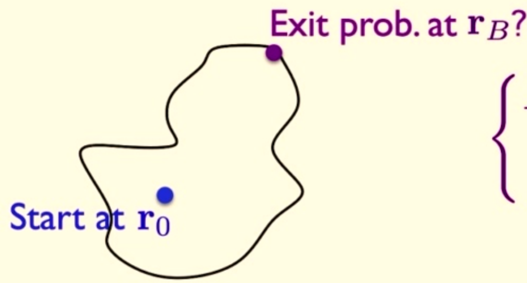
define $\mathcal{P}(\mathbf{r}) \equiv \int_0^\infty P(\mathbf{r},t)\,dt$





Connection with Electrostatics
Exit prob. at $\mathbf{r}_B$?
Start at $\mathbf{r}_0$
$\begin{cases} -\delta(\mathbf{r}-\mathbf{r}_0) & = D\nabla^2\mathcal{P}(\mathbf{r}) \\ \mathcal{P}(r)\big|_{\mathbf{r}_B} & = 0 \end{cases}$
exit probability at $\mathbf{r}_B$
$\mathcal{E}(\mathbf{r}_B) = -\int_0^\infty D\frac{\partial P(\mathbf{r},t)}{\partial\hat{\mathbf{n}}}\Big|_{\mathbf{r}_B} dt \;\; = -D\frac{\partial\mathcal{P}(\mathbf{r})}{\partial\hat{\mathbf{n}}}\Big|_{\mathbf{r}_B}$
define $\mathcal{P}(\mathbf{r}) \equiv \int_0^\infty P(\mathbf{r},t)\,dt$
$\mathcal{E}(\mathbf{r}_B) =$ electric field at $\mathbf{r}_B$ on grounded conductor with point charge of magnitude $1/(D\Omega_d)$ at $\mathbf{r}_0$

Connection with Electrostatics



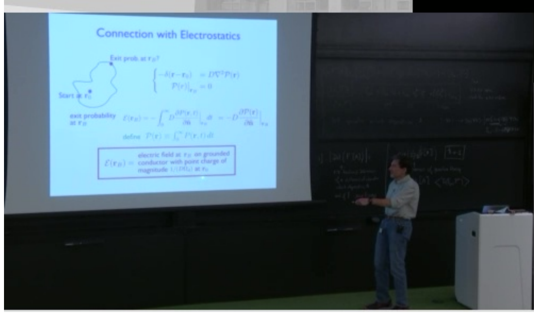
$$\begin{cases} -\delta(\mathbf{r}-\mathbf{r}_0) & = D\nabla^2\mathcal{P}(\mathbf{r}) \\ \mathcal{P}(r)\big|_{\mathbf{r}_B} & = 0 \end{cases}$$

exit probability at $\mathbf{r}_B$

$$\mathcal{E}(\mathbf{r}_B) = -\int_0^\infty D\frac{\partial P(\mathbf{r},t)}{\partial\hat{\mathbf{n}}}\Big|_{\mathbf{r}_B} dt \;\; = -D\frac{\partial\mathcal{P}(\mathbf{r})}{\partial\hat{\mathbf{n}}}\Big|_{\mathbf{r}_B}$$

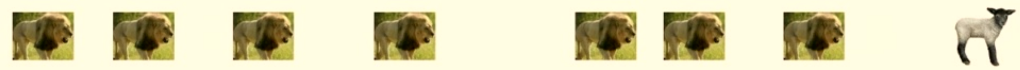
define $\mathcal{P}(\mathbf{r}) \equiv \int_0^\infty P(\mathbf{r},t)\,dt$

$\mathcal{E}(\mathbf{r}_B) =$ electric field at $\mathbf{r}_B$ on grounded conductor with point charge of magnitude $1/(D\Omega_d)$ at $\mathbf{r}_0$



Stochastic Hunting in One Dimension

Krapivsky & SR (1999)
ben-Avraham et al (2003)



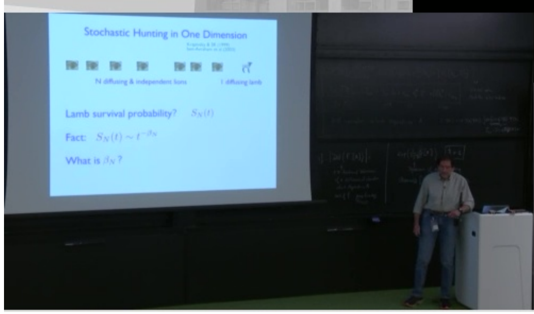
N diffusing & independent lions

1 diffusing lamb

Lamb survival probability? $S_N(t)$

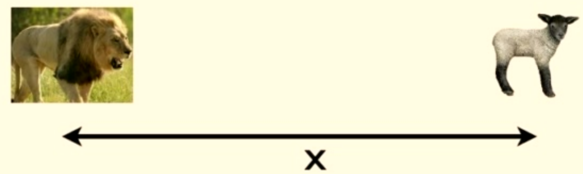
Fact: $S_N(t) \sim t^{-\beta_N}$

What is β_N ?



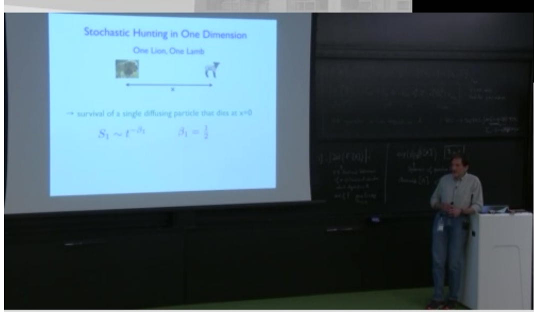
Stochastic Hunting in One Dimension

One Lion, One Lamb



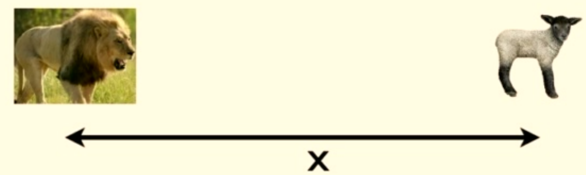
→ survival of a single diffusing particle that dies at x=0

$$S_1 \sim t^{-\beta_1} \qquad \beta_1 = \frac{1}{2}$$



Stochastic Hunting in One Dimension

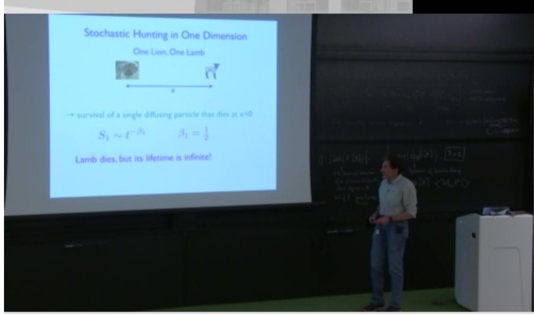
One Lion, One Lamb

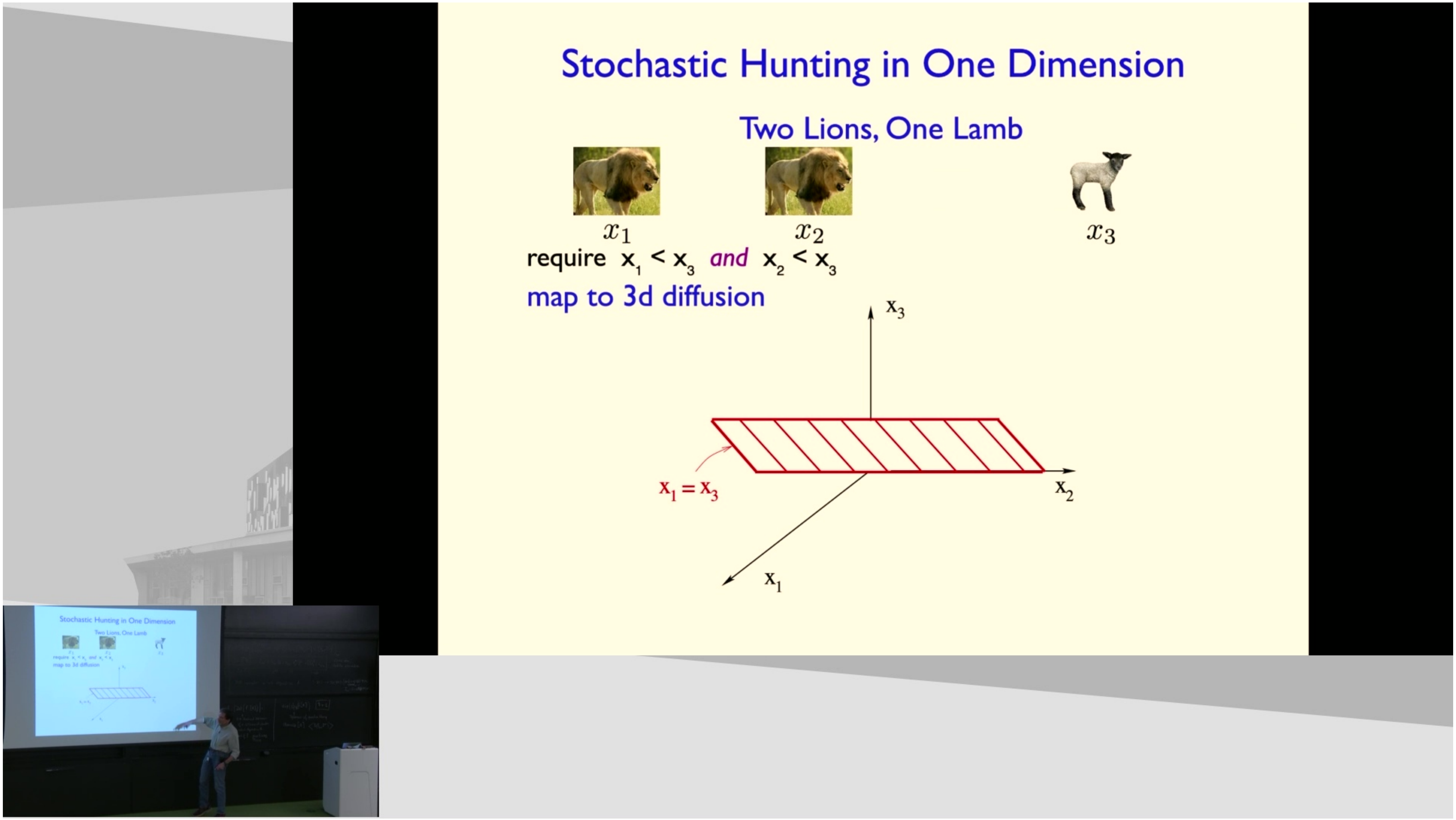


→ survival of a single diffusing particle that dies at x=0

$$S_1 \sim t^{-\beta_1} \qquad \beta_1 = \frac{1}{2}$$

Lamb dies, but its lifetime is infinite!



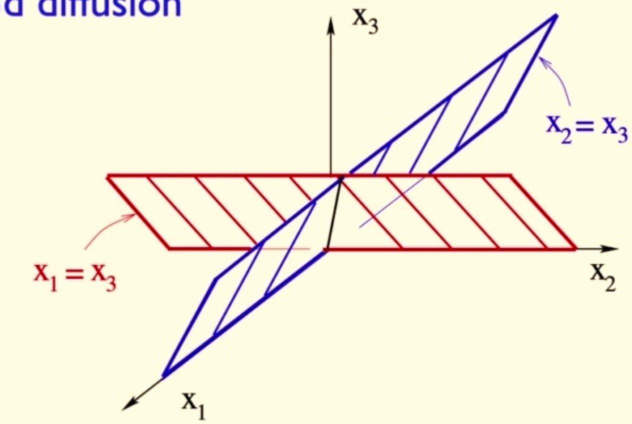

Stochastic Hunting in One Dimension
Two Lions, One Lamb
x_1
x_2
x_3
require $x_1 < x_3$ *and* $x_2 < x_3$
map to 3d diffusion
x_3
$x_1 = x_3$
x_2
x_1

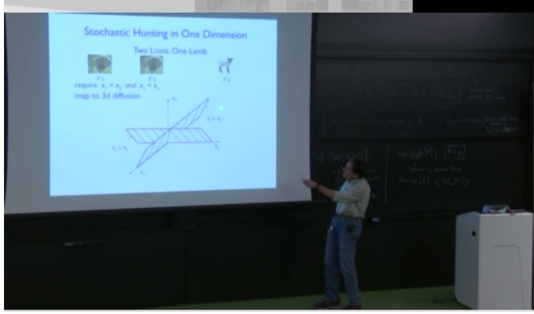
Stochastic Hunting in One Dimension

Two Lions, One Lamb

require $x_1 < x_3$ *and* $x_2 < x_3$

map to 3d diffusion





Stochastic Hunting in One Dimension

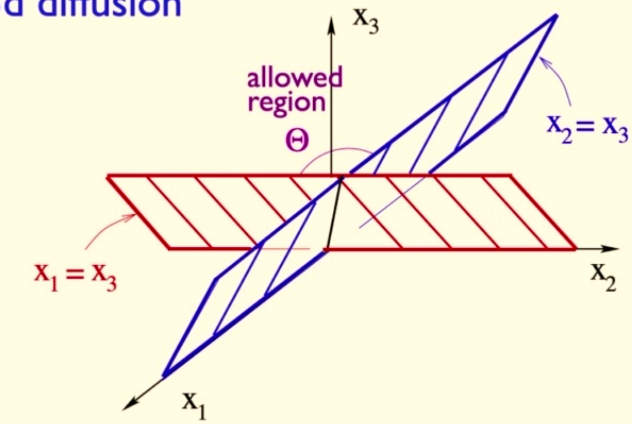
Two Lions, One Lamb

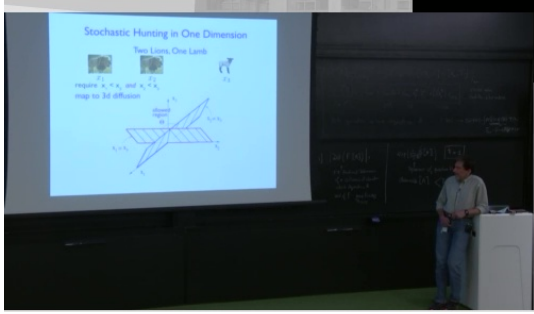
  

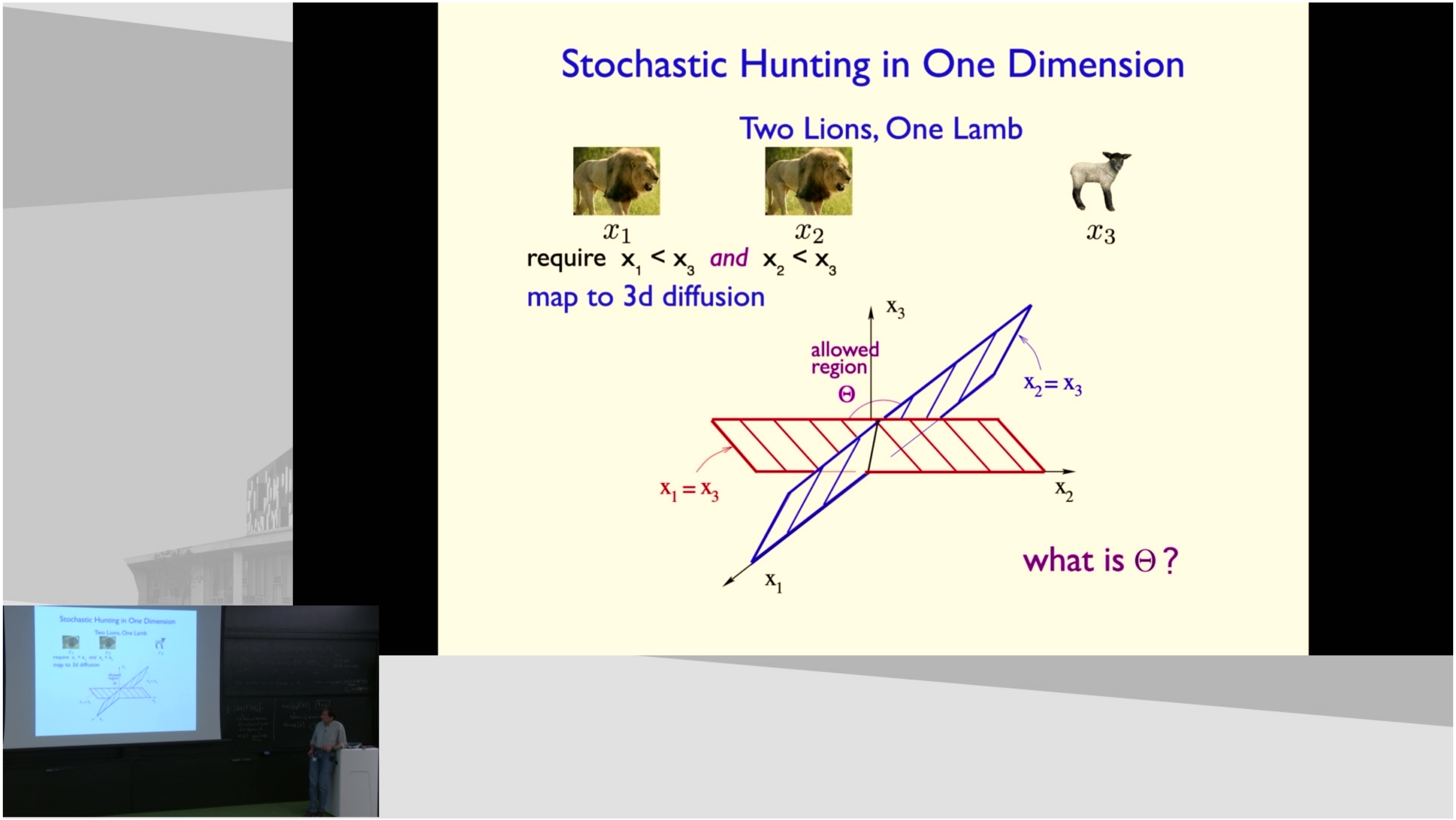
x_1 x_2 x_3

require $x_1 < x_3$ *and* $x_2 < x_3$

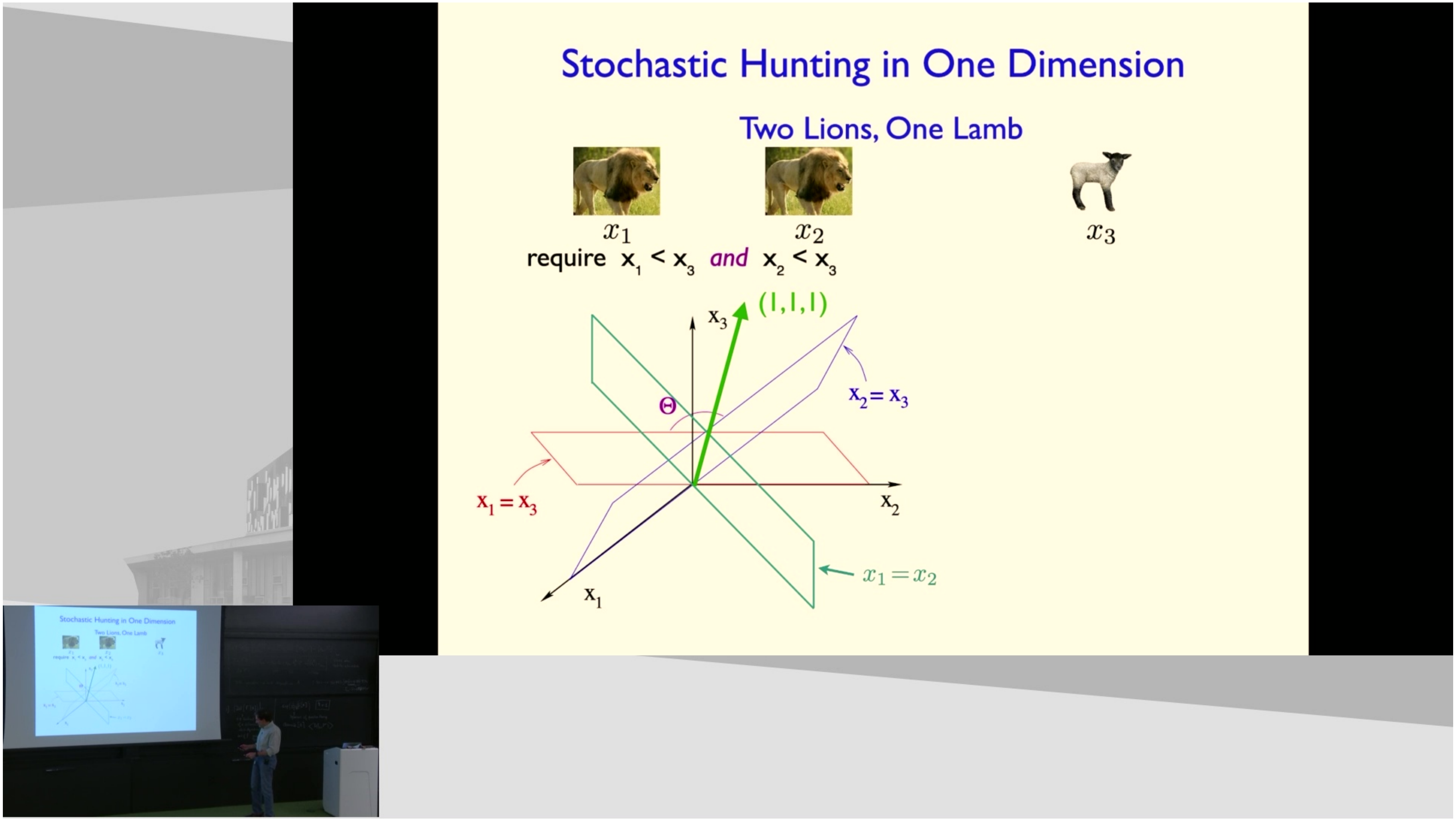
map to 3d diffusion

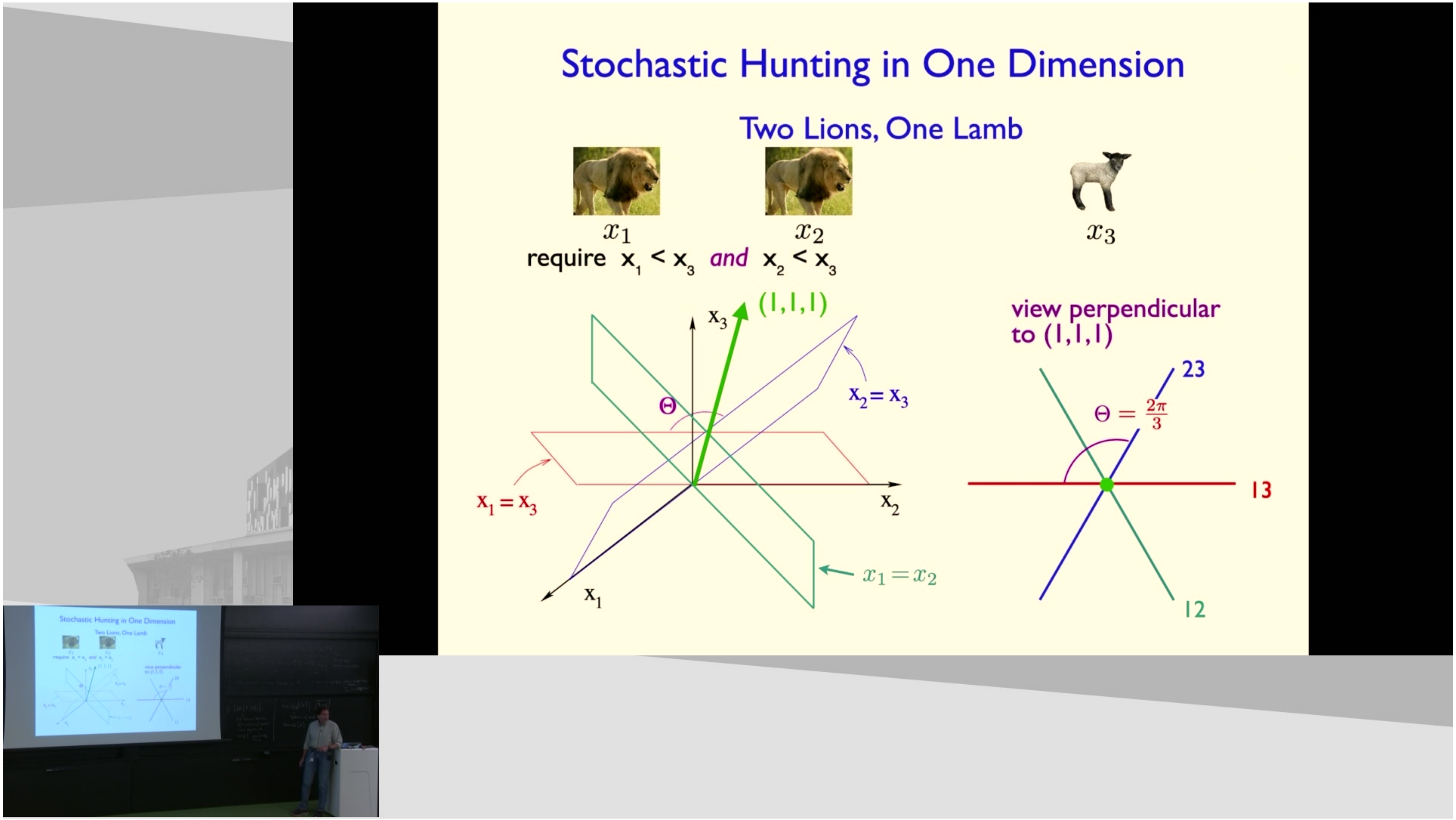






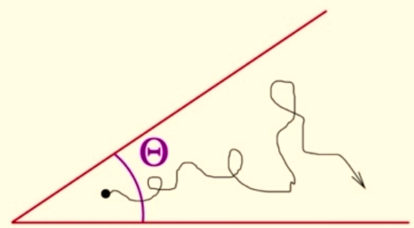
Stochastic Hunting in One Dimension
Two Lions, One Lamb
x_1
x_2
x_3
require $x_1 < x_3$ *and* $x_2 < x_3$
map to 3d diffusion
x_3
allowed region
Θ
$x_2 = x_3$
$x_1 = x_3$
x_2
x_1
what is Θ ?


Stochastic Hunting in One Dimension
Two Lions, One Lamb
x_1
x_2
x_3
require $x_1 < x_3$ *and* $x_2 < x_3$
(1,1,1)
x_3
$x_2 = x_3$
Θ
$x_1 = x_3$
x_2
$x_1 = x_2$
x_1


Stochastic Hunting in One Dimension
Two Lions, One Lamb
x_1
x_2
x_3
require $x_1 < x_3$ and $x_2 < x_3$
(1,1,1)
x_3
Θ
$x_2 = x_3$
$x_1 = x_3$
x_2
x_1
$x_1 = x_2$
view perpendicular to (1,1,1)
23
$\Theta = \frac{2\pi}{3}$
13
12

Stochastic Hunting in One Dimension

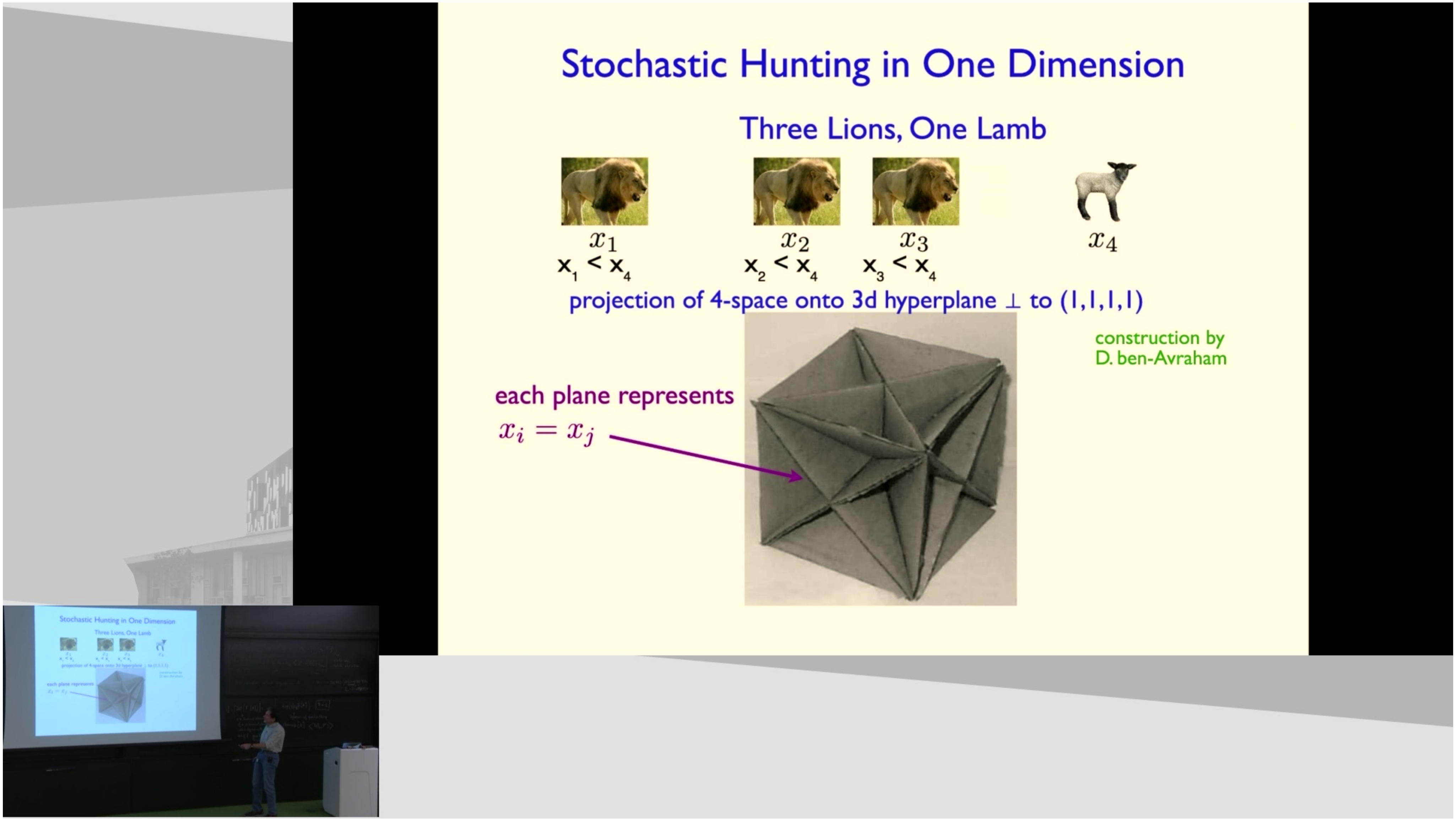
Diffusion in Absorbing Wedge

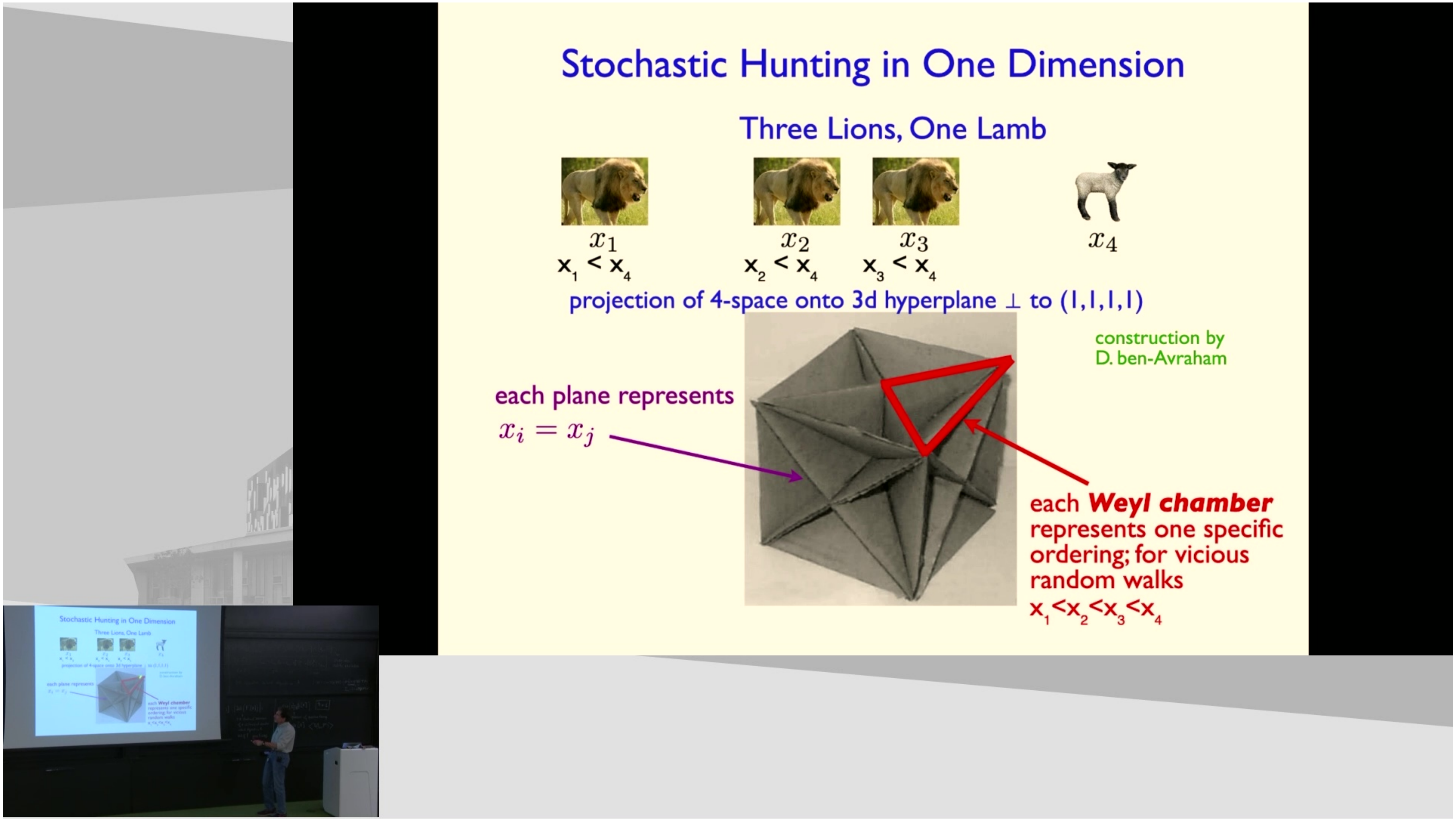


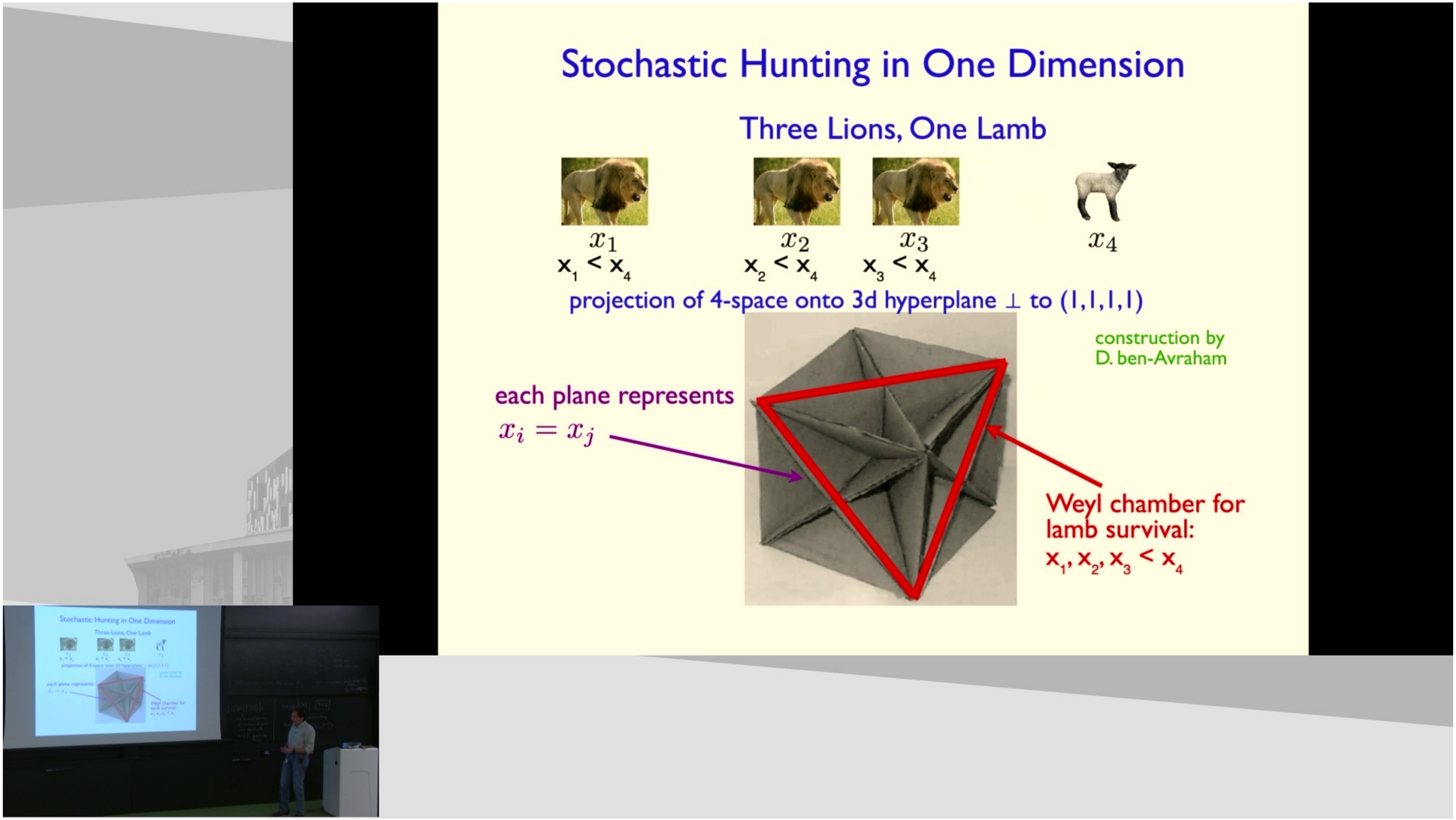
$$S(t) \sim t^{-\pi/2\Theta} \to t^{-3/4}$$

Carslaw & Jaeger (1959)

Lamb dies, but its lifetime is infinite!


Stochastic Hunting in One Dimension
Three Lions, One Lamb
x1 x2 x3 x4
x1 < x4
x2 < x4
x3 < x4
projection of 4-space onto 3d hyperplane ⊥ to (1,1,1,1)
construction by
D. ben-Avraham
each plane represents
xi = xj


Stochastic Hunting in One Dimension
Three Lions, One Lamb
x_1
$x_1 < x_4$
x_2
$x_2 < x_4$
x_3
$x_3 < x_4$
x_4
projection of 4-space onto 3d hyperplane ⊥ to (1,1,1,1)
construction by D. ben-Avraham
each plane represents $x_i = x_j$
each Weyl chamber represents one specific ordering; for vicious random walks $x_1<x_2<x_3<x_4$


Stochastic Hunting in One Dimension
Three Lions, One Lamb
x_1
x_2
x_3
x_4
$x_1 < x_4$
$x_2 < x_4$
$x_3 < x_4$
projection of 4-space onto 3d hyperplane ⊥ to (1,1,1,1)
construction by D. ben-Avraham
each plane represents
$x_i = x_j$
Weyl chamber for lamb survival:
$x_1, x_2, x_3 < x_4$

Stochastic Hunting in One Dimension

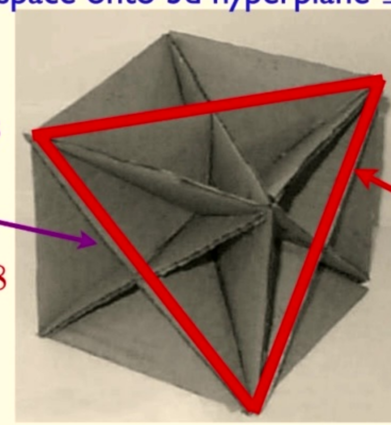
Three Lions, One Lamb

x_1 x_2 x_3 x_4

$x_1 < x_4$ $x_2 < x_4$ $x_3 < x_4$

projection of 4-space onto 3d hyperplane $\perp$ to (1,1,1,1)

construction by D. ben-Avraham

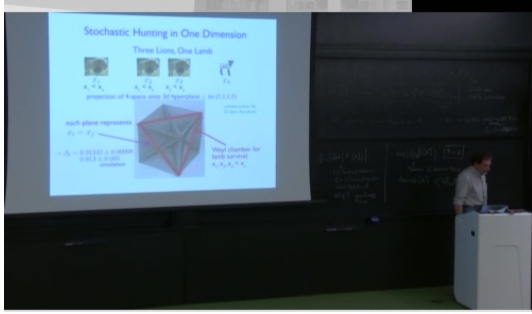


each plane represents $x_i = x_j$

$\rightarrow \beta_3 = 0.91342 \pm 0.00008$

0.913 ± 0.005 simulation

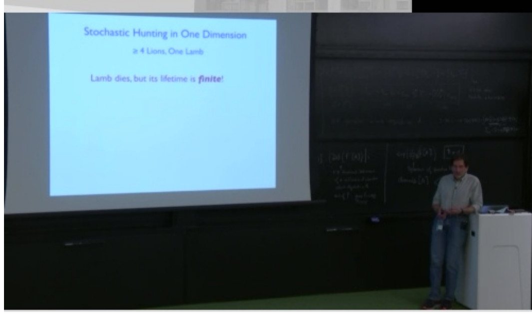
Weyl chamber for lamb survival: $x_1, x_2, x_3 < x_4$

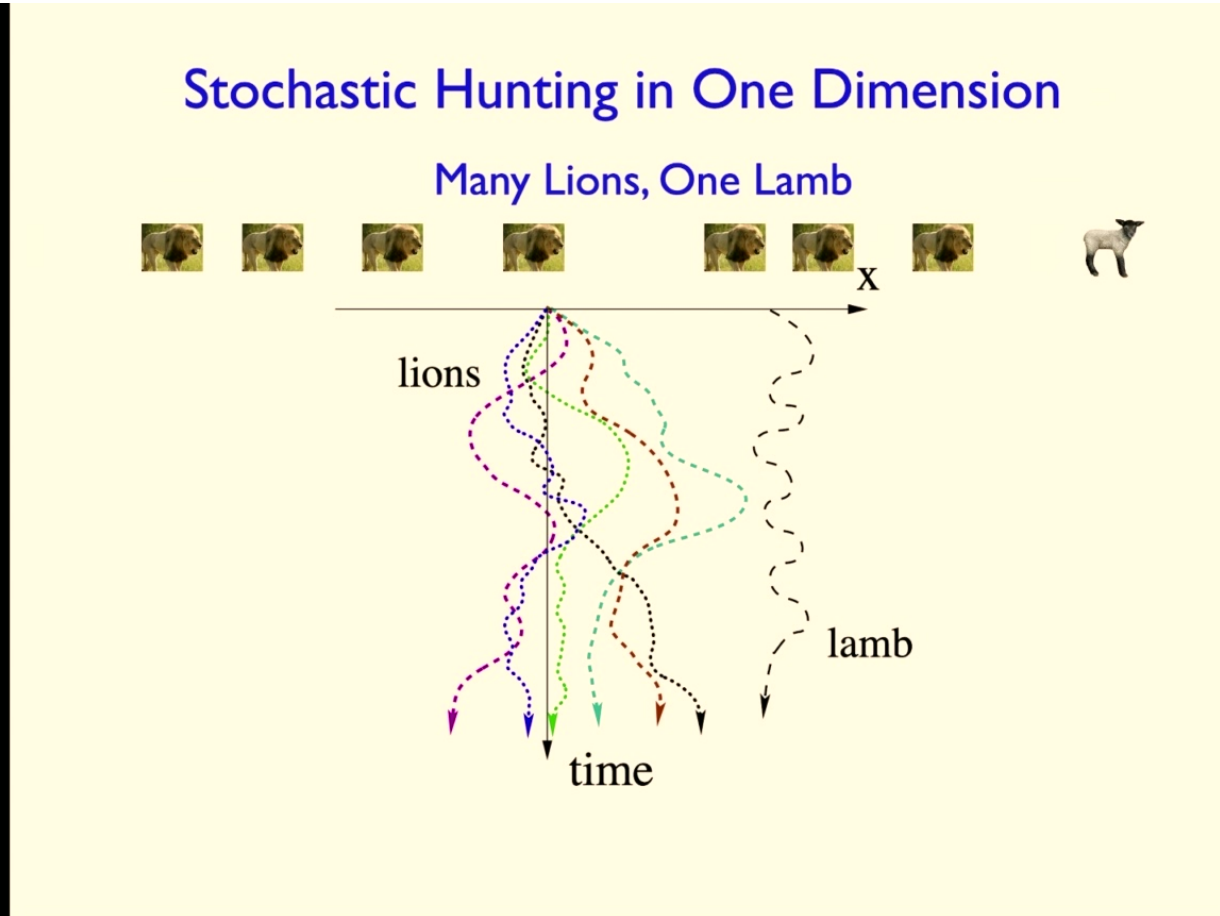


Stochastic Hunting in One Dimension

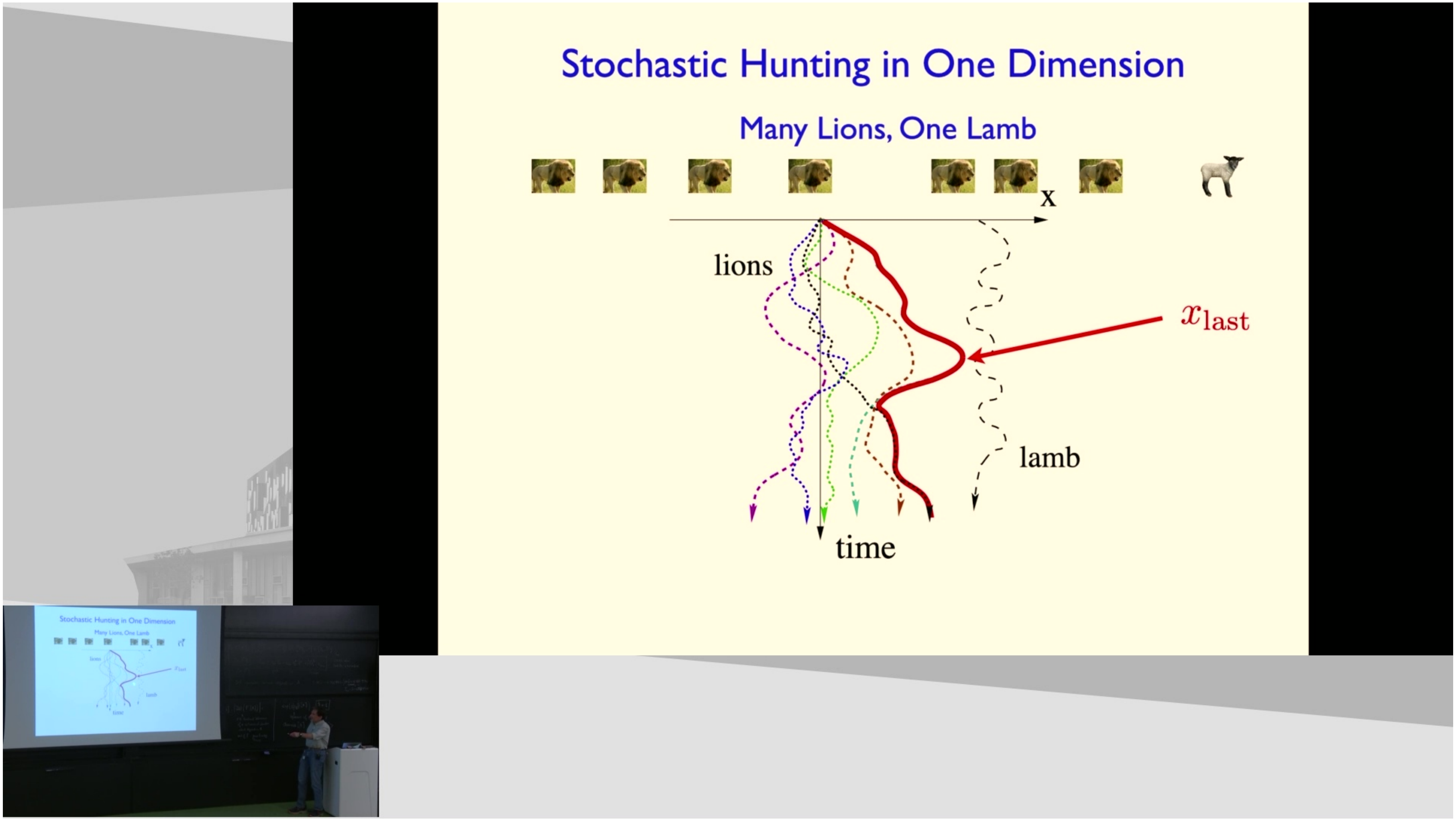
≥ 4 Lions, One Lamb

Lamb dies, but its lifetime is ***finite***!



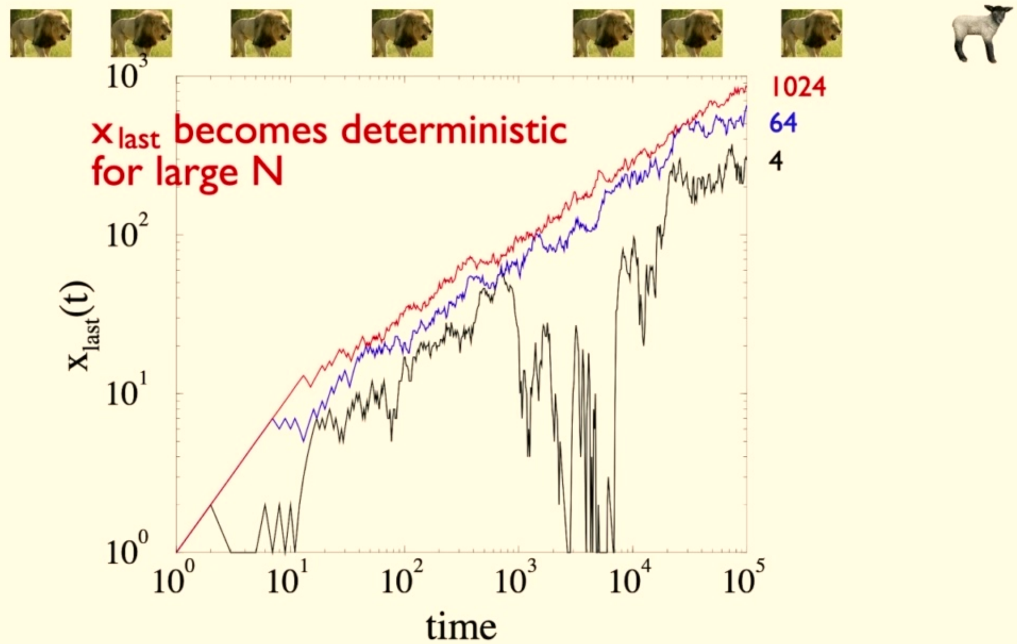


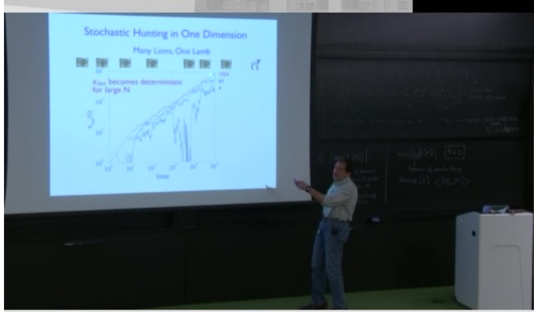
Stochastic Hunting in One Dimension
Many Lions, One Lamb
x
lions
lamb
time


Stochastic Hunting in One Dimension
Many Lions, One Lamb
x
lions
$x_{\rm last}$
lamb
time

Stochastic Hunting in One Dimension

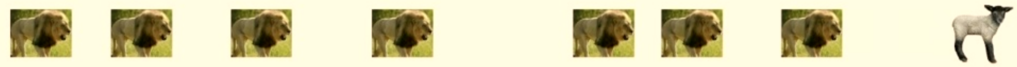
Many Lions, One Lamb





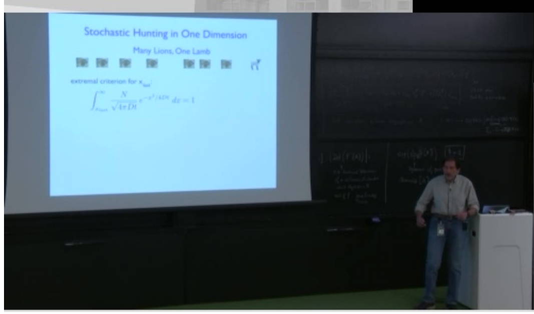
Stochastic Hunting in One Dimension

Many Lions, One Lamb



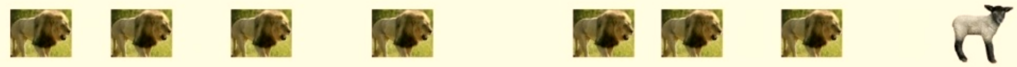
extremal criterion for x_{last}:

$$\int_{x_{\rm last}}^{\infty} \frac{N}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt}\, dx = 1$$



Stochastic Hunting in One Dimension

Many Lions, One Lamb



extremal criterion for x_{last}:

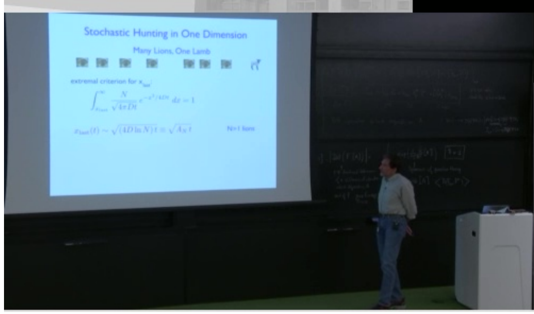
$$\int_{x_{\text{last}}}^{\infty} \frac{N}{\sqrt{4\pi D t}}\, e^{-x^2/4Dt}\, dx = 1$$

$$x_{\text{last}}(t) \sim \sqrt{(4D\ln N)\,t} \equiv \sqrt{A_N\, t}$$ N≫1 lions

$$x_{\text{last}}(t) \sim \sqrt{2D\ln(c_0^2 Dt)\,t}$$ N=∞ lions

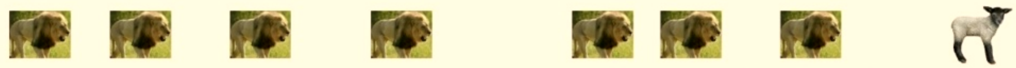
constant density for x<0

only $N \sim c_0\sqrt{Dt}$ are "dangerous"



Stochastic Hunting in One Dimension

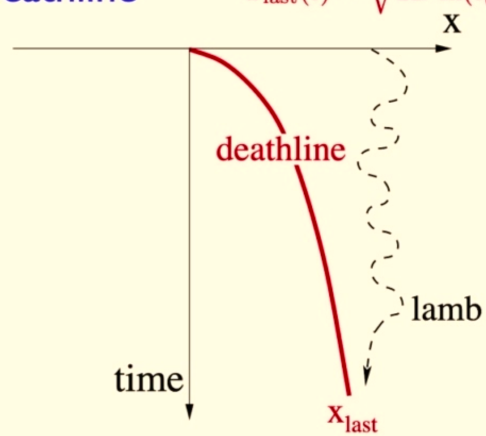
Many Lions, One Lamb



Effective Problem:
Deterministic Deathline

$$x_{\rm last}(t) \sim \sqrt{(4D\ln N)\,t} \equiv \sqrt{A_N\,t}$$

$$x_{\rm last}(t) \sim \sqrt{2D\ln(c_0^2 Dt)\,t} \qquad N = \infty$$



Stochastic Hunting in One Dimension

Many Lions, One Lamb

lamb probability distribution:

$$\frac{\partial p(x,t)}{\partial t} - \frac{x_{\text{last}}}{2t}\frac{\partial p(x,t)}{\partial x} = D\,\frac{\partial^2 p(x,t)}{\partial x^2} \qquad (0 \le x < \infty)$$

scaling ansatz: $p(x,t) \sim t^{-\beta_N - 1/2}\,F(\xi) \qquad \xi = x/x_{\text{last}}$

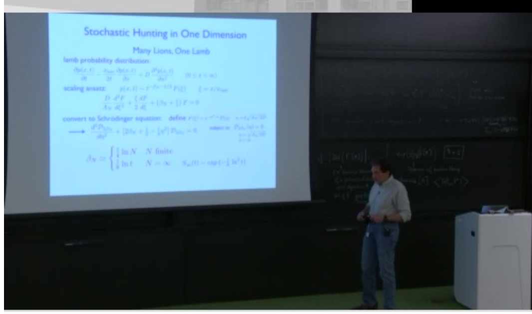
$$\frac{D}{A_N}\frac{d^2F}{d\xi^2} + \frac{\xi}{2}\frac{dF}{d\xi} + \left(\beta_N + \tfrac{1}{2}\right)F = 0$$

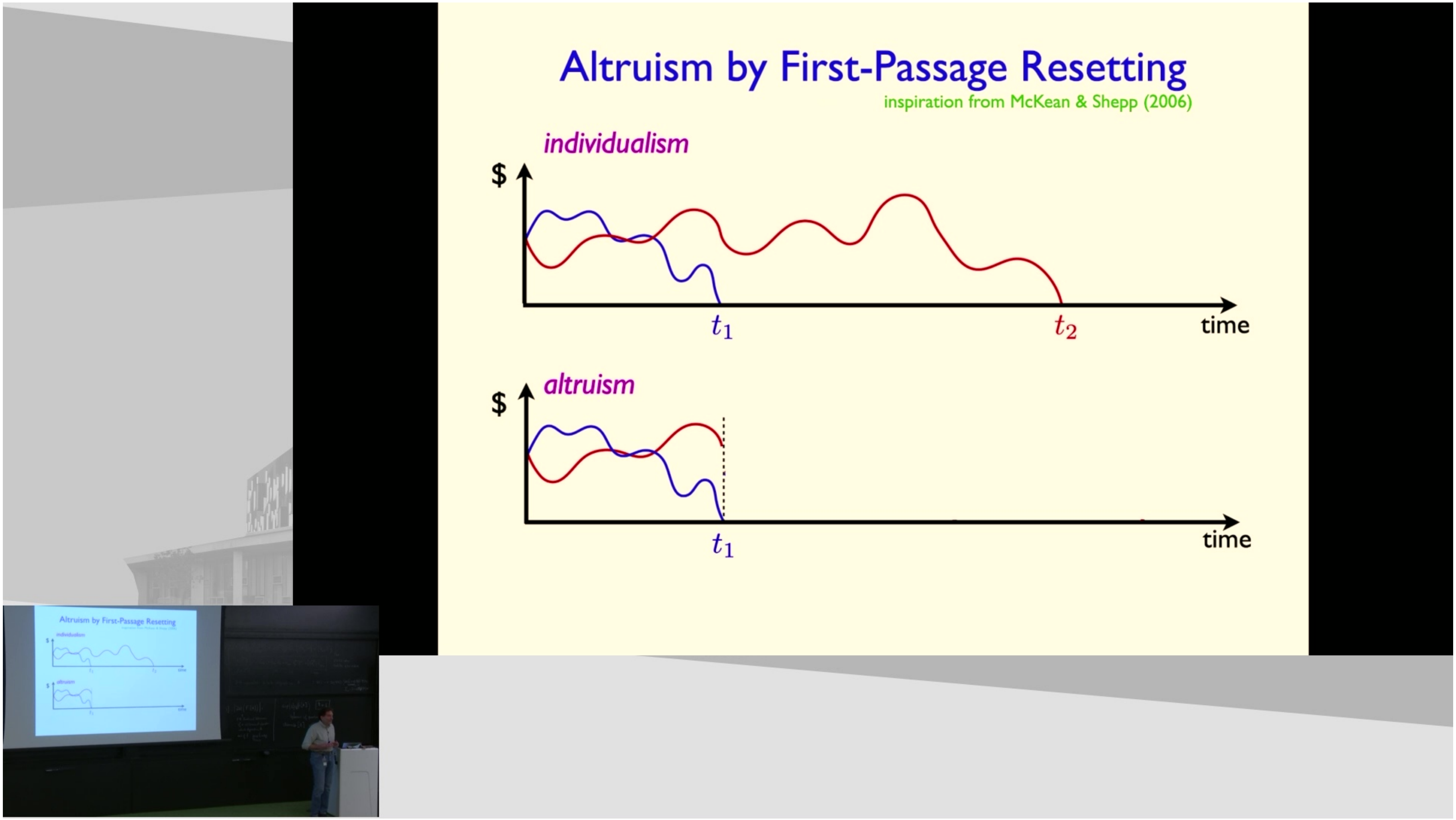
convert to Schrödinger equation: **define** $F(\xi) = e^{-\eta^2/4}\,\mathcal{D}(\eta) \qquad \eta = \xi\sqrt{A_N/2D}$

$$\longrightarrow \frac{d^2\mathcal{D}_{2\beta_N}}{d\eta^2} + \left[2\beta_N + \tfrac{1}{2} - \tfrac{1}{4}\eta^2\right]\mathcal{D}_{2\beta_N} = 0$$

subject to $\mathcal{D}_{2\beta_N}(\eta) = 0$ at $\eta = \sqrt{A_N/2D}$, $\eta = \infty$

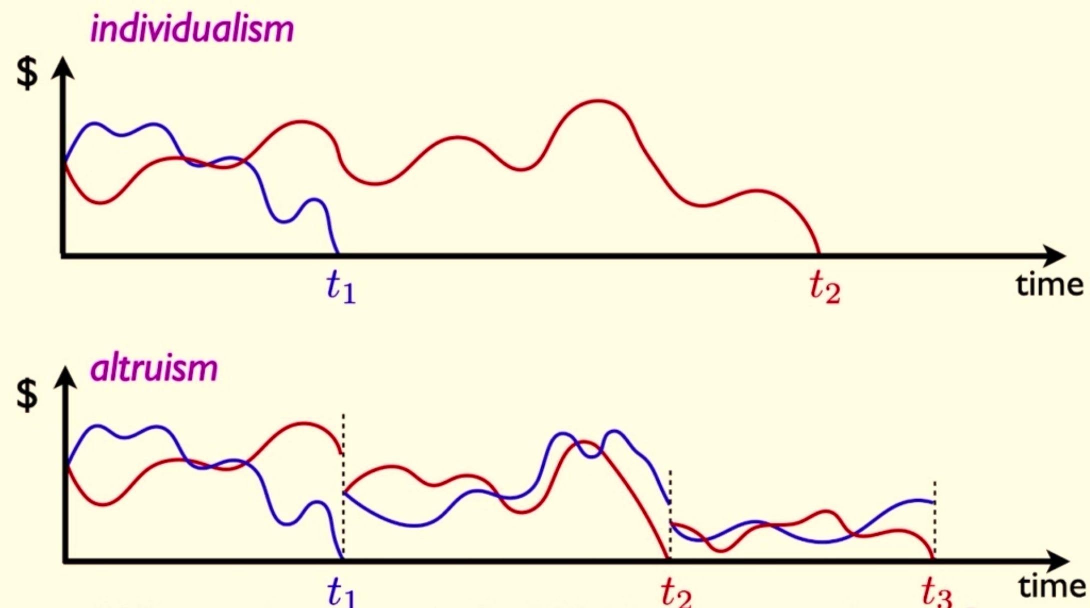
$$\beta_N \simeq \begin{cases} \frac{1}{4}\ln N & N \text{ finite} \\ \frac{1}{8}\ln t & N = \infty \end{cases} \qquad S_\infty(t) \sim \exp\left(-\tfrac{1}{8}\,\ln^2 t\right)$$



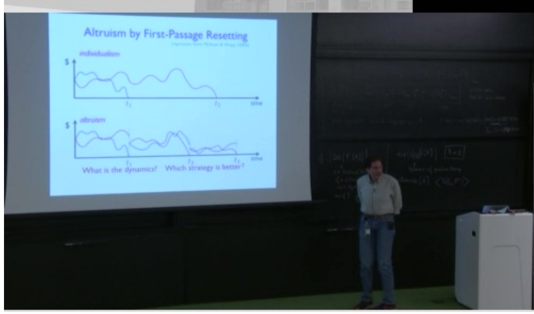

Altruism by First-Passage Resetting
inspiration from McKean & Shepp (2006)
individualism
$
t_1
t_2
time
altruism
$
t_1
time

Altruism by First-Passage Resetting

inspiration from McKean & Shepp (2006)



What is the dynamics? Which strategy is better?



Dynamics of Individualism

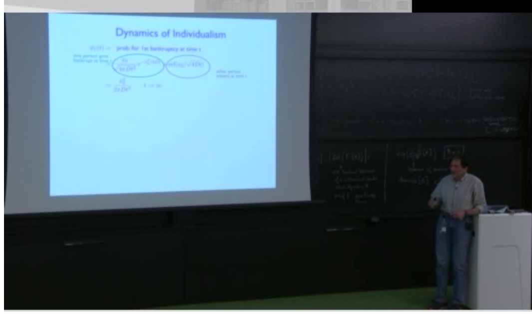
$\phi_1(t) =$ prob. for 1st bankruptcy at time t

one person goes bankrupt at time t

$$= \frac{x_0}{\sqrt{4\pi D t^3}}\, e^{-x_0^2/4Dt} \times \mathrm{erf}(x_0/\sqrt{4Dt})$$

other person solvent at time t

$$\simeq \frac{x_0^2}{2\pi D t^2} \qquad t \to \infty$$



Dynamics of Individualism

$\phi_1(t) =$ **prob. for 1st bankruptcy at time t**

$$= \frac{x_0}{\sqrt{4\pi D t^3}}\, e^{-x_0^2/4Dt} \times \mathrm{erf}(x_0/\sqrt{4Dt})$$

one person goes bankrupt at time t

other person solvent at time t

$$\simeq \frac{x_0^2}{2\pi D t^2} \qquad t \to \infty$$

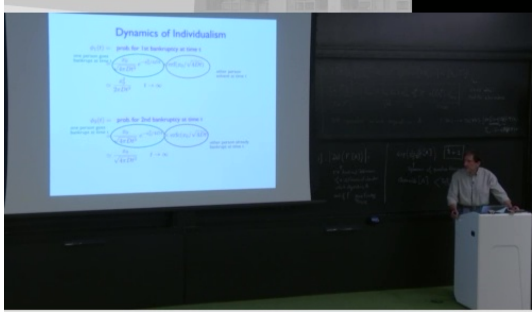
$\phi_2(t) =$ **prob. for 2nd bankruptcy at time t**

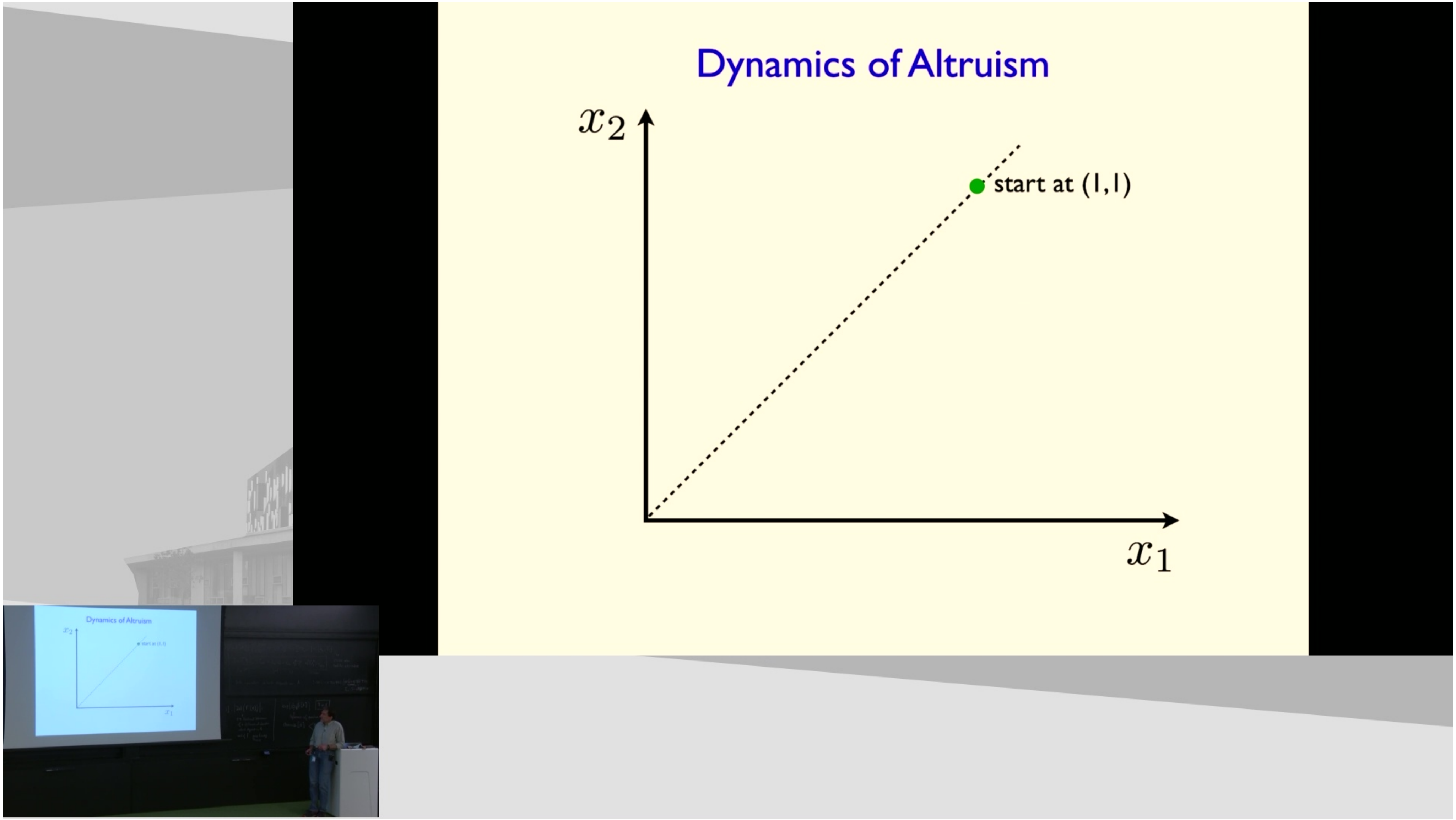
$$= \frac{x_0}{\sqrt{4\pi D t^3}}\, e^{-x_0^2/4Dt} \times \mathrm{erfc}(x_0/\sqrt{4Dt})$$

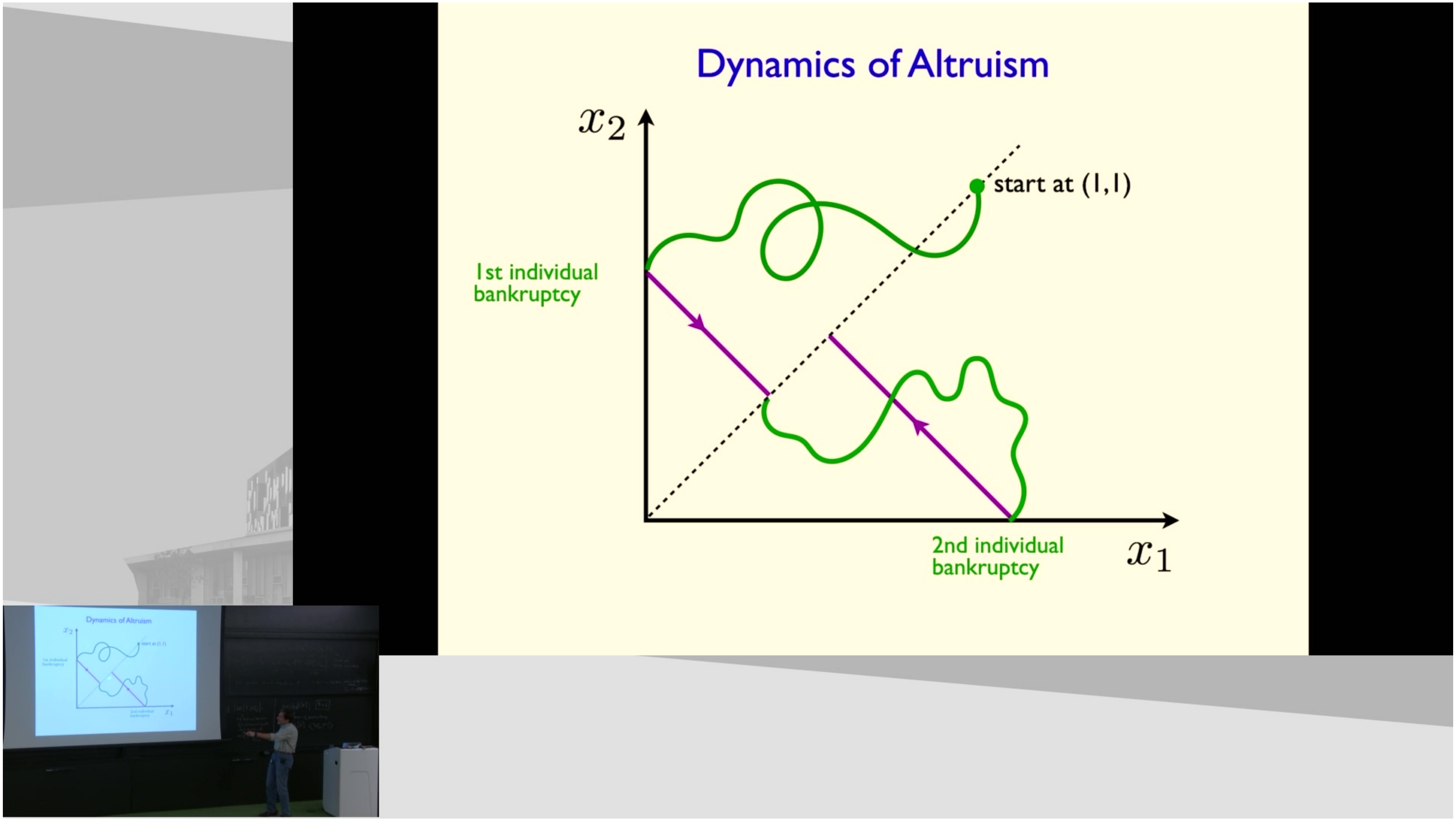
one person goes bankrupt at time t

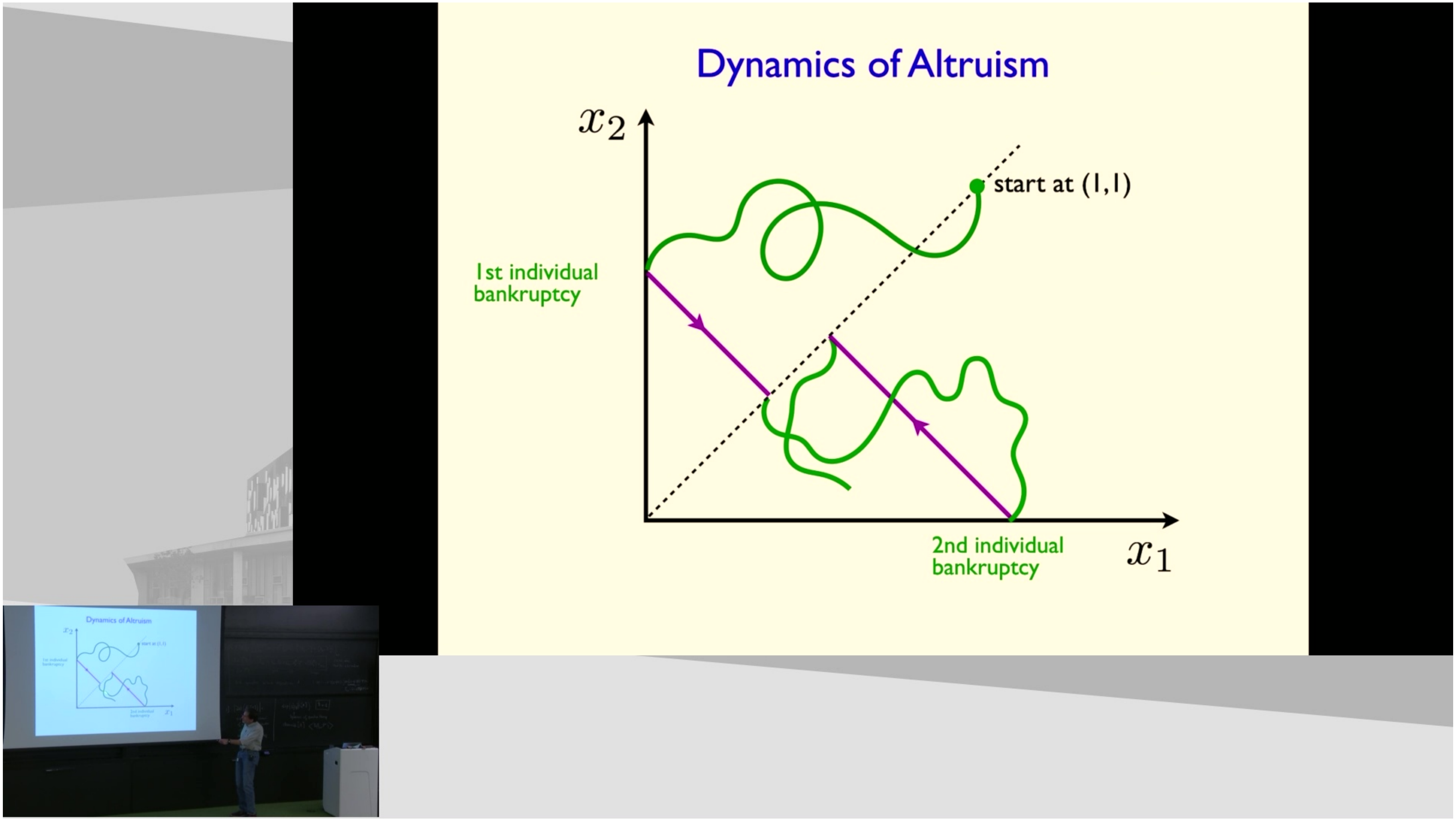
other person already bankrupt at time t

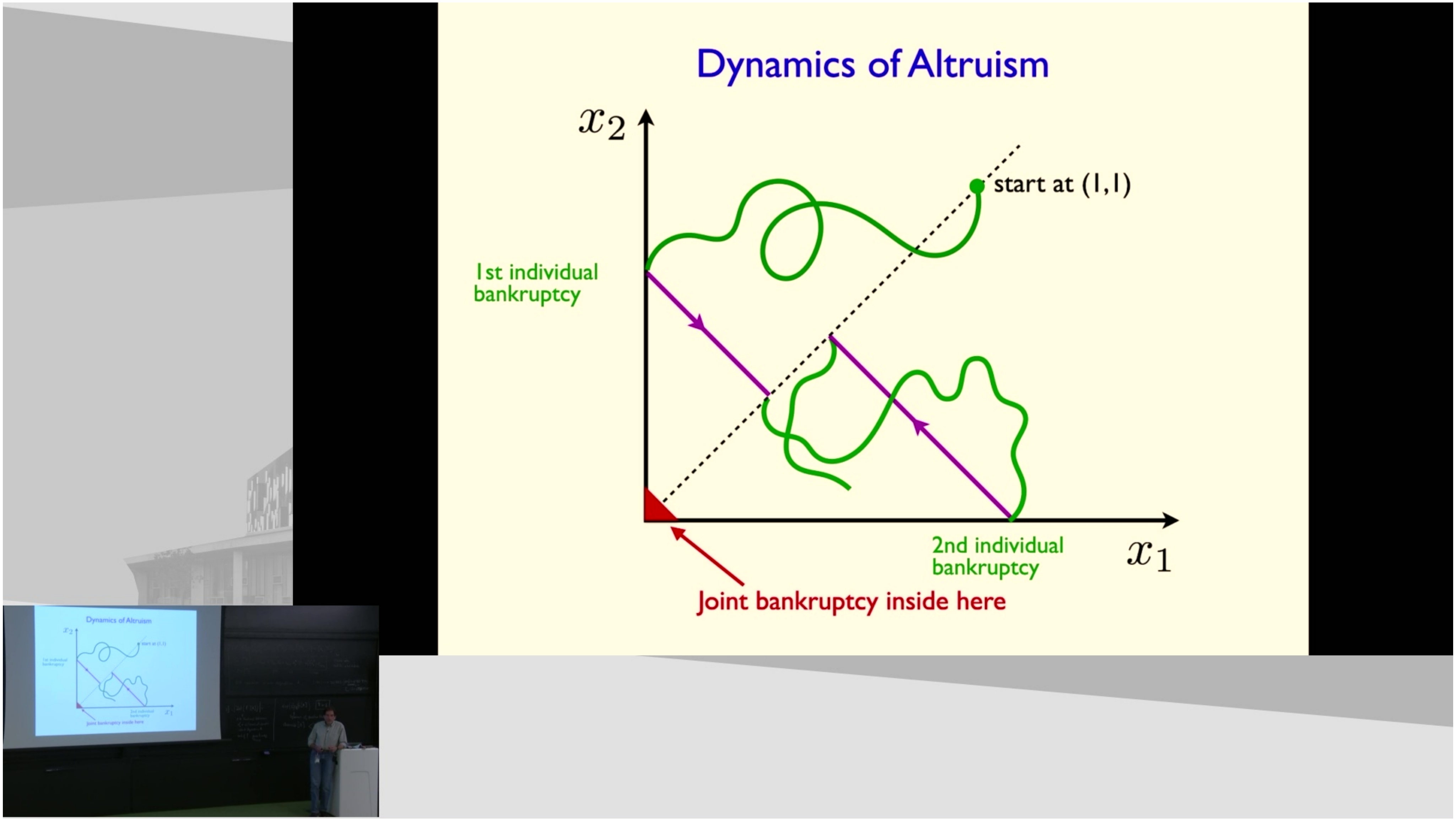
$$\simeq \frac{x_0}{\sqrt{4\pi D t^3}} \qquad t \to \infty$$




Dynamics of Altruism
x_2
start at (1,1)
x_1


Dynamics of Altruism
x2
start at (1,1)
1st individual bankruptcy
2nd individual bankruptcy
x1


Dynamics of Altruism
x_2
start at (1,1)
1st individual bankruptcy
2nd individual bankruptcy
x_1



Dynamics of Altruism
x_2
start at (1,1)
1st individual bankruptcy
2nd individual bankruptcy
x_1
Joint bankruptcy inside here

Dynamics of Altruism

$\mathcal{F}(x) =$ wealth distribution of solvent ~~agent at 1st~~ bankruptcy

probability that one person goes bankrupt at time t

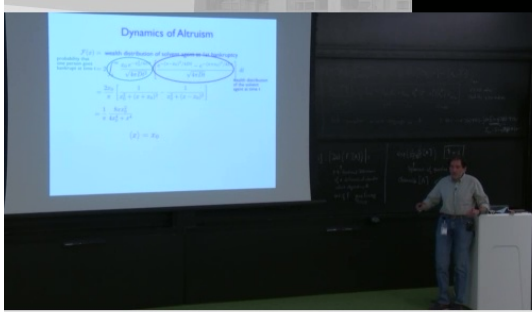
$$= 2\int_0^\infty \frac{x_0\, e^{-x_0^2/4Dt}}{\sqrt{4\pi D t^3}} \left[\frac{e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt}}{\sqrt{4\pi D t}}\right] dt$$

wealth distribution of the solvent agent at time t

$$= \frac{2x_0}{\pi}\left[\frac{1}{x_0^2+(x+x_0)^2} - \frac{1}{x_0^2+(x-x_0)^2}\right]$$

$$= \frac{1}{\pi}\,\frac{8x x_0^2}{4x_0^4+x^4}$$

$$\langle x \rangle = x_0$$



Dynamics of Altruism

$\mathcal{F}(x) =$ wealth distribution of solvent agent at 1st bankruptcy

probability that one person goes bankrupt at time t

$$= 2\int_0^\infty \frac{x_0\, e^{-x_0^2/4Dt}}{\sqrt{4\pi D t^3}} \left[\frac{e^{-(x-x_0)^2/4Dt} - e^{-(x+x_0)^2/4Dt}}{\sqrt{4\pi D t}}\right] dt$$

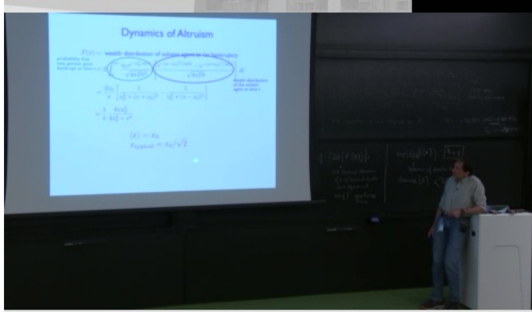
wealth distribution of the solvent agent at time t

$$= \frac{2x_0}{\pi}\left[\frac{1}{x_0^2+(x+x_0)^2} - \frac{1}{x_0^2+(x-x_0)^2}\right]$$

$$= \frac{1}{\pi}\,\frac{8x x_0^2}{4x_0^4+x^4}$$

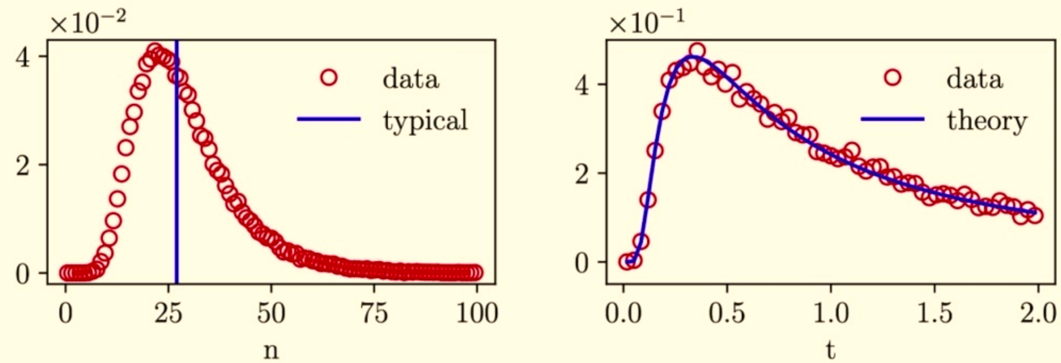
$$\langle x\rangle = x_0$$

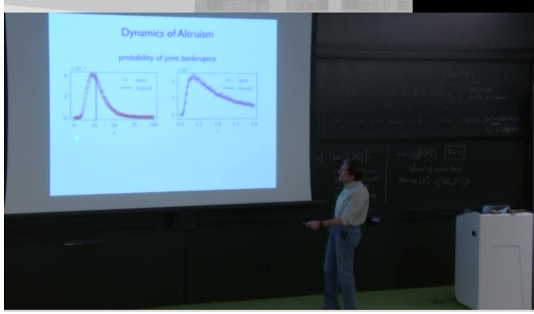
$$x_{\text{typical}} = x_0/\sqrt{2}$$



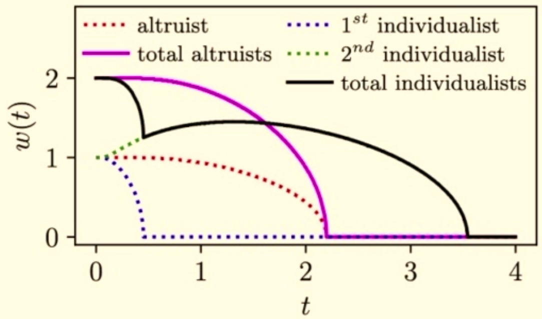
Dynamics of Altruism

probability of joint bankruptcy

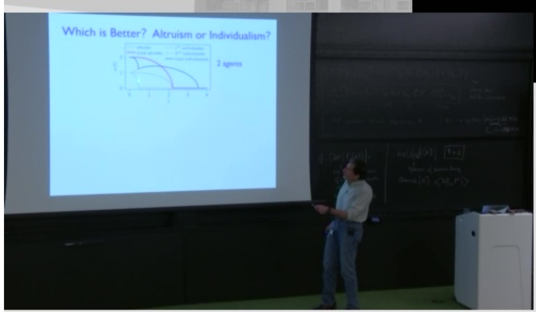




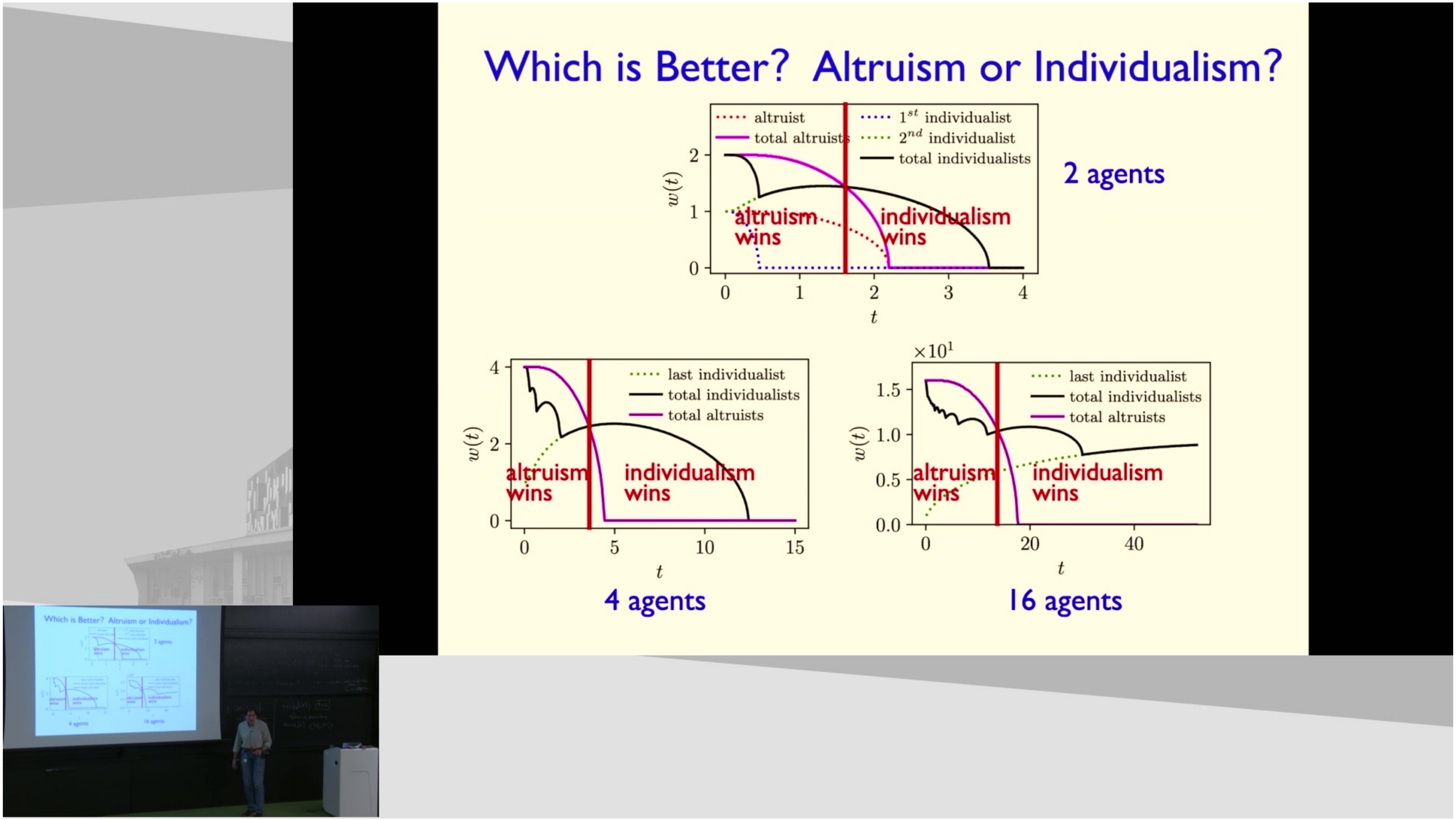
Which is Better? Altruism or Individualism?



2 agents

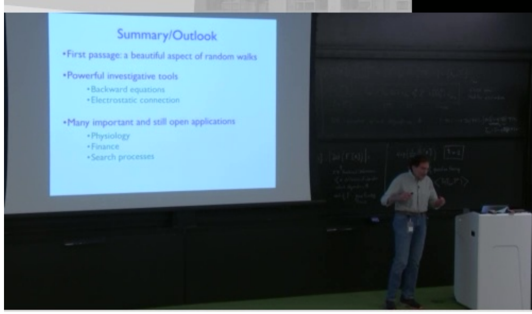


Which is Better? Altruism or Individualism?



Summary/Outlook

- First passage: a beautiful aspect of random walks
- Powerful investigative tools
  - Backward equations
  - Electrostatic connection
- Many important and still open applications
  - Physiology
  - Finance
  - Search processes



Thanks for listening!