Title: Entanglement Linear Response â€” Extracting the Quantum Hall Conductance from a Single Bulk Wavefunction and Beyond

Speakers: Ruihua Fan

Series: Quantum Matter

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Abstract: In this talk, I will introduce the so-called entanglement linear response, i.e., response under entanglement generated unitary dynamics. As an application, I will show how it can be applied to certain anomalies in 1D CFTs. Moreover, I will apply it to extract the quantum Hall conductance from a wavefunction and how it embraces a previous work on the chiral central charge. This gives a new connection between entanglement, anomaly and topological response. If time permits, I will also talk about how it inspires some generalizations of the real-space Chern number formula in free fermion systems.

Zoom link: https://pitp.zoom.us/j/96535214681?pwd=MldXRkRjZ1J6WS95WXQ0cG03cWdCZz09

Entanglement linear response —

Extracting Quantum Hall Conductance From a Single Wavefunction

and Beyond

Ruihua Fan @ Perimeter Institute (Virtual)

Rahul Sahay

Ashvin Vishwanath

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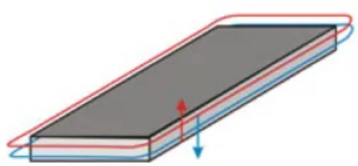
Yingfei Gu

RF, arXiv: 2206.02823
RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710
RF, P. Zhang, Y. Gu, arXiv: 2211.04510

Introduction

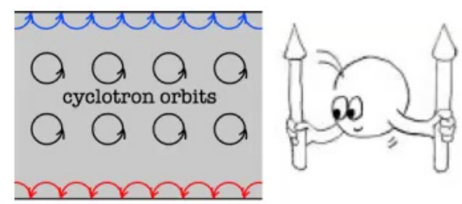
- Gapped systems can exhibit a variety of topological phenomena at the zero temperature (ground states). Below are two well-understood examples:

Invertible (short-range entangled) phases



E.g., Kitaev chain, IQH, topological insulators,...

Topological orders



E.g., FQH, gapped spin liquid,...

- It is widely accepted that these phenomena arise from the entanglement of the ground state. But how to quantify this connection?

Introduction

- The first answer was found for the 2D topological orders [Hamma, Ionicioiu, Zarnardi (2004); Kitaev, Preskill (2005); Levin, Wen (2005)]
  - Topological entanglement entropy => Anyon total quantum dimension
- What about other topological invariants? Such as the quantum Hall conductance, chiral central charge ....
- One can certainly extend this program to gapless systems, e.g., CFTs, Fermi liquid....

Quantum order ⟷ Entanglement

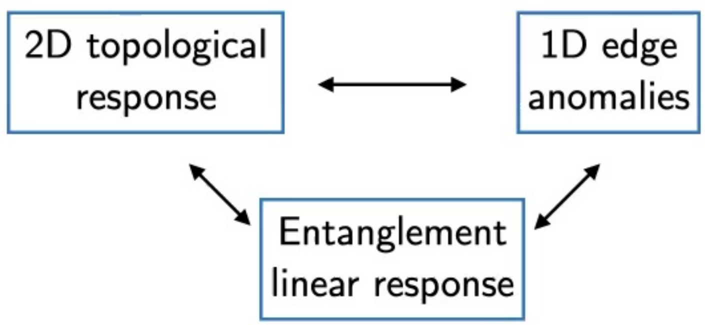
Q: what happened when entanglement met locality?

Introduction

- The motivation comes from various directions:
- Non-perturbative information/constraints on quantum many-body systems.
- Entanglement-based numerical study, i.e., tensor-networks
  - Compatibility between the TN architecture and the entanglement structure of the many-body wavefunction [Dubail&Read, 13]
- Hamiltonian-free characterization of the quantum orders
  - Notion of topological phases in open systems [Altman, Yimu, RF, Vishwanath, to appear]
- Entanglement in QFTs
  - Distill universal data from the UV divergence and ambiguities

Introduction

- We want to propose a general idea, **entanglement response**, to study the above questions systematically.
- It is about understanding an <u>intrinsic</u> dynamics generated by the state itself, known as the modular flow in mathematics and high energy physics.
- As one application, we show how it can be applied to extract the quantum Hall conductance, or more generally, get the following triangle



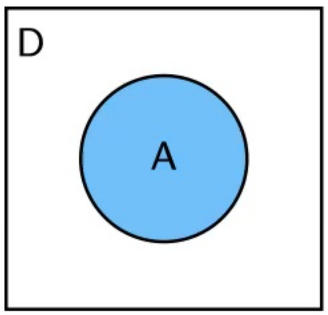
Outline

- Review
  - topological entanglement entropy
  - entanglement (modular) Hamiltonian
- Entanglement linear response
- Proposal for the Hall conductance (and chiral central charge)
  - Setup, various justifications
  - Wiedemann-Franz law
- Summary & Outlooks

[Kim, Shi, Kato, Albert, arXiv: 2110.06932, 2110.10400]

Review

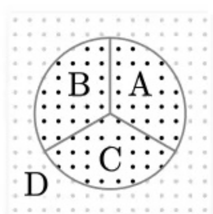
- Topological entanglement entropy (TEE) $\gamma \geq 0$



$$|\psi\rangle \in H_A \otimes H_D, \quad \rho_A = \mathrm{Tr}_D |\psi\rangle\langle\psi|$$

$$S_A = -\,\mathrm{Tr}\rho_A \ln \rho_A = \alpha L_A - \boxed{\gamma} \qquad \gamma = \ln \mathscr{D}$$

- A better definition is to consider some linear superposition (not quite just bipartite entanglement)



$$-\gamma = S_A + S_B + S_C - S_{AB} - S_{AC} - S_{BC} + S_{ABC} \leq 0$$

Review

- To understand the essence for γ being topological, and for the purpose of generalization, let us take a detour to introduce
- Entanglement Hamiltonian (Half sided modular Hamiltonian)

$$K_A = -\ln \rho_A$$

- The reduced density matrix becomes a thermal state $\rho_A = e^{-K_A}$
- The von Neumann entropy becomes the thermal energy

$$S_A = \mathrm{Tr}\, K_A e^{-K_A} = \alpha L_A - \gamma$$

Review

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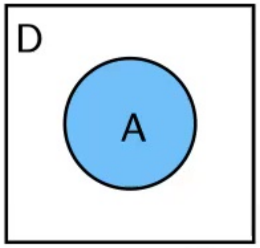
Review

- Entanglement Hamiltonian is rather different from the physical Hamiltonian:
  - Area law => non-uniform in space
  - Pure state => conversion property

$$K_A |\psi\rangle = K_D |\psi\rangle$$



$\rho_A = e^{-K_A}$

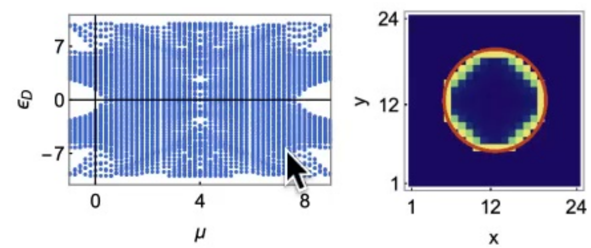
It follows from the Schmidt decomposition:

$$|\psi\rangle = \sum_n \sqrt{\lambda_n} |n_A\rangle \otimes |n_D\rangle$$

This property is like a symmetry, that has no analogue with physical Hamiltonians.

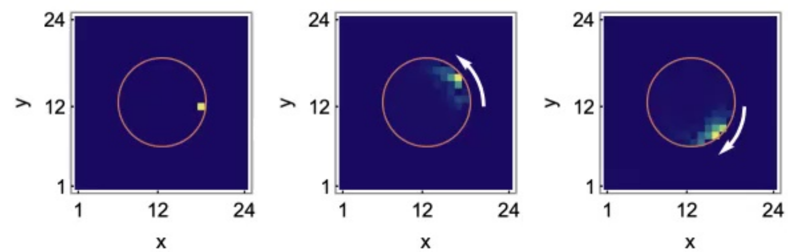
Review

- Entanglement Hamiltonian also has certain similarity with the physical (edge) Hamiltonian
  - Spectrum [Kitaev, Preskill; Li, Haldane; ...]



  - Dynamics, called the modular flow

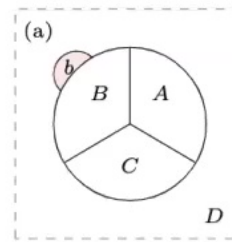
$$|\psi\rangle \mapsto |\psi(s)\rangle = e^{-isK_A}|\psi\rangle$$



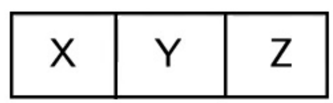
- These are important intuitions for our later discussion.

Review

- TEE can be regarded as the study of thermal energy at this fictitious equilibrium. This quantity is
    - topological: invariant under the change of shapes without changing the topology
    - universal: invariant under local deformation of the state



- The key ingredients for showing these two properties:
    - $K_A|\psi\rangle = K_{\bar{A}}|\psi\rangle$
    - $\langle O_{r_1} O_{r_2}\rangle \approx \langle O_{r_1}\rangle\langle O_{r_1}\rangle \quad (|r_1 - r_2| \gg \xi)$
    - $\langle K_{XY} + K_{YZ}\rangle \approx \langle K_{XYZ} + K_Y\rangle$



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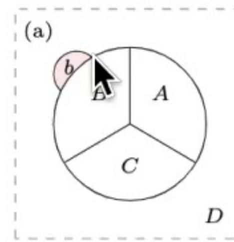

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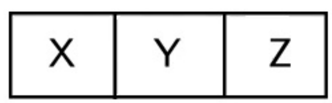
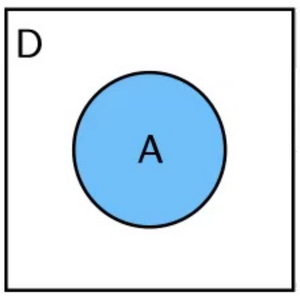


Review

- When the system possesses global symmetries, one can also measure the symmetry defect (in addition to the von Neumann entropy, which is a "geometric defect") [Chen, Tu, Meng, Cheng, arXiv: 2203.08847]



$$\log |\operatorname{Tr} U_A(g) e^{-K_A}| = -\alpha_g L_A + \boxed{\gamma_g}$$

topological disorder parameter

- Again, it is the study of properties at the fictitious thermal equilibrium.
- Generalization becomes clear.

Entanglement linear response

- We call this scheme, the entanglement response:

$$|\psi\rangle \mapsto |\psi(s)\rangle = e^{-isK_X}|\psi\rangle$$

$$\langle O_Y(s)\rangle_{\psi(s)} = \langle\psi|\, e^{isK_X}\, O_Y(s)\, e^{-isK_X}\,|\psi\rangle$$

- To start, let us consider the response at the linear order in the modular time s, i.e., entanglement linear response:

$$\frac{d}{ds}\langle O_Y(s)\rangle_{\psi(s)}\Big|_{s=0} = ?$$

- Claim: entanglement linear response => physical topological response.

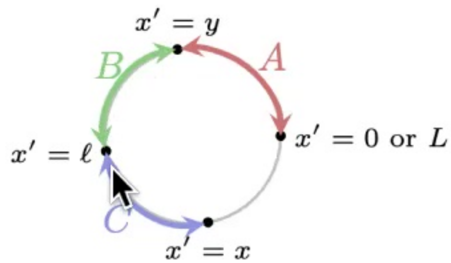
U(1) symmetry defect => quantum Hall conductance

geometric defect => chiral central charge

[RF, arXiv: 2206.02823]

Application in 1D: anomaly

- Application: pure 1D CFT calculation => the U(1) chiral anomaly.
- CFT with a U(1) symmetry: $J(x)J(0) \sim \frac{k_L}{x^2}, \quad \tilde{J}(x)\tilde{J}(0) \sim \frac{k_R}{x^2}$
- We repeat the entanglement linear response exercise



$$K_{AB} = 4\pi \int_{AB} \frac{\sin\frac{\ell - x}{2} \sin\frac{x}{\pi}}{\sin\frac{\ell}{2}} T_{00}(x) dx$$

$$e^{i\mu Q_{BC}} \mapsto V_\mu(x) V_{-\mu}(y), \quad h = \frac{k_L \mu^2}{2(2\pi)^2}, \tilde{h} = \frac{k_R \mu^2}{2(2\pi)^2}$$

- A simple limit: ABC form the entire circle

$$\frac{d}{ds} \ln\langle e^{i\mu Q_{BC}} \rangle \Big|_{s=0} = -\frac{k_L - k_R}{4\pi} \mu^2$$

$k_L = k_R$ for a genuine 1D system

[RF, arXiv: 2206.02823
RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710
also see Y. Zou, et.al, arXiv: 2206.00027]

Proposal for quantum Hall conductance

- Setup:
  - Divide the plane into four parts: A, B, C and D
  - Apply modular flow on AB:

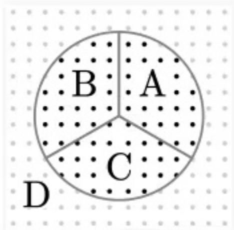
$$|\psi\rangle \mapsto |\psi(s)\rangle = e^{-isK_{AB}}|\psi\rangle$$

  - Measure the charge response in region BC:

$$\langle\psi|\, e^{isK_{AB}}\, Q_{BC}\, e^{-isK_{AB}}\,|\psi\rangle$$

- Proposal: the linear response yields the Hall conductance

$$\sigma_{xy} = \frac{i}{2}\langle\psi|[K_{AB}, Q_{BC}^2]|\psi\rangle$$



chiral central charge
[Kim et. al, arXiv: 2110.06932]

$$\frac{\pi}{3}c_- = i\langle\psi|[K_{AB}, K_{BC}]|\psi\rangle$$

[RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710]

Proposal for quantum Hall conductance

- There have been many efforts on quantum Hall conductance
  - Free fermions: TKNN formula [TKNN (1982)]; Fredholm index formula [Bellissard, Elst, Schulz-Baldes (1994), Avron, Seiler, Simon (1990), Kitaev (2005)]
  - Interacting systems: a nontrivial generalization of the Fredholm index formula [Bachmann, Bols, Roeck, Fraas (2020), Kapustin, Sopenko (2020)]
  - A related but different topological invariant: many-body Chern number [Shiozaki, Shapourian, Gomi, Ryu (2017), Dehghani, Cian, Hafezi, Barkeshli (2020)]
- Our formula is superfacially different from all of them, but they should be secretly related.
- E.g., in free-fermion systems, our formula is indeed related to the real-space Chern number formula [RF, P. Zhang, Y. Gu, arXiv: 2211.04510].

$$2\pi i \mathrm{Tr}[(PfP)^n, (PgP)^m] = \nu(P) \oint_C f^n dg^m$$

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$$2\pi i \operatorname{Tr}[(PfP)^n, (PgP)^m] = \nu(P) \oint_C f^n dg^m$$

Justifications

$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi\,|\,[K_{AB}, Q_{BC}^2]\,|\,\psi\rangle$$

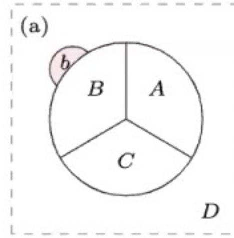
- Basic logic:
  - Step I: Our formula $\Sigma(\psi; A, B, C)$ satisfies the same general properties as the quantum Hall conductance: additive, CRT, topological, universal
  - Step II: Provide an examples to show that it is actually nonzero
- We will also provide a mechanism to show that why it gives universal results

[RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710]

Justification I: General properties

$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi|[K_{AB}, Q_{BC}^2]|\psi\rangle$$

- Our formula $\Sigma(\psi; A, B, C)$ satisfies the same general properties as the quantum Hall conductance:
  - Additivity $\Sigma(\psi_1 \otimes \psi_2; A, B, C) = \Sigma(\psi_1; A, B, C) + \Sigma(\psi_2; A, B, C)$
  - CRT
  - Topological and Universal



- There are three key ingredients to show these:
  - $K_A|\psi\rangle = K_{\bar{A}}|\psi\rangle$
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[RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710]

Justification I: General properties

$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi|[K_{AB}, Q_{BC}^2]|\psi\rangle$$

- Example: Reflection property (actually stronger than reflection)

$$\Sigma(\psi; A, B, C) = -\Sigma(\psi; B, A, C)$$

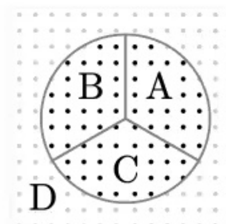
- Equivalently, we can show

$$\langle[K_{AB}, Q_{BC}^2 + Q_{AC}^2]\rangle_\psi = 0$$

- It is important to note that K_{AB} conserves the total charge Q_{ABC}

$$\langle[K_{AB}, Q_{BC}^2 + Q_{AC}^2 - Q_{ABC}^2]\rangle_\psi = \langle[K_{AB}, Q_C^2 - 2Q_AQ_B]\rangle_\psi = 0$$

QED



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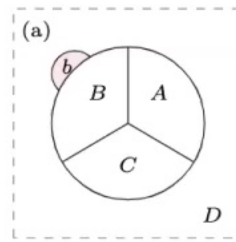
$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi|[K_{AB}, Q_{BC}^2]|\psi\rangle$$

- Our formula $\Sigma(\psi; A, B, C)$ satisfies the same general properties as the quantum Hall conductance:
  - Additivity $\quad \Sigma(\psi_1 \otimes \psi_2; A, B, C) = \Sigma(\psi_1; A, B, C) + \Sigma(\psi_2; A, B, C)$
  - CRT
  - Topological and Universal



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X	Y	Z

[RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710]

Justification II: bulk-edge correspondence

$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi\,|\,[K_{AB}, Q_{BC}^2]\,|\,\psi\rangle$$

- Because $\Sigma(\psi; A, B, C)$ is universal, we can deform the state to a nice one without changing its value.

- In particular, we want to consider a state with [Kitaev, Preskill; Li, Haldane, Swingle, Senthil; Chandran, Hermanns, Regnault, Bernevig; Qi, Katsura, Ludwig]

$$K_D = \epsilon H_{CFT}$$

- We consider the expectation value of a U(1) defect operator $\ln\langle e^{i\mu Q_{BC}}\rangle$, and can obtain Q_{BC}^2 from Taylor expansion.

- We want to show that

$$\frac{d}{ds}\ln\langle e^{i\mu Q_{BC}}\rangle\Big|_{s=0} = -\,\sigma_{xy}\mu^2, \quad \sigma_{xy} = \frac{k_L - k_R}{2\pi}$$

Justification II: bulk-edge correspondence

- Back to the two-dimensional case, the expectation value of a U(1) defect operator is given by the Charged Cardy formula

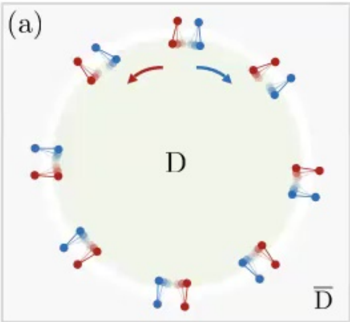
$$\ln\langle e^{i\mu Q_D}\rangle = -\frac{(k_L + k_R)\mu^2}{4\pi}\frac{L_D}{\epsilon} + \cdots$$

- We can separate the area-law coefficient into two pieces

$$\alpha_{chiral} = \frac{k_L\mu^2}{4\pi\epsilon} \qquad \alpha_{anti-chiral} = \frac{k_R\mu^2}{4\pi\epsilon}$$

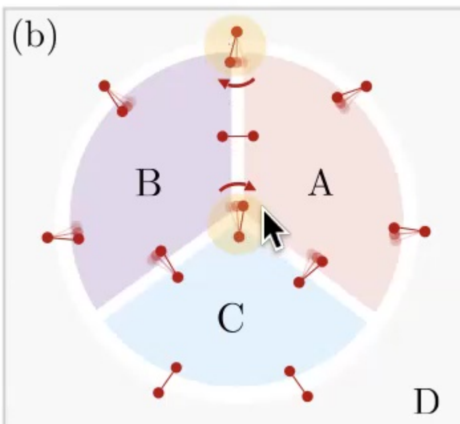
- We interpret them as the line density of the chiral and anti-chiral charged modes.

- Their difference is encoded in the motion under modular flow.



Justification II: bulk-edge correspondence

- Let us apply the above picture and understand what happens to $\langle e^{i\mu Q_{BC}} \rangle$ under the modular flow generated by K_{AB}.
- Only the two triple contact points make nonzero contributions



$$\frac{d}{ds} \ln\langle e^{i\mu Q_{BC}} \rangle \Big|_{s=0} = -2(\alpha_{chiral} - \alpha_{anti-chiral})\nu$$

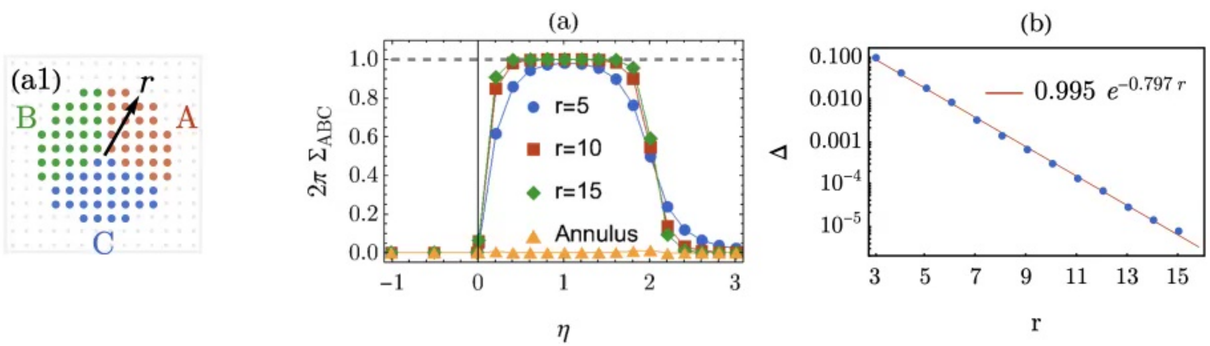
$$K_D = \epsilon H_{CFT} \quad \alpha_{anti-chiral} = \frac{k_R \mu^2}{4\pi\epsilon} \quad \alpha_{chiral} = \frac{k_L \mu^2}{4\pi\epsilon}$$

$$\frac{d}{ds} \ln\langle e^{i\mu Q_{BC}} \rangle \Big|_{s=0} = -\frac{k_L - k_R}{2\pi}\mu^2 = -\sigma_{xy}\mu^2$$

Justification III: Numerics

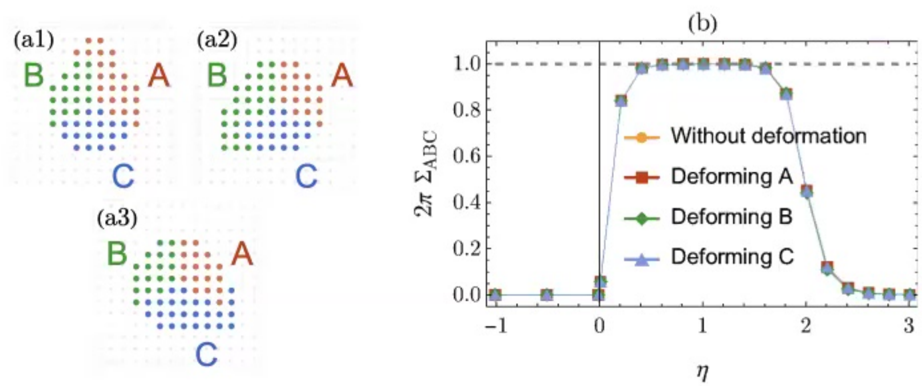
$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi\,|\,[K_{AB}, Q_{BC}^2]\,|\,\psi\rangle$$

- We also provide numerical justification using free fermion lattice model (the pi-flux model with weak disorders)
- It vanishes identically in the time-reversal symmetric phase, detects the transition, converges to the quantized value exponentially fast in the subsystem size



Justification III: Numerics

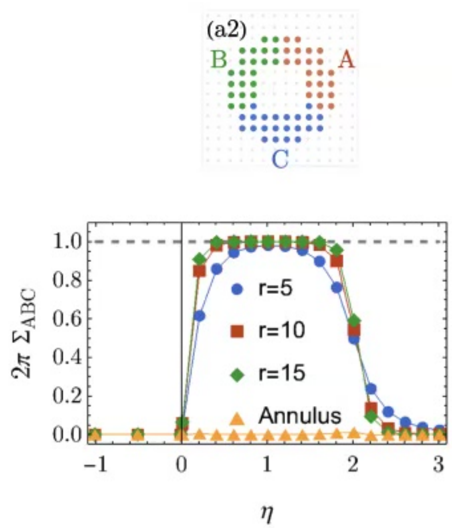
- Deforming the shapes of A,B,C does not change the result across the entire phase diagram.
- Note that the size of the blob we add here is comparable with the correlation length.



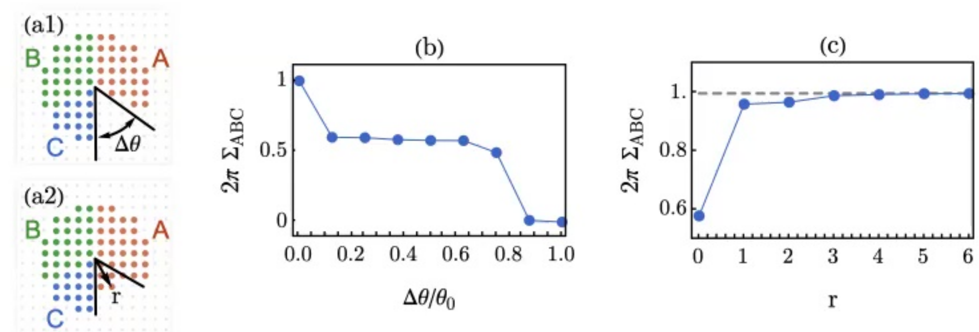
Justification III: Numerics

- Changing the topology has a significant effect

Annulus



Incomplete disk



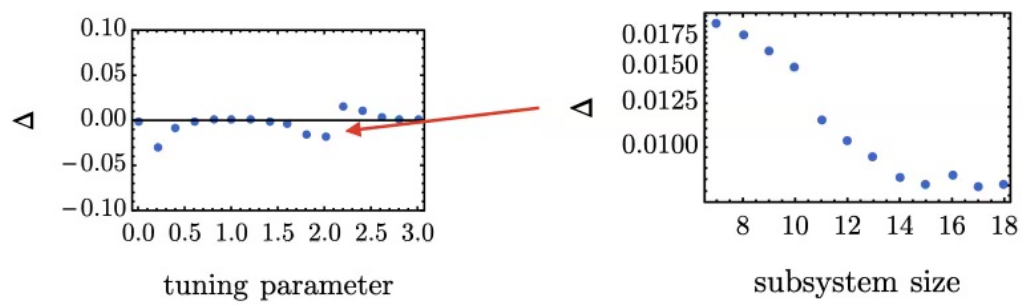
Entanglement Wiedemann-Franz law

- In free fermion systems with single-charge fermions, one can combine the result for the chiral central charge and quantum Hall conductance to obtain

$$\langle\psi|[K_{AB}, K_{BC} - \frac{\pi^2}{3}Q_{BC}^2]|\psi\rangle = 0$$

- An entanglement version of the Wiedemann–Franz law

- A numerical calculation on the left-hand side seems to suggest that it holds even at the critical point



Justification 0: free-fermion systems

- In free-fermion systems, 2pt function $P_{jk} = \langle c_k^\dagger c_j \rangle$ determines everything, including entanglement

$$\hat{K}_X = \sum_{jk} (K_X)_{jk}\, c_j^\dagger c_k, \quad K_X = \log \frac{1 - P_X}{P_X}, \quad P_X = XPX$$

- We need to understand commutators of restricted projectors, e.g.,

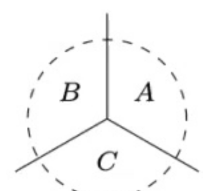
$$\langle [\hat{K}_{AB}, \hat{K}_{BC}] \rangle_\psi \sim \mathrm{Tr}\, P_{ABC}[P_{AB}^m, P_{BC}^n]$$

- Up to combinatorial manipulations (and the existence of a gap), it is equivalent to showing the following

$$i\,\mathrm{Tr}[(PAP)^m, (PBP)^n] = \frac{m!n!}{(m+n)!} \frac{\nu(P)}{2\pi i}$$

- The "smooth" version of this formula (the $m = n = 1$ case is related to GMP algebra)

$$\boxed{2\pi i\,\mathrm{Tr}[(PfP)^n, (PgP)^m] = \nu(P) \oint_C f^n dg^m}$$

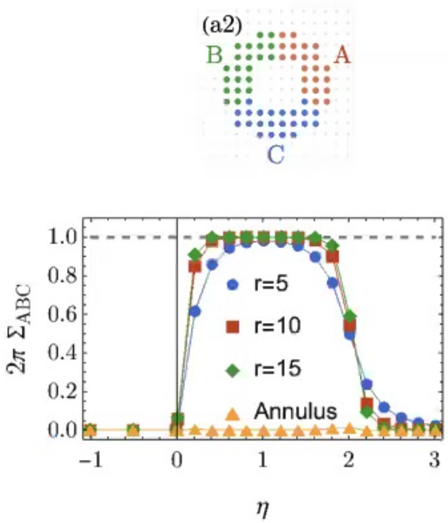


[RF, P. Zhang, Y. Gu, arXiv: 2211.04510; Kitaev, private communication]

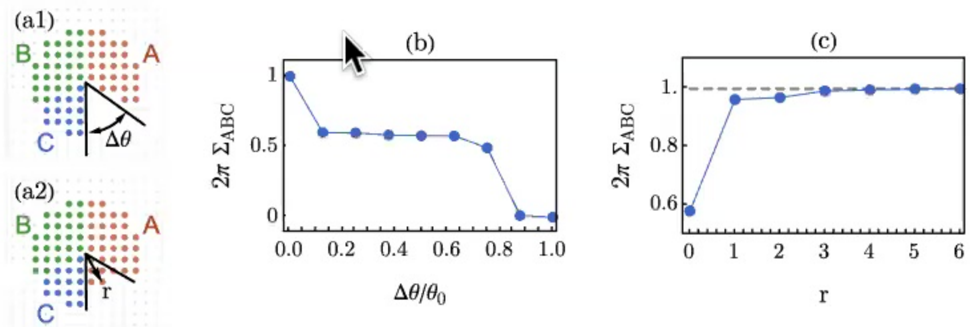
Justification III: Numerics

- Changing the topology has a significant effect

Annulus



Incomplete disk



Justification 0: free-fermion systems

- In free-fermion systems, 2pt function $P_{jk} = \langle c_k^\dagger c_j \rangle$ determines everything, including entanglement

$$\hat{K}_X = \sum_{jk} (K_X)_{jk} c_j^\dagger c_k, \quad K_X = \log \frac{1 - P_X}{P_X}, \quad P_X = XPX$$

- We need to understand commutators of restricted projectors, e.g.,

$$\langle [\hat{K}_{AB}, \hat{K}_{BC}] \rangle_\psi \sim \mathrm{Tr}\, P_{ABC}[P_{AB}^m, P_{BC}^n]$$

- Up to combinatorial manipulations (and the existence of a gap), it is equivalent to showing the following

$$i \mathrm{Tr}[(PAP)^m, (PBP)^n] = \frac{m!n!}{(m+n)!} \frac{\nu(P)}{2\pi i}$$

- The "smooth" version of this formula (the $m = n = 1$ case is related to GMP algebra)

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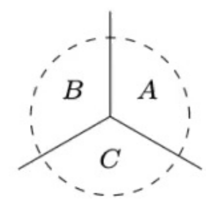
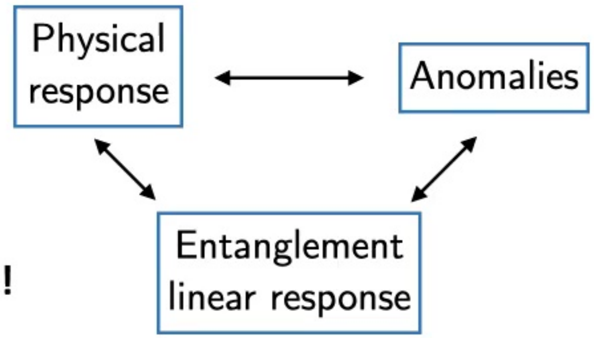


[RF, P. Zhang, Y. Gu, arXiv: 2211.04510; Kitaev, private communication]

Summary & Outlook

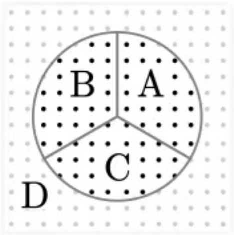
- New links between transport and entanglement? How to generalize it to other symmetries & higher dimensions?
- What is the implication of such formulas in tensor networks? E.g., can one show that PEPS must give vanishing results?
- Implications in TQFT? Formulating and calculating the formulas require new thoughts.
- Better understanding on the assumptions we used? Important quantum information questions on its own.



Thanks for your attention!

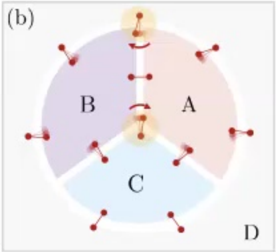
Summary & Outlook

- We propose the entanglement linear response, as a systematic framework to understand the topological phenomena and quantum entanglement.
- As an application, we show how to find for the Hall conductance and chiral central charge.
- We can understand them via the bulk-edge correspondence, i.e., the modular flow evolves the edge degrees of freedom.



$$\sigma_{xy} = \frac{i}{2}\langle\psi|[K_{AB}, Q_{BC}^2]|\psi\rangle$$

$$\frac{\pi}{3}c_- = i\langle\psi|[K_{AB}, K_{BC}]|\psi\rangle$$



Justification II: bulk-edge correspondence

- Back to the two-dimensional case, the expectation value of a U(1) defect operator is given by the Charged Cardy formula

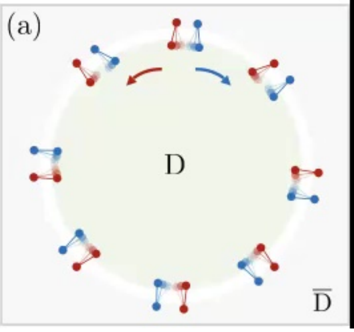
$$\ln\langle e^{i\mu Q_D}\rangle = -\frac{(k_L + k_R)\mu^2}{4\pi}\frac{L_D}{\epsilon} + \cdots$$

- We can separate the area-law coefficient into two pieces

$$\alpha_{chiral} = \frac{k_L \mu^2}{4\pi\epsilon} \qquad \alpha_{anti-chiral} = \frac{k_R \mu^2}{4\pi\epsilon}$$

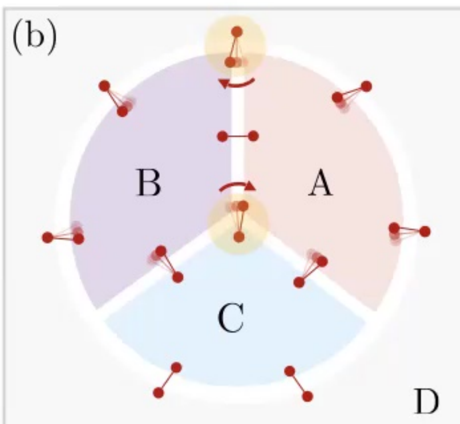
- We interpret them as the line density of the chiral and anti-chiral charged modes.

- Their difference is encoded in the motion under modular flow.



Justification II: bulk-edge correspondence

- Let us apply the above picture and understand what happens to $\langle e^{i\mu Q_{BC}}\rangle$ under the modular flow generated by K_{AB}.
- Only the two triple contact points make nonzero contributions



$$\frac{d}{ds}\ln\langle e^{i\mu Q_{BC}}\rangle\Big|_{s=0} = -2(\alpha_{chiral} - \alpha_{anti-chiral})\nu$$

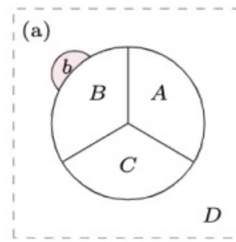
$$K_D = \epsilon H_{CFT} \quad \alpha_{anti-chiral} = \frac{k_R\mu^2}{4\pi\epsilon} \qquad \alpha_{chiral} = \frac{k_L\mu^2}{4\pi\epsilon}$$

$$\frac{d}{ds}\ln\langle e^{i\mu Q_{BC}}\rangle\Big|_{s=0} = -\frac{k_L - k_R}{2\pi}\mu^2 = -\sigma_{xy}\mu^2$$

Justification I: General properties

$$\Sigma(\psi; A, B, C) = \frac{i}{2}\langle\psi|[K_{AB}, Q_{BC}^2]|\psi\rangle$$

- Our formula $\Sigma(\psi; A, B, C)$ satisfies the same general properties as the quantum Hall conductance:
  - Additivity $\Sigma(\psi_1 \otimes \psi_2; A, B, C) = \Sigma(\psi_1; A, B, C) + \Sigma(\psi_2; A, B, C)$
  - CRT
  - Topological and Universal



- There are three key ingredients to show these:
  - $K_A|\psi\rangle = K_{\bar{A}}|\psi\rangle$
  - $\langle O_{r_1} O_{r_2}\rangle \approx \langle O_{r_1}\rangle\langle O_{r_1}\rangle \quad (|r_1 - r_2| \gg \xi)$
  - $K_{XY} + K_{YZ}|\psi\rangle \approx K_{XYZ} + K_Y|\psi\rangle$

X	Y	Z

[RF, R. Sahay, A. Vishwanath, arXiv: 2208.11710]

Justification III: Numerics

- Deforming the shapes of A,B,C does not change the result across the entire phase diagram.
- Note that the size of the blob we add here is comparable with the correlation length.