Title: An operator-algebraic formulation of self-testing

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Series: Perimeter Institute Quantum Discussions

Date: November 16, 2022 - 11:00 AM

URL: https://pirsa.org/22110100

Abstract: We give a new definition of self-testing for correlations in terms of states on C*-algebras. We show that this definition is equivalent to the standard definition for any class of finite-dimensional quantum models which is closed under submodels and direct sums, provided that the correlation is extremal and has a full-rank model in the class. This last condition automatically holds for the class of POVM quantum models, but does not necessarily hold for the class of projective models by a result of Mancinska and Kaniewski. For extremal binary correlations and for extremal synchronous correlations, we show that any self-test for projective models is a self-test for POVM models. The question of whether there is a self-test for projective models which is not a self-test for POVM models remains open. An advantage of our new definition is that it extends naturally to commuting operator models. We show that an extremal correlation is a self-test for finite-dimensional quantum models if and only if it is a self-test for finite-dimensional commuting operator models, and also observe that many known finite-dimensional self-tests are in fact self-tests for infinite-dimensional commuting operator models.

Zoom link: https://pitp.zoom.us/j/95783943431?pwd=SDFyQVVZR1d4WlVNSDZ4OENzSmJQUT09

An operator-algebraic formulation of self-testing

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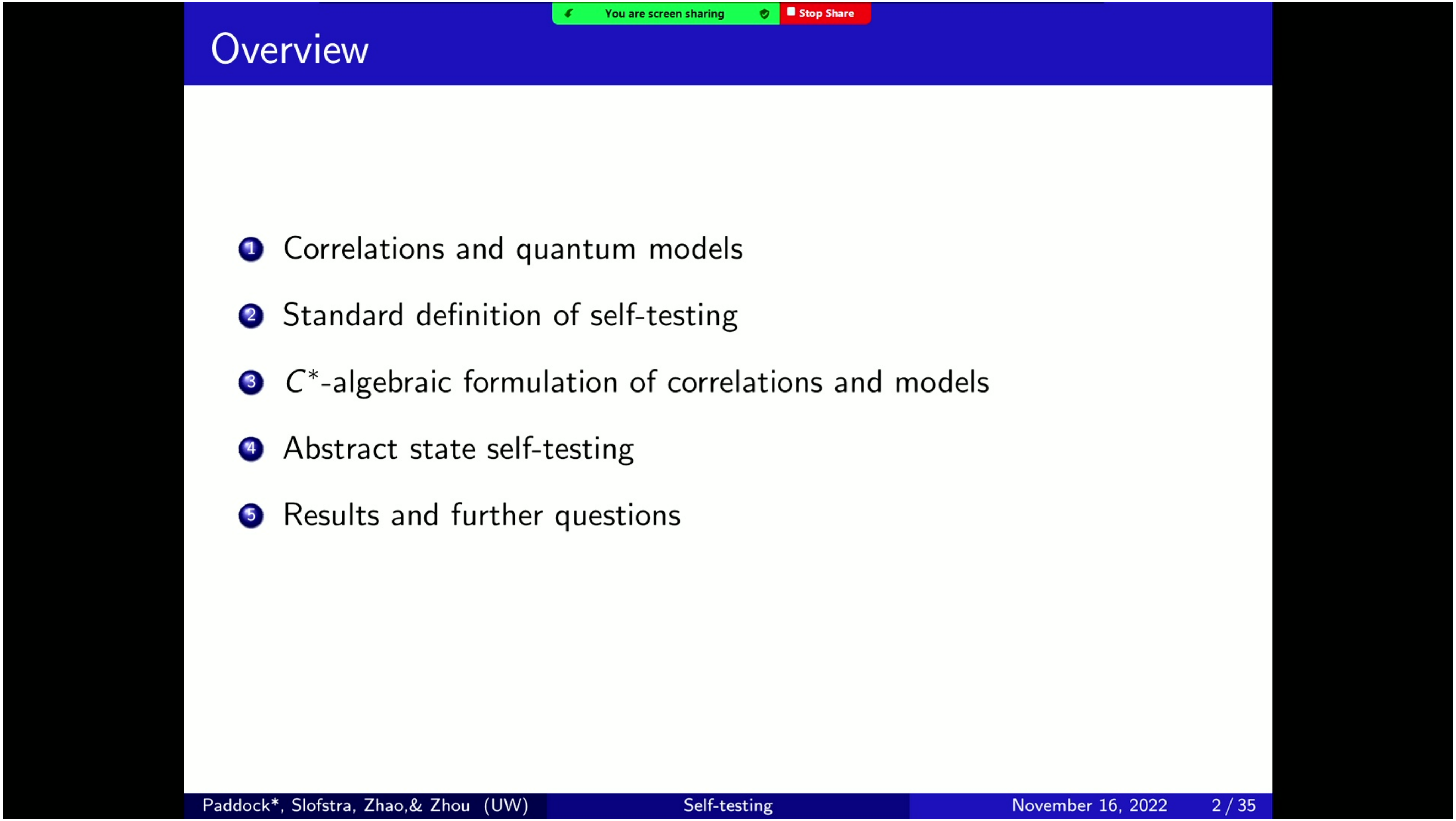
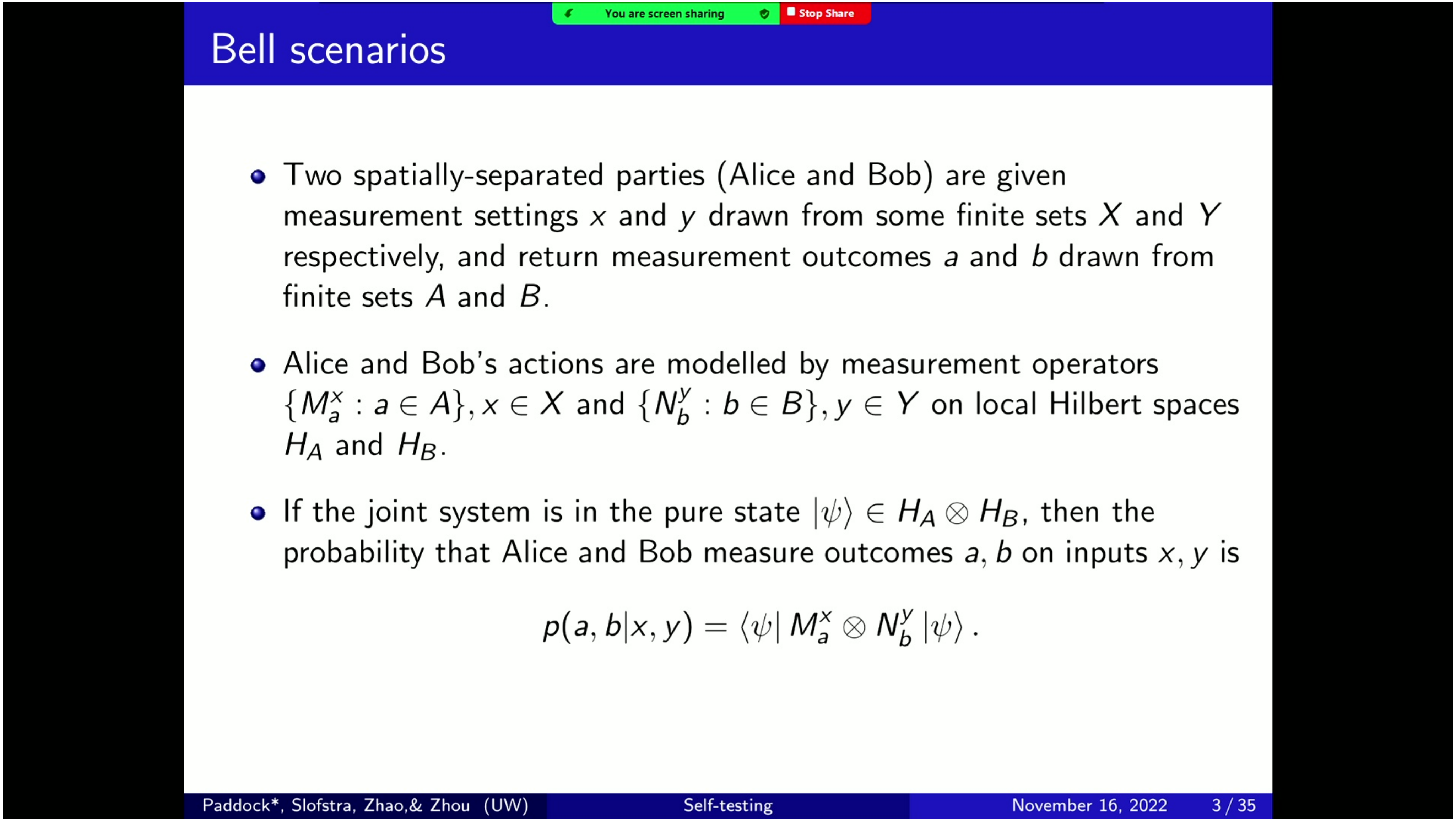
University of Waterloo

November 16, 2022



Overview

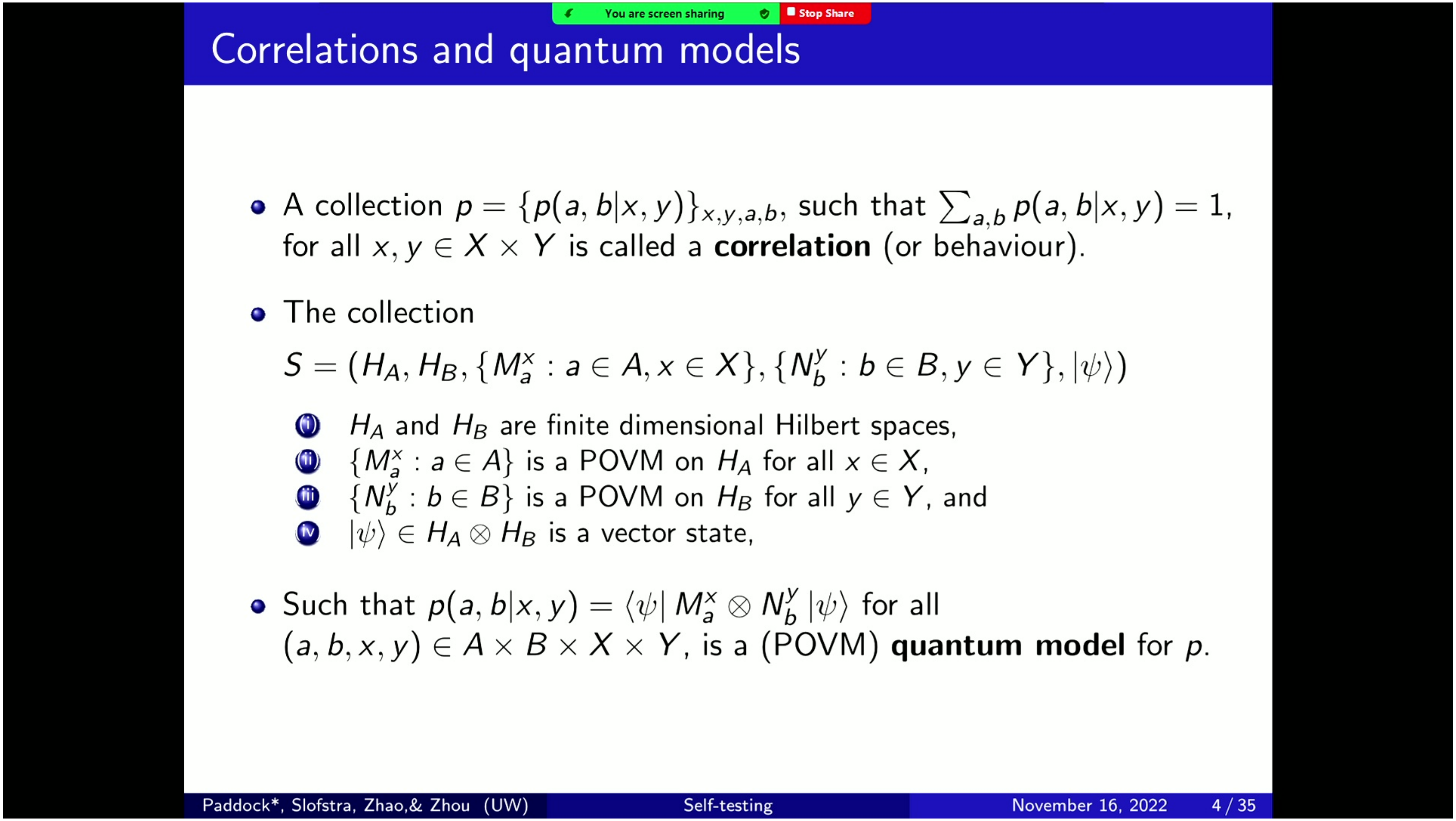
1. Correlations and quantum models
2. Standard definition of self-testing
3. C^*-algebraic formulation of correlations and models
4. Abstract state self-testing
5. Results and further questions



Bell scenarios

- Two spatially-separated parties (Alice and Bob) are given measurement settings x and y drawn from some finite sets X and Y respectively, and return measurement outcomes a and b drawn from finite sets A and B.
- Alice and Bob's actions are modelled by measurement operators $\{M_a^x : a \in A\}, x \in X$ and $\{N_b^y : b \in B\}, y \in Y$ on local Hilbert spaces H_A and H_B.
- If the joint system is in the pure state $|\psi\rangle \in H_A \otimes H_B$, then the probability that Alice and Bob measure outcomes a, b on inputs x, y is

$$p(a, b|x, y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle .$$

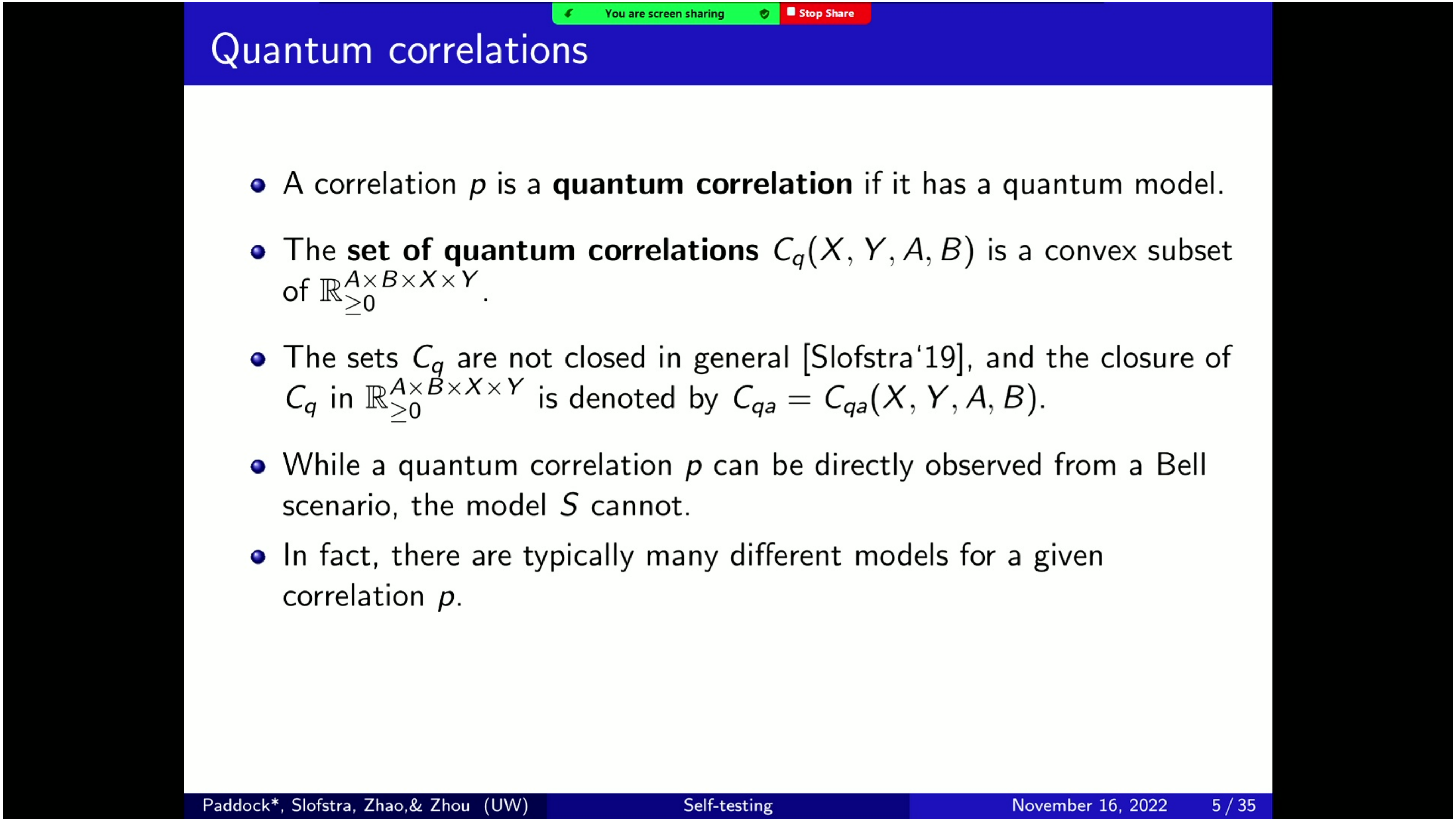


Correlations and quantum models

- A collection $p = \{p(a,b|x,y)\}_{x,y,a,b}$, such that $\sum_{a,b} p(a,b|x,y) = 1$, for all $x, y \in X \times Y$ is called a **correlation** (or behaviour).
- The collection

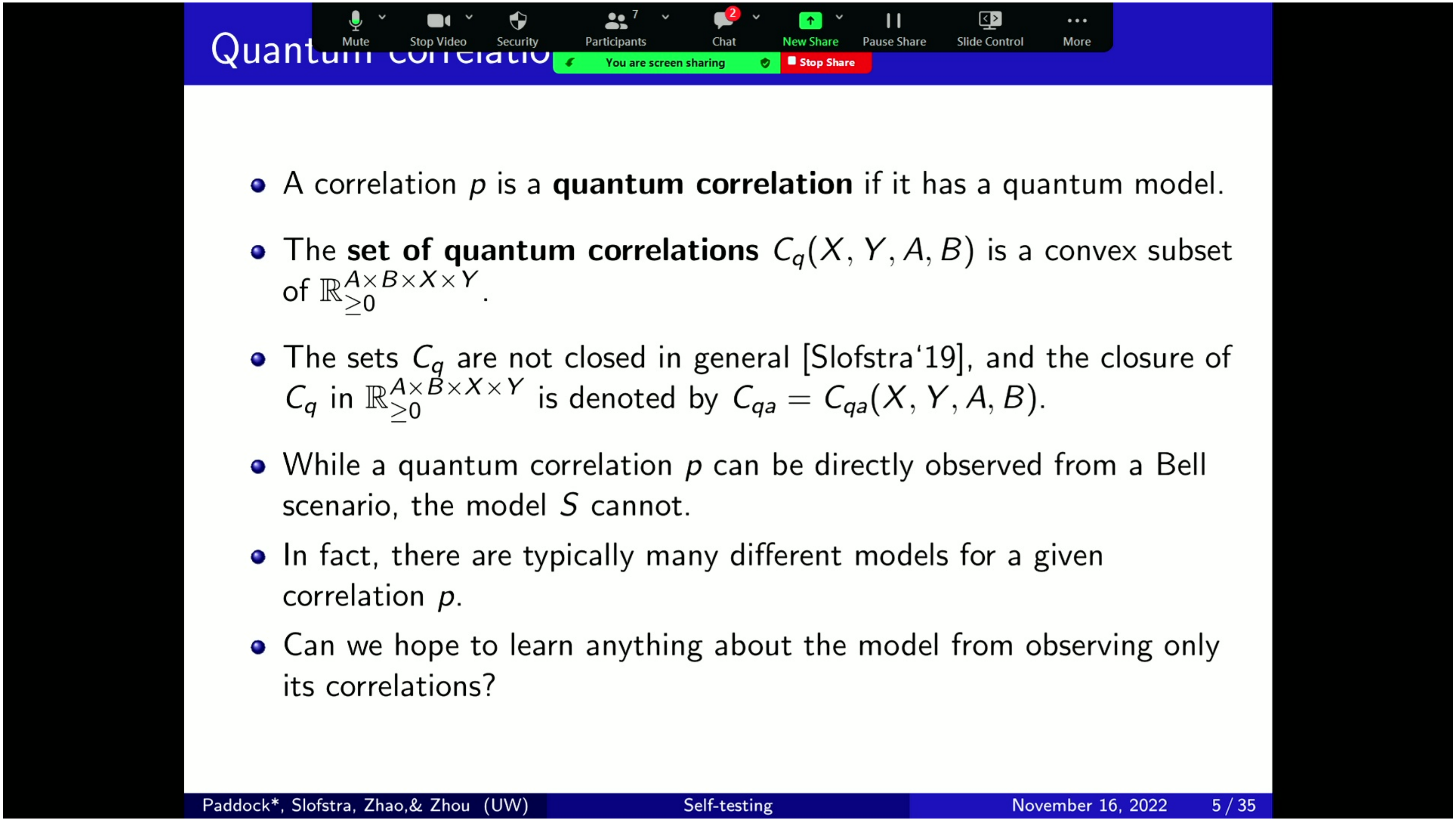
  $$S = (H_A, H_B, \{M_a^x : a \in A, x \in X\}, \{N_b^y : b \in B, y \in Y\}, |\psi\rangle)$$

  1. H_A and H_B are finite dimensional Hilbert spaces,
  2. $\{M_a^x : a \in A\}$ is a POVM on H_A for all $x \in X$,
  3. $\{N_b^y : b \in B\}$ is a POVM on H_B for all $y \in Y$, and
  4. $|\psi\rangle \in H_A \otimes H_B$ is a vector state,
- Such that $p(a,b|x,y) = \langle\psi| M_a^x \otimes N_b^y |\psi\rangle$ for all $(a,b,x,y) \in A \times B \times X \times Y$, is a (POVM) **quantum model** for p.



Quantum correlations

- A correlation p is a **quantum correlation** if it has a quantum model.
- The **set of quantum correlations** $C_q(X, Y, A, B)$ is a convex subset of $\mathbb{R}_{\geq 0}^{A \times B \times X \times Y}$.
- The sets C_q are not closed in general [Slofstra'19], and the closure of C_q in $\mathbb{R}_{\geq 0}^{A \times B \times X \times Y}$ is denoted by $C_{qa} = C_{qa}(X, Y, A, B)$.
- While a quantum correlation p can be directly observed from a Bell scenario, the model S cannot.
- In fact, there are typically many different models for a given correlation p.



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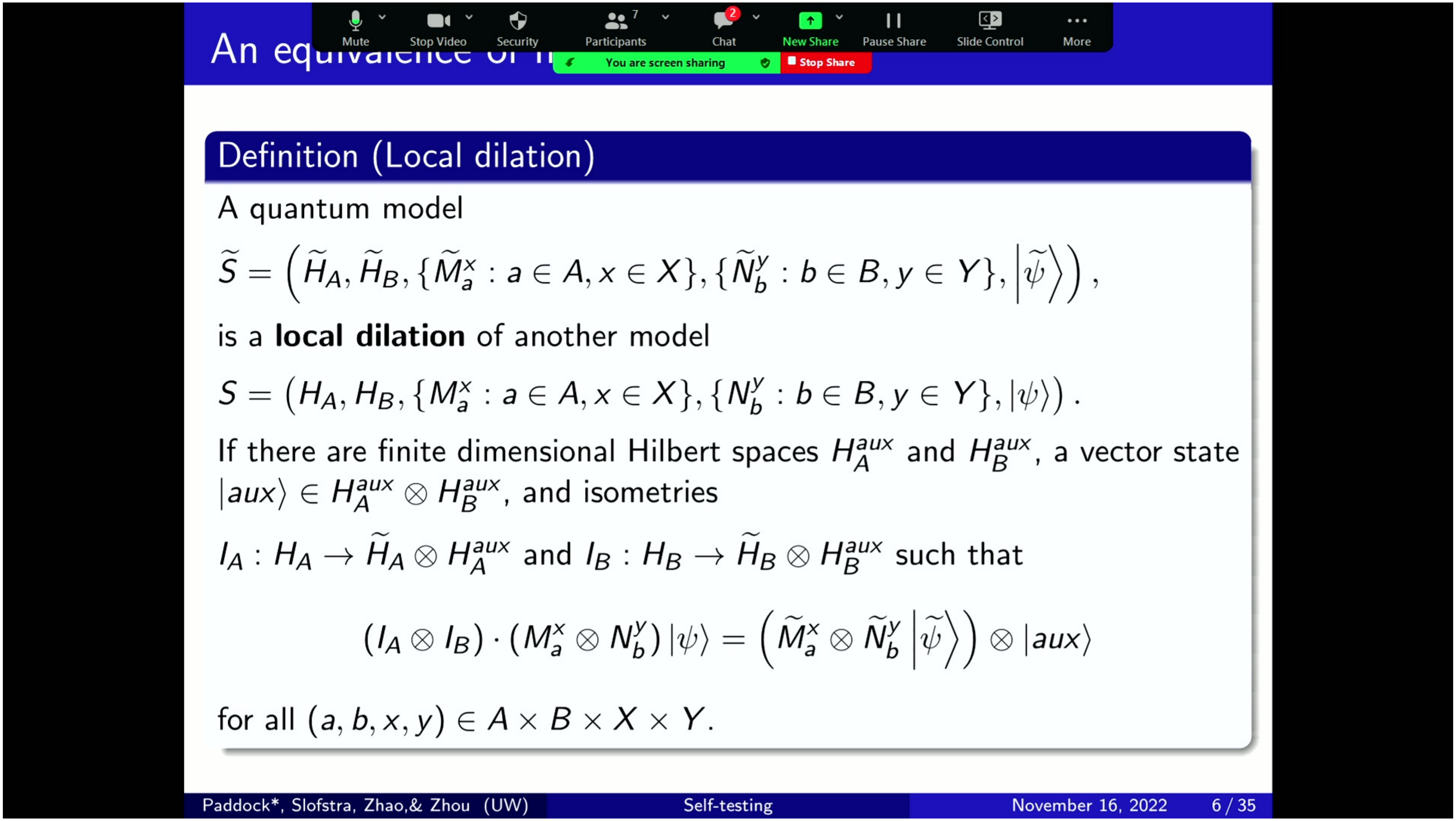
- While a quantum correlation p can be directly observed from a Bell scenario, the model S cannot.
- In fact, there are typically many different models for a given correlation p.
- Can we hope to learn anything about the model from observing only its correlations?



An equivalence of

Definition (Local dilation)

A quantum model

$$\widetilde{S} = \left(\widetilde{H}_A, \widetilde{H}_B, \{\widetilde{M}^x_a : a \in A, x \in X\}, \{\widetilde{N}^y_b : b \in B, y \in Y\}, \left|\widetilde{\psi}\right\rangle\right),$$

is a **local dilation** of another model

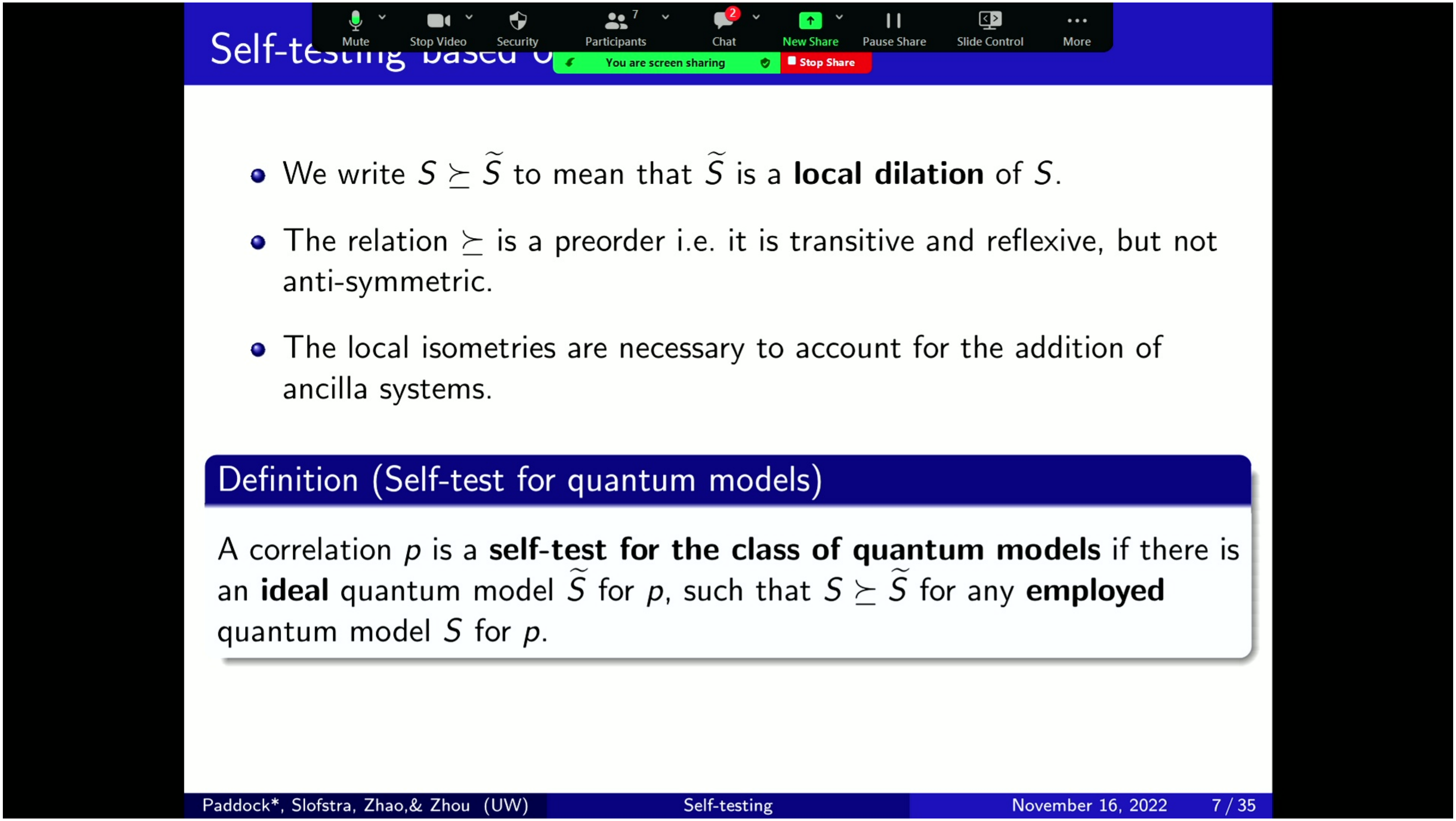
$$S = \left(H_A, H_B, \{M^x_a : a \in A, x \in X\}, \{N^y_b : b \in B, y \in Y\}, |\psi\rangle\right).$$

If there are finite dimensional Hilbert spaces H^{aux}_A and H^{aux}_B, a vector state $|aux\rangle \in H^{aux}_A \otimes H^{aux}_B$, and isometries

$I_A : H_A \to \widetilde{H}_A \otimes H^{aux}_A$ and $I_B : H_B \to \widetilde{H}_B \otimes H^{aux}_B$ such that

$$(I_A \otimes I_B) \cdot (M^x_a \otimes N^y_b) |\psi\rangle = \left(\widetilde{M}^x_a \otimes \widetilde{N}^y_b \left|\widetilde{\psi}\right\rangle\right) \otimes |aux\rangle$$

for all $(a, b, x, y) \in A \times B \times X \times Y$.

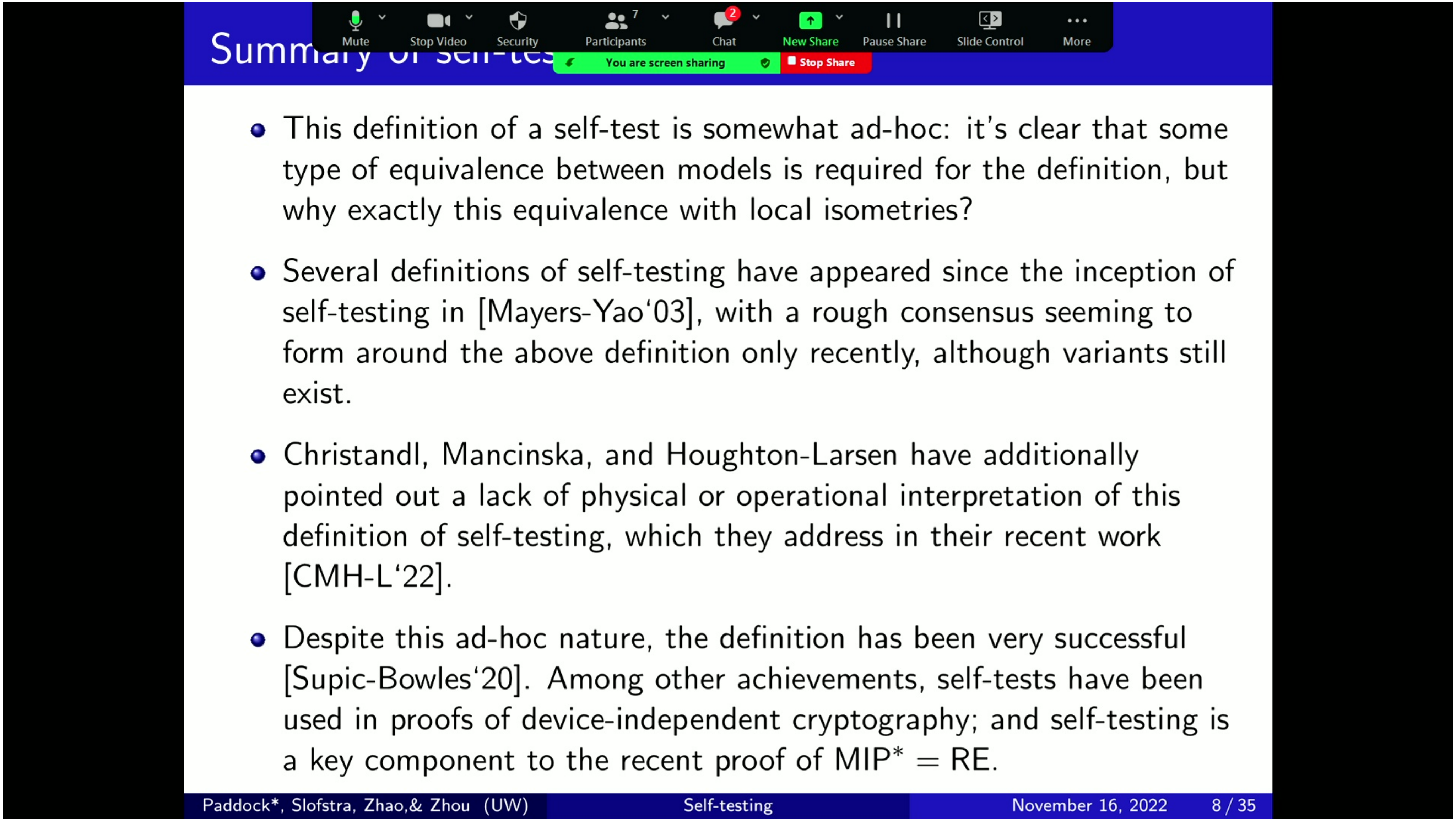


Self-testing based o

- We write $S \succeq \widetilde{S}$ to mean that $\widetilde{S}$ is a **local dilation** of S.
- The relation $\succeq$ is a preorder i.e. it is transitive and reflexive, but not anti-symmetric.
- The local isometries are necessary to account for the addition of ancilla systems.

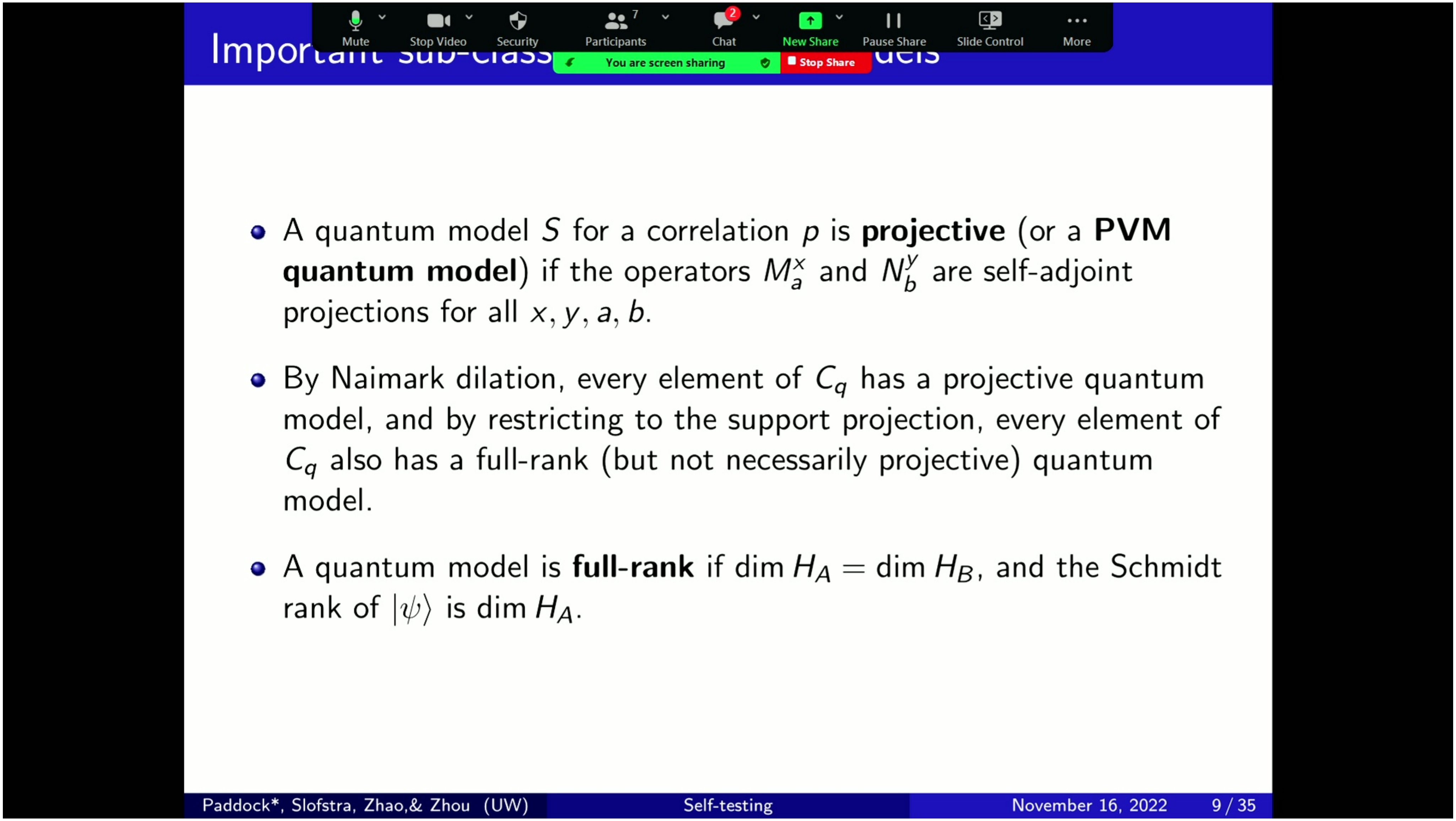
Definition (Self-test for quantum models)

A correlation p is a **self-test for the class of quantum models** if there is an **ideal** quantum model $\widetilde{S}$ for p, such that $S \succeq \widetilde{S}$ for any **employed** quantum model S for p.



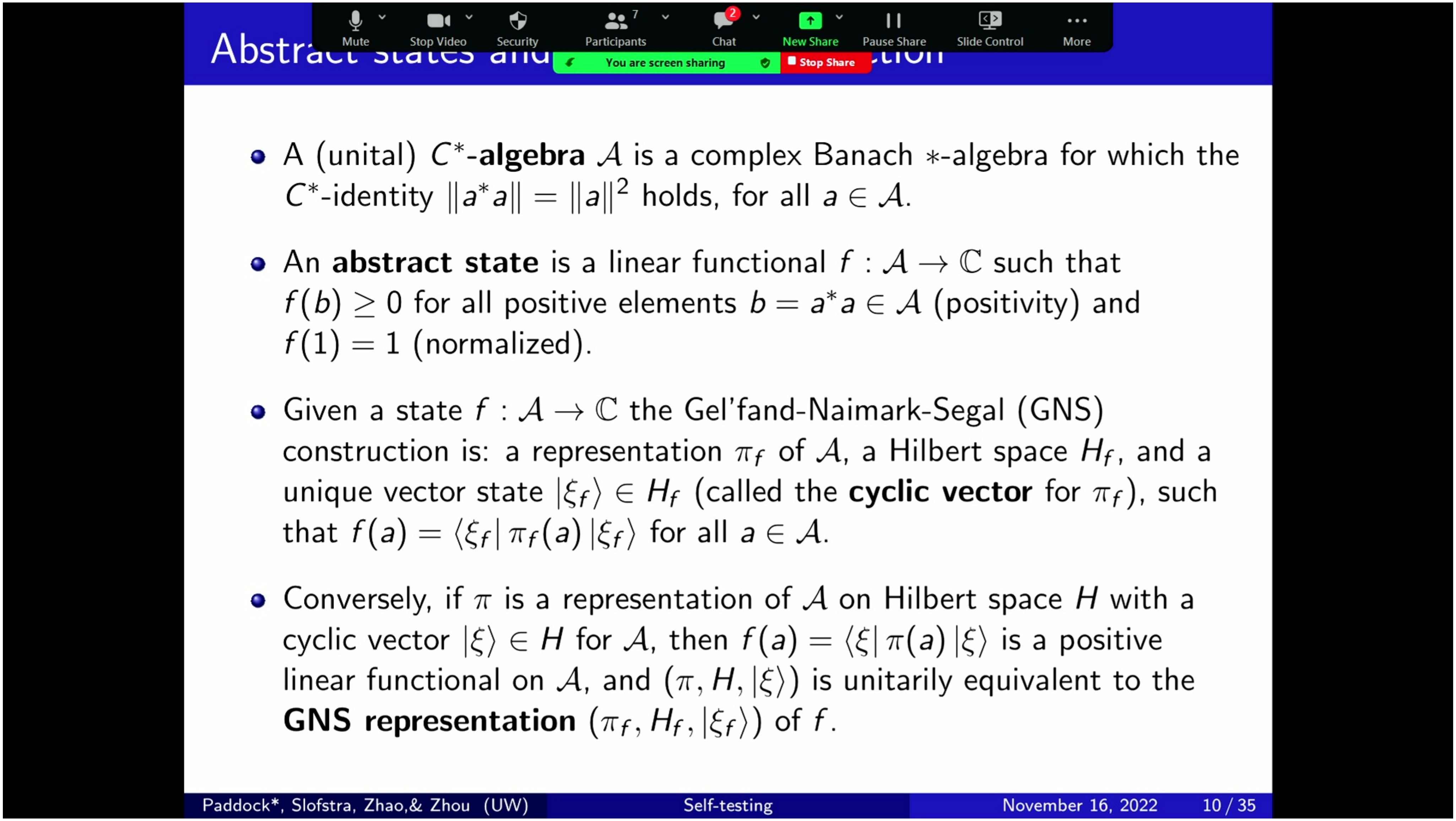
Summary of self-tes

- This definition of a self-test is somewhat ad-hoc: it's clear that some type of equivalence between models is required for the definition, but why exactly this equivalence with local isometries?
- Several definitions of self-testing have appeared since the inception of self-testing in [Mayers-Yao'03], with a rough consensus seeming to form around the above definition only recently, although variants still exist.
- Christandl, Mancinska, and Houghton-Larsen have additionally pointed out a lack of physical or operational interpretation of this definition of self-testing, which they address in their recent work [CMH-L'22].
- Despite this ad-hoc nature, the definition has been very successful [Supic-Bowles'20]. Among other achievements, self-tests have been used in proofs of device-independent cryptography; and self-testing is a key component to the recent proof of $\mathrm{MIP}^* = \mathrm{RE}$.



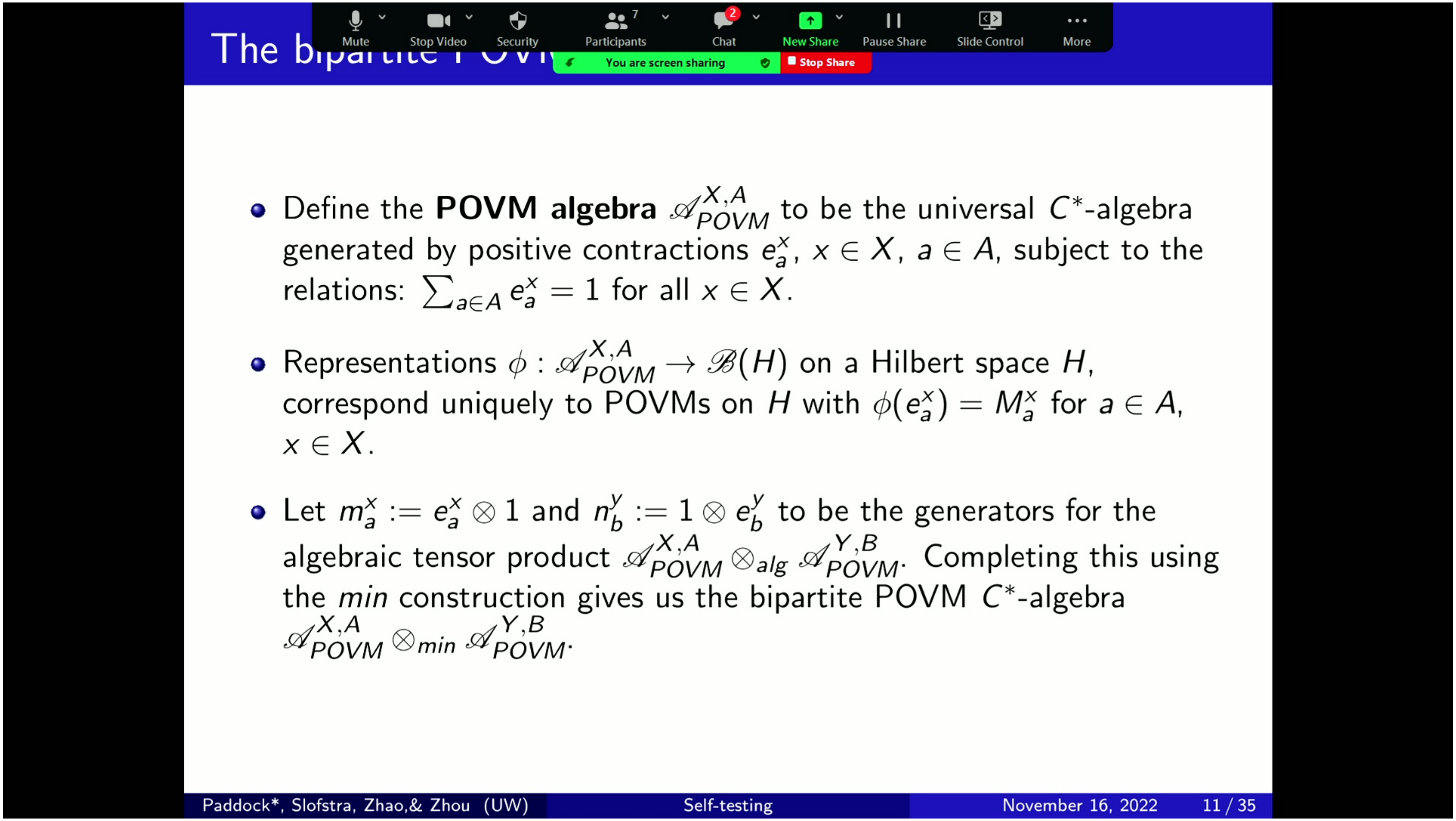
Important sub-class dels

- A quantum model S for a correlation p is **projective** (or a **PVM quantum model**) if the operators M_a^x and N_b^y are self-adjoint projections for all x, y, a, b.
- By Naimark dilation, every element of C_q has a projective quantum model, and by restricting to the support projection, every element of C_q also has a full-rank (but not necessarily projective) quantum model.
- A quantum model is **full-rank** if $\dim H_A = \dim H_B$, and the Schmidt rank of $|\psi\rangle$ is $\dim H_A$.



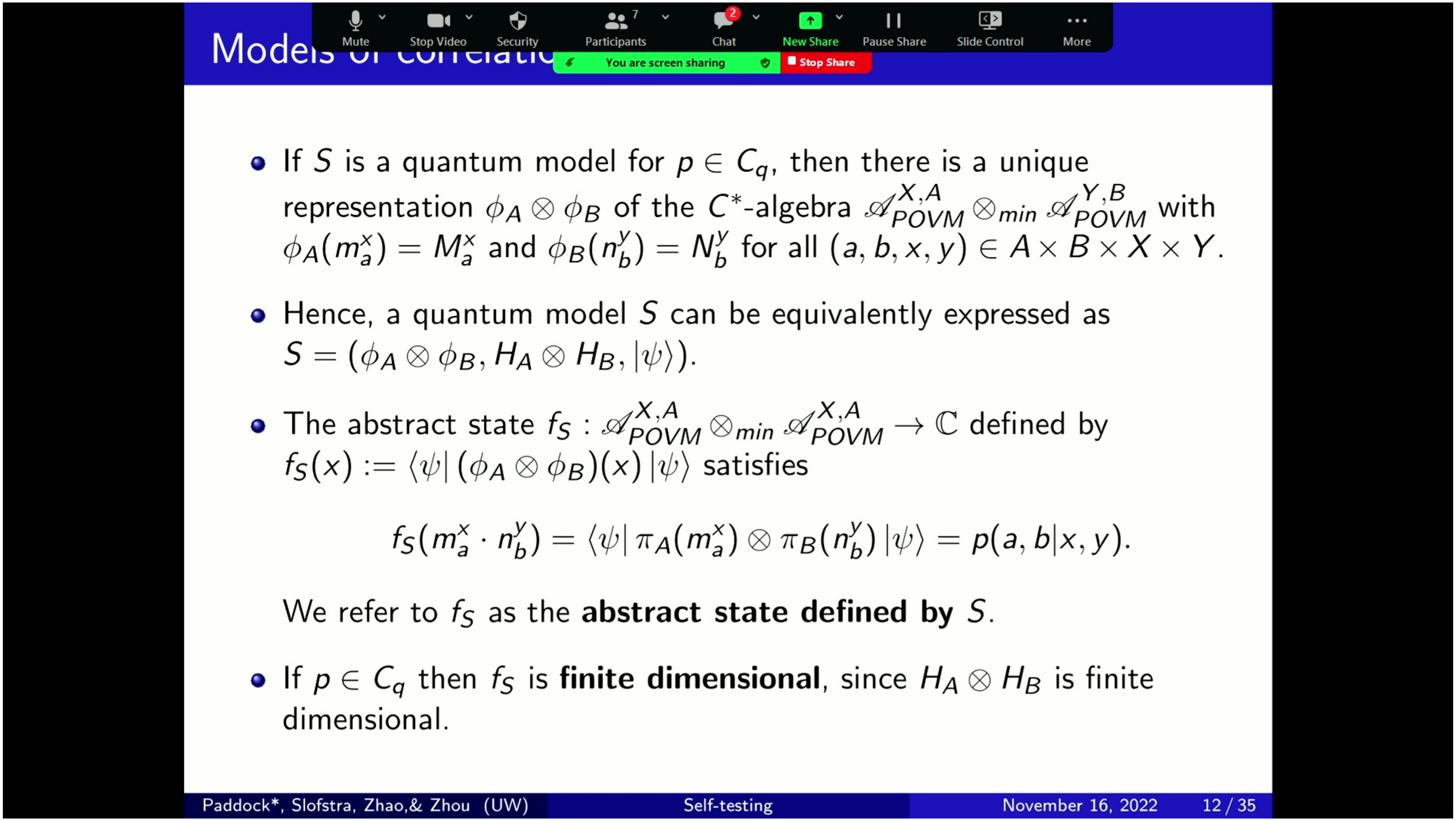
Abstract states and ction

- A (unital) C^*-**algebra** $\mathcal{A}$ is a complex Banach $*$-algebra for which the C^*-identity $\|a^*a\| = \|a\|^2$ holds, for all $a \in \mathcal{A}$.
- An **abstract state** is a linear functional $f : \mathcal{A} \to \mathbb{C}$ such that $f(b) \geq 0$ for all positive elements $b = a^*a \in \mathcal{A}$ (positivity) and $f(1) = 1$ (normalized).
- Given a state $f : \mathcal{A} \to \mathbb{C}$ the Gel'fand-Naimark-Segal (GNS) construction is: a representation π_f of $\mathcal{A}$, a Hilbert space H_f, and a unique vector state $|\xi_f\rangle \in H_f$ (called the **cyclic vector** for π_f), such that $f(a) = \langle \xi_f | \pi_f(a) | \xi_f \rangle$ for all $a \in \mathcal{A}$.
- Conversely, if π is a representation of $\mathcal{A}$ on Hilbert space H with a cyclic vector $|\xi\rangle \in H$ for $\mathcal{A}$, then $f(a) = \langle \xi | \pi(a) | \xi \rangle$ is a positive linear functional on $\mathcal{A}$, and $(\pi, H, |\xi\rangle)$ is unitarily equivalent to the **GNS representation** $(\pi_f, H_f, |\xi_f\rangle)$ of f.



The bipartite POV

- Define the **POVM algebra** $\mathscr{A}_{POVM}^{X,A}$ to be the universal C^*-algebra generated by positive contractions e_a^x, $x \in X$, $a \in A$, subject to the relations: $\sum_{a \in A} e_a^x = 1$ for all $x \in X$.
- Representations $\phi : \mathscr{A}_{POVM}^{X,A} \to \mathscr{B}(H)$ on a Hilbert space H, correspond uniquely to POVMs on H with $\phi(e_a^x) = M_a^x$ for $a \in A$, $x \in X$.
- Let $m_a^x := e_a^x \otimes 1$ and $n_b^y := 1 \otimes e_b^y$ to be the generators for the algebraic tensor product $\mathscr{A}_{POVM}^{X,A} \otimes_{alg} \mathscr{A}_{POVM}^{Y,B}$. Completing this using the *min* construction gives us the bipartite POVM C^*-algebra $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$.

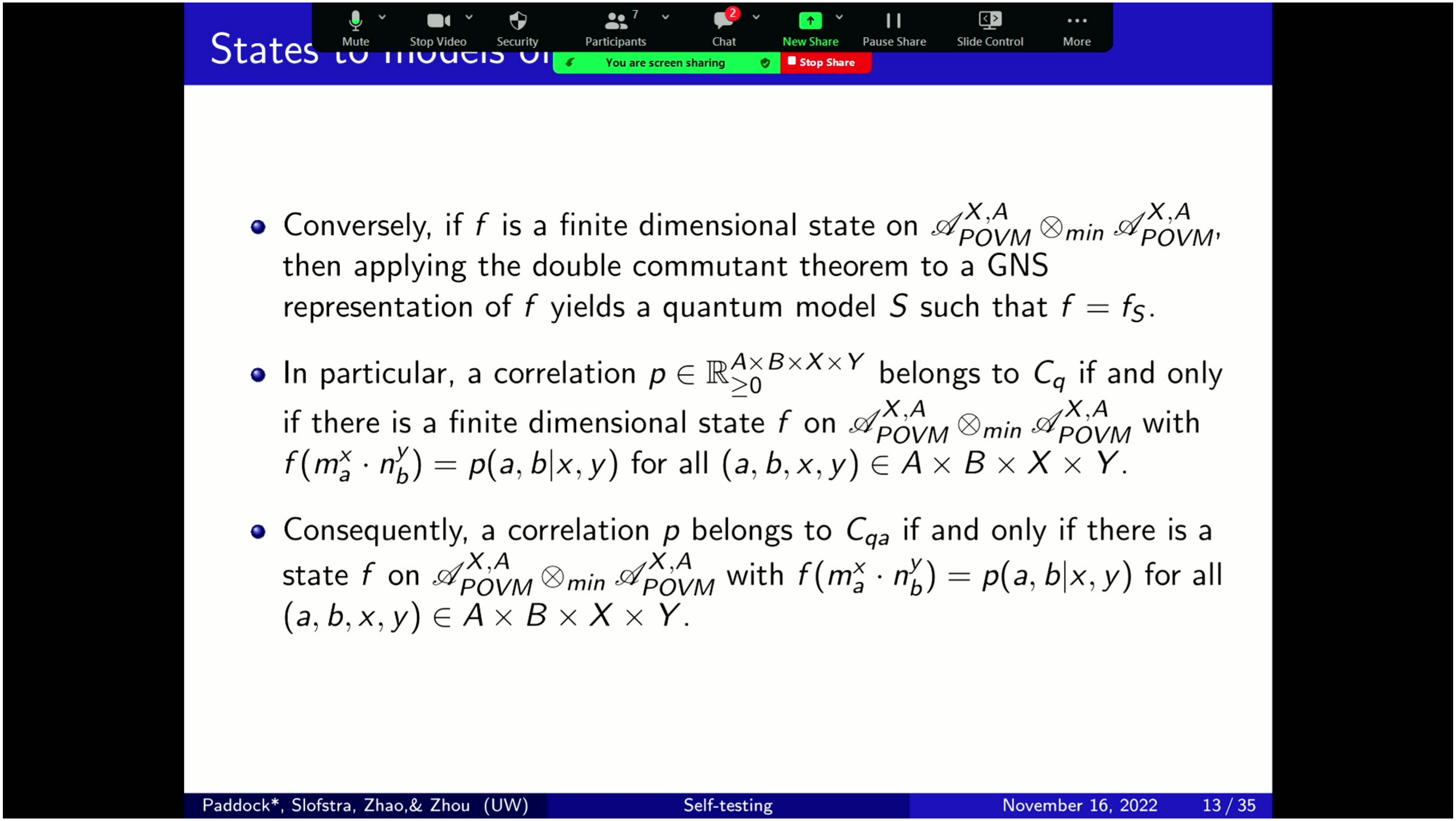


Models of correlatio

- If S is a quantum model for $p \in C_q$, then there is a unique representation $\phi_A \otimes \phi_B$ of the C^*-algebra $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$ with $\phi_A(m^x_a) = M^x_a$ and $\phi_B(n^y_b) = N^y_b$ for all $(a,b,x,y) \in A \times B \times X \times Y$.
- Hence, a quantum model S can be equivalently expressed as $S = (\phi_A \otimes \phi_B, H_A \otimes H_B, |\psi\rangle)$.
- The abstract state $f_S : \mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{X,A}_{POVM} \to \mathbb{C}$ defined by $f_S(x) := \langle\psi|\,(\phi_A \otimes \phi_B)(x)\,|\psi\rangle$ satisfies

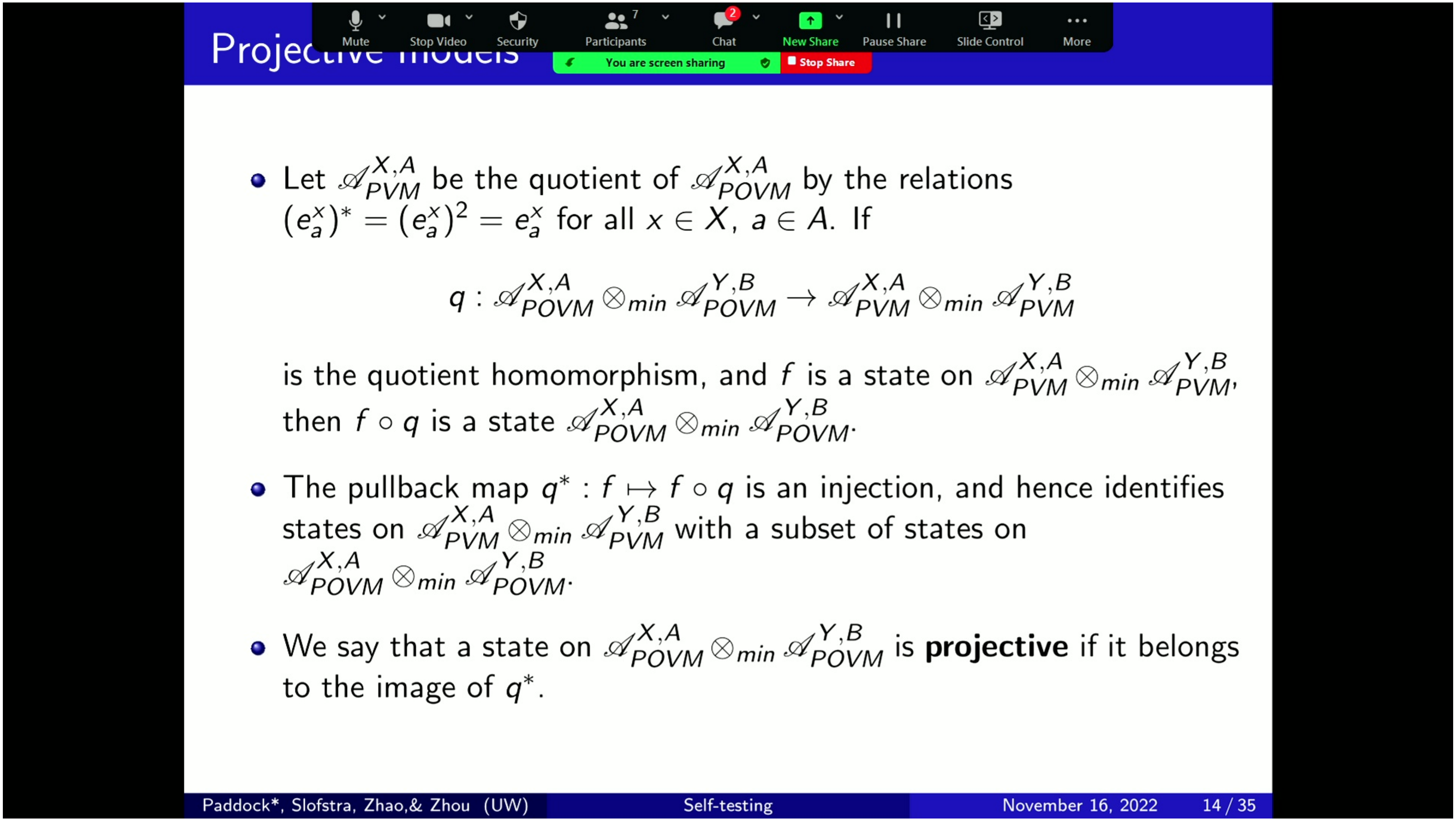
$$f_S(m^x_a \cdot n^y_b) = \langle\psi|\,\pi_A(m^x_a) \otimes \pi_B(n^y_b)\,|\psi\rangle = p(a,b|x,y).$$

  We refer to f_S as the **abstract state defined by** S.
- If $p \in C_q$ then f_S is **finite dimensional**, since $H_A \otimes H_B$ is finite dimensional.



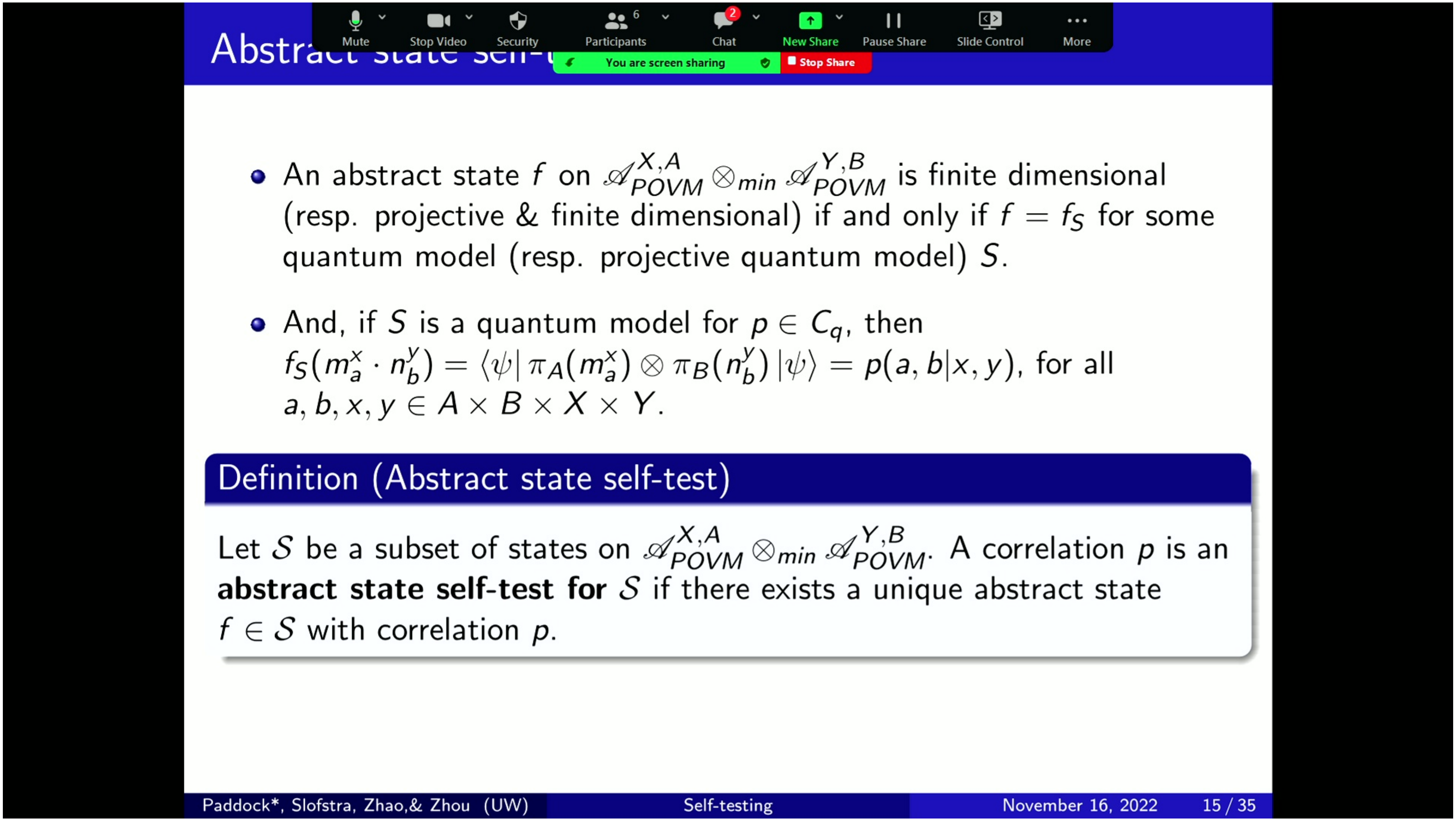
States to models of

- Conversely, if f is a finite dimensional state on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{X,A}$, then applying the double commutant theorem to a GNS representation of f yields a quantum model S such that $f = f_S$.
- In particular, a correlation $p \in \mathbb{R}_{\geq 0}^{A \times B \times X \times Y}$ belongs to C_q if and only if there is a finite dimensional state f on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{X,A}$ with $f(m_a^x \cdot n_b^y) = p(a, b|x, y)$ for all $(a, b, x, y) \in A \times B \times X \times Y$.
- Consequently, a correlation p belongs to C_{qa} if and only if there is a state f on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{X,A}$ with $f(m_a^x \cdot n_b^y) = p(a, b|x, y)$ for all $(a, b, x, y) \in A \times B \times X \times Y$.



Projective models

- Let $\mathscr{A}^{X,A}_{PVM}$ be the quotient of $\mathscr{A}^{X,A}_{POVM}$ by the relations $(e^x_a)^* = (e^x_a)^2 = e^x_a$ for all $x \in X$, $a \in A$. If
$$q : \mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM} \to \mathscr{A}^{X,A}_{PVM} \otimes_{min} \mathscr{A}^{Y,B}_{PVM}$$
is the quotient homomorphism, and f is a state on $\mathscr{A}^{X,A}_{PVM} \otimes_{min} \mathscr{A}^{Y,B}_{PVM}$, then $f \circ q$ is a state $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$.
- The pullback map $q^* : f \mapsto f \circ q$ is an injection, and hence identifies states on $\mathscr{A}^{X,A}_{PVM} \otimes_{min} \mathscr{A}^{Y,B}_{PVM}$ with a subset of states on $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$.
- We say that a state on $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$ is **projective** if it belongs to the image of q^*.

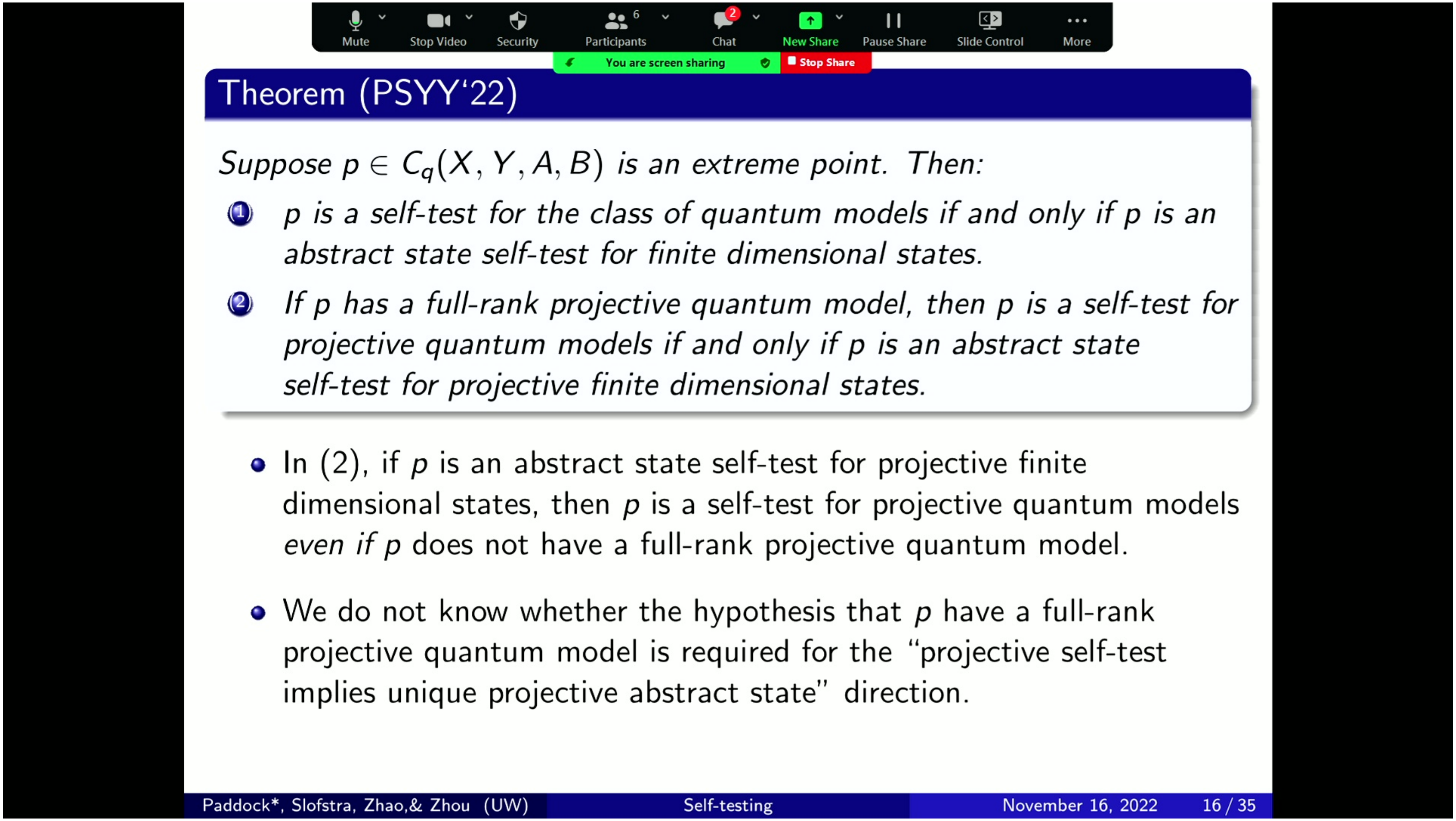


Abstract state self-

- An abstract state f on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$ is finite dimensional (resp. projective & finite dimensional) if and only if $f = f_S$ for some quantum model (resp. projective quantum model) S.
- And, if S is a quantum model for $p \in C_q$, then $f_S(m_a^x \cdot n_b^y) = \langle\psi|\,\pi_A(m_a^x) \otimes \pi_B(n_b^y)\,|\psi\rangle = p(a,b|x,y)$, for all $a,b,x,y \in A \times B \times X \times Y$.

Definition (Abstract state self-test)

Let $\mathcal{S}$ be a subset of states on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$. A correlation p is an **abstract state self-test for** $\mathcal{S}$ if there exists a unique abstract state $f \in \mathcal{S}$ with correlation p.

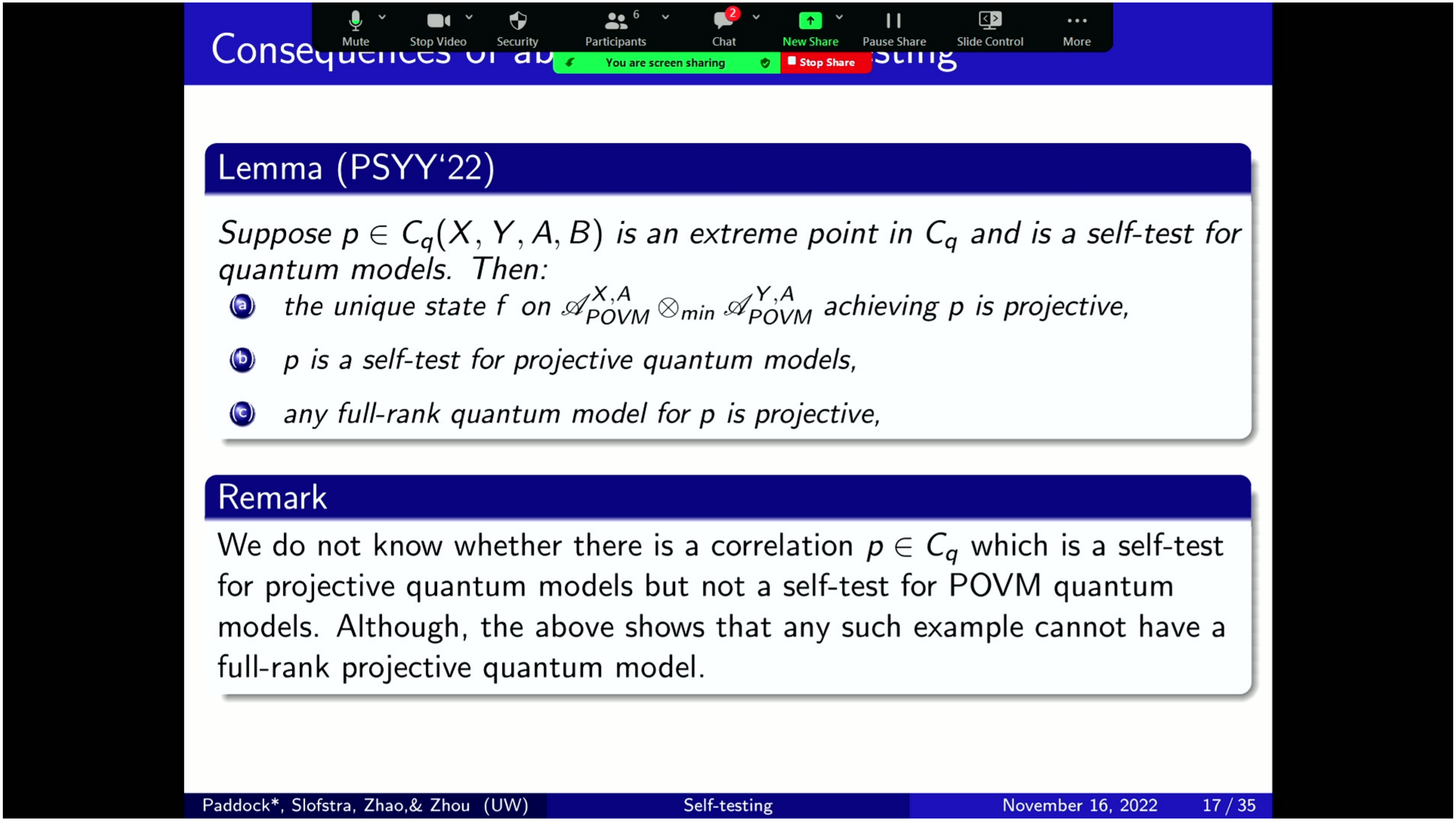


Theorem (PSYY'22)

Suppose $p \in C_q(X, Y, A, B)$ *is an extreme point. Then:*

1. p *is a self-test for the class of quantum models if and only if* p *is an abstract state self-test for finite dimensional states.*
2. *If* p *has a full-rank projective quantum model, then* p *is a self-test for projective quantum models if and only if* p *is an abstract state self-test for projective finite dimensional states.*

- In (2), if p is an abstract state self-test for projective finite dimensional states, then p is a self-test for projective quantum models *even if* p does not have a full-rank projective quantum model.
- We do not know whether the hypothesis that p have a full-rank projective quantum model is required for the "projective self-test implies unique projective abstract state" direction.



Consequences of ab sting

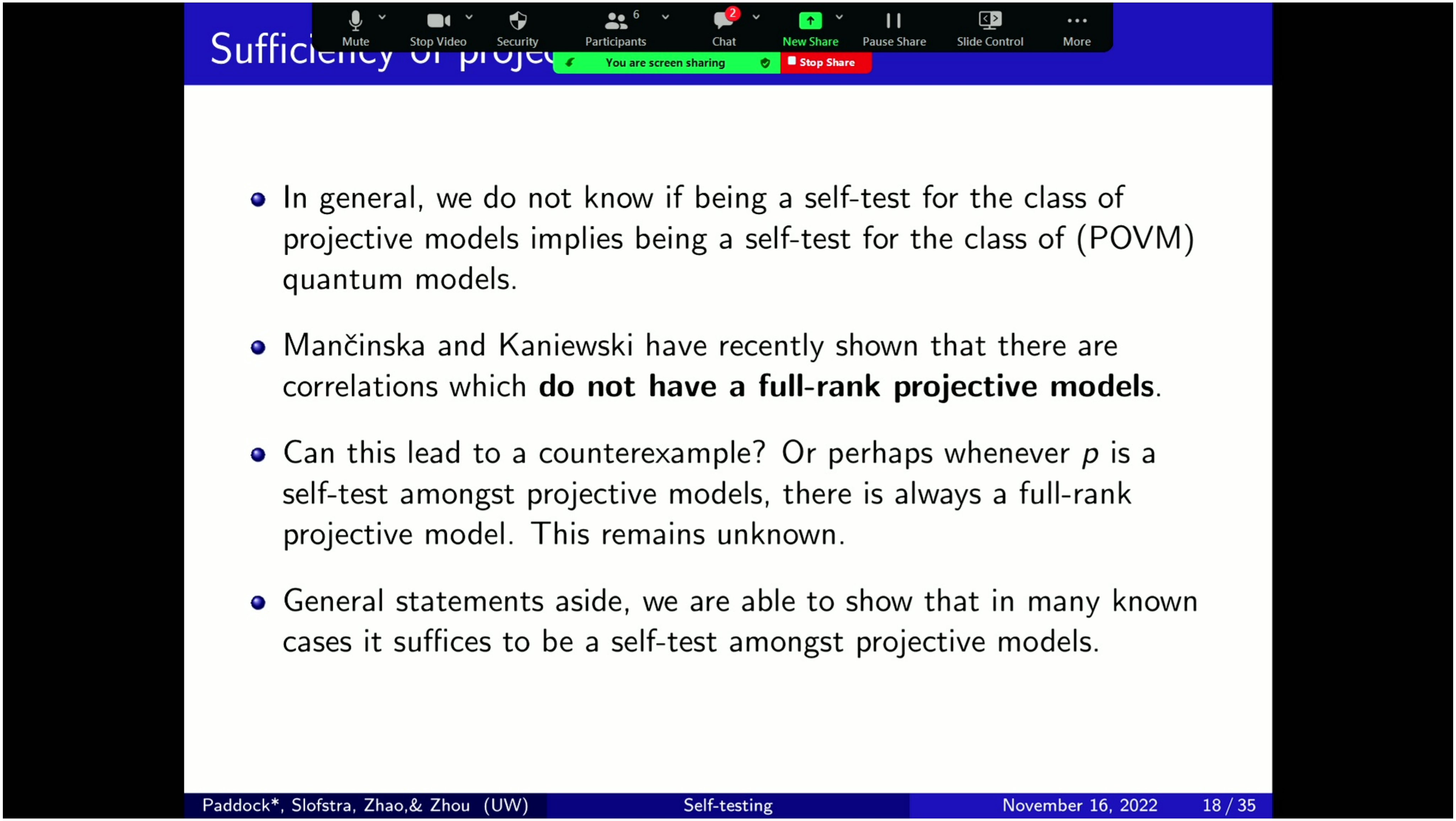
Lemma (PSYY'22)

Suppose $p \in C_q(X, Y, A, B)$ *is an extreme point in* C_q *and is a self-test for quantum models. Then:*

- ⓐ *the unique state* f *on* $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,A}_{POVM}$ *achieving* p *is projective,*
- ⓑ p *is a self-test for projective quantum models,*
- ⓒ *any full-rank quantum model for* p *is projective,*

Remark

We do not know whether there is a correlation $p \in C_q$ which is a self-test for projective quantum models but not a self-test for POVM quantum models. Although, the above shows that any such example cannot have a full-rank projective quantum model.



Sufficiency of projec

- In general, we do not know if being a self-test for the class of projective models implies being a self-test for the class of (POVM) quantum models.
- Mančinska and Kaniewski have recently shown that there are correlations which **do not have a full-rank projective models**.
- Can this lead to a counterexample? Or perhaps whenever *p* is a self-test amongst projective models, there is always a full-rank projective model. This remains unknown.
- General statements aside, we are able to show that in many known cases it suffices to be a self-test amongst projective models.

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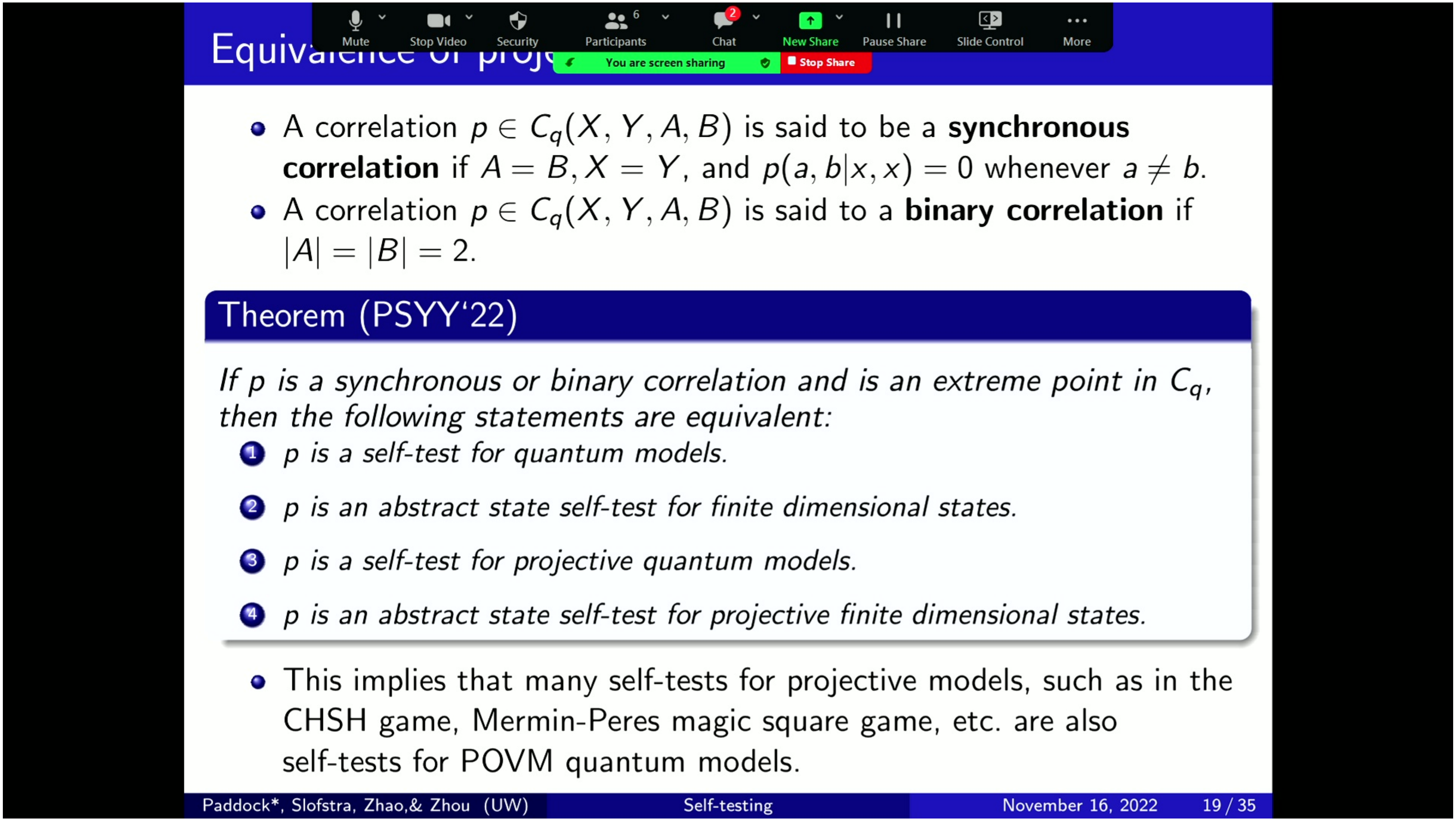


Equivalence of proj

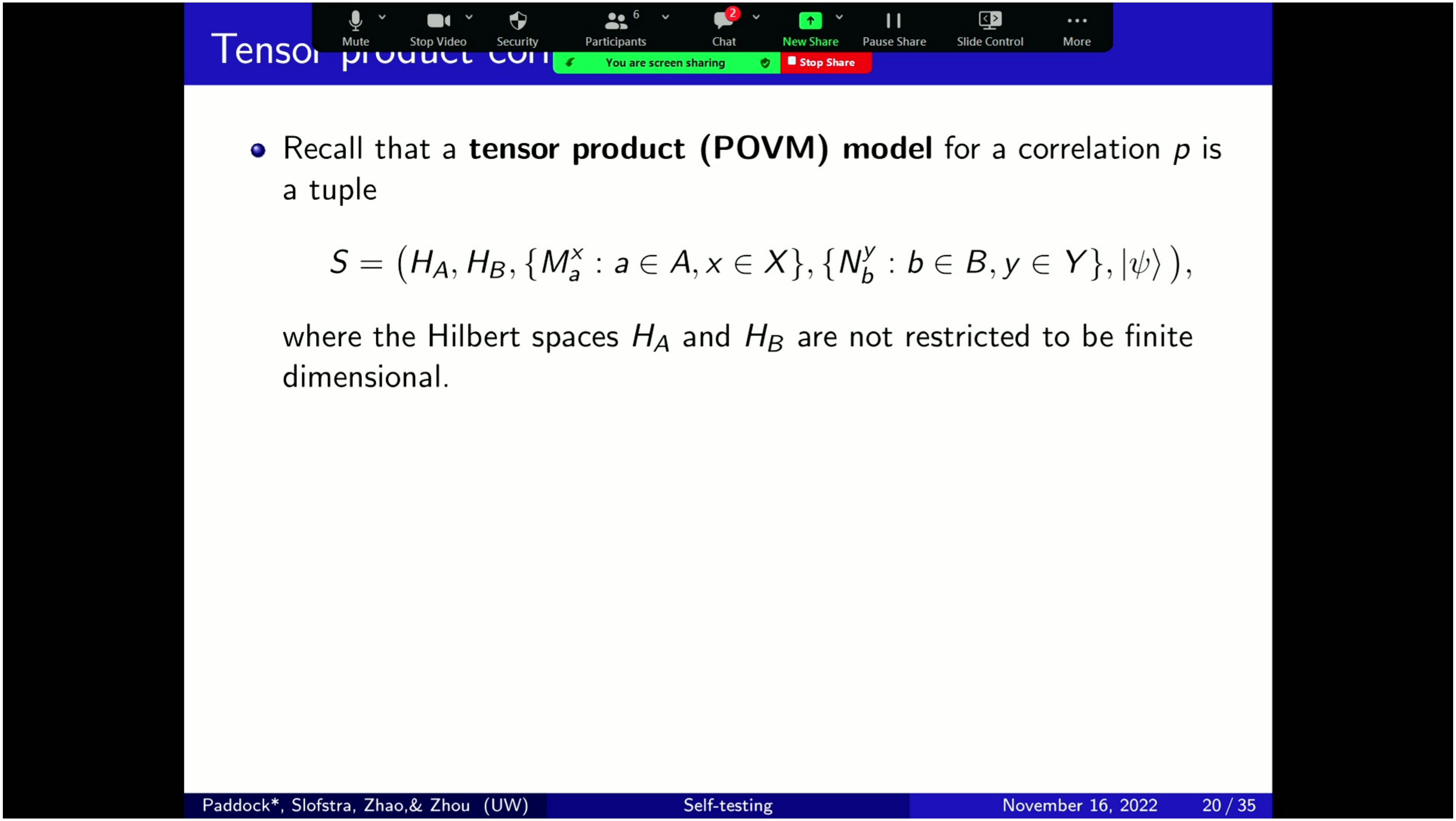
- A correlation $p \in C_q(X, Y, A, B)$ is said to be a **synchronous correlation** if $A = B, X = Y$, and $p(a, b|x, x) = 0$ whenever $a \neq b$.
- A correlation $p \in C_q(X, Y, A, B)$ is said to a **binary correlation** if $|A| = |B| = 2$.

Theorem (PSYY'22)

If p is a synchronous or binary correlation and is an extreme point in C_q, then the following statements are equivalent:

1. *p is a self-test for quantum models.*
2. *p is an abstract state self-test for finite dimensional states.*
3. *p is a self-test for projective quantum models.*
4. *p is an abstract state self-test for projective finite dimensional states.*

- This implies that many self-tests for projective models, such as in the CHSH game, Mermin-Peres magic square game, etc. are also self-tests for POVM quantum models.

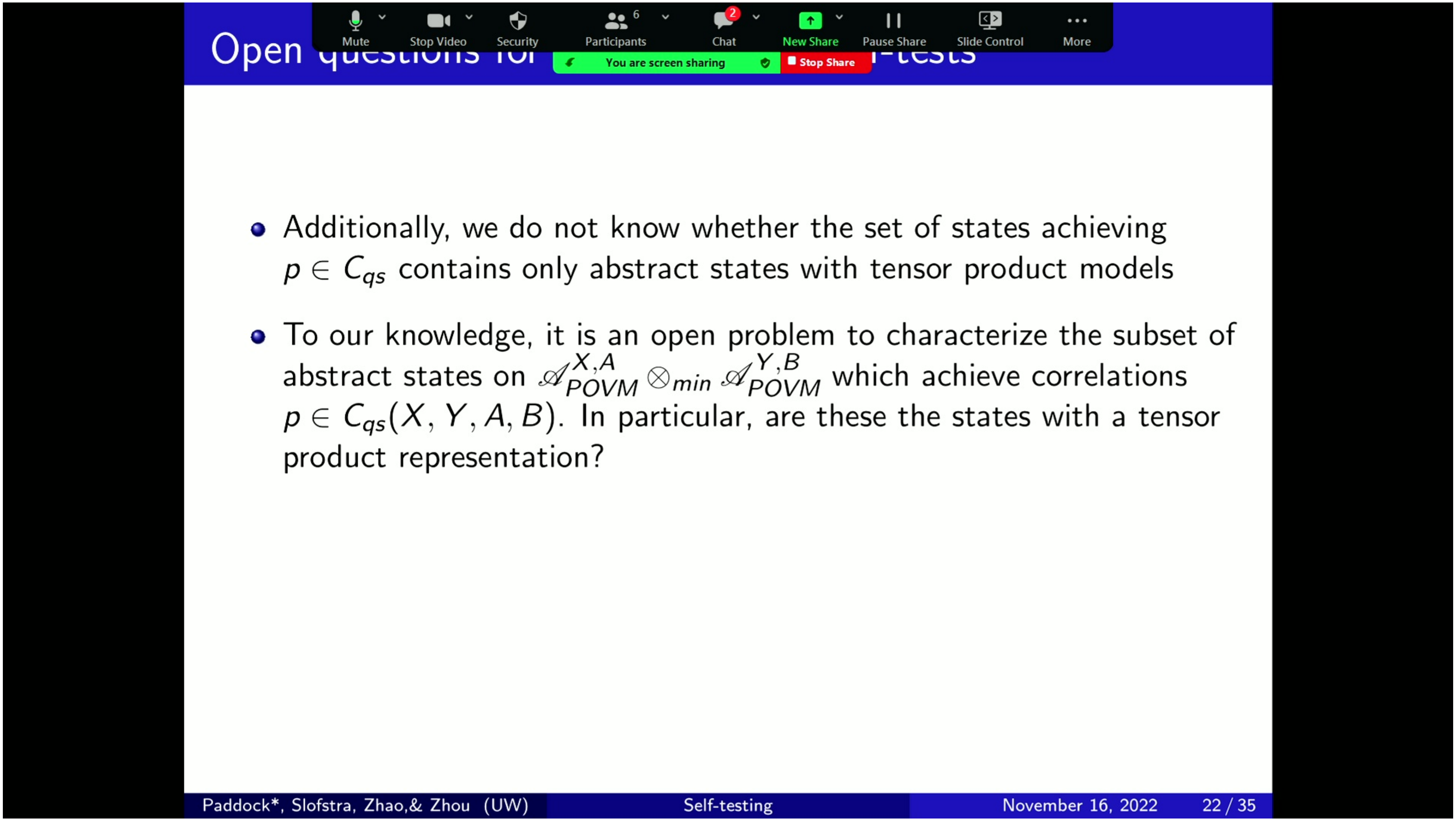


Tenso product con

- Recall that a **tensor product (POVM) model** for a correlation p is a tuple

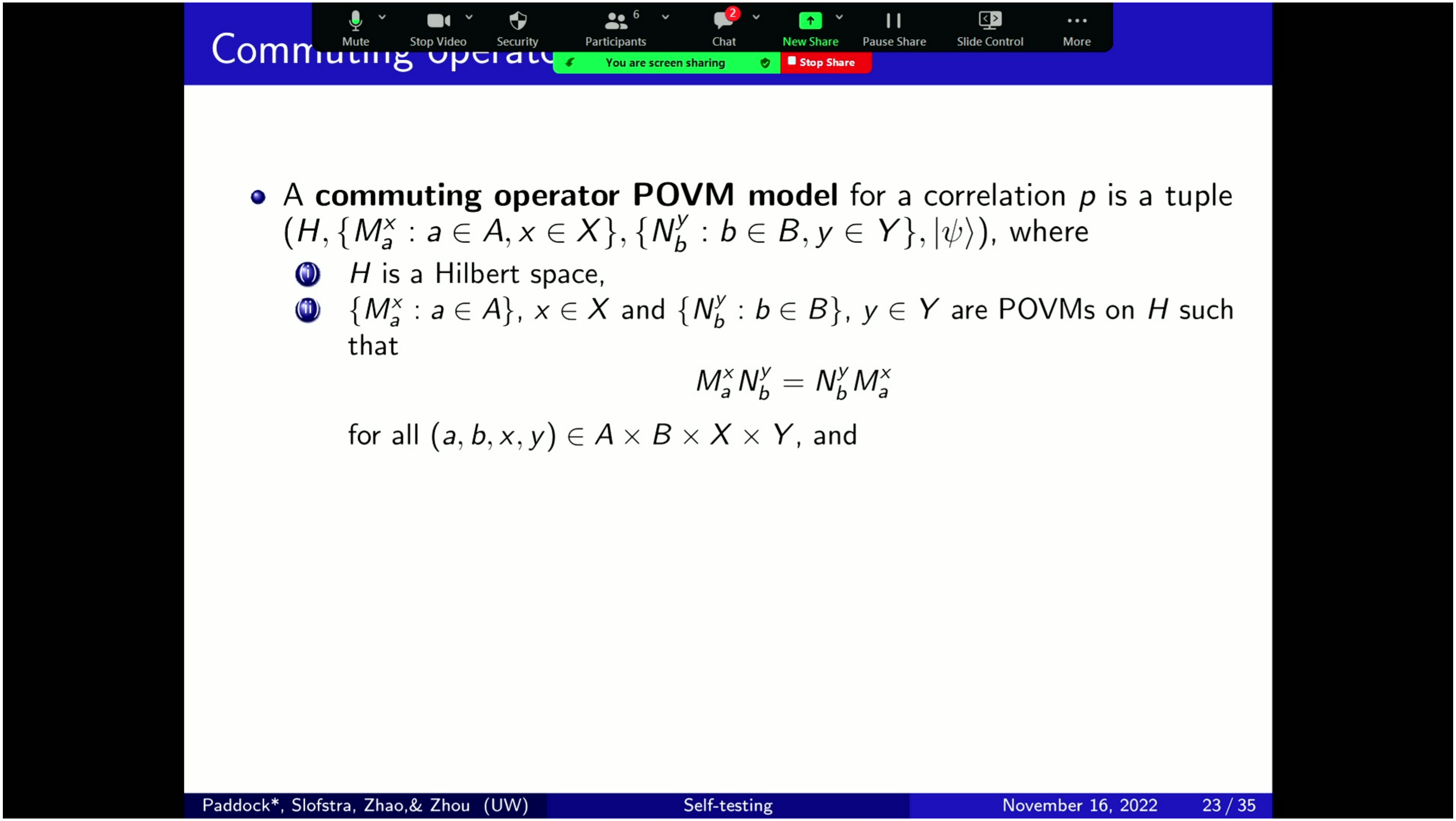
$$S = \big(H_A, H_B, \{M_a^x : a \in A, x \in X\}, \{N_b^y : b \in B, y \in Y\}, |\psi\rangle\big),$$

where the Hilbert spaces H_A and H_B are not restricted to be finite dimensional.



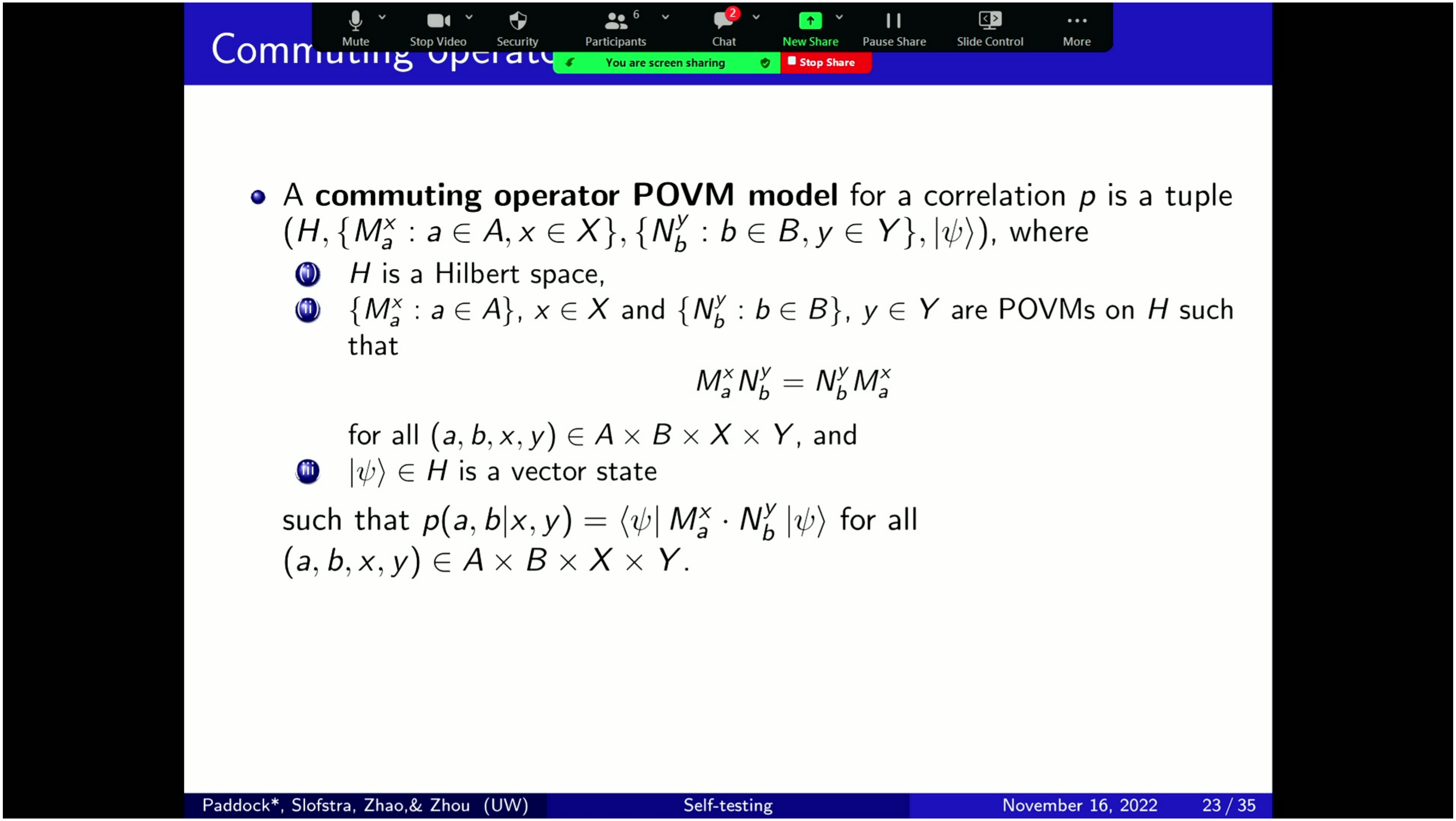
Open questions for -tests

- Additionally, we do not know whether the set of states achieving $p \in C_{qs}$ contains only abstract states with tensor product models
- To our knowledge, it is an open problem to characterize the subset of abstract states on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$ which achieve correlations $p \in C_{qs}(X, Y, A, B)$. In particular, are these the states with a tensor product representation?



Commuting operat

- A **commuting operator POVM model** for a correlation p is a tuple $(H, \{M_a^x : a \in A, x \in X\}, \{N_b^y : b \in B, y \in Y\}, |\psi\rangle)$, where
  1. (i) H is a Hilbert space,
  2. (ii) $\{M_a^x : a \in A\}$, $x \in X$ and $\{N_b^y : b \in B\}$, $y \in Y$ are POVMs on H such that
  $$M_a^x N_b^y = N_b^y M_a^x$$
  for all $(a, b, x, y) \in A \times B \times X \times Y$, and



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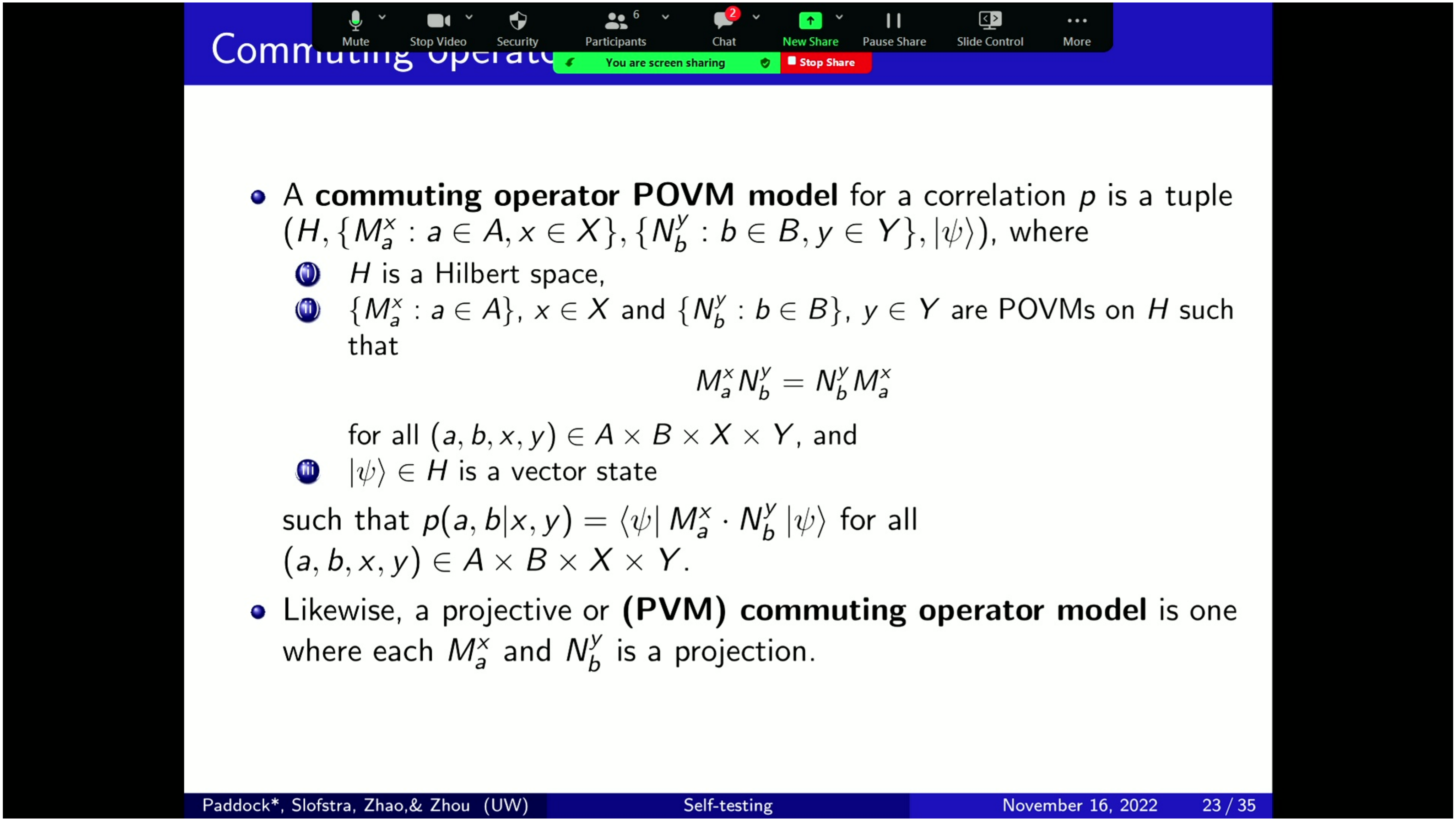
  $$M_a^x N_b^y = N_b^y M_a^x$$
  for all $(a, b, x, y) \in A \times B \times X \times Y$, and
  3. $|\psi\rangle \in H$ is a vector state

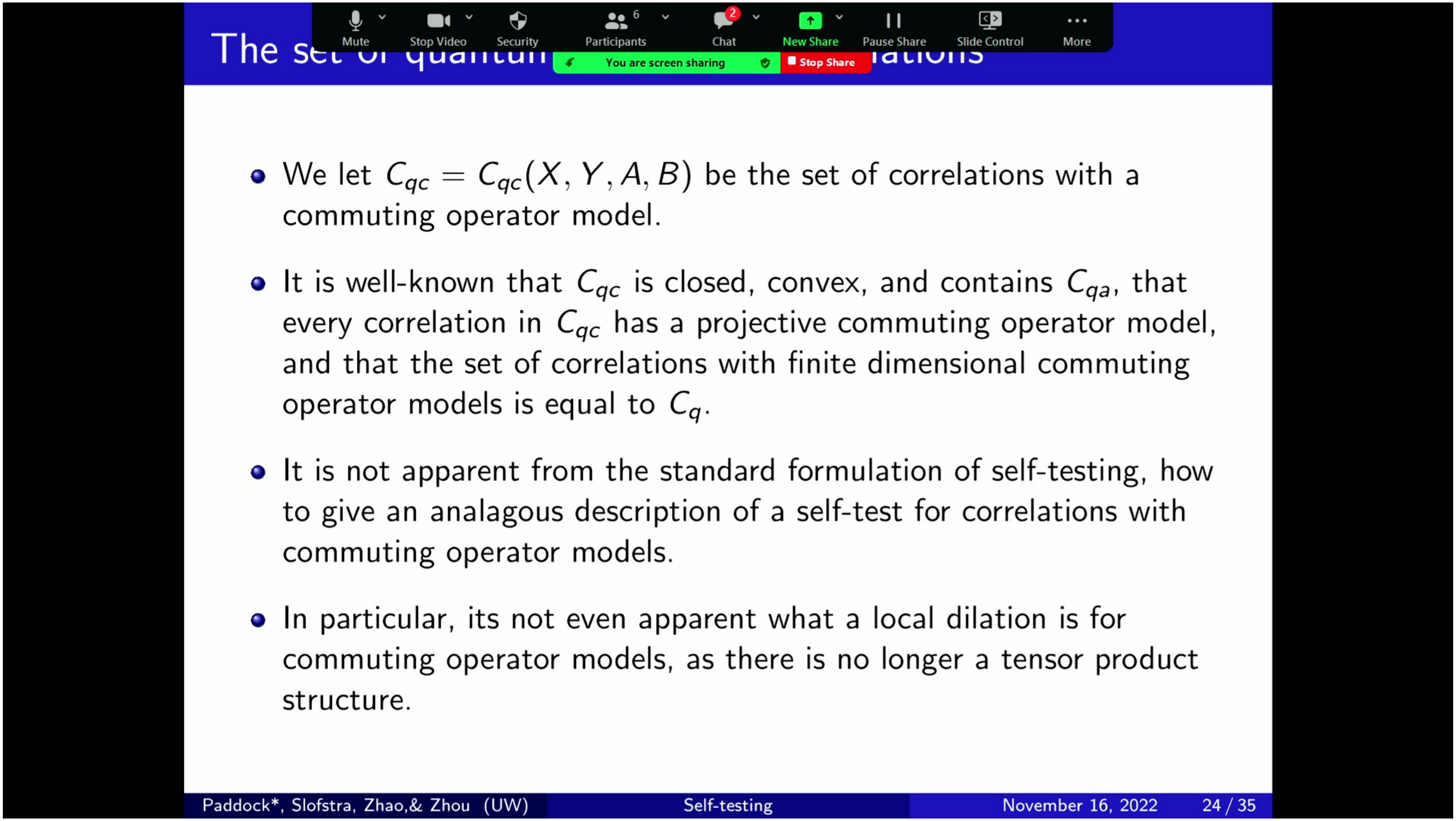
  such that $p(a, b|x, y) = \langle\psi| M_a^x \cdot N_b^y |\psi\rangle$ for all $(a, b, x, y) \in A \times B \times X \times Y$.



Commuting operato

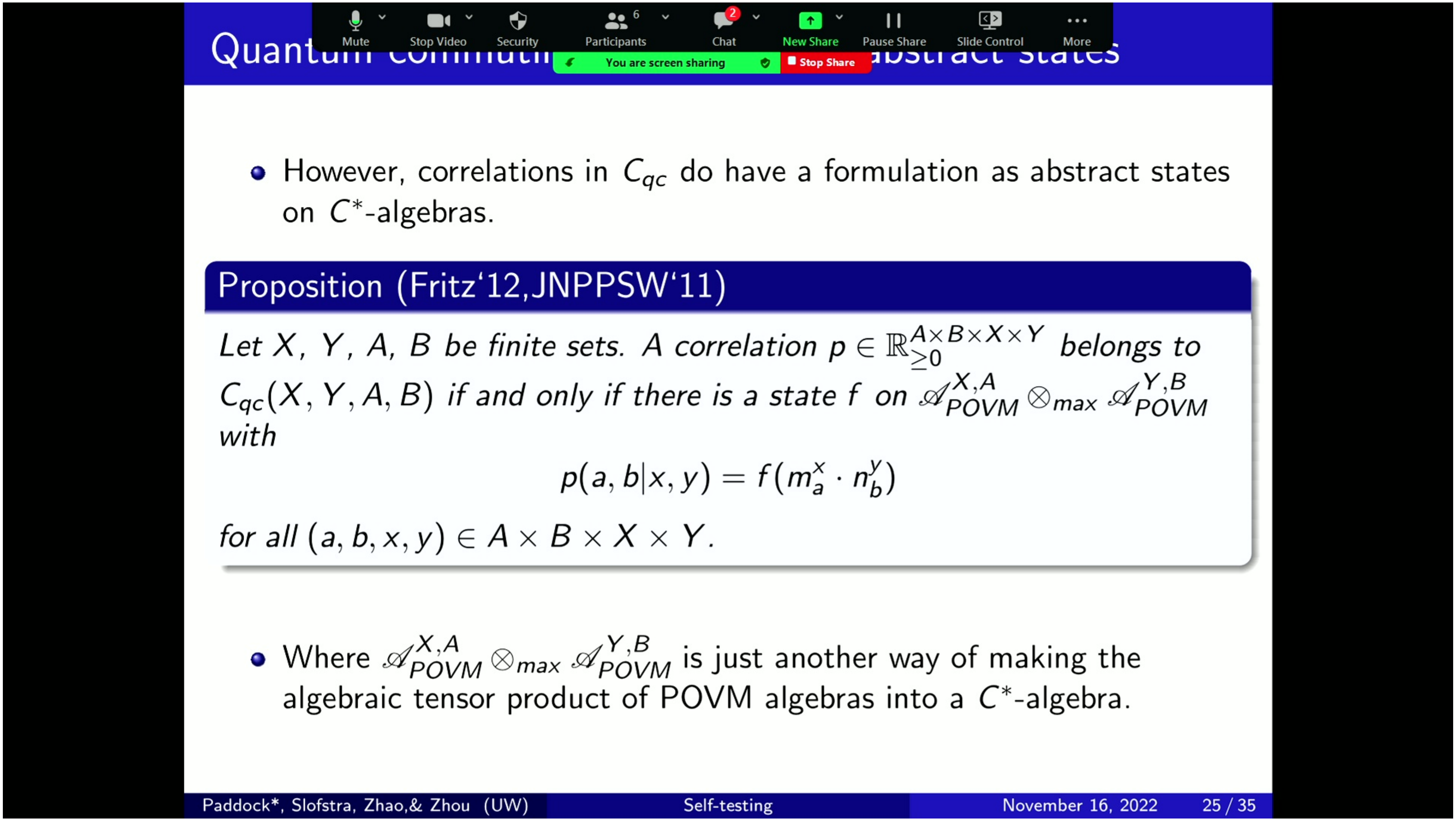
- A **commuting operator POVM model** for a correlation p is a tuple $(H, \{M_a^x : a \in A, x \in X\}, \{N_b^y : b \in B, y \in Y\}, |\psi\rangle)$, where
  - (i) H is a Hilbert space,
  - (ii) $\{M_a^x : a \in A\}$, $x \in X$ and $\{N_b^y : b \in B\}$, $y \in Y$ are POVMs on H such that
$$M_a^x N_b^y = N_b^y M_a^x$$
for all $(a, b, x, y) \in A \times B \times X \times Y$, and
  - (iii) $|\psi\rangle \in H$ is a vector state

  such that $p(a, b|x, y) = \langle\psi| M_a^x \cdot N_b^y |\psi\rangle$ for all $(a, b, x, y) \in A \times B \times X \times Y$.
- Likewise, a projective or **(PVM) commuting operator model** is one where each M_a^x and N_b^y is a projection.



The set of quantum lations

- We let $C_{qc} = C_{qc}(X, Y, A, B)$ be the set of correlations with a commuting operator model.
- It is well-known that C_{qc} is closed, convex, and contains C_{qa}, that every correlation in C_{qc} has a projective commuting operator model, and that the set of correlations with finite dimensional commuting operator models is equal to C_q.
- It is not apparent from the standard formulation of self-testing, how to give an analagous description of a self-test for correlations with commuting operator models.
- In particular, its not even apparent what a local dilation is for commuting operator models, as there is no longer a tensor product structure.



Quantum commuti abstract states

- However, correlations in C_{qc} do have a formulation as abstract states on C^*-algebras.

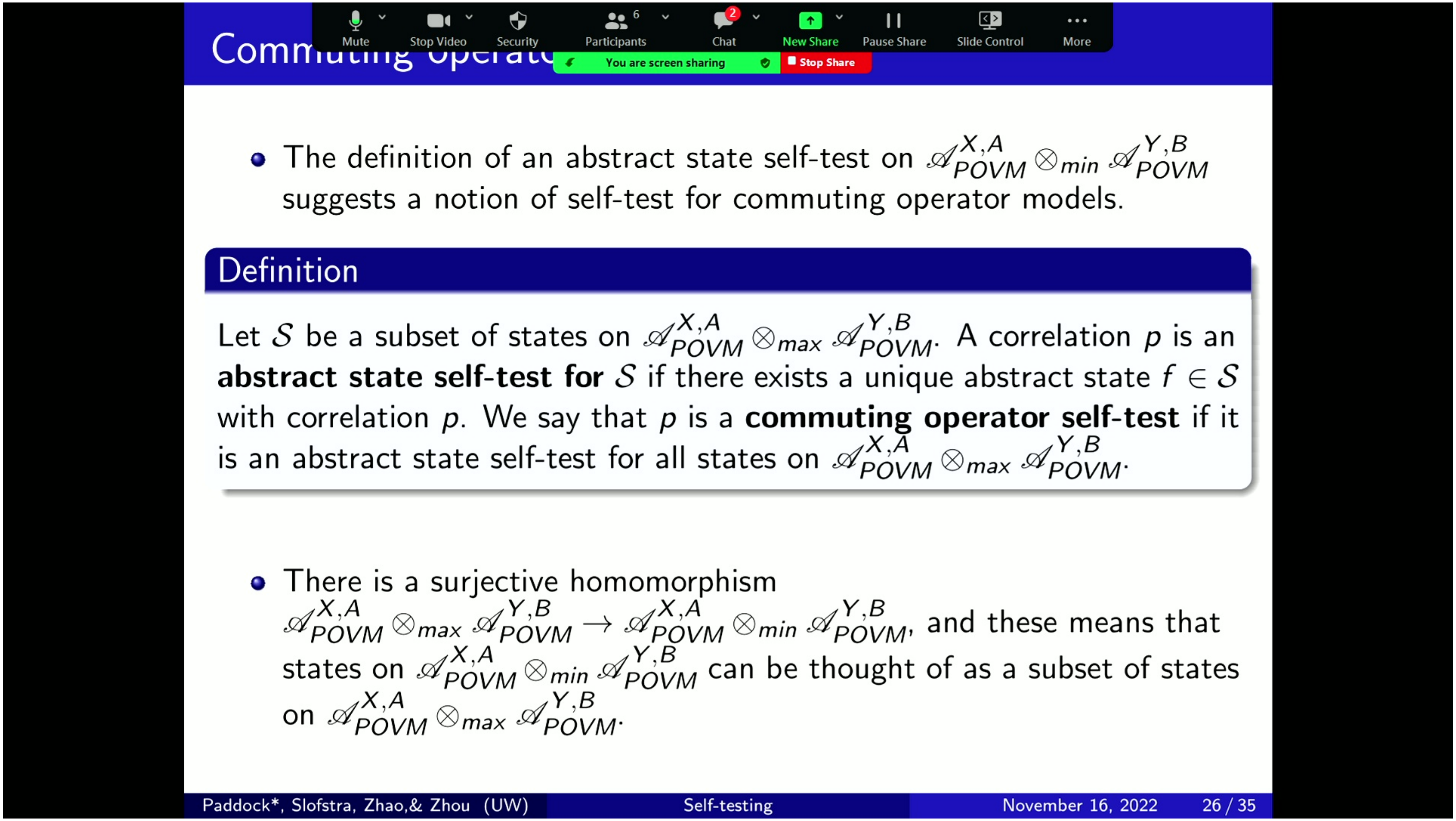
Proposition (Fritz'12,JNPPSW'11)

Let X, Y, A, B be finite sets. A correlation $p \in \mathbb{R}_{\geq 0}^{A\times B\times X\times Y}$ belongs to $C_{qc}(X,Y,A,B)$ if and only if there is a state f on $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B}$ with

$$p(a,b|x,y) = f(m_a^x \cdot n_b^y)$$

for all $(a,b,x,y) \in A \times B \times X \times Y$.

- Where $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B}$ is just another way of making the algebraic tensor product of POVM algebras into a C^*-algebra.



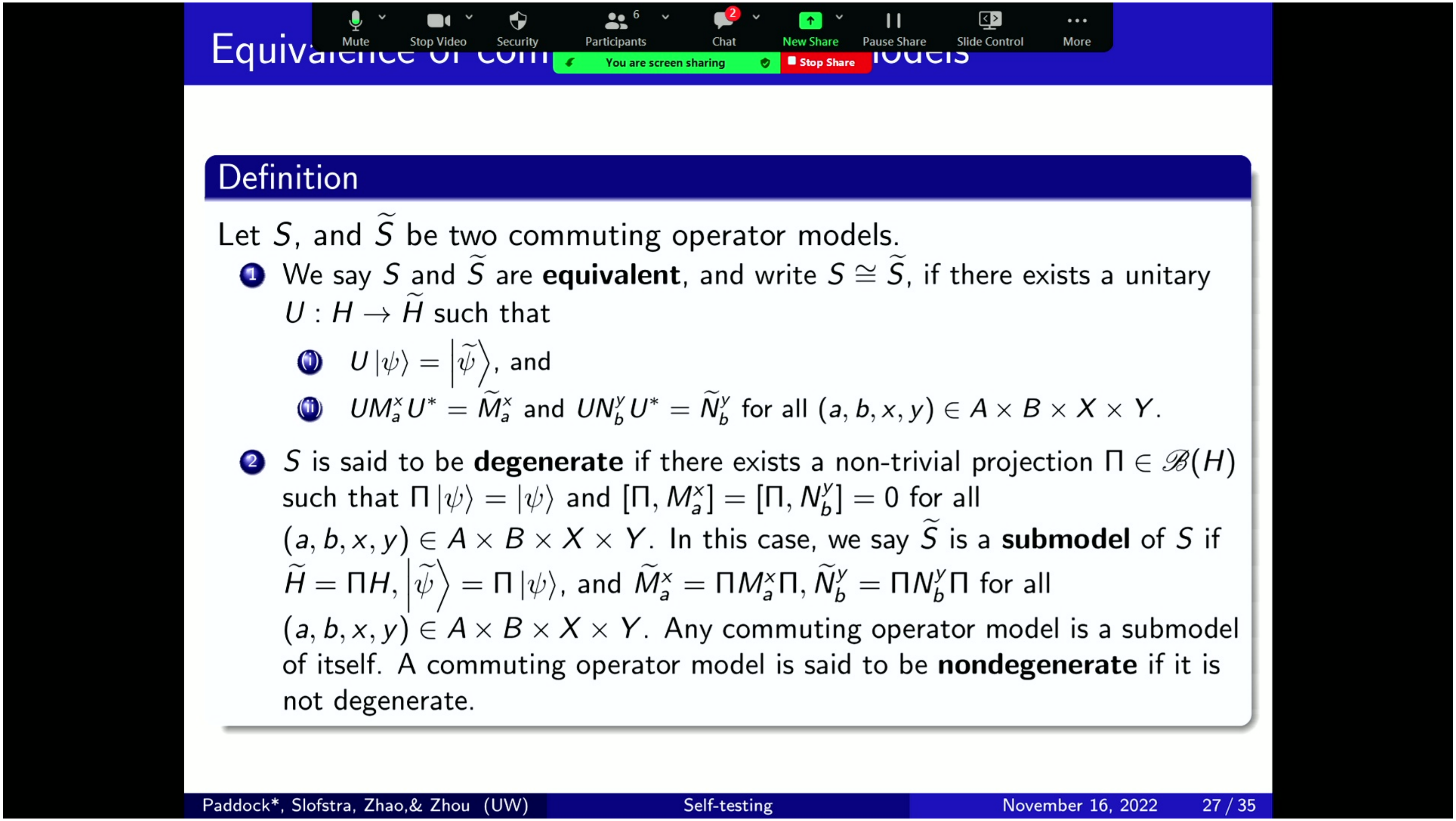
Commuting operat

- The definition of an abstract state self-test on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$ suggests a notion of self-test for commuting operator models.

Definition

Let $\mathcal{S}$ be a subset of states on $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B}$. A correlation p is an **abstract state self-test for** $\mathcal{S}$ if there exists a unique abstract state $f \in \mathcal{S}$ with correlation p. We say that p is a **commuting operator self-test** if it is an abstract state self-test for all states on $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B}$.

- There is a surjective homomorphism $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B} \to \mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$, and these means that states on $\mathscr{A}_{POVM}^{X,A} \otimes_{min} \mathscr{A}_{POVM}^{Y,B}$ can be thought of as a subset of states on $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B}$.

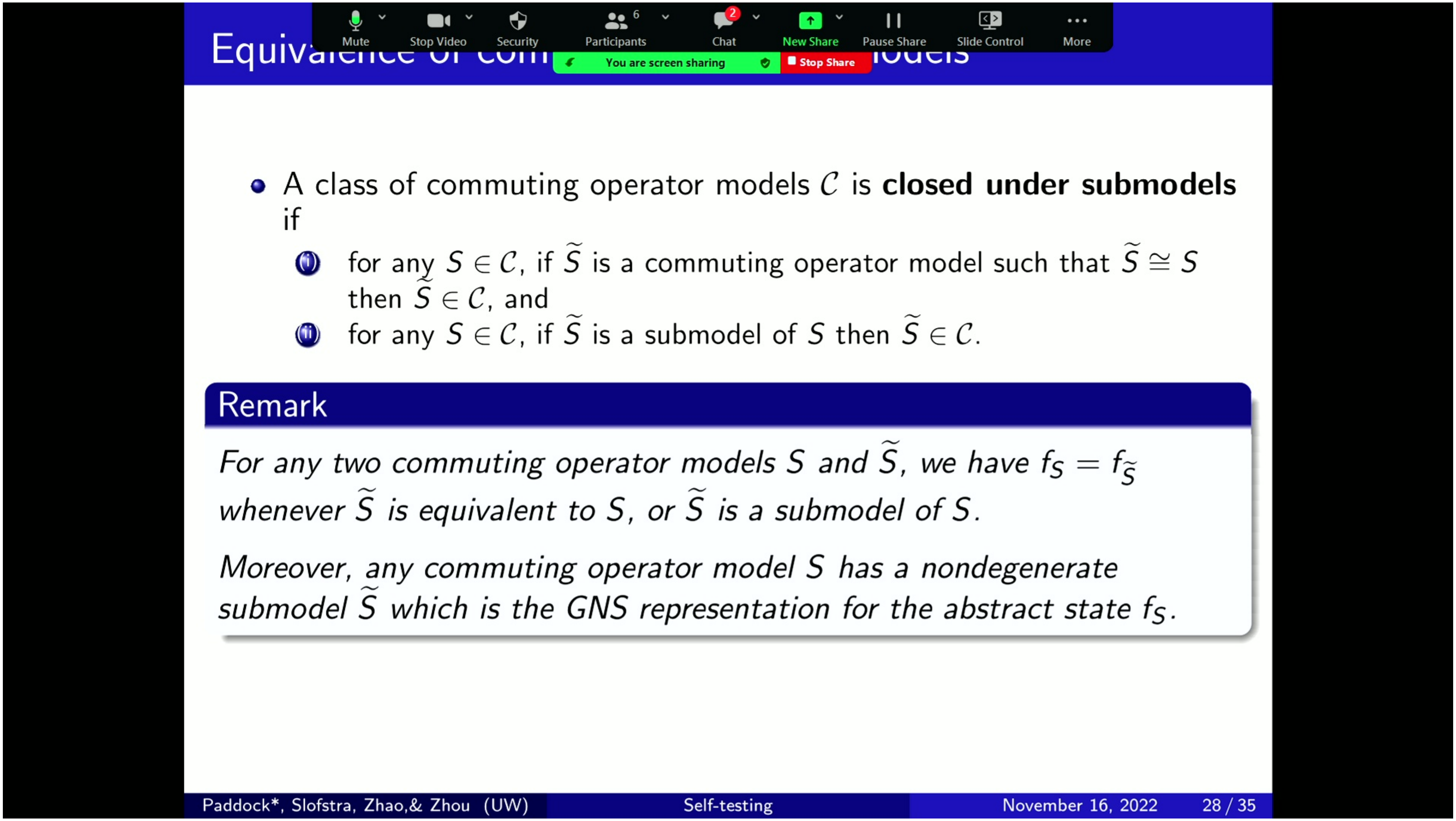


Equivalence of com odels

Definition

Let S, and $\widetilde{S}$ be two commuting operator models.

1. We say S and $\widetilde{S}$ are **equivalent**, and write $S \cong \widetilde{S}$, if there exists a unitary $U : H \to \widetilde{H}$ such that
   1. $U\left|\psi\right\rangle = \left|\widetilde{\psi}\right\rangle$, and
   2. $UM_a^x U^* = \widetilde{M}_a^x$ and $UN_b^y U^* = \widetilde{N}_b^y$ for all $(a, b, x, y) \in A \times B \times X \times Y$.
2. S is said to be **degenerate** if there exists a non-trivial projection $\Pi \in \mathscr{B}(H)$ such that $\Pi\left|\psi\right\rangle = \left|\psi\right\rangle$ and $[\Pi, M_a^x] = [\Pi, N_b^y] = 0$ for all $(a, b, x, y) \in A \times B \times X \times Y$. In this case, we say $\widetilde{S}$ is a **submodel** of S if $\widetilde{H} = \Pi H$, $\left|\widetilde{\psi}\right\rangle = \Pi\left|\psi\right\rangle$, and $\widetilde{M}_a^x = \Pi M_a^x \Pi$, $\widetilde{N}_b^y = \Pi N_b^y \Pi$ for all $(a, b, x, y) \in A \times B \times X \times Y$. Any commuting operator model is a submodel of itself. A commuting operator model is said to be **nondegenerate** if it is not degenerate.



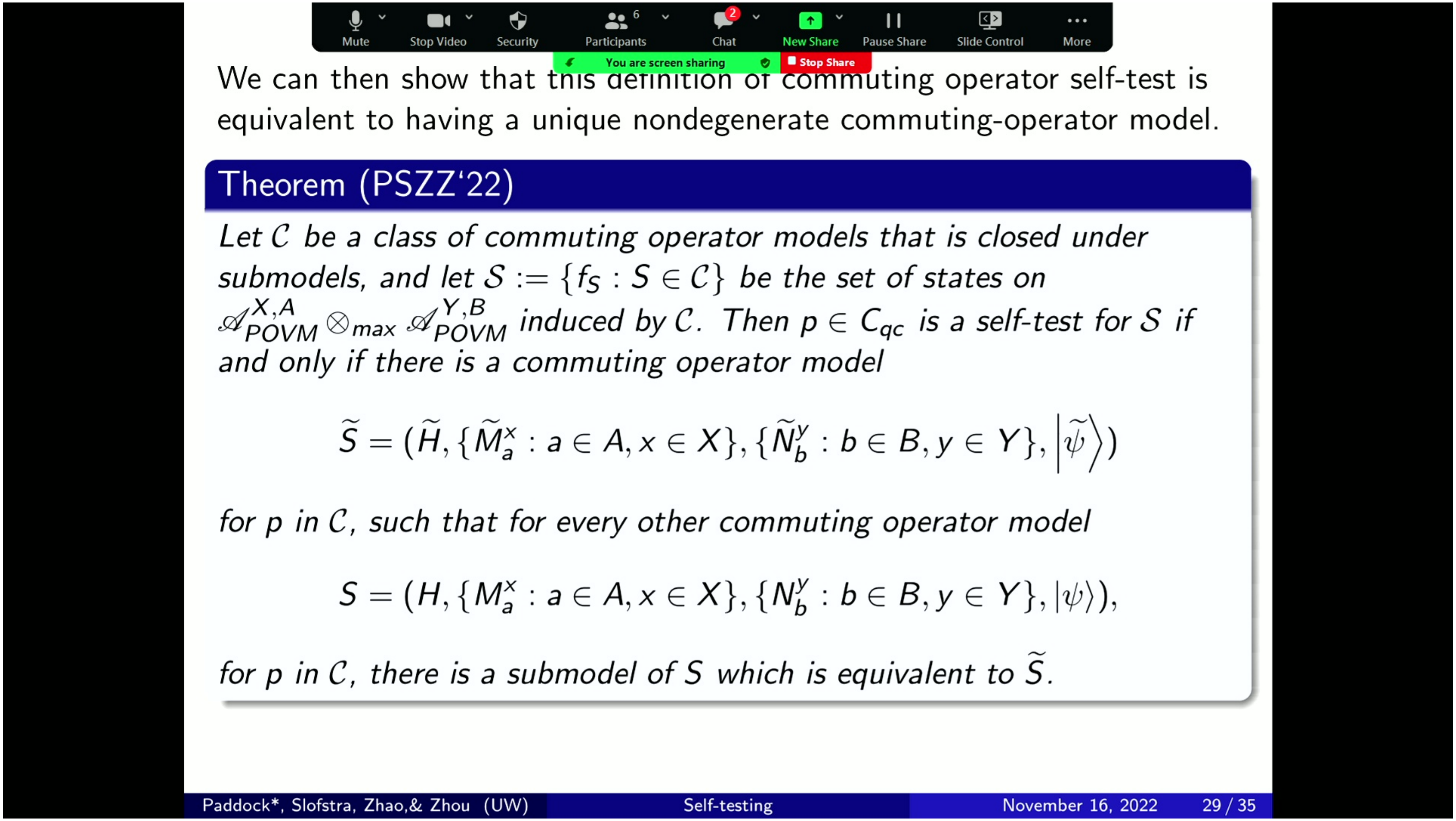
Equivalence of com odels

- A class of commuting operator models $\mathcal{C}$ is **closed under submodels** if
    - (i) for any $S \in \mathcal{C}$, if $\widetilde{S}$ is a commuting operator model such that $\widetilde{S} \cong S$ then $\widetilde{S} \in \mathcal{C}$, and
    - (ii) for any $S \in \mathcal{C}$, if $\widetilde{S}$ is a submodel of S then $\widetilde{S} \in \mathcal{C}$.

Remark

For any two commuting operator models S and $\widetilde{S}$, we have $f_S = f_{\widetilde{S}}$ whenever $\widetilde{S}$ is equivalent to S, or $\widetilde{S}$ is a submodel of S.

Moreover, any commuting operator model S has a nondegenerate submodel $\widetilde{S}$ which is the GNS representation for the abstract state f_S.



We can then show that this definition of commuting operator self-test is equivalent to having a unique nondegenerate commuting-operator model.

Theorem (PSZZ'22)

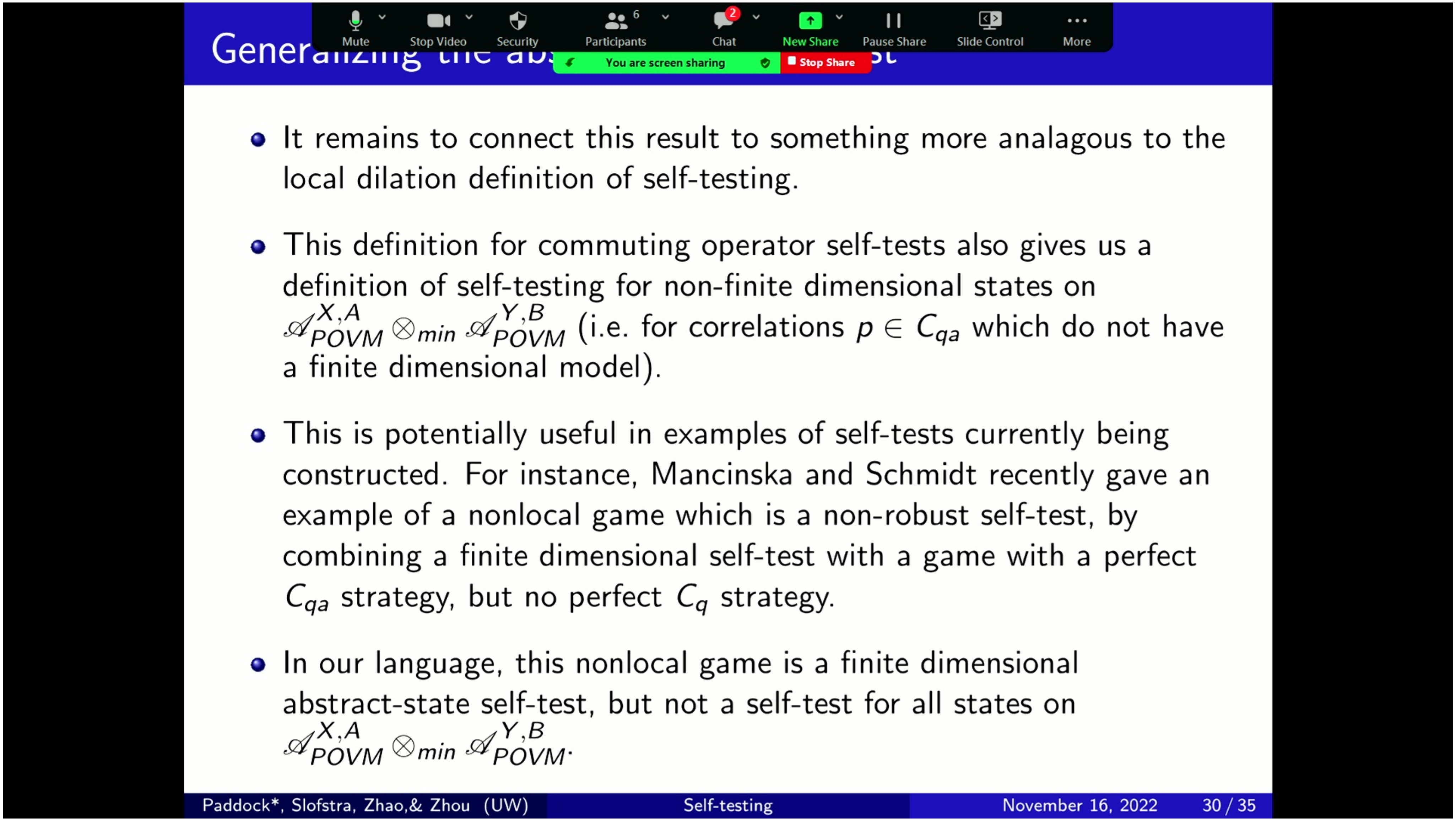
Let $\mathcal{C}$ be a class of commuting operator models that is closed under submodels, and let $\mathcal{S} := \{f_S : S \in \mathcal{C}\}$ be the set of states on $\mathscr{A}_{POVM}^{X,A} \otimes_{max} \mathscr{A}_{POVM}^{Y,B}$ induced by $\mathcal{C}$. Then $p \in C_{qc}$ is a self-test for $\mathcal{S}$ if and only if there is a commuting operator model

$$\widetilde{S} = (\widetilde{H}, \{\widetilde{M}_a^x : a \in A, x \in X\}, \{\widetilde{N}_b^y : b \in B, y \in Y\}, \left|\widetilde{\psi}\right\rangle)$$

for p in $\mathcal{C}$, such that for every other commuting operator model

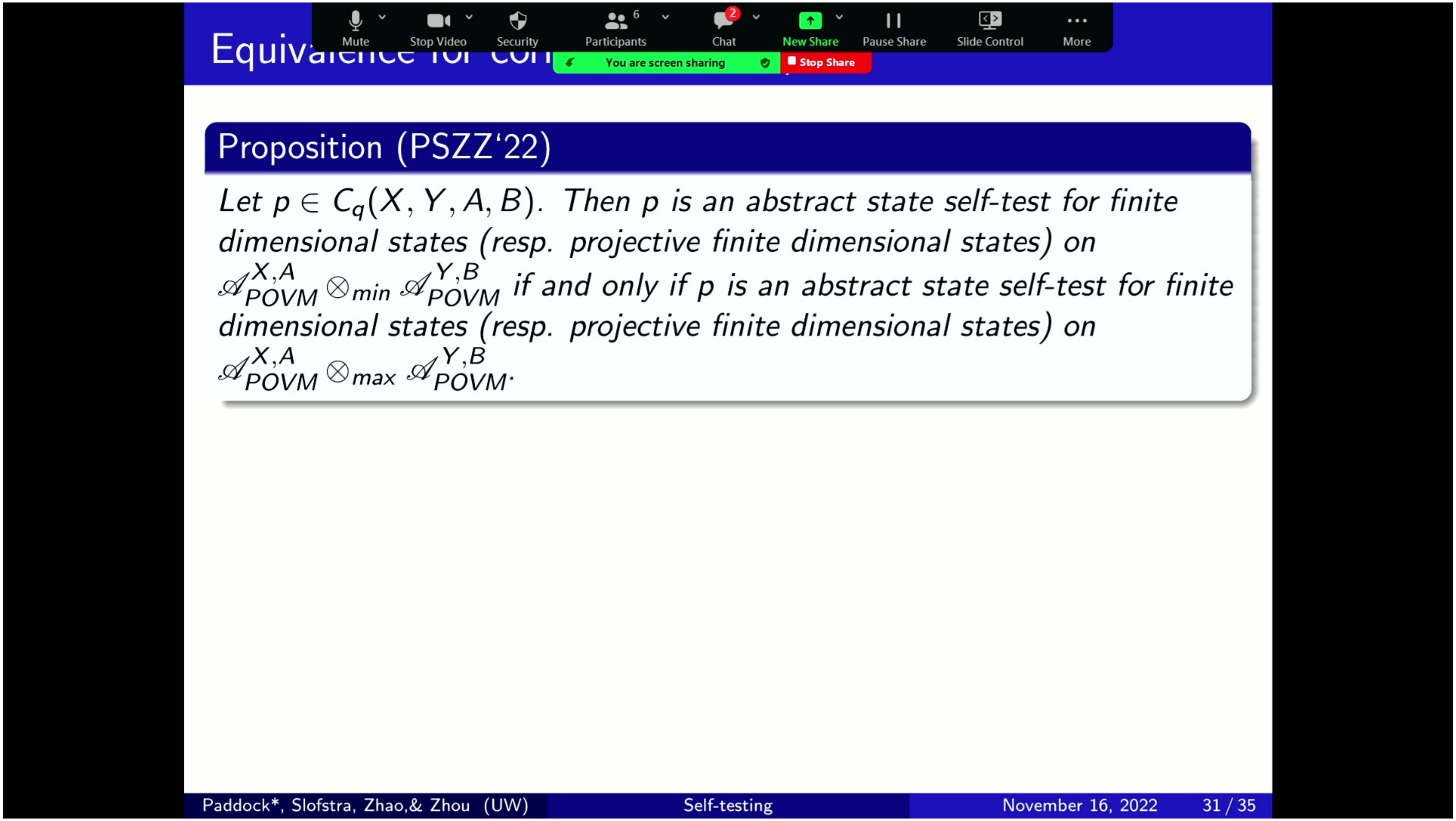
$$S = (H, \{M_a^x : a \in A, x \in X\}, \{N_b^y : b \in B, y \in Y\}, |\psi\rangle),$$

for p in $\mathcal{C}$, there is a submodel of S which is equivalent to $\widetilde{S}$.



Generalizing the ab st

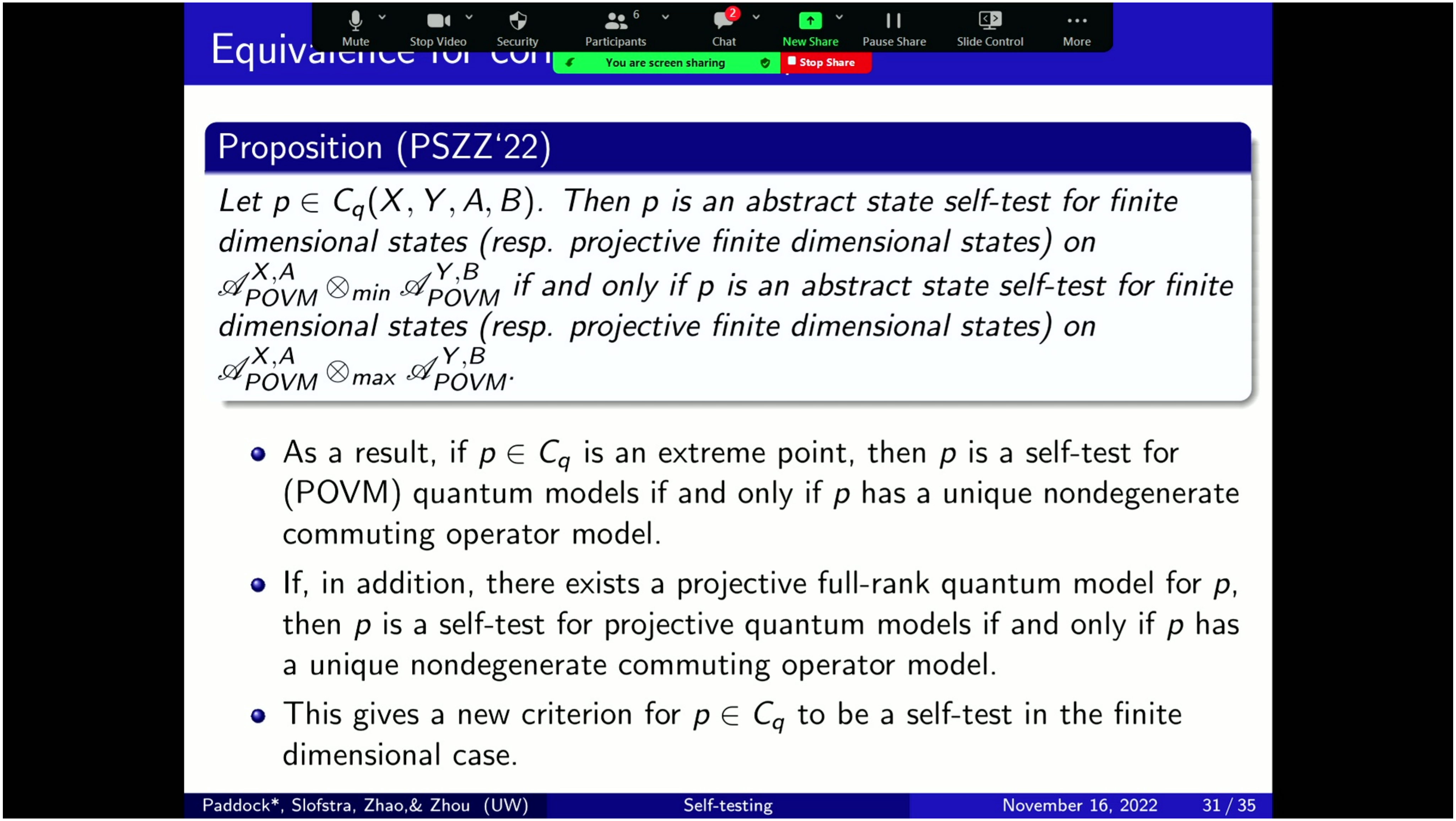
- It remains to connect this result to something more analagous to the local dilation definition of self-testing.
- This definition for commuting operator self-tests also gives us a definition of self-testing for non-finite dimensional states on $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$ (i.e. for correlations $p \in C_{qa}$ which do not have a finite dimensional model).
- This is potentially useful in examples of self-tests currently being constructed. For instance, Mancinska and Schmidt recently gave an example of a nonlocal game which is a non-robust self-test, by combining a finite dimensional self-test with a game with a perfect C_{qa} strategy, but no perfect C_q strategy.
- In our language, this nonlocal game is a finite dimensional abstract-state self-test, but not a self-test for all states on $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$.



Equivalence for con

Proposition (PSZZ'22)

Let $p \in C_q(X, Y, A, B)$. *Then* p *is an abstract state self-test for finite dimensional states (resp. projective finite dimensional states) on* $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$ *if and only if* p *is an abstract state self-test for finite dimensional states (resp. projective finite dimensional states) on* $\mathscr{A}^{X,A}_{POVM} \otimes_{max} \mathscr{A}^{Y,B}_{POVM}$.

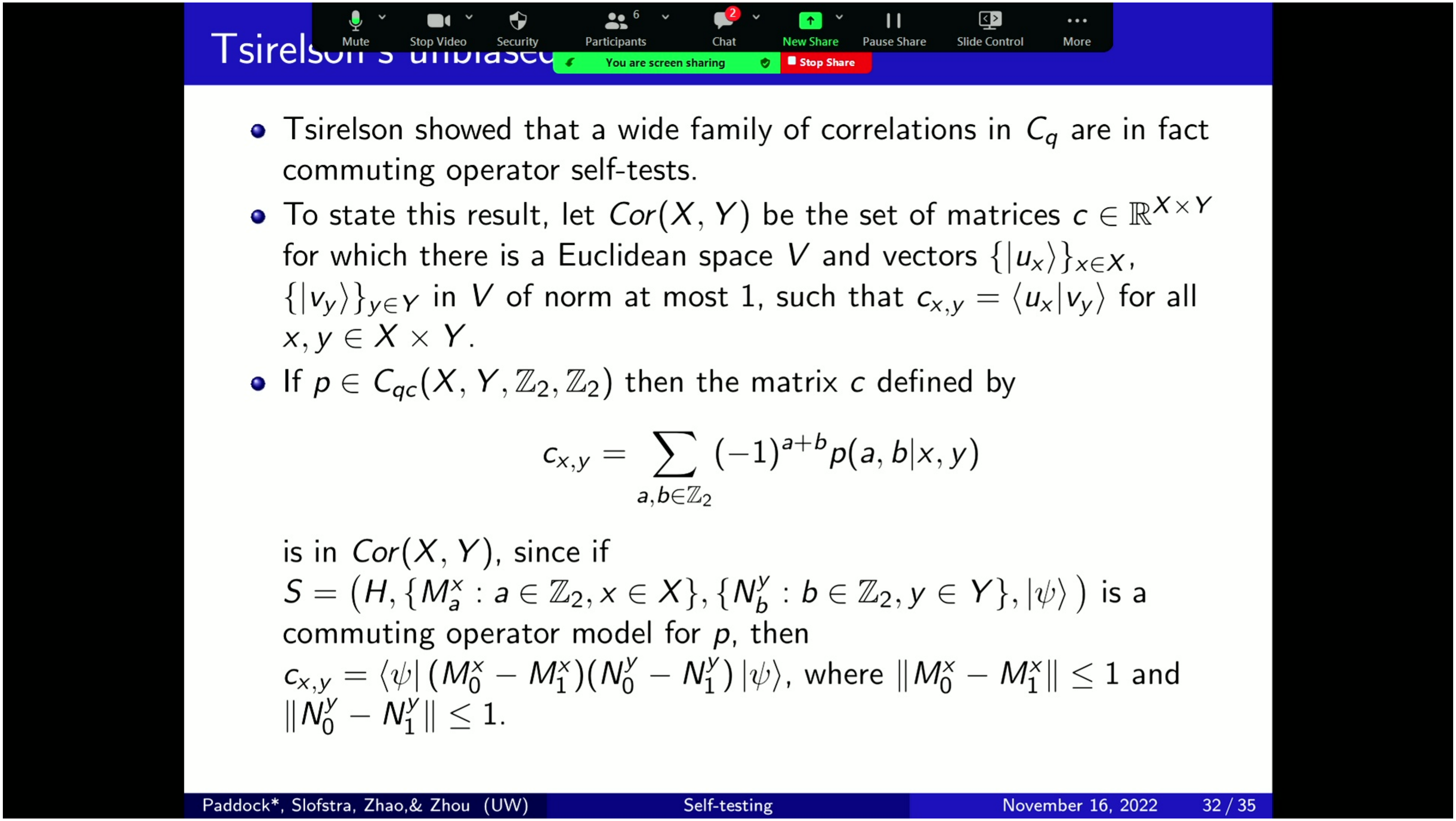


Equivalence for con

Proposition (PSZZ'22)

Let $p \in C_q(X, Y, A, B)$. Then p is an abstract state self-test for finite dimensional states (resp. projective finite dimensional states) on $\mathscr{A}^{X,A}_{POVM} \otimes_{min} \mathscr{A}^{Y,B}_{POVM}$ if and only if p is an abstract state self-test for finite dimensional states (resp. projective finite dimensional states) on $\mathscr{A}^{X,A}_{POVM} \otimes_{max} \mathscr{A}^{Y,B}_{POVM}$.

- As a result, if $p \in C_q$ is an extreme point, then p is a self-test for (POVM) quantum models if and only if p has a unique nondegenerate commuting operator model.
- If, in addition, there exists a projective full-rank quantum model for p, then p is a self-test for projective quantum models if and only if p has a unique nondegenerate commuting operator model.
- This gives a new criterion for $p \in C_q$ to be a self-test in the finite dimensional case.

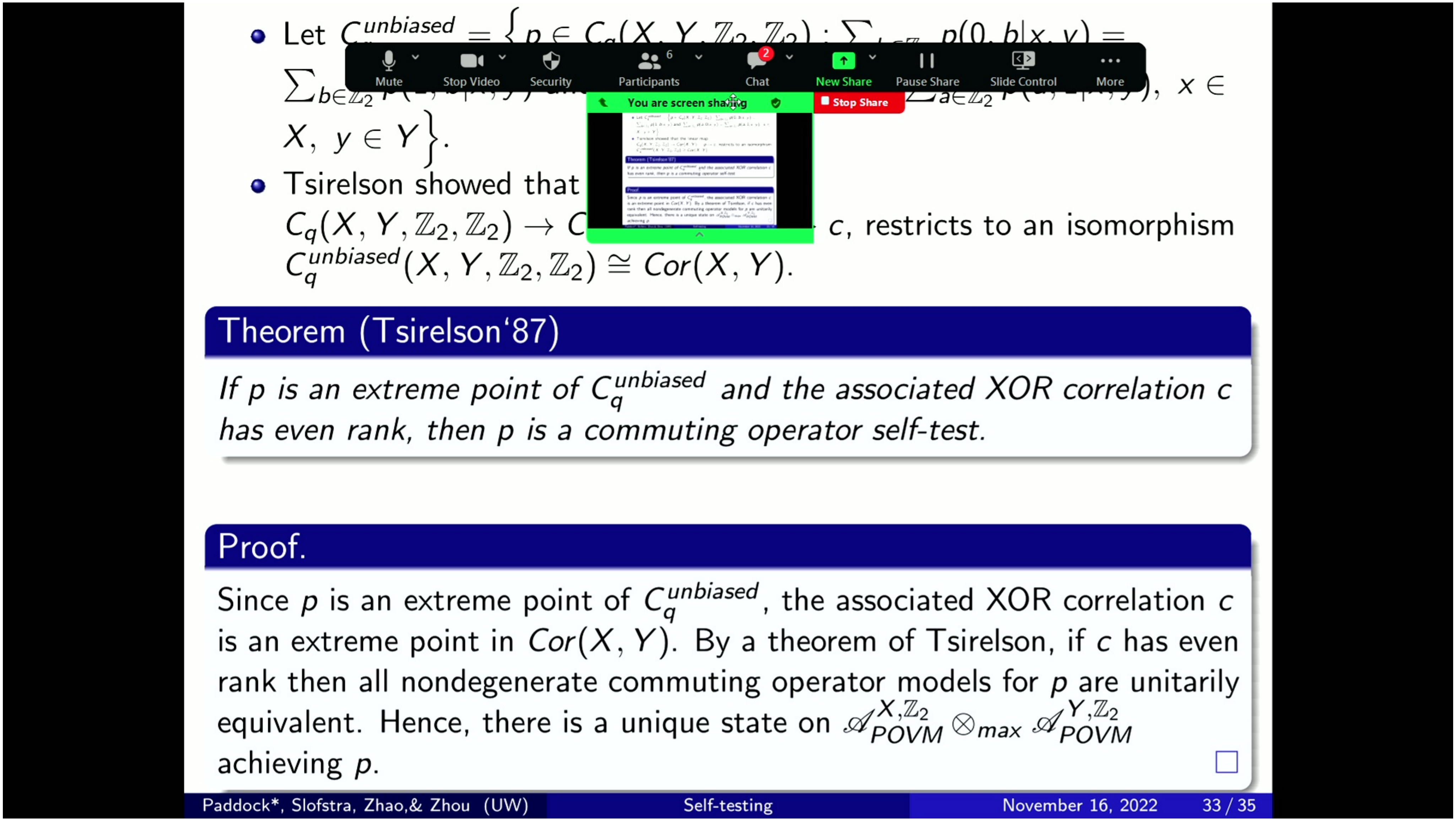


Tsirelson's unbiased

- Tsirelson showed that a wide family of correlations in C_q are in fact commuting operator self-tests.
- To state this result, let $Cor(X, Y)$ be the set of matrices $c \in \mathbb{R}^{X \times Y}$ for which there is a Euclidean space V and vectors $\{|u_x\rangle\}_{x \in X}$, $\{|v_y\rangle\}_{y \in Y}$ in V of norm at most 1, such that $c_{x,y} = \langle u_x | v_y \rangle$ for all $x, y \in X \times Y$.
- If $p \in C_{qc}(X, Y, \mathbb{Z}_2, \mathbb{Z}_2)$ then the matrix c defined by

$$c_{x,y} = \sum_{a,b \in \mathbb{Z}_2} (-1)^{a+b} p(a, b|x, y)$$

is in $Cor(X, Y)$, since if $S = \big(H, \{M_a^x : a \in \mathbb{Z}_2, x \in X\}, \{N_b^y : b \in \mathbb{Z}_2, y \in Y\}, |\psi\rangle\big)$ is a commuting operator model for p, then $c_{x,y} = \langle\psi| (M_0^x - M_1^x)(N_0^y - N_1^y) |\psi\rangle$, where $\|M_0^x - M_1^x\| \leq 1$ and $\|N_0^y - N_1^y\| \leq 1$.



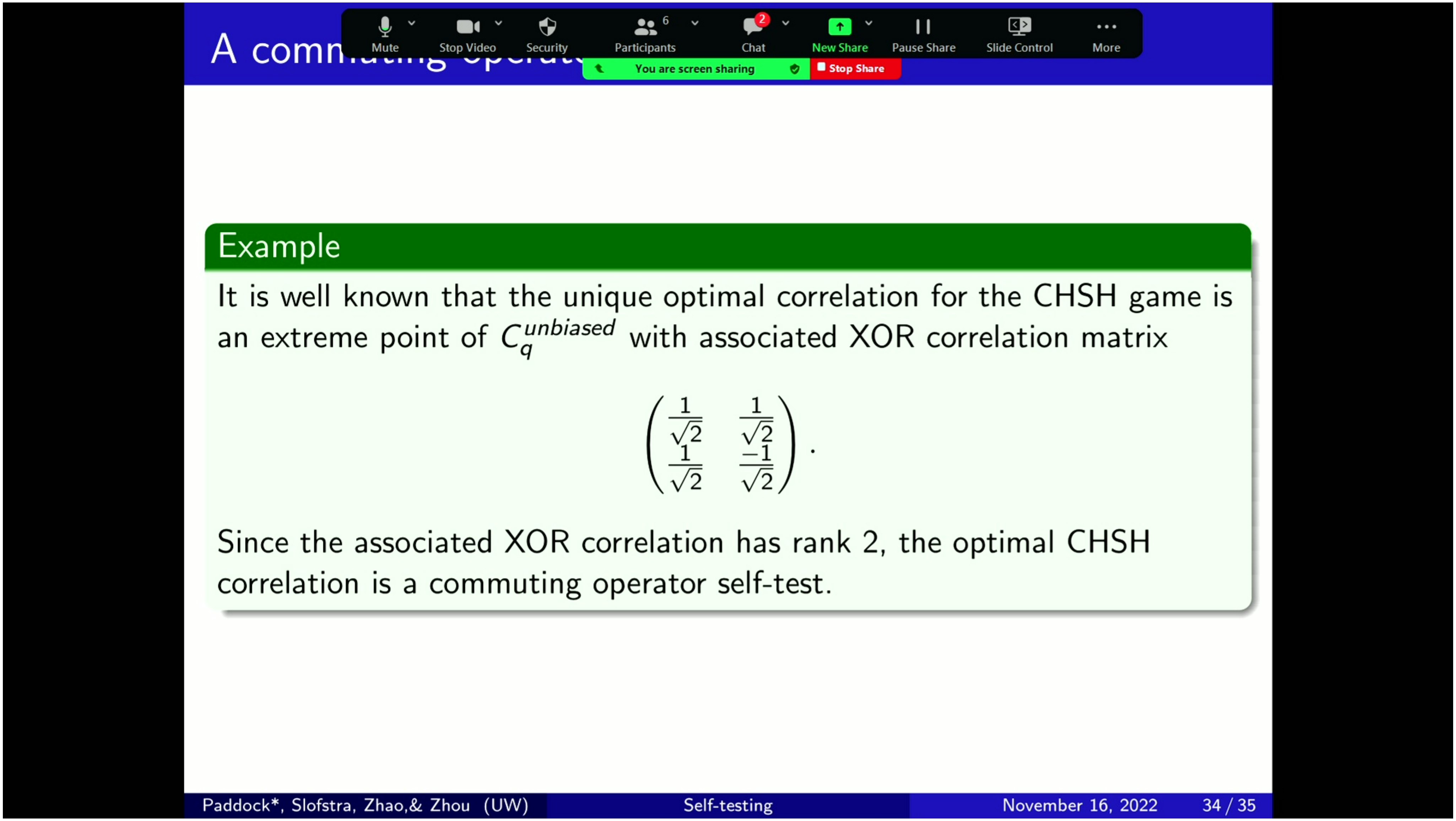
- Let $C_q^{unbiased} = \Big\{p \in C_q(X, Y, \mathbb{Z}_2, \mathbb{Z}_2) : \sum_{a\in\mathbb{Z}} p(0, b|x, y) =$ $\sum_{b\in\mathbb{Z}_2}$ … $\sum_{a\in\mathbb{Z}_2}$ …$)$, $x \in X,\ y \in Y\Big\}$.
- Tsirelson showed that … $C_q(X, Y, \mathbb{Z}_2, \mathbb{Z}_2) \to C$ … c, restricts to an isomorphism $C_q^{unbiased}(X, Y, \mathbb{Z}_2, \mathbb{Z}_2) \cong Cor(X, Y)$.

Theorem (Tsirelson‘87)

If p is an extreme point of $C_q^{unbiased}$ and the associated XOR correlation c has even rank, then p is a commuting operator self-test.

Proof.

Since p is an extreme point of $C_q^{unbiased}$, the associated XOR correlation c is an extreme point in $Cor(X, Y)$. By a theorem of Tsirelson, if c has even rank then all nondegenerate commuting operator models for p are unitarily equivalent. Hence, there is a unique state on $\mathscr{A}_{POVM}^{X,\mathbb{Z}_2} \otimes_{max} \mathscr{A}_{POVM}^{Y,\mathbb{Z}_2}$ achieving p. ☐



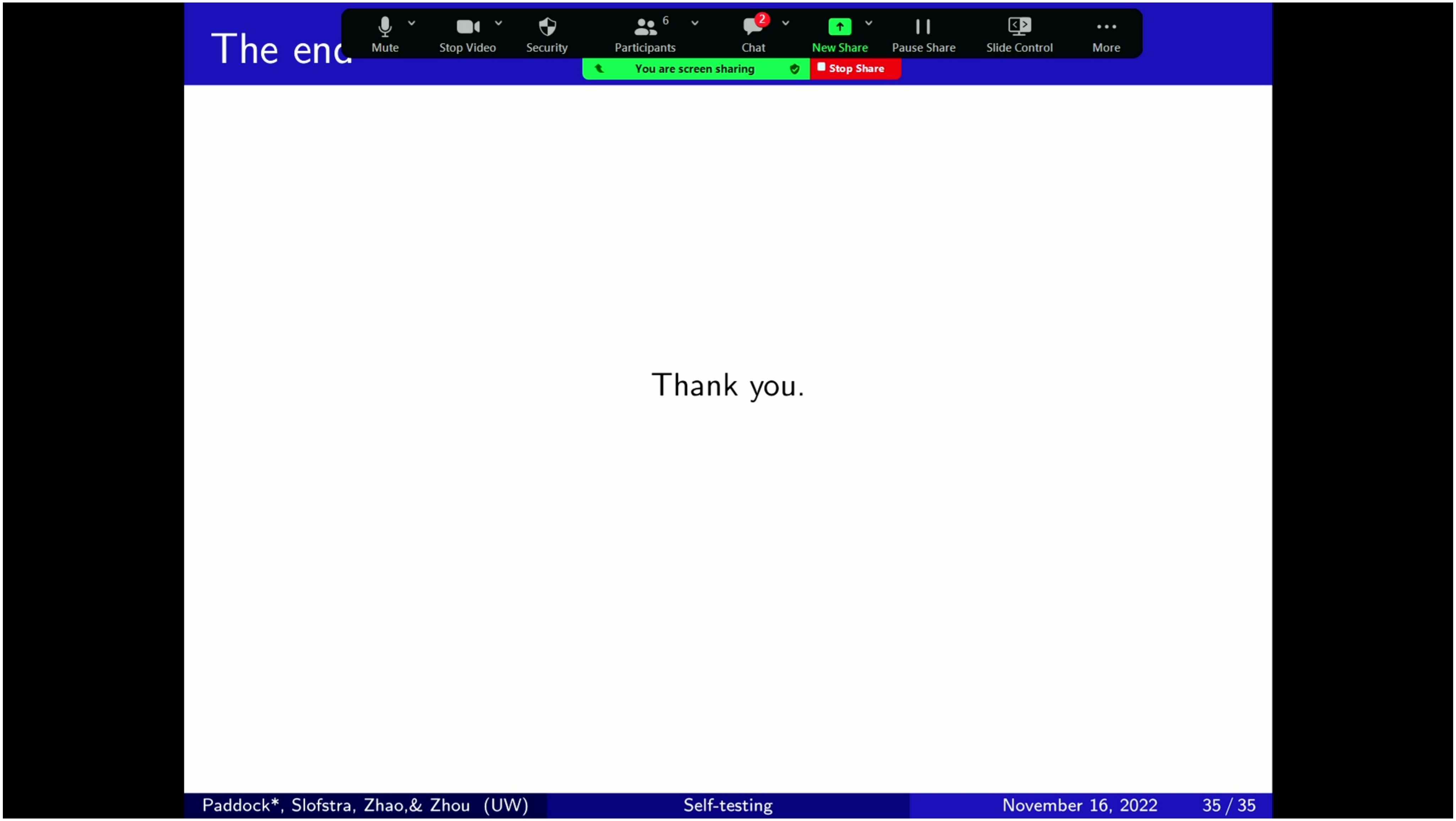
A commuting operat

Example

It is well known that the unique optimal correlation for the CHSH game is an extreme point of $C_q^{unbiased}$ with associated XOR correlation matrix

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}.$$

Since the associated XOR correlation has rank 2, the optimal CHSH correlation is a commuting operator self-test.



The end

Thank you.