Title: Newtonâ€™s Cradle Spectra

Speakers: Barbara Soda

Series: Perimeter Institute Quantum Discussions

Date: November 09, 2022 - 11:00 AM

URL: https://pirsa.org/22110081

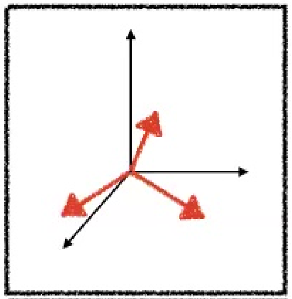
Abstract: We present broadly applicable nonperturbative results on the behavior of eigenvalues and eigenvectors under the addition of self-adjoint operators and under the multiplication of unitary operators, in finite-dimensional Hilbert spaces. To this end, we decompose these operations into elementary 1-parameter processes in which the eigenvalues move similarly to the spheres in Newton's cradle. As special cases, we recover level repulsion and Cauchy interlacing. We discuss two examples of applications. Applied to adiabatic quantum computing, we obtain new tools to relate algorithmic complexity to computational slowdown through gap narrowing. Applied to information theory, we obtain a generalization of Shannon sampling theory, the theory that establishes the equivalence of continuous and discrete representations of information. The new generalization of Shannon sampling applies to signals of varying information density and finite length.

Zoom link: https://pitp.zoom.us/j/94120657832?pwd=SmpsWFhhVCtyeXM3a0pVQU9lMGFLdz09

What happens when we add two Hamiltonians?

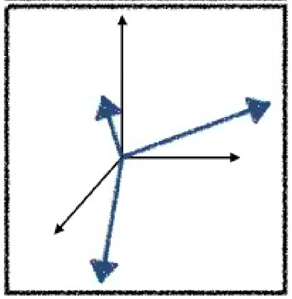
- Suppose we know the eigenvalues and eigenvectors of a Hamiltonian H_0.

$$H_0 = \sum_i e_i \left|e_i\right\rangle \left\langle e_i\right|$$



- We also know the eigenvectors and eigenvalues of H'.

$$H' = \sum_i f_i \left|f_i\right\rangle \left\langle f_i\right|$$



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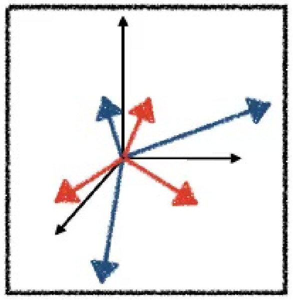


- Typically, we approach this problem *perturbatively*.
- *Nonperturbative* results:
  - Wigner-von Neumann on level repulsion
  - Weyl's inequalities
  - …too few!

New nonperturbative result: Newton's cradle spectra

Simpler problem: addition of a projector

- What happens to the eigenvalues when we add a rank-1 projector?

$$S(\mu) = S_0 + \mu \left| v \right\rangle \left\langle v \right|$$

- If we know the answer, we can iteratively add a full interaction Hamiltonian:

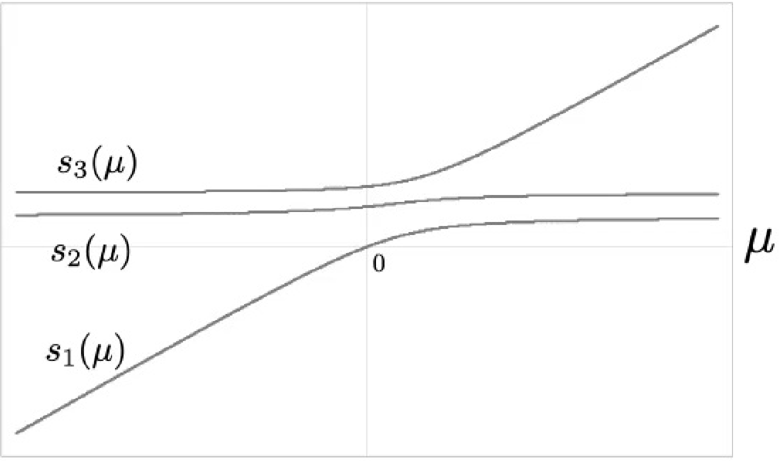
$$S\left(\mu\right) = S_0 + \sum_j \mu_j \left| v_j \right\rangle \left\langle v_j \right|$$

Properties of the Newton cradle of spectra

Nonperturbative:
Newton cradle spectra

$$\frac{d\mu(s)}{ds} = \left(\sum_{m=1}^{N} \frac{|v_m|^2}{s - s_m}\right)^{-2} \sum_{r=1}^{N} \frac{|v_r|^2}{(s - s_r)^2}$$

positive



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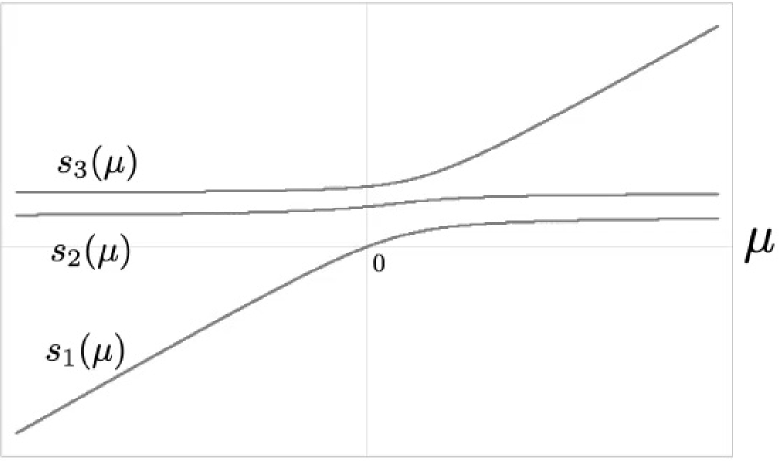

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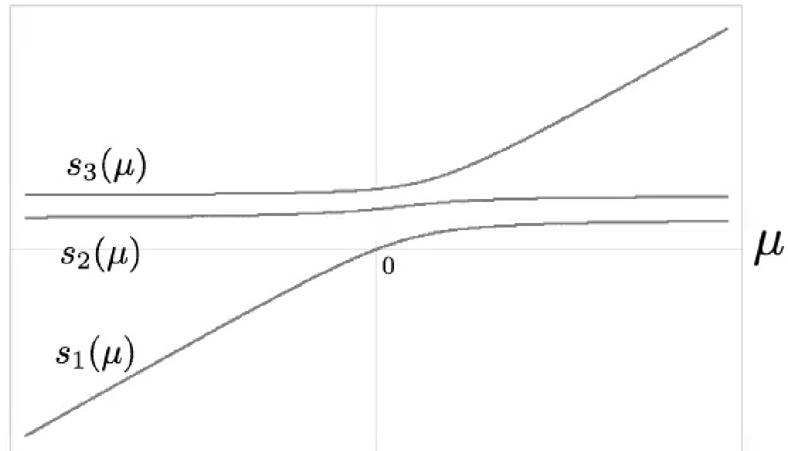

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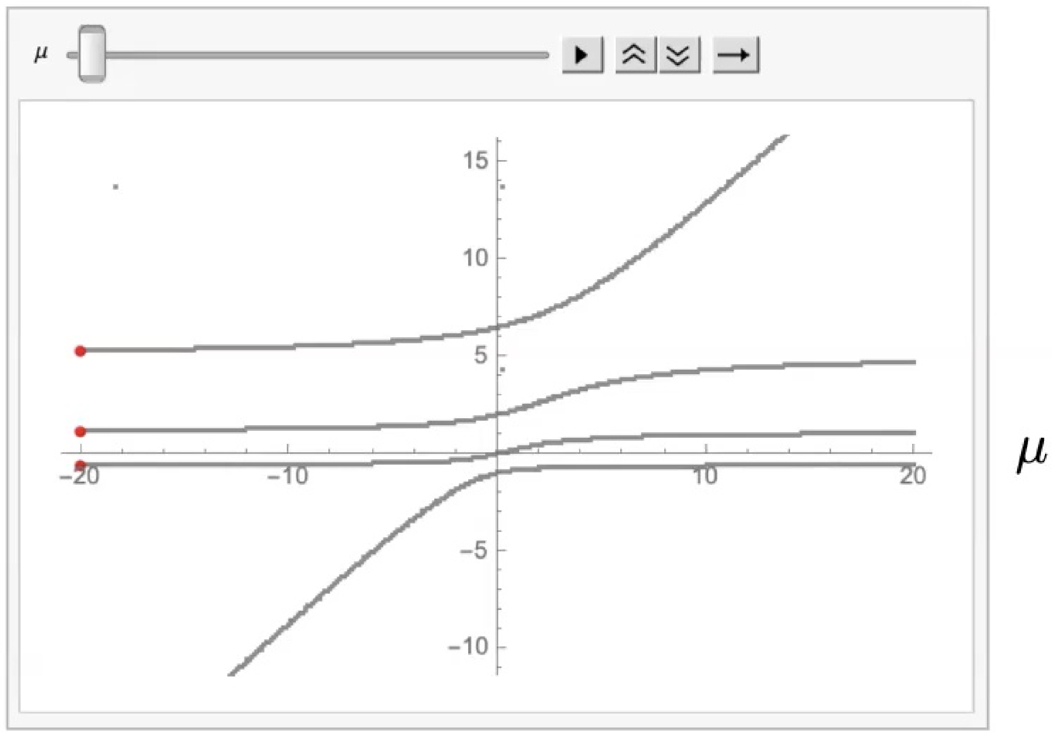
Example with 4 eigenvalues.

Animation of eigenvalues of $S(\mu) = S_0 + \mu \, |v\rangle \langle v|$



Newton cradle-like motion as a function of the coupling constant μ.

Properties of the Newton cradle of spectra

- The spectra cover the entire real line exactly once :

$$\bigcup_{n=1,\ldots,N} \bigcup_{\mu\in\mathbb{R}\cup\infty} \{s_n(\mu)\} = \mathbb{R}\cup\infty$$

- Therefore, for any $s \in \mathbb{R}$, there is a μ for which s is an eigenvalue of $S(\mu)$:

$$S(\mu)\,|s\rangle = s\,|s\rangle$$

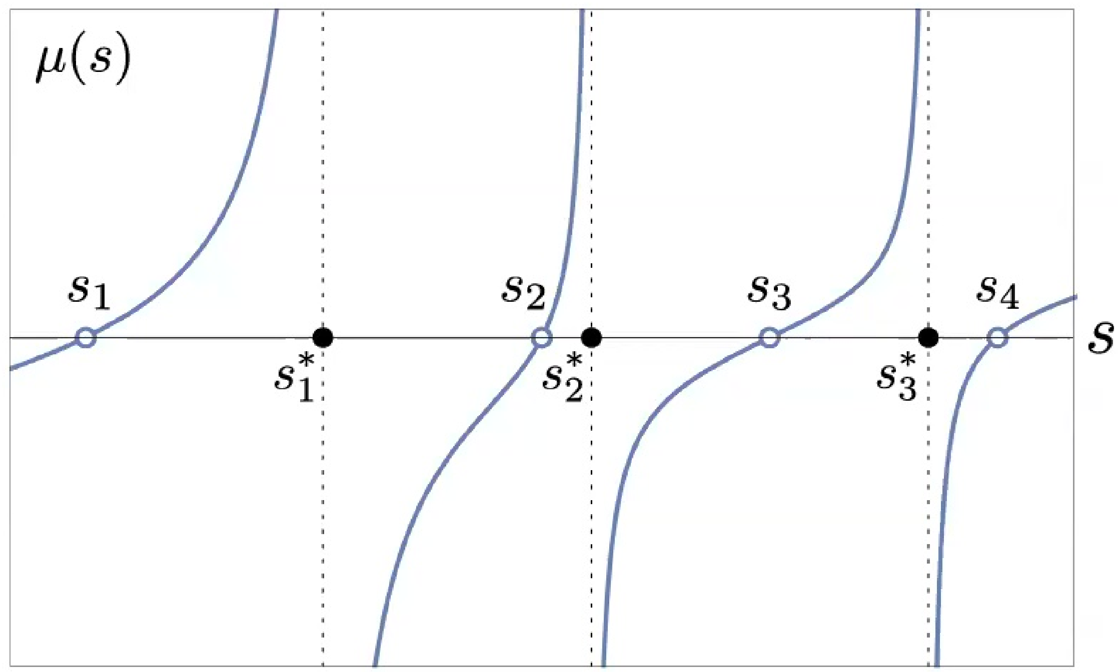
- We can calculate the μ exactly using this formula:

$$\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n}\right)^{-1}$$

Coupling constant μ as a function of an eigenvalue s:

$$\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n} \right)^{-1}$$

Plot of the dependence:



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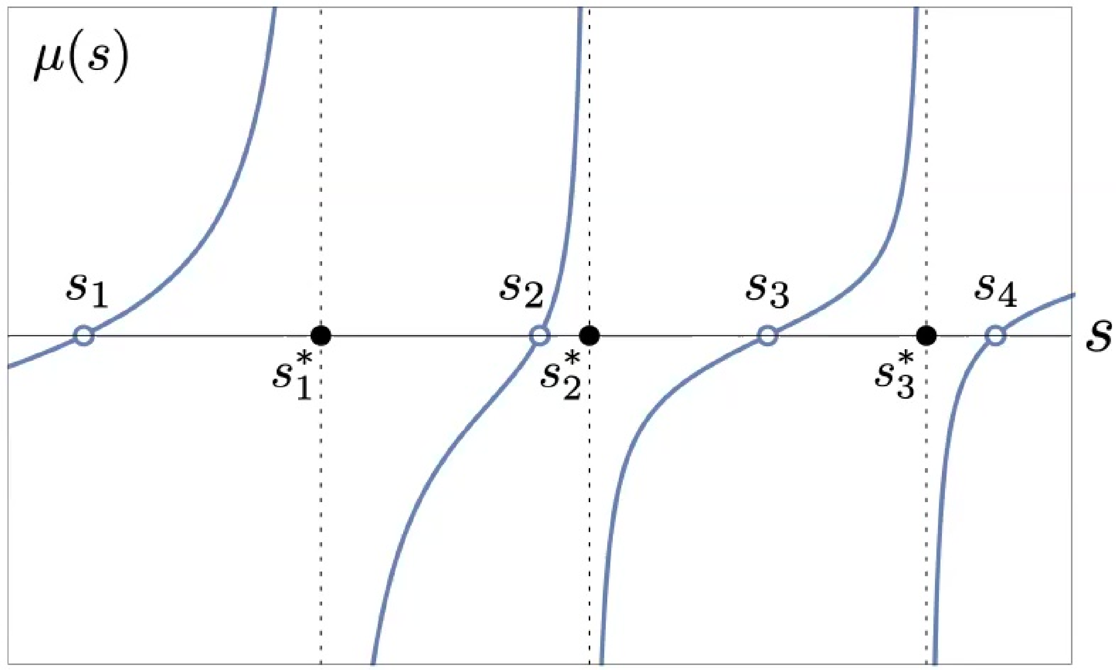

Special case: Cauchy's interlacing theorem

Theorem paraphrased:

Start with a NxN self-adjoint matrix M, which has eigenvalues $s_1, ..., s_N$.

E.g.

$$M = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$ **has eigenvalues** s_1, s_2, s_3, s_4.

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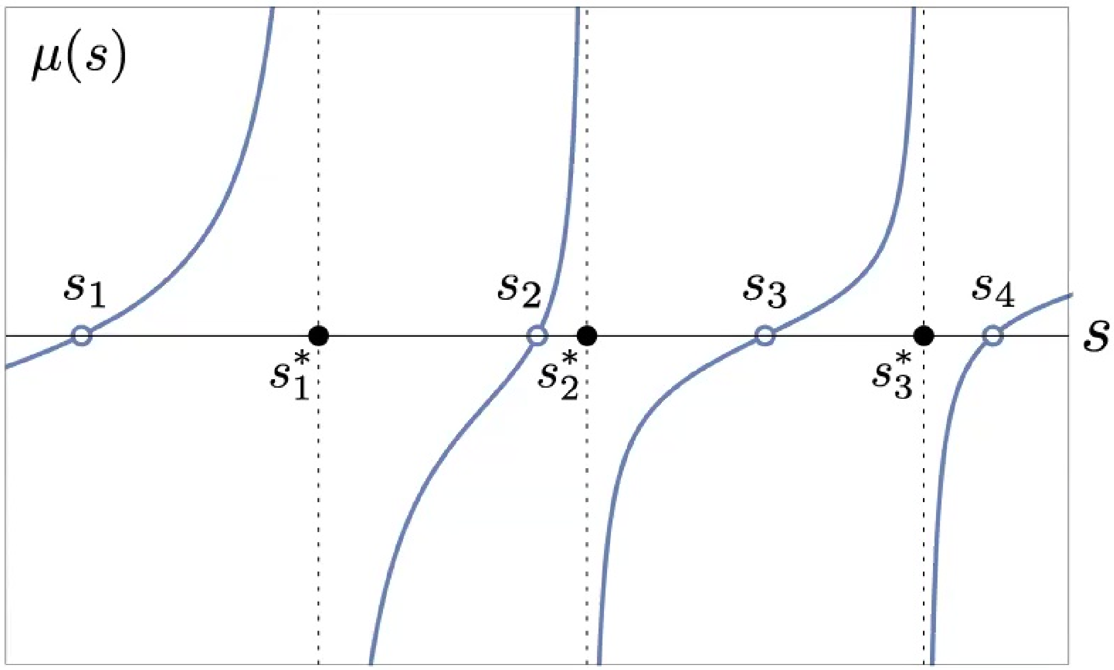
If we cross out a row and a column, the new eigenvalues interlace the old ones.

$$M' = \begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$ has eigenvalues: s'_1, s'_2, s'_3

$$s_1 \leq s'_1 \leq s_2 \leq s'_2 \leq s_3 \leq s'_3 \leq s_4$$

$$\mu(s) = \left(\sum_{i=1}^{N} \frac{|v_n|^2}{s - s_n} \right)^{-1} \xrightarrow{\mu \to \infty} \sum_{m=1}^{N} \frac{|v_m|^2}{s_n^* - s_m} = 0$$



Evolution of eigenvectors

- Assume $|s\rangle$ is an eigenvector corresponding to the eigenvalue s:

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- The new eigenvector $|s\rangle$ expressed in the original eigenbasis (of S_0):

$$|s\rangle = \frac{e^{i\varphi(s)}}{\mathcal{N}} \sum_{n=1}^{N} \frac{v_n}{|s - s_n|} |s_n\rangle$$

Normalization: $\mathcal{N} = \sqrt{\sum_k \frac{|v_k|^2}{(s - s_k)^2}}$

Phase arbitrary: $\varphi(s)$

Simple: $|\langle s|s_n\rangle| \propto \frac{|v_n|}{|s - s_n|}$

Newton's cradle of unitary operators

- We start with a unitary matrix U_0.
- We know its eigenvalues and vectors:

$$U_0 = \sum_i u_i \left|u_i\right\rangle \left\langle u_i\right|$$

- e.g. if it acts on a 3-dim Hilbert space $\mathcal{H}$,

  3 eigenvalues u_1, u_2, u_3.

- We get a behaviour analogous to Newton's cradle when we act with a U(1) operator family:

$$U(\alpha) := \left(\mathbb{1} + \left(e^{i\alpha} - 1\right) \left|w\right\rangle \left\langle w\right|\right) U_0 \quad , \alpha \in [0, 2\pi)$$

The connection between two Newton cradles

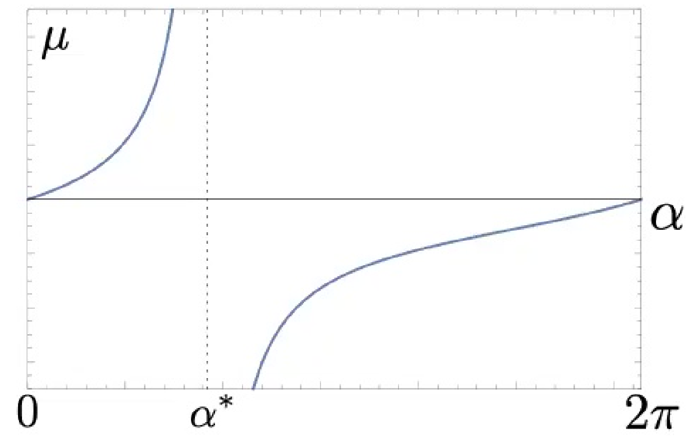
- The self-adjoint and the unitary Newton cradles are related by Cayley transforms:

$$U(\alpha) = (S(\mu) - i\mathbb{1})(S(\mu) + i\mathbb{1})^{-1}\,, \quad S(\mu) = -i(U(\alpha) + \mathbb{1})(U(\alpha) - \mathbb{1})^{-1}$$

- Moebius transform for the eigenvalues: $u_j = \dfrac{s_j - i}{s_j + i}$

- Relationship between μ and α :

$$\mu(\alpha) = \left(\sum_{m=1}^{N} \frac{|v_m|^2}{s_m^2 + 1} \cot\left(\frac{\alpha}{2}\right) - \sum_{k=1}^{N} \frac{|v_k|^2 s_k}{s_k^2 + 1} \right)^{-1}$$



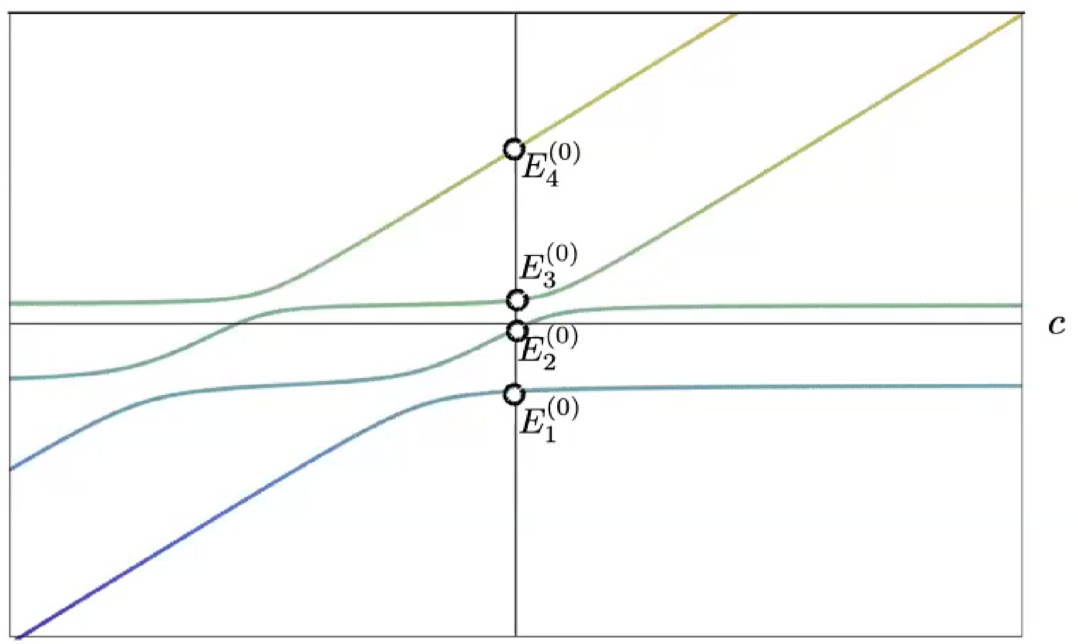
- Addition of self-adjoint operators translates nicely to multiplication of unitaries, but not the exponentiated ones, instead the Cayley transformed unitaries.

$$\boxed{H \to e^{iHt}}$$

$$\boxed{H \to (H - i\mathbb{1})(H + i\mathbb{1})^{-1}}$$

Special case: Level repulsion

Eigenvalues of $A + cB$, where c is the coupling strength, as a function of c:



Plot of eigenvalues of $A + cB$ as a function of c.

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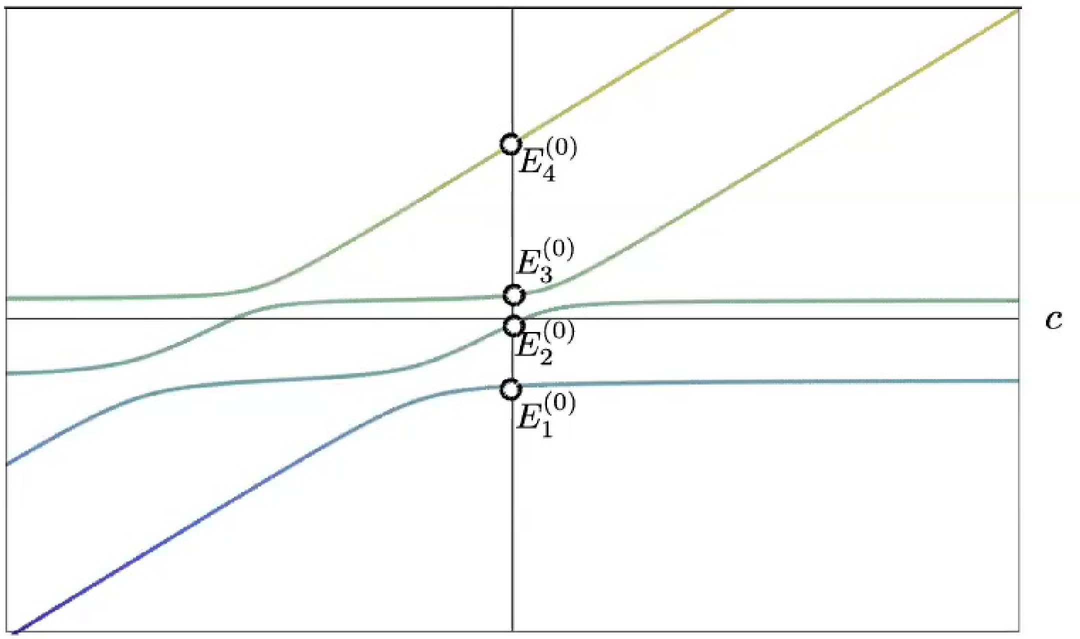
$$B = \sum_i b_i \left|v_i\right\rangle \left\langle v_i\right|$$

Add projectors $b_i \left|v_i\right\rangle \left\langle v_i\right|$ one by one.



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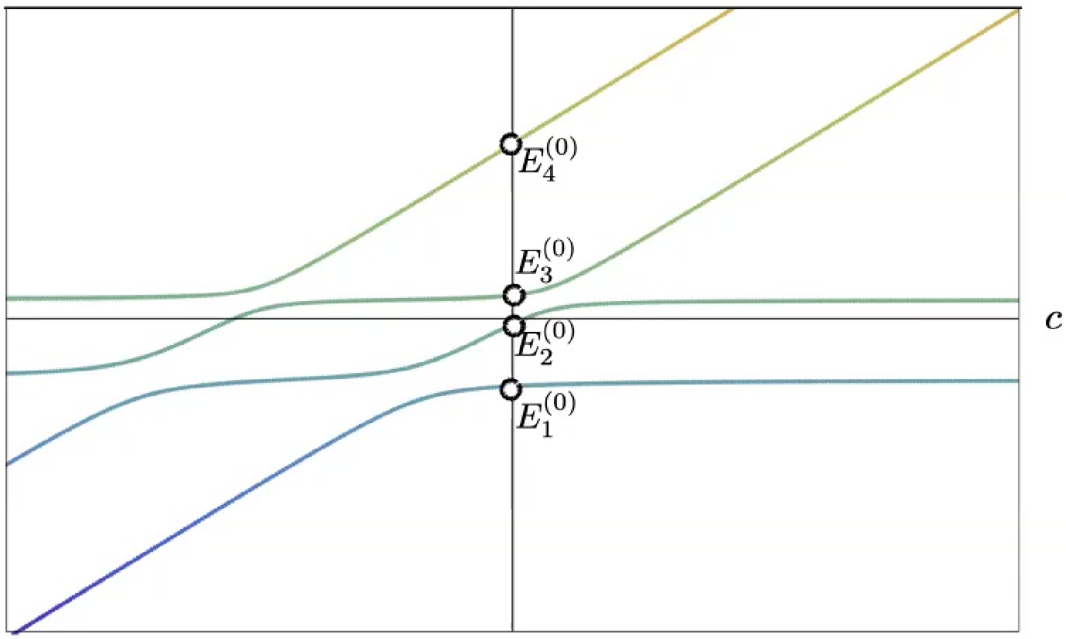
As long as the overlap:

$$\langle a_i(b)|v_i\rangle \neq 0$$

-> no level crossing!



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Summary so far

- Newton's cradle spectra shows how eigenvalues and eigenvectors change when we add a 1-dimensional projector to a Hamiltonian.
  - Eigenvalues move like Newton's cradle.
  - Eigenvector of the highest eigenvalue rotates into the projector subspace as the coupling constant $\mu \to \infty$.
- We can decompose any Hamiltonian into a sum of 1-dim. projectors ->
  -> new strategy for understanding addition of Hamiltonians.
- New understanding of Cauchy interlacing, level repulsion…

Stimulated Unruh Effect and Acceleration-Induced Transparency

Acceleration effects in quantum physics

- Acceleration doesn't fit naturally into quantum physics.
- An example where it does: Unruh effect (extremely small).
- Is there more?

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Why?

- Acceleration related to gravity, through the equivalence principle.

Here:

- New acceleration-induced phenomena,
- Increased measurability of the Unruh effect,
- Insights into acceleration (gravity).

Light-matter interactions

Resonant	Non-resonant
• The usual	• The unusual
• Absorption, emission	• Unruh effect, Hawking effect
• Strong	• Weak
• Can be stimulated (e.g. laser)	• Activated by acceleration or gravity

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New: Can be modulated with acceleration/gravity ! $\Rightarrow$ "Acceleration-induced transparency"	**New:** Can be stimulated ! $\Rightarrow$ Non-resonant effects become more measurable!

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The Unruh effect can be stimulated

- First we **stimulate the Unruh effect** by the presence of photons in the quantum field
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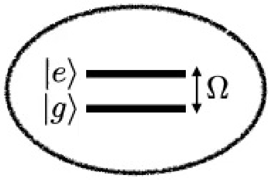
- First we **stimulate the Unruh effect** by the presence of photons in the quantum field
- Problem: this also stimulates the resonant terms
- Solution: **acceleration-induced transparency**
    - suppresses the resonant terms to zero
- Outcome: we are left with strong non-resonant terms, in particular:
  Strong Unruh effect

The standard Unruh effect

- UDW detector:



$$H_0^{(D)} = \Omega\ \hat{\sigma}_z$$

- Scalar quantum field:

$$H_0^{(F)} = \int d^3k\, \omega_{\mathbf{k}}\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$$

- Interaction Hamiltonian: $H_{int} = G\, \hat{\sigma}_x(\tau) \otimes \hat{\phi}(t(\tau), \mathbf{x}(\tau))$

$$H_{int} \propto \int d\mathbf{k} \left(\underbrace{\sigma^- a^\dagger_{\mathbf{k}} + \sigma^+ a_{\mathbf{k}}}_{\text{RESONANT}} + \underbrace{\sigma^- a_{\mathbf{k}} + \sigma^+ a^\dagger_{\mathbf{k}}}_{\text{NON-RESONANT}} \right)$$

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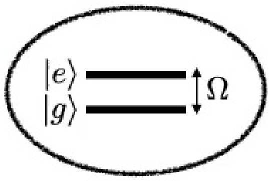

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- Amplitude for $|g\rangle \otimes |0\rangle \to |e\rangle \otimes |1\rangle$: $\mathcal{A}(\mathbf{k}) = \langle e|\, \langle 1_{\mathbf{k}}|\, i \int d\tau H_{int}(\tau)\, |g\rangle\, |0\rangle$

$$\Rightarrow \quad \mathcal{A}(\mathbf{k}) = \frac{i\, G}{(2\pi)^{3/2}\sqrt{\omega_{\mathbf{k}}}} I_+(\mathbf{k}),$$

where $I_\mp(\Omega, k) = \int \mathrm{d}\tau\, e^{i\Omega\tau \mp i k^\mu x_\mu(\tau)}$.

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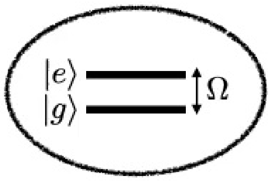
The stimulated Unruh effect (Fock state field)

- UDW detector:



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- Interaction Hamiltonian: $H_{int} = G\, \hat{\sigma}_x(\tau) \otimes \hat{\phi}(t(\tau), \mathbf{x}(\tau))$
- The transition: $|g\rangle \otimes |n_{\mathbf{k}}\rangle \to |e\rangle \otimes |\phi'\rangle$
- Amplitude for transition: $|g, n_{\mathbf{k}}\rangle \to G\Big[\sqrt{n+1} I_+(k)\, |e, (n+1)_{\mathbf{k}}\rangle + \sqrt{n} I_-(k)\, |e, (n-1)_{\mathbf{k}}\rangle\Big]$

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Standard vs. stimulated Unruh effect

	Standard	Stimulated (Fock state)
Transition:	$\|g\rangle \otimes \|0\rangle \rightarrow \|e\rangle \otimes \|1\rangle$	$\|g\rangle \otimes \|n_{\mathbf{k}}\rangle \rightarrow \|e\rangle \otimes \|(n+1)_{\mathbf{k}}\rangle$
Probability:	$p(\mathbf{k}) = \frac{G^2}{(2\pi)^3\omega_{\mathbf{k}}}\|I_+(\mathbf{k})\|^2$	$p(\mathbf{k}) = \frac{G^2}{(2\pi)^3\omega_{\mathbf{k}}}\left(n\|I_-(\mathbf{k})\|^2 + (n+1)\|I_+(\mathbf{k})\|^2\right)$

The resonant terms can be suppressed

Standard: $p(\mathbf{k}) \propto |I_+(\mathbf{k})|^2$

Stimulated (Fock state): $p(\mathbf{k}) \propto (n+1)|I_+(\mathbf{k})|^2 + n|I_-(\mathbf{k})|^2$

Resonant term contribution

- Trajectory-dependent time integrals:

$$I_{\mp}(\Omega, k) = \int \mathrm{d}\tau \, e^{i\Omega\tau \mp ik^{\mu}x_{\mu}(\tau)}$$

- We know how to eliminate $I_+(\mathbf{k})$: in inertial motion $I_+(\Omega, k) = 0,$

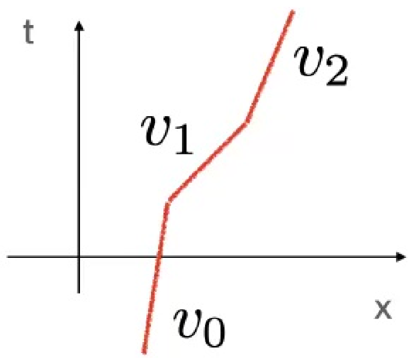
$$I_-(\Omega, k) \neq 0.$$

- Can we get rid of $I_-(\mathbf{k})$ somehow?

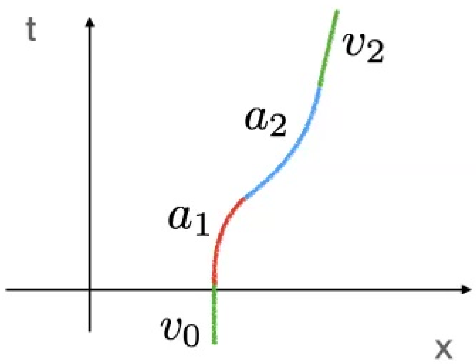
Acceleration-induced transparency

- Q: Can we tune the trajectory so that for some Ω the time integral $I_-(\Omega, k) = \int \mathrm{d}\tau\, e^{i\Omega\tau - ik^\mu x_\mu(\tau)} = 0$?

- Yes!



- Proved also true for a smoother trajectory:



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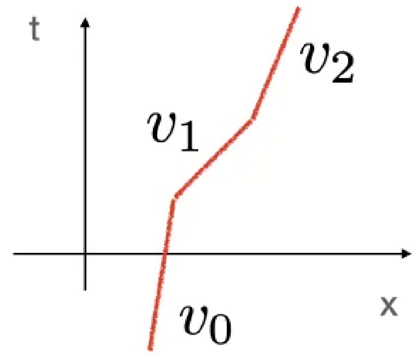

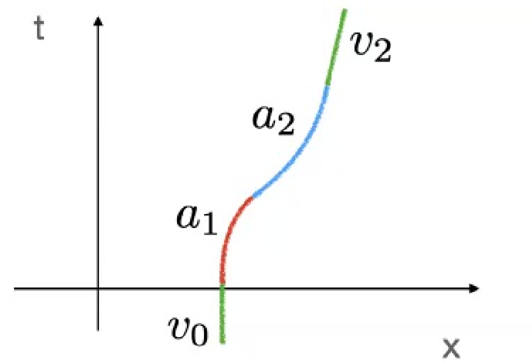

- The detector with gap Ω will not get excited due to resonant terms.
- At the same time: $|I_+(\Omega, \mathbf{k})| > 0$.
- If the detector gets excited, it is due to the non-resonant terms.

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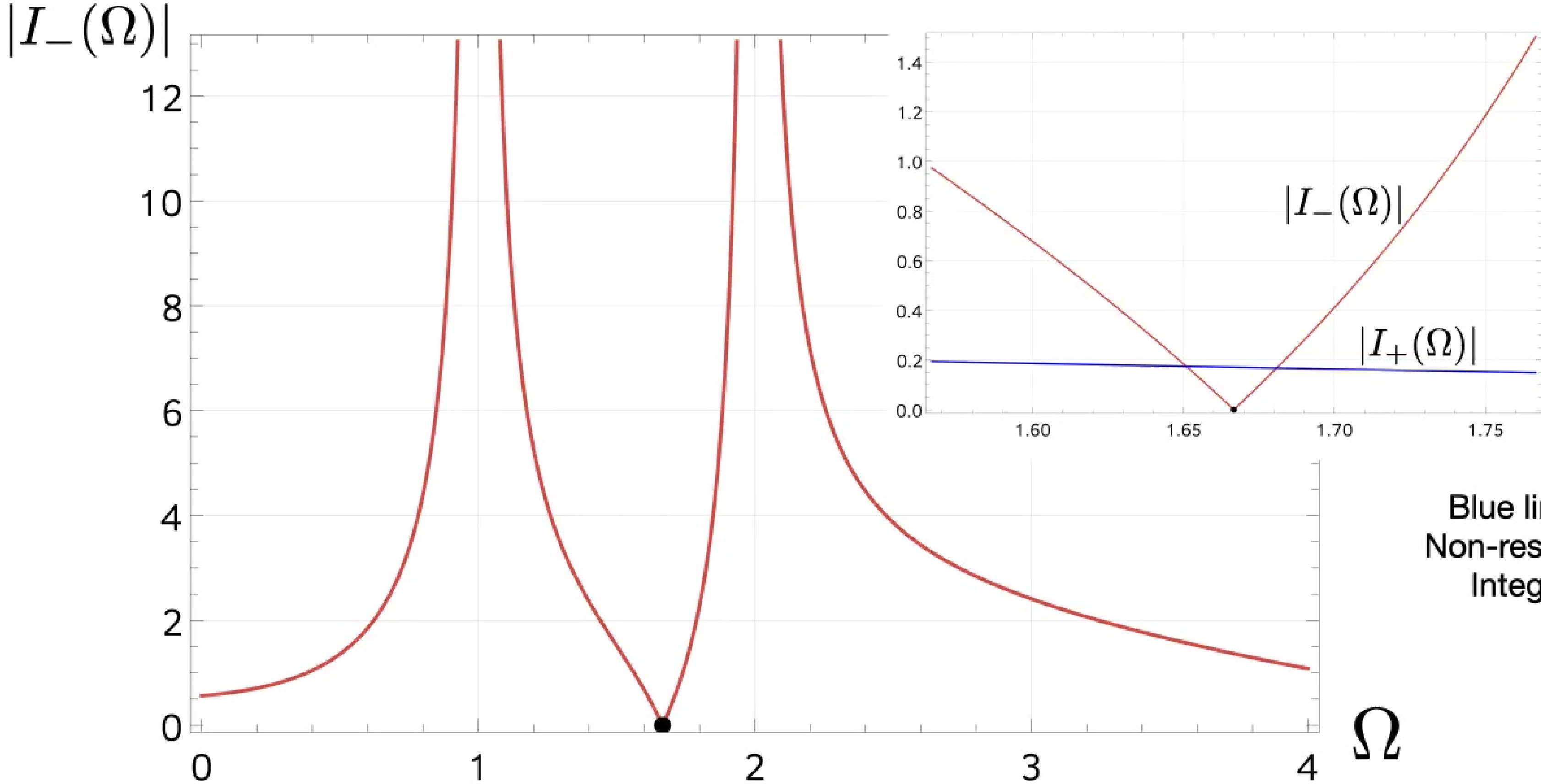
Acceleration-induced transparency



Blue line = Non-resonant Integral

Summary

[1] Also the non-resonant transitions can be stimulated !

Stimulated transitions are n times stronger :

$\sigma^+ a$ $\quad$ $\sigma^+ a^\dagger$

$$p(\mathbf{k}) = \frac{G^2}{(2\pi)^3\omega_{\mathbf{k}}}|I_+(\mathbf{k})|^2 \quad \longrightarrow \quad p(\mathbf{k}) = \frac{G^2}{(2\pi)^3\omega_{\mathbf{k}}}\left(n|I_-(\mathbf{k})|^2 + (n+1)|I_+(\mathbf{k})|^2\right)$$

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[2] Acceleration-induced transparency can suppress the resonant terms !

$\Longrightarrow$

Strong non-resonant terms dominate.

Strong forward (or time-reversed) Unruh effect.

A new method to measure the Unruh effect in lab.

- New effects:

  [1] Stimulated forward and time-reversed Unruh effects,

  [2] Acceleration-induced transparency,

- Combining them leads to improved measurability of the Unruh effect.

Stimulated Unruh effect = $n \times$ Standard Unruh effect

- E.g. for a 100 mW laser pointer: $n \approx 10^{16}$

The new effects suggest for gravity:

- Resonant terms are potentially strongly affected by gravity, including the possibility of gravity-induced translucency or even transparency.

- Non-resonant terms, e.g. of Hawking and other horizon radiation, can be stimulated (e.g. by accretion disk luminosity?).

Paper: *Acceleration-Induced Effects in Stimulated Light-Matter Interactions*, B. Šoda, V. Sudhir, and A. Kempf, *Phys. Rev. Lett.* 128, 163603

Parametrically Induced Decoupling

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Setup: Total Hamiltonian: $H = H_0 + H_{int}(t)$

$$H_{\text{int}} = \mu\left(t\right)\left|v\left(t\right)\right\rangle\left\langle v\left(t\right)\right| .$$ (simple, gives results)

Interaction picture: $H_{\text{int}}\left(t\right) = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t}$

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Setup: Total Hamiltonian: $H = H_0 + H_{int}(t)$

$$H_{\text{int}} = \mu(t) \left| v(t) \right\rangle \left\langle v(t) \right| .$$ (simple, gives results)

Interaction picture: $H_{\text{int}}(t) = e^{iH_0 t} H_{\text{int}} e^{-iH_0 t}$

$$H_0 = \sum_{j=1}^{N} s_j \left| s_j \right\rangle \left\langle s_j \right| \longrightarrow H_{\text{int}}(t) = \mu(t) \sum_{j,k} e^{i(s_j - s_k)t} \langle s_j | v(t) \rangle \langle v(t) | s_k \rangle \left| s_j \right\rangle \left\langle s_k \right| .$$

Parametrically induced decoupling

Time evolution: $U_I(t) = e^{i\int_{t_0}^{t} d\tau H_{\text{int}}(\tau)} \approx \mathbf{1} + i\int_{t_0}^{t} d\tau H_{\text{int}}(\tau) + \ldots,$ (Weak coupling approx.)

Probability of excitation: $p_{\text{exc.}} = |\langle e|U_I(t)|g\rangle|^2$

$$p_{\text{exc.}} = \left|\int_{t_0}^{t} d\tau \mu(\tau)\, e^{i\Omega\tau} v_1(\tau)\, v_0^*(\tau)\right|^2$$ (gap = Ω)

Problem: find $f(t)$ such that $p_{\text{exc.}} = \left|\int_{t_0}^{t} d\tau e^{i\Omega\tau} f(\tau)\right|^2 = 0.$

Parametrically induced decoupling

We already know a solution $f(t)$ - from acceleration-induced transparency.

$$f(\tau) = e^{-ik^{\mu}x_{\mu}(\tau)}$$

There, $p_{\text{exc.}} \propto \left| \int d\tau e^{i\Omega\tau - ik^{\mu}x_{\mu}(\tau)} \right|^2 = 0$, for a suitable choice of $x_{\mu}(\tau)$.

Parametrically induced decoupling

Example. Varying the projector in time. $v_i(t) = e^{if_i(\tau)} v_i^{(0)}$

Parametrically induced decoupling

Example. Varying the coupling in time: $\mu(t)$

Projector(s) constant in time: $|v_1\rangle\langle v_1|\,,|v_2\rangle\langle v_2|$

Summary

Showed two new tools for interactions:

1) Newton's cradle spectra

2) Parametrically induced decoupling

(generalization of acceleration-induced transparency in light-matter int.)

Their purpose:

1) New insights into non-perturbative interactions. (Newton's cradle spectra)

2) A way to turn off (some) transitions using parametric control over

the interaction Hamiltonian.