Title: Non-Fermi liquids and quantum criticality in multipolar Kondo systems

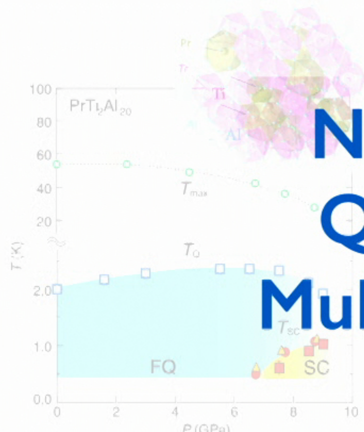
Speakers: Yong-Baek Kim

Collection: Quantum Matter Workshop

Date: November 15, 2022 - 9:30 AM

URL: https://pirsa.org/22110070

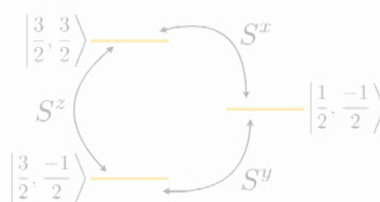
Abstract: We discuss emergent non-Fermi liquid behaviors in multipolar Kondo systems, where conduction electrons interact with the non-Kramers local moments carrying higher-rank multipolar moments such as quadrupolar and octupolar moments. We first show that unexpected non-Fermi liquid states arise in the single impurity multipolar Kondo system using the renormalization group and conformal field theory. Next, we study the competition between the Kondo and RKKY interactions in the Bose-Fermi Kondo systems, where the RKKY interaction between multipolar moments is represented by a bosonic degree of freedom. We present the renormalization group solution of this problem and describe the quantum critical behaviors. If time permits, we also discuss possible superconducting states arising from the multipolar Kondo interactions. We compare the theoretical results with existing experimental data on some cubic f-electron systems.

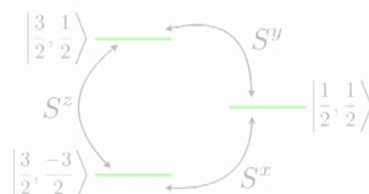


Non-Fermi Liquids and Quantum Criticality in Multipolar Kondo Systems

Yong Baek Kim
University of Toronto

Perimeter Institute
November 15 , 2022





NSERC CRSNG
SIMONS FOUNDATION
John Simon Guggenheim Memorial Foundation

Outline

1. Non-Fermi liquids in multipolar Kondo systems

   Single multipolar local moment coupled to conduction electrons

   (Focus on cubic Pr^{3+} systems)

   RG and Conformal Field Theory

2. Competition with RKKY and quantum criticality

   Bose-Fermi Kondo model and quantum criticality

3. Two-stage Kondo destruction and Fermi surface reconstruction: cubic Ce^{3+} systems

Collaborators

Adarsh Patri
U of Toronto
=> MIT

SangEun Han
U of Toronto

Daniel Schulz
U of Toronto

A. Patri, YBK, PRX (2020), arXiv:2005.08973

A. Patri, I. Khait, YBK, PR Research (2020), arXiv:1904.02717

Kondo problem

S. Han, D. Schultz, YBK, arXiv:2206.02808 (2022)

S. Han, D. Schultz, YBK, arXiv:2207.07661 (2022)

Kondo + RKKY

Multipolar local moments

Many f-electron localized ions carry multipolar moments

Example:

Pr^{3+}

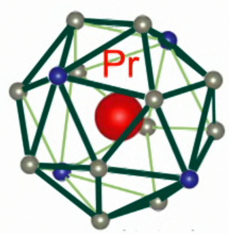
$4f^2$

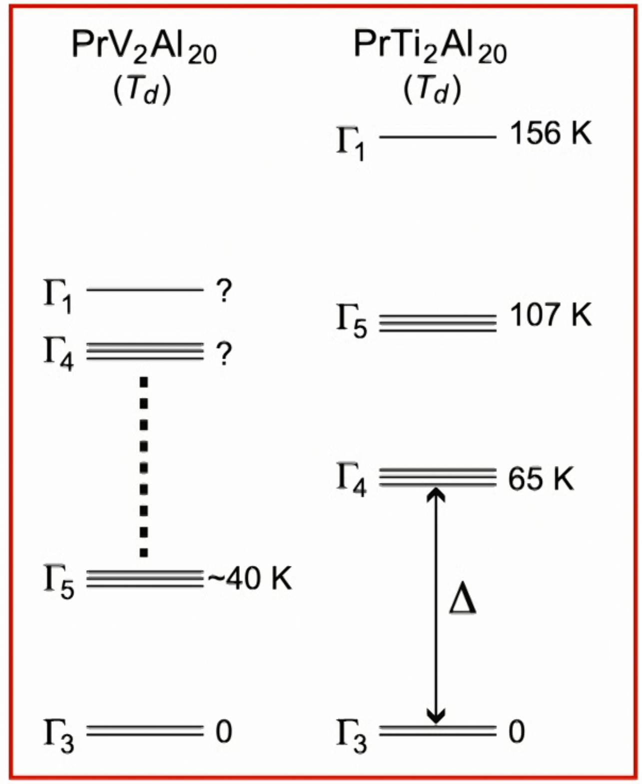
J=4

Crystal Electric Field Splitting

Point Group Symmetry

T_d





Multipolar local moments

Many f-electron localized ions carry multipolar moments

Example:

Pr^{3+}

$4f^2$

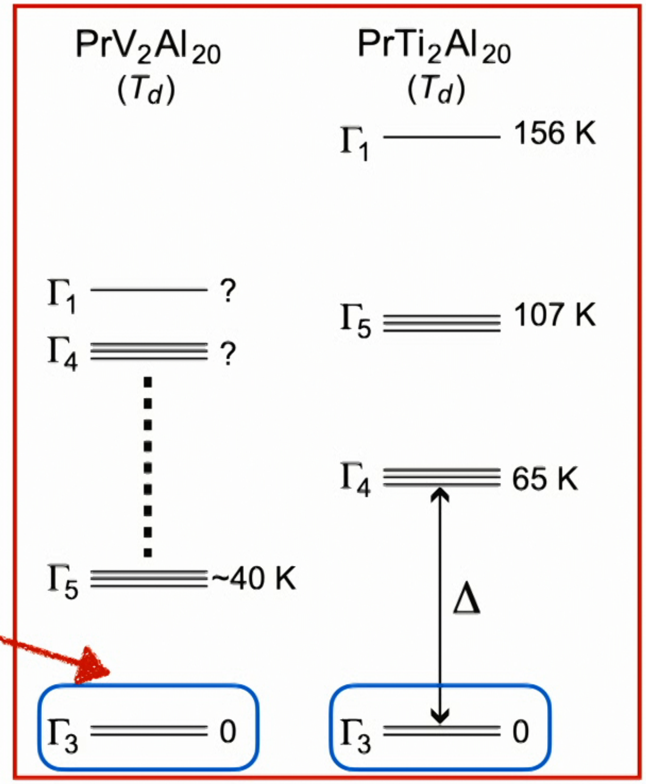
J=4

Crystal Electric Field Splitting

Point Group Symmetry T_d

Non-Kramers Doublet

$\Gamma_3^{(1)}$	$\frac{1}{2}\sqrt{\frac{7}{6}}\|4\rangle - \frac{1}{2}\sqrt{\frac{5}{3}}\|0\rangle + \frac{1}{2}\sqrt{\frac{7}{6}}\|-4\rangle$
$\Gamma_3^{(2)}$	$\sqrt{\frac{1}{2}}\|2\rangle + \sqrt{\frac{1}{2}}\|-2\rangle$



Non-Kramers Doublet: Pr^{3+} $|+\rangle \equiv \Gamma_3^{(1)}$ $|-\rangle \equiv \Gamma_3^{(2)}$

$\langle \pm | J_\alpha | \pm \rangle = 0$ No dipole moment !

But, finite matrix elements for

$O_{22} = \frac{\sqrt{3}}{2}(J_x^2 - J_y^2)$ $O_{20} = \frac{1}{2}(3J_z^2 - J^2)$ Quadrupolar

$T_{xyz} = \frac{\sqrt{15}}{6}\overline{J_x J_y J_z}$ Octupolar

Pseudo-spin basis

$|\uparrow\rangle \equiv \frac{1}{\sqrt{2}}(|\Gamma_3^{(1)}\rangle + i\,|\Gamma_3^{(2)}\rangle)$

$|\downarrow\rangle \equiv \frac{1}{\sqrt{2}}(i\,|\Gamma_3^{(1)}\rangle + |\Gamma_3^{(2)}\rangle)$

$S^x = -\frac{1}{4}\mathcal{O}_{22}$

$S^y = -\frac{1}{4}\mathcal{O}_{20}$

$S^z = \frac{1}{3\sqrt{5}}\mathcal{T}_{xyz}$

$[S^\alpha, S^\beta] = i\epsilon_{\alpha\beta\gamma}S^\gamma$

(S^x, S^y) Quadrupolar — time-reversal even

S^z Octupolar — time-reversal odd

Multipolar local moments with non-Kramers doublet

$PrTi_2Al_{20}$ PrV_2Al_{20}

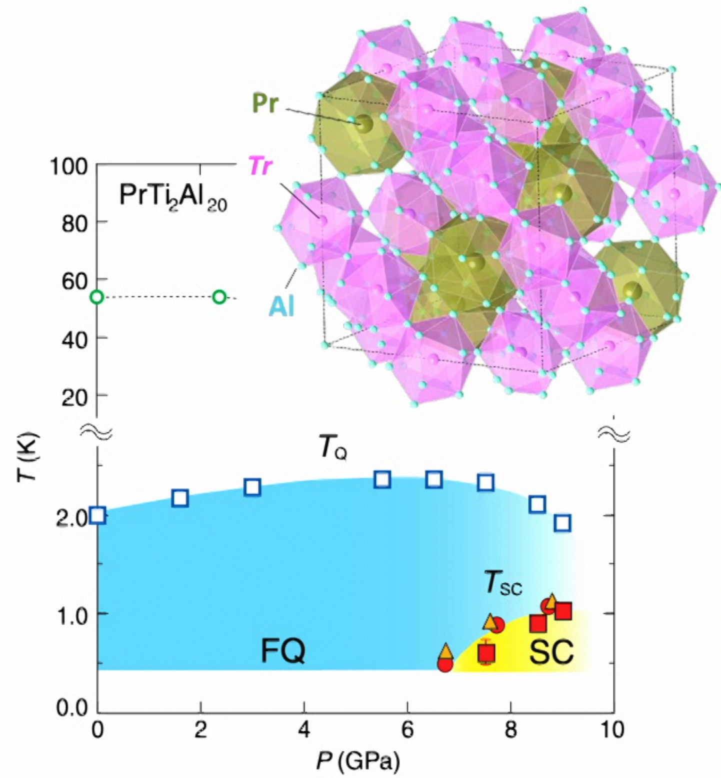
$PrIr_2Zn_{20}$ $PrRh_2Zn_{20}$

$PrNi_2Cd_{20}$ $PrPd_2Cd_{20}$

Pr^{3+} $4f^2$

S. Nakatsuji, A. Sakai,
P. Gegenwart,
T. Onimaru,
B. Maple, …

Pr: Diamond Lattice



Non-Kramers Doublet: Pr^{3+} $|+\rangle \equiv \Gamma_3^{(1)}$ $|-\rangle \equiv \Gamma_3^{(2)}$

Unusual forms of the Kondo and RKKY interaction

First, look at the single impurity Kondo problem !

Pseudo-spin basis

$$|\uparrow\rangle \equiv \frac{1}{\sqrt{2}}(|\Gamma_3^{(1)}\rangle + i\,|\Gamma_3^{(2)}\rangle)$$

$$|\downarrow\rangle \equiv \frac{1}{\sqrt{2}}(i\,|\Gamma_3^{(1)}\rangle + |\Gamma_3^{(2)}\rangle)$$

$$S^x = -\frac{1}{4}\mathcal{O}_{22}$$

$$S^y = -\frac{1}{4}\mathcal{O}_{20}$$

$$S^z = \frac{1}{3\sqrt{5}}\mathcal{T}_{xyz}$$

$$[S^\alpha, S^\beta] = i\epsilon_{\alpha\beta\gamma}S^\gamma$$

(S^x, S^y) Quadrupolar — time-reversal even

S^z Octupolar — time-reversal odd

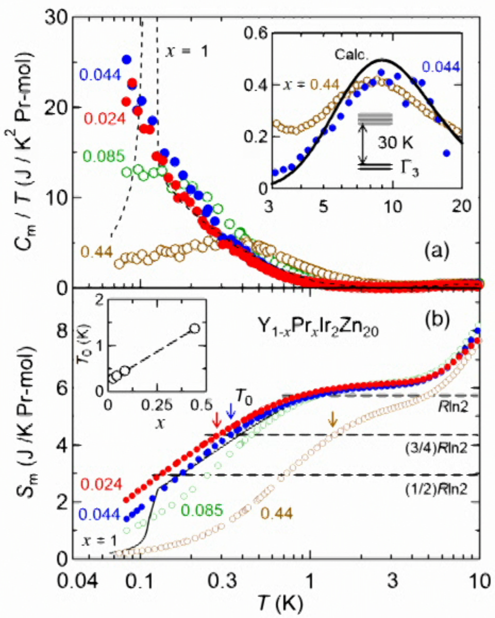
$PrIr_2Zn_{20}$ → Dilution → $Y_{1-x}Pr_xIr_2Zn_{20}$

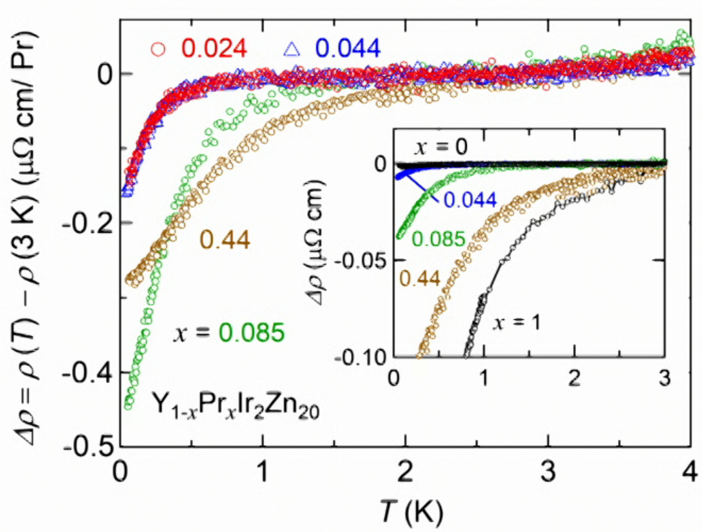
Quadrupolar ordering → Non-Fermi liquid

PHYSICAL REVIEW LETTERS **121,** 077206 (2018)

Single-Site Non-Fermi-Liquid Behaviors in a Diluted $4f^2$ System $Y_{1-x}Pr_xIr_2Zn_{20}$

Y. Yamane,[1] T. Onimaru,[1,*] K. Wakiya,[1,†] K. T. Matsumoto,[1,‡] K. Umeo,[2] and T. Takabatake[1]
[1]Graduate School of Advanced Sciences of Matter, Hiroshima University, Higashi-Hiroshima 739-8530, Japan
[2]Cryogenic and Instrumental Analysis Division, N-BARD, Hiroshima University, Higashi-Hiroshima 739-8526, Japan





How to model conduction electrons ?

Local symmetry around Pr ion is T_d

Fermi pockets around
zone center
(quantum oscillations, S. Nagashima et al, 2014)

Conduction electron orbitals can be
classified in terms of
irreducible representation of T_d

A_1	$xyz,\ x^4+y^4+z^4,\ x^2y^2z^2$
A_2	$x^4(y^2-z^2)+y^4(z^2-x^2)+z^4(x^2-y^2)$
E	$\{x^2-y^2,\ 2z^2-x^2-y^2\}$
T_1	$\{x(z^2-y^2),\ y(z^2-x^2),\ z(x^2-y^2)\}$
T_2	$\{x,\ y,\ z\},\ \{xy,\ xz,\ yz\},$

The "Kondo" model

S^x, S^y, S^z couple to fermion spin-orbital 'currents' that transform exactly in the same way

For example,

$$H_3 = K_3 S^z \Big[\sigma^x_{\alpha\beta}(c^\dagger_{p_y,\alpha} c_{p_z,\beta} + h.c.) + \sigma^y_{\alpha\beta}(c^\dagger_{p_z,\alpha} c_{p_x,\beta} + h.c.) + \sigma^z_{\alpha\beta}(c^\dagger_{p_x,\alpha} c_{p_y,\beta} + h.c.) \Big]$$

$\alpha, \beta = \uparrow, \downarrow$

K_1, K_2, K_3 three independent couplings

Entangled fluctuations of both orbital and spin !

Perform RG computations up to 3rd order:
two stable fixed points

A. Patri, I. Khait, YBK, PR Research 2, 013257 (2020)

The "Kondo" model

S^x, S^y, S^z couple to fermion spin-orbital 'currents' that transform exactly in the same way

For example,

$$H_3 = K_3 S^z \Big[\sigma^x_{\alpha\beta}(c^\dagger_{p_y,\alpha} c_{p_z,\beta} + h.c.) + \sigma^y_{\alpha\beta}(c^\dagger_{p_z,\alpha} c_{p_x,\beta} + h.c.) + \sigma^z_{\alpha\beta}(c^\dagger_{p_x,\alpha} c_{p_y,\beta} + h.c.) \Big]$$

$\alpha, \beta = \uparrow, \downarrow$

K_1, K_2, K_3 three independent couplings

L=1, S=1/2 → $|j, j_z\rangle = |j = 3/2, j_z\rangle,\ |j = 1/2, j_z\rangle$

$\vert p_x, \uparrow\rangle$	$\vert p_x, \downarrow\rangle$
$\vert p_y, \uparrow\rangle$	$\vert p_y, \downarrow\rangle$
$\vert p_z, \uparrow\rangle$	$\vert p_z, \downarrow\rangle$

Spin-orbital entangled basis

$\vert 3/2, 3/2\rangle$	$\vert 3/2, 1/2\rangle$
$\vert 3/2, -1/2\rangle$	$\vert 3/2, -3/2\rangle$
$\vert 1/2, 1/2\rangle$	$\vert 1/2, -1/2\rangle$

Fixed point I

Only j=3/2 electrons are involved

$$H_{\text{tot}} = S^x\left(\tau_A^x + \tau_B^x\right) - S^y\left(\tau_A^z + \tau_B^z\right) + S^z\left(\tau_A^y + \tau_B^y\right)$$

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle \quad \left|\frac{3}{2}, \frac{-1}{2}\right\rangle$$

$$(\uparrow_A, \quad \downarrow_A)$$

conduction electron pseudospin for channel A $\quad \vec{\tau}_A$

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle \quad \left|\frac{3}{2}, \frac{-3}{2}\right\rangle$$

$$(\uparrow_B, \quad \downarrow_B)$$

conduction electron pseudospin for channel B $\quad \vec{\tau}_B$

A. Patri, YBK, PRX, arXiv:2005.08973 (2020)

Fixed point I Two channel Kondo model !

Only j=3/2 electrons are involved

$$H_{\text{tot}} = S^x \left(\tau_A^x + \tau_B^x\right) - S^y \left(\tau_A^z + \tau_B^z\right) + S^z \left(\tau_A^y + \tau_B^y\right)$$

$\left|\frac{3}{2}, \frac{3}{2}\right\rangle \quad \left|\frac{3}{2}, \frac{-1}{2}\right\rangle$

$(\uparrow_A, \quad \downarrow_A)$

conduction electron pseudospin for channel A $\quad \vec{\tau}_A$

$\left|\frac{3}{2}, \frac{1}{2}\right\rangle \quad \left|\frac{3}{2}, \frac{-3}{2}\right\rangle$

$(\uparrow_B, \quad \downarrow_B)$

conduction electron pseudospin for channel B $\quad \vec{\tau}_B$

A. Patri, YBK, PRX, arXiv:2005.08973 (2020)

Fixed point I   Two channel Kondo model !

Only j=3/2 electrons are involved

$$H_{\text{tot}} = S^x\left(\tau_A^x + \tau_B^x\right) - S^y\left(\tau_A^z + \tau_B^z\right) + S^z\left(\tau_A^y + \tau_B^y\right)$$

$\left|\frac{3}{2},\frac{3}{2}\right\rangle \quad \left|\frac{3}{2},\frac{-1}{2}\right\rangle$

$(\uparrow_A, \quad \downarrow_A)$

conduction electron pseudospin for channel A $\quad \vec{\tau}_A$

$\left|\frac{3}{2},\frac{1}{2}\right\rangle \quad \left|\frac{3}{2},\frac{-3}{2}\right\rangle$

$(\uparrow_B, \quad \downarrow_B)$

conduction electron pseudospin for channel B $\quad \vec{\tau}_B$

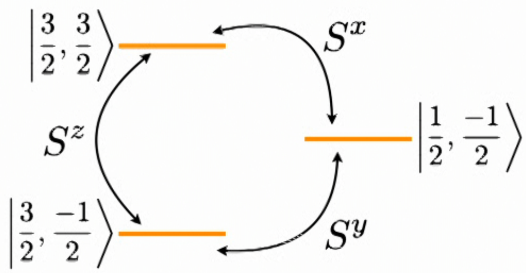
$C_v \propto T\ \ln T$

$\rho \propto \sqrt{T}$

A. Patri, YBK, PRX, arXiv:2005.08973 (2020)

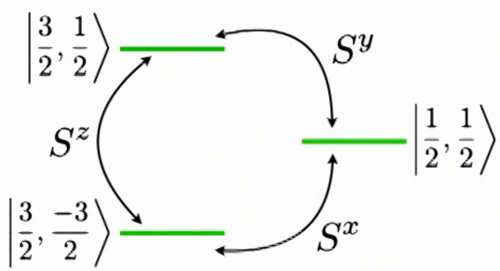
Fixed point II

All of j=3/2 and j=1/2 electrons are involved



Two channels

But each channel involves
three flavors of conduction electrons - SU(3) 'current'



This is not the usual Kondo problem

A. Patri, YBK, PRX, arXiv:2005.08973 (2020)

Fixed point II

From the perturbative RG

Approaching the fixed point by fixing the ratios of the couplings

$$K_1 = -\frac{1}{2\sqrt{6}}g_k,\ K_2 = -\frac{1}{12\sqrt{2}}g_k,\ K_3 = -\frac{1}{4\sqrt{3}}g_k$$

$$\frac{dg_k}{dl} = \frac{g_k^2}{4} - \frac{g_k^3}{4} \approx -\frac{1}{4}(g_k - g_k^*)$$

$g_k^* = 1$ at the fixed point

$1+\Delta$ dimension of the leading irrelevant operator

$\Delta = 1/4$ $\quad C_v \propto T^{2\Delta} = T^{1/2}$ $\quad \rho \propto T^{\Delta} = T^{1/4}$

Fixed point II

True nature of this new fixed point ?

Rewriting the Kondo coupling

$$H_k = g_k \sum_{m=1,2} \vec{\psi}^{\dagger}_m(0) \left[\frac{S^x}{2}\frac{\lambda^4}{2} + \frac{S^y}{2}\frac{\lambda^6}{2} + \frac{S^z}{2}\frac{\lambda^2}{2} \right] \vec{\psi}_m(0)$$

$\lambda^a \qquad a = 1, ..., 8$ **3x3 SU(3) Gell-Mann matrices**

$$\vec{\psi}^{\dagger}_{m=1} = \left(-c^{\dagger}_{\frac{3}{2},\frac{3}{2}}, -c^{\dagger}_{\frac{3}{2},\frac{-1}{2}}, c^{\dagger}_{\frac{1}{2},\frac{-1}{2}} \right)$$

$$\vec{\psi}^{\dagger}_{m=2} = \left(c^{\dagger}_{\frac{3}{2},-\frac{3}{2}}, c^{\dagger}_{\frac{3}{2},\frac{1}{2}}, c^{\dagger}_{\frac{1}{2},\frac{1}{2}} \right)$$

Fixed point II

True nature of this new fixed point ?

Rewriting the Kondo coupling

$$H_k = g_k \sum_{m=1,2} \vec{\psi}^\dagger_m(0) \left[\frac{S^x}{2}\frac{\lambda^4}{2} + \frac{S^y}{2}\frac{\lambda^6}{2} + \frac{S^z}{2}\frac{\lambda^2}{2} \right] \vec{\psi}_m(0)$$

$\lambda^a \qquad a = 1, ..., 8$ **3x3 SU(3) Gell-Mann matrices**

$$\vec{\psi}^\dagger_{m=1} = \left(-c^\dagger_{\frac{3}{2},\frac{3}{2}}, -c^\dagger_{\frac{3}{2},\frac{-1}{2}}, c^\dagger_{\frac{1}{2},\frac{-1}{2}} \right)$$

$$\vec{\psi}^\dagger_{m=2} = \left(c^\dagger_{\frac{3}{2},-\frac{3}{2}}, c^\dagger_{\frac{3}{2},\frac{1}{2}}, c^\dagger_{\frac{1}{2},\frac{1}{2}} \right)$$

$g_k^* = 1$

at the perturbative fixed point

Fixed point II

True nature of this new fixed point ?

Rewriting the Kondo coupling

$$H_k = g_k \sum_{m=1,2} \vec{\psi}^\dagger_m(0) \left[\frac{S^x}{2}\frac{\lambda^4}{2} + \frac{S^y}{2}\frac{\lambda^6}{2} + \frac{S^z}{2}\frac{\lambda^2}{2} \right] \vec{\psi}_m(0)$$

$\lambda^a \qquad a = 1, ..., 8$ **3x3 SU(3) Gell-Mann matrices**

$$\vec{\psi}^\dagger_{m=1} = \left(-c^\dagger_{\frac{3}{2},\frac{3}{2}}, -c^\dagger_{\frac{3}{2},\frac{-1}{2}}, c^\dagger_{\frac{1}{2},\frac{-1}{2}} \right)$$

$$\vec{\psi}^\dagger_{m=2} = \left(c^\dagger_{\frac{3}{2},-\frac{3}{2}}, c^\dagger_{\frac{3}{2},\frac{1}{2}}, c^\dagger_{\frac{1}{2},\frac{1}{2}} \right)$$

$g^*_k = 1$

at the perturbative fixed point

This is not an SU(3) Kondo problem …

Current Algebra and Conformal Field Theory (Results)

Useful to generalize this to *k* channels Here $k = 2$

$$g_k^* = \frac{2}{k+3} \qquad \Delta = \frac{2}{4k+2}$$

$1 + \Delta$ dimension of the leading irrelevant operator

$$C_v \propto T^{2\Delta} \qquad \rho \propto T^{\Delta}$$

Exact $k = 2$

$$g_k^* = 2/5$$

$$\Delta = 1/5$$

$$C_v/T \propto T^{-3/5}$$

$$\rho \propto T^{1/5}$$

A. Patri, YBK, PRX, arXiv:2005.08973 (2020)

Current Algebra and Conformal Field Theory (Results)

Useful to generalize this to k channels   Here $k = 2$

$$g_k^* = \frac{2}{k+3} \qquad \Delta = \frac{2}{4k+2}$$

$1 + \Delta$ dimension of the leading irrelevant operator

$$C_v \propto T^{2\Delta} \qquad \rho \propto T^{\Delta}$$

Perturbative regime $k \gg 1$

$$g_k^* \to 2/k \qquad \Delta \to 1/2k$$

$$k = 2 \qquad g_k^* \to 1 \qquad \Delta \to 1/4$$

recover the perturbative results !

Exact $k = 2$

$$g_k^* = 2/5$$

$$\Delta = 1/5$$

$$C_v/T \propto T^{-3/5}$$

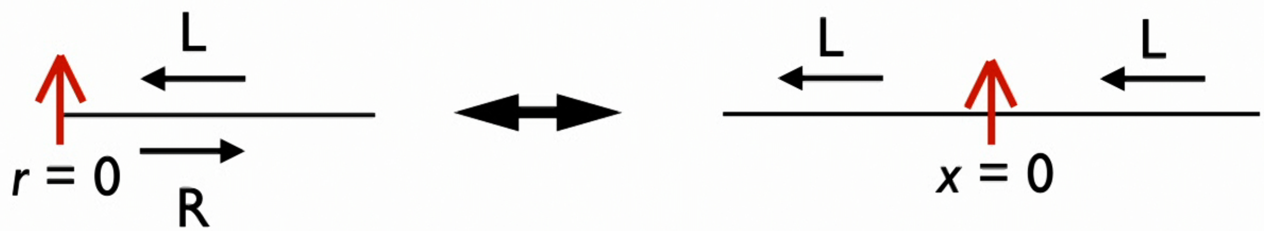
$$\rho \propto T^{1/5}$$

A. Patri, YBK, PRX, arXiv:2005.08973 (2020)

Current Algebra and Conformal Field Theory Solution (Schematic)

Consider δ-function interaction and s-wave scattering

Mapping to 1+1 dimensional (chiral) model



$$\psi_L(x,\tau) = \psi_L(z = \tau + ix)$$

$$\psi_R(x,\tau) = \psi_R(z^* = \tau - ix)$$

$$\psi_L(0,\tau) = \psi_R(0,\tau)$$

$$\psi_R(x,\tau) \equiv \psi_L(-x,\tau)$$

Current Algebra and Conformal Field Theory Solution (Schematic)

Bulk free theory (before coupling to impurity)

Current algebra and conformal embedding

$$\mathrm{U}(1)\times\mathrm{SU}(3)_2\times\mathrm{SU}(2)_3$$

Charge Spin-Orbital Flavor(channel)

3 = three spin-orbital bands in each channel

2 = two channels

A. Patri, YBK, arXiv:2005.08973 (2020)

Current Algebra and Conformal Field Theory Solution (Schematic)

Bulk free theory (before coupling to impurity)

Current algebra and conformal embedding

$$\mathrm{U}(1)\times\mathrm{SU}(3)_2\times\mathrm{SU}(2)_3$$

Charge Spin-Orbital Flavor(channel)

3 = three spin-orbital bands in each channel

2 = two channels

$$H_0 = \frac{1}{12} : JJ : (z) + \frac{1}{5} : J^a J^a : (z) + \frac{1}{5} : J^A J^A : (z)$$

Charge Spin-Orbital Flavor(channel)

$a = 1, ..., 8$ $A = 1, ..., 3$

A. Patri, YBK, arXiv:2005.08973 (2020)

Current Algebra and Conformal Field Theory Solution (Schematic)

Bulk free theory (before coupling to impurity)

Current algebra and conformal embedding

$$\mathrm{U}(1)\times\mathrm{SU}(3)_2\times\mathrm{SU}(2)_3$$

Charge Spin-Orbital Flavor(channel)

But only three of eight generators of SU(3) are coupled to the impurity

$$H_K = g_k \left(J^4(z)\frac{S^x}{2} + J^6(z)\frac{S^y}{2} + J^2(z)\frac{S^z}{2} \right)$$ Kondo coupling

We need a different conformal embedding …

A. Patri, YBK, arXiv:2005.08973 (2020)

Current Algebra and Conformal Field Theory Solution (Schematic)

$\mathrm{U}(1)\times\mathrm{SU}(3)_2\times\mathrm{SU}(2)_3$ Bulk free theory

Charge Spin-Orbital Flavor(channel) Conformal embedding

Coset construction

$$\mathrm{SU}(3)_2 = [\text{3-state Potts model}] \times \widetilde{\mathrm{SU}}(2)_8$$

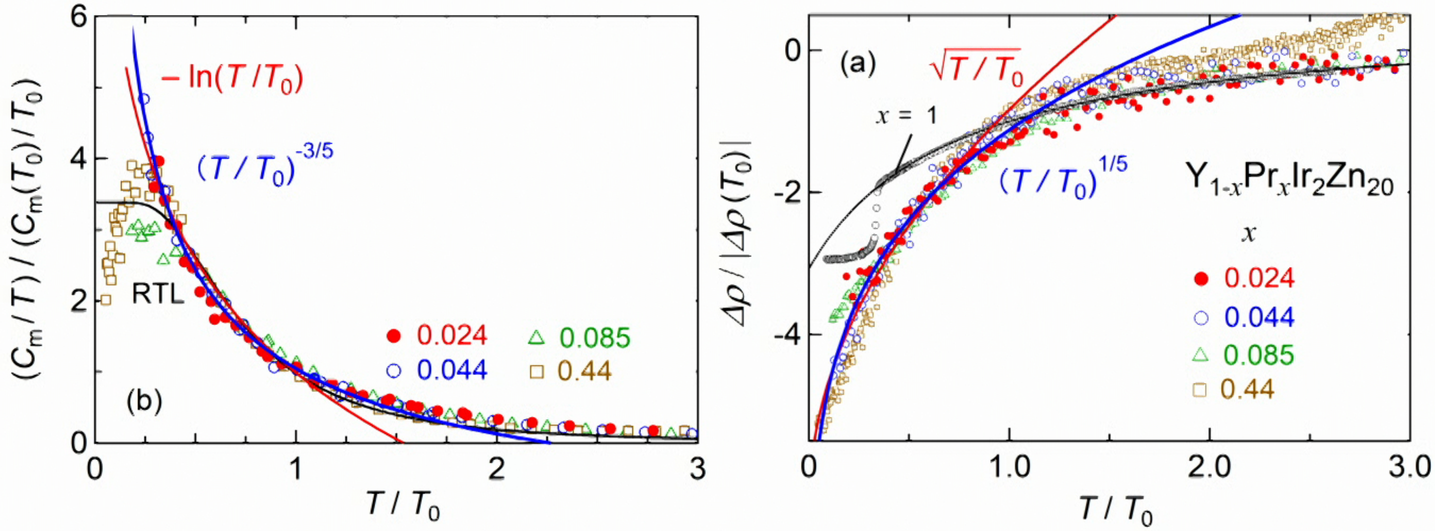
$$J^2, J^4, J^6 \in \widetilde{\mathrm{SU}}(2)_8$$

Kondo coupling imposes (conformally invariant) boundary conditions on the currents

↓ Boundary CFT (John Cardy)
Affleck + Ludwig (1991)

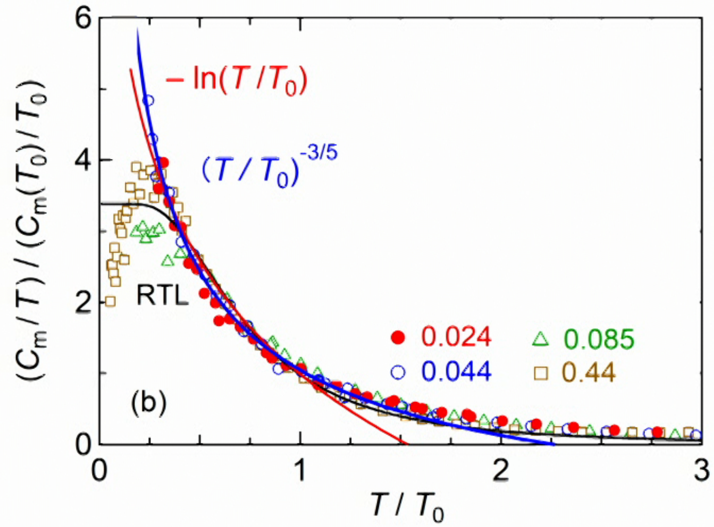
Rearrangement of the conformal towers

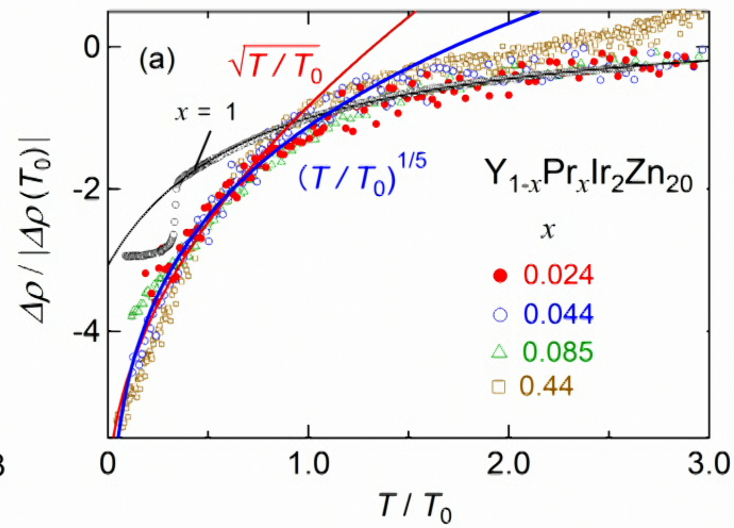
Experiments ?



Y. Yamane, T. Onimaru (2018, 2020)

Experiments ?





Y. Yamane, T. Onimaru (2018, 2020)

The fits are encouraging ! We need sharper predictions …

Need more systematic understanding as to
Quadrupolar ordering (non-diluted) → "Kondo" regime (diluted)

Competition with RKKY: Bose-Fermi Kondo model

$$H = H_{\text{Fermi Kondo}} + H_{\text{Bose Kondo}}$$

$$H_{\text{RKKY}} = \sum_{ij} \left[J_{ij}^{Q} (S_i^x S_j^x + S_i^y S_j^y) + J_{ij}^{O} S_i^z S_j^z \right]$$

$$\rightarrow \quad H_{\text{Bose Kondo}} = g_Q (S^x \phi_0^x + S^y \phi_0^y) + g_O S^z \phi_0^z + \sum_{\mathbf{k}} \left[\Omega_{Q\mathbf{k}} (\phi_{\mathbf{k}}^{x\dagger} \phi_{\mathbf{k}}^x + \phi_{\mathbf{k}}^{y\dagger} \phi_{\mathbf{k}}^y) + \Omega_{O\mathbf{k}} \phi_{\mathbf{k}}^{z\dagger} \phi_{\mathbf{k}}^z \right]$$

Boson density of states

$$D_i(\omega) = C_i |\omega|^{1-\epsilon_i} \text{sgn}(\omega) \qquad i = Q, O$$

This is a local approximation of the RKKY interaction in the spirit of DMFT

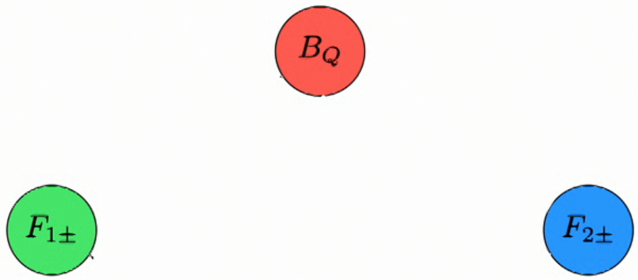
J. L. Smith, Q. Si (96,97), A. M. Sengupta (97)

S. Han, D. Schultz, YBK, arXiv:2206.02808 (2022)

Renormalization Group Analysis

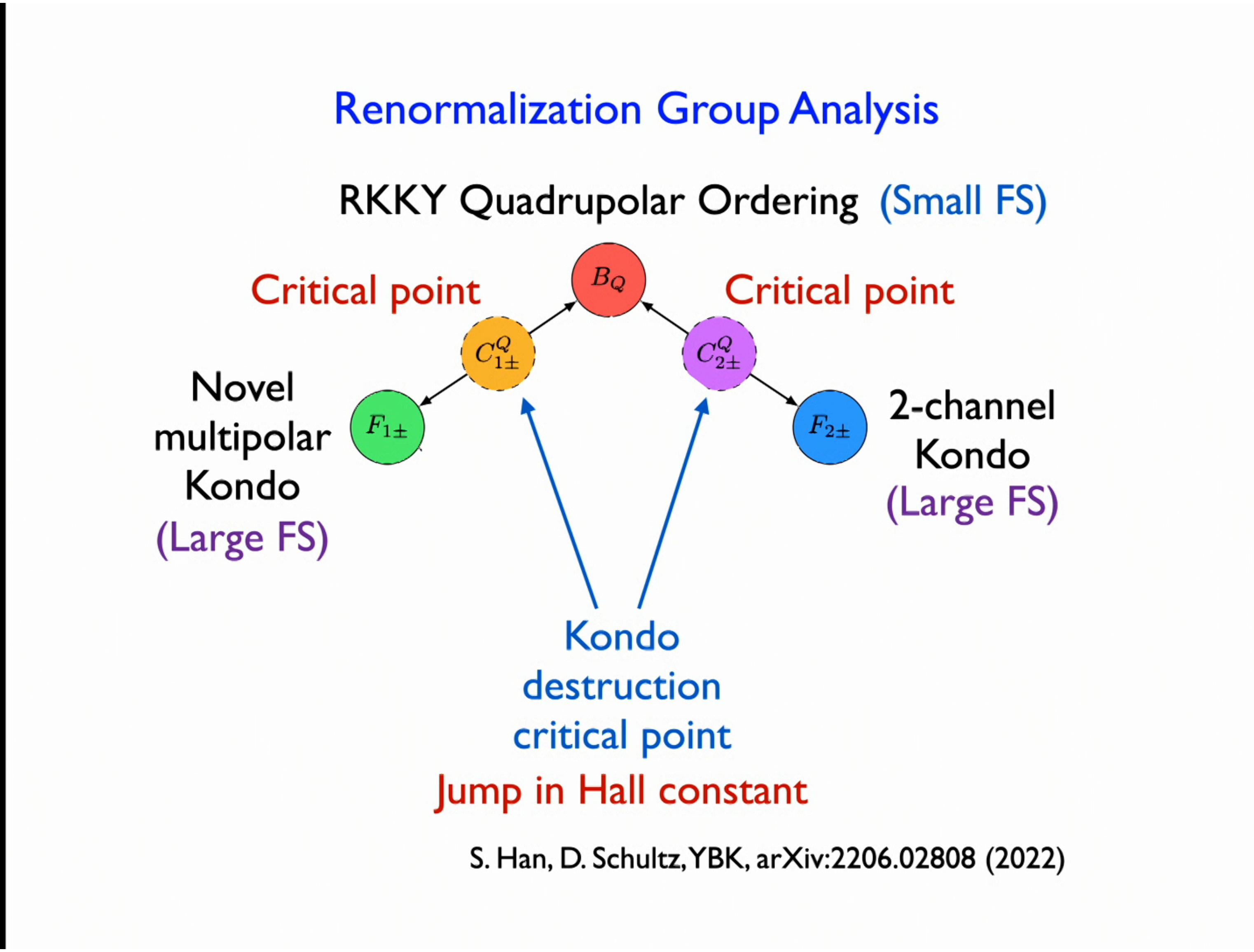
RKKY Quadrupolar Ordering (Small FS)

Novel multipolar Kondo (Large FS)

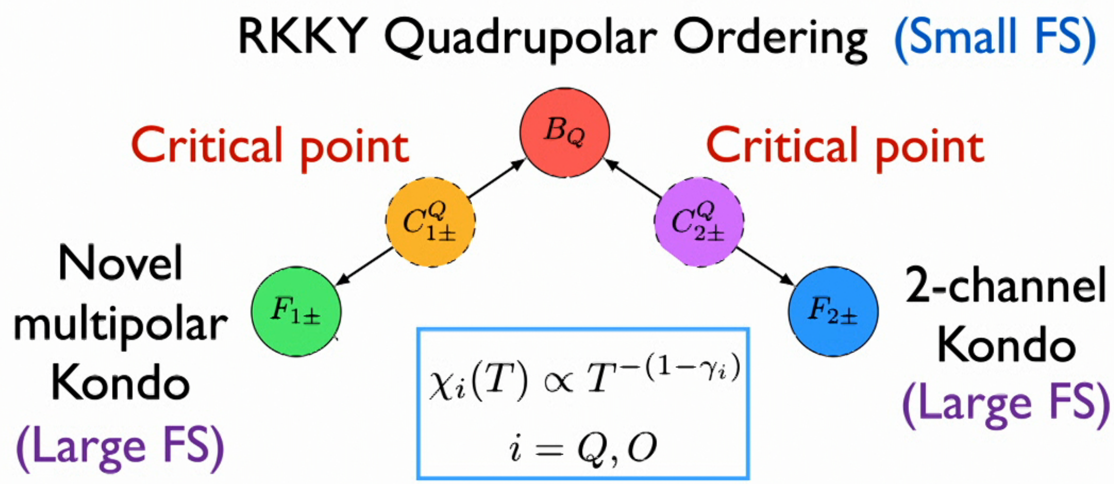


2-channel Kondo (Large FS)

S. Han, D. Schultz, YBK, arXiv:2206.02808 (2022)


Renormalization Group Analysis
RKKY Quadrupolar Ordering (Small FS)
B_Q
Critical point
Critical point
$C^Q_{1\pm}$
$C^Q_{2\pm}$
Novel multipolar Kondo (Large FS)
$F_{1\pm}$
$F_{2\pm}$
2-channel Kondo (Large FS)
Kondo destruction critical point
Jump in Hall constant
S. Han, D. Schultz, YBK, arXiv:2206.02808 (2022)

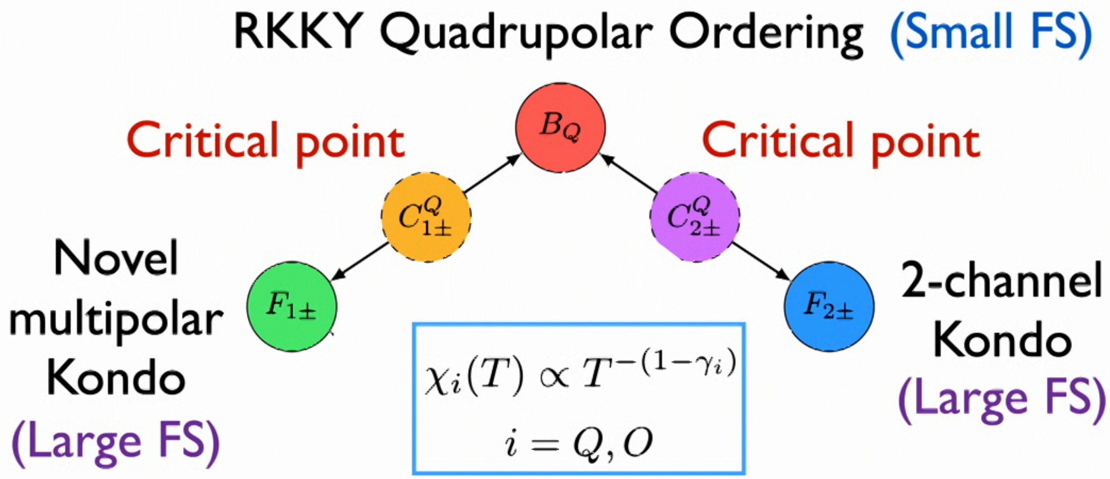
Renormalization Group Analysis



Quadrupolar susceptibility $\chi^Q(\tau) = \langle T_\tau S^{x,y}(\tau) S^{x,y}(0) \rangle \propto \left(\frac{\tau_0}{|\tau|} \right)^{\gamma_Q}$

Octupolar susceptibility $\chi^O(\tau) = \langle T_\tau S^{z}(\tau) S^{z}(0) \rangle \propto \left(\frac{\tau_0}{|\tau|} \right)^{\gamma_O}$

Renormalization Group Analysis



$$\chi_i(T) \propto T^{-(1-\gamma_i)}$$
$$i = Q, O$$

Quadrupolar susceptibility $\chi^Q(\tau) = \langle T_\tau S^{x,y}(\tau) S^{x,y}(0)\rangle \propto \left(\frac{\tau_0}{|\tau|}\right)^{\gamma_Q}$

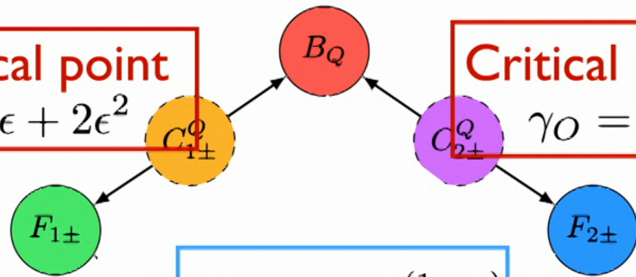
Octupolar susceptibility $\chi^O(\tau) = \langle T_\tau S^{z}(\tau) S^{z}(0)\rangle \propto \left(\frac{\tau_0}{|\tau|}\right)^{\gamma_O}$

Renormalization Group Analysis

RKKY Quadrupolar Ordering (Small FS)

Critical point
$\gamma_O = 2\epsilon + 2\epsilon^2$

Critical point
$\gamma_O = 2\epsilon + 3\epsilon^2/2$



Novel multipolar Kondo (Large FS)

2-channel Kondo (Large FS)

$$\chi_i(T) \propto T^{-(1-\gamma_i)}$$
$$i = Q, O$$

Measurement via elastic constants (ultrasound)

$$(C_{11} - C_{12}) = (C^0_{11} - C^0_{12}) - (s^2_Q)\chi'_Q,$$
$$C_{44} = C^0_{44} - (s^2_O h^2)\chi'_O,$$